Robustness of a semiparametric estimator of a copula

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Abstract

Robust properties of a semiparametric method for estimating the parameter of a copula are investigated using a simulation study, and compared with the maximum likelihood estimator[MLE] and an estimator based on the Inference Function Method [IFM]. The semiparameric method estimates the marginal distributions nonparametrically and hence the form of the marginal distribution need not be known. By contrast, MLE and IFM require the exact form of the marginal distributions; it is reasonable to expect that incorrect specification of the marginal distribution would almost certainly lead to inconsistent estimators. The simulation results show that, when the marginal distributions are correctly specified as normal for MLE and IFM, the semiparametric method is slightly less efficient than the MLE and IFM. However, if the marginal distributions are incorrectly specified as normal for MLE and/or IFM, the semiparametric method is considerably better than the MLE and IFM. Based on these results, the semiparametric estimator appears to be an excellent competitor to, if not better than, the MLE and IFM for estimating the parameter of the copula.

JEL Classification: C13, C14, C32

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1 Introduction

There has been a growing interest in modelling multivariate observations using flexible functional forms for distribution functions and in estimating parameters that capture dependence between the different components of the vector of variables. It has been recognized that the traditional approach based on multivariate normal distribution is limited in scope because it can capture only a very limited range of distributional shapes. Further, it is also known that the maximum likelihood method based on multivariate normal is sensitive to departures away from multinormal. Thus, there is a genuine need for methods of modelling multivariate data that address the flexibility and robustness issues. In response to the demand for flexibility, methods based on *copula* have been the subject of extensive study in the recent literature; for example see the recent books Joe (1997) and Nelson (1999).

In what follows, we shall restrict our discussion to bivariate observations only for simplicity. Let (X_1, X_2) be a continuous bivariate random variable, $H(x_1, x_2)$ denote the *cdf* of (X_1, X_2), and let F_k and f_k denote the marginal *cdf* and *pdf* respectively of X_k, $(k = 1, 2)$. Then, a well-known result says (for example, see Joe 1997) that there is a unique function $C(u_1, u_2)$, termed the copula, such that

$$H(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\}. \tag{1}$$

It turns out that the copula C is the joint distribution of (U_1, U_2) where $U_k = F_k(X_k), k = 1, 2$; clearly, U_1 and U_2 are uniformly distributed on $(0, 1)$. Thus, any continuous bivariate distribution is uniquely defined by its marginal distributions and its copula; conversely, given the marginal distributions and the copula, there is a unique bivariate distribution with the same marginal distributions and copula. This suggests the possibility of estimating the marginal distributions and the copula separately. In fact, this flexibility has played an important role for the recent interest in copulas. For example, it is possible to specify a gamma distribution for X_1, a t-distribution for X_2, and a copula to capture the joint behaviour of the two variables. The shapes of the marginal distributions of X_1 and X_2 do not play a role in the specification of the copula. For example, if $Y_k = h_k(X_k)$ where h_k is continuous and increasing $(i = 1, 2)$, then the copula of (X_1, X_2) is the same as that for (Y_1, Y_2). Thus, the copula captures features that are invariant under monotonic transformations of the marginal variables. Features that are not invariant under such transformations would be captured by the marginal distributions. Thus, copulas offer a flexible approach to modelling multivariate observations. In this setting, one's interest may be on the complete joint distribution of (X_1, X_2), or on the copula with the marginal distribution being a nuisance function. In this paper, we are interested in the latter.

Several approaches to estimating copulas, including maximum likelihood, estimating equation, semi-parametric and nonparametric methods have been suggested. In this paper we evaluate the performance of a semi-parametric method introduced by Genest, Ghoudi and Rivest (1995). An attractive feature of this method is that it estimates the marginal distributions nonparametrically by the empirical distribution function [*edf*], thus allowing the distribution of the marginals to be quite free and not restricted by parametric families. Once this is done, the interdependence between the margins is estimated using a parametric family of copulas. This approach is particularly suitable for our purposes because by estimating the nuisance function, namely the marginal distributions, nonparametrically the validity of the estimator of the copula would not be compromised due to possible misspecification of the marginal distribution functions. In this paper, we evaluate the performance, including robustness, of the semi-parametric estimator for a range of realistic settings. Our simulations studies suggest that the semi-parametric method has excellent robustness properties. For example, the method of maximum likelihood is slightly better than the semiparametric method if the full likelihood, which includes the marginal distributions as well, is correctly specified. Otherwise, which is likely to be the case in most practical situations, the semiparametric method is substantially better.

2 Specification and estimation of copulas

Let (X_1, X_2) denote a continuous bivariate random variable, $F_k(x; \alpha_k)$ and $f_k(x_k; \alpha_k)$ be the *cdf* and *pdf* respectively of X_k, $U_k = F_k(X_k; \alpha_k)$, $C(x_1, x_2; \theta)$ denote the copula, $c(x_1, x_2; \theta)$ denote the *pdf* corresponding to $C(x_1, x_2; \theta)$, $\xi = (\alpha_1', \alpha_2', \theta')'$ and $H(x_1, x_2; \xi)$ and $h(x_1, x_2; \xi)$ denote the *cdf* and *pdf* of (X_1, X_2) respectively. The parameters α_1 and α_2 may be vectors; further, for the most part, we consider the case when θ is a scalar, although an extension to the vector case would be obvious. In this paper, we are interested in estimating θ; thus α_1 and α_2 are treated as nuisance parameters. Let us first mention briefly the methods of estimating θ that are considered here.

2.1 Maximum likelihood

In view of (1) the joint density function $h(x_1, x_2; \xi)$ of (X_1, X_2) can be expressed as follows:

$$h(x_1, x_2; \xi) = c\{F_1(x_1; \alpha_1), F_2(x_2; \alpha_2); \theta\} f_1(X_1; \alpha_1) f_2(X_2; \alpha_2). \tag{2}$$

Let (X_{1i}, X_{2i}), $i = 1, \ldots, n$, be n *iid* observations on (X_1, X_2). Therefore, the loglikelihood function takes the form

$$L(\xi) = \sum_{i=1}^{n} \log[c\{F_1(X_{1i}, \alpha_1), F_2(X_{2i}; \alpha_2); \theta\} f_1(X_{1i}; \alpha_1) f_2(X_{2i}; \alpha_2)]. \tag{3}$$

Hence the maximum likelihood estimator [*MLE*] of ξ, which we denote by ξ^{**} is the global maximizer of $L(\xi)$. Then, we have that $\sqrt{n}(\xi^{**} - \xi_0)$ converges to a normal distribution with mean zero, where ξ_0 is the true value. If the model is correctly specified so that $L(\xi)$ is the correct loglikelihood, then as a general rule, the *MLE* is the preferred first option, at least in large samples.

2.2 Inference function method [IFM]

In this method, the parameters are estimated in two stages. In the first stage, α_k is estimated using $X_{k1} \ldots, X_{kn}$, and let the estimator be denoted by $\hat{\alpha}_k$ $(k = 1, 2)$. Then, in the second stage, θ is estimated with $F_k(x_k; \hat{\alpha}_k)$ being treated as the true distribution of X_k ($k = 1, 2$). While there are several ways of implementing such a method, the one that is adopted here is to substitute $\hat{\alpha}_k$ for α_k in the loglikelihood. Thus, the IFM estimate of θ is the maximizer of

$$\sum_{i=1}^{n} \log[c\{F_1(X_{1i}, \hat{\alpha}_1), F_2(X_{2i}; \hat{\alpha}_2); \theta\}]; \tag{4}$$

let us denote this estimator by $\hat{\theta}$. Under a reasonable set of regularity conditions, we have that $\sqrt{n}(\hat{\theta} - \theta_0)$ is asymptotically normal with mean zero; for example, see Joe (1997, Chapter 10).

2.3 Semiparametric method

The *MLE* and *IFM* methods just mentioned are completely parametric because they require the model to be specified up to a finite number of unknown parameters. A possible shortcoming of these two methods of estimating θ is that they are likely to be inconsistent, and possibly inefficient, if the marginal distributions are misspecified. Since, the marginal distributions are seen as nuisance functions, ideally the method of estimation should be insensitive to misspecification of the marginal distributions. To this end, we relax the assumption that the marginal distribution of X_k is known up to the finite-dimensional parameter α_k $(k = 1, 2)$. Instead, we allow the marginal distributions to be arbitrary. Estimation is carried out in two stages as in IFM, but the difference is that the marginal distributions are estimated nonparametrically by their sample empirical distributions. More specifically, let F_k denote the *cdf* of X_k and let $\tilde{F}_k$ denote the *empirical cdf* of $X_{k1}, \ldots X_{kn}$, $(k = 1, 2)$. Then, θ is estimated by the maximizer of

$$\sum_{i=1}^{n} \log[c\{\tilde{F}_1(X_{1i}), \tilde{F}_2(X_{2i}); \theta\}]. \tag{5}$$

Let us denote the resulting semiparametric estimator by $\tilde{\theta}$. It has been shown that $\sqrt{n}(\tilde{\theta} - \theta_0)$ is asymptotically $N(0, \nu^2)$; this result holds irrespective or whether or not we know the marginal distributions. A large sample 95% confidence interval for θ is $\tilde{\theta} \pm 1.96\hat{\nu}$ where $\hat{\nu}$ is a consistent estimator of ν given in section 3 of Genest et al (1995).

2.4 A bench-mark estimator

In order to evaluate the performance of the foregoing estimators, we introduce the following estimator. Let F_1 and F_2 be as in the previous subsection. Let us suppose that these distribution functions are known. Now, θ is estimated by the maximizer of the loglikelihood

$$\sum_{i=1}^{n} \log[c\{F_1(X_{1i}), F_2(X_{2i}); \theta\}]. \tag{6}$$

Let us denote the resulting estimator by θ^*. Note the difference between (5) and (6) is that in (5) F_k is replaced by $\tilde{F}_k$. Although, the marginal distributions are unknown in practice, this hypothetical scenario, where F_k is assumed to be known, represents the ideal situation which can be used as a benchmark for comparative purposes, because we would not expect ML/IF/Semiparametric estimators to perform better than θ^*.. The difference between the efficiencies of θ^* and the estimators in the previous subsections quantify the loss due to the functional form of the marginal distribution being unknown.

3 Simulation study

A simulation study was carried out to compare the different estimators mentioned in the previous section for a range of copulas and marginal distributions. There are two mains objective of the study: (1) Evaluate the efficiency-robustness of the semiparametric estimator against violations of the assumed marginal distributions, and (2) estimate the coverage rate of the confidence interval $\tilde{\theta} \pm 1.96\hat{\nu}$.

3.1 Design of the simulation

The following seven copulas are studied; for six of them, the parameter θ is a scalar and for the seventh one it has two components. More details about these copulas may be found in Joe (1997) and Nelson (1999).

1. *Ali-Mikhail-Haq [AMH] Family of copulas:* $C(u, v; \theta) = uv/\{1 - \theta(1-u)(1-v)\}$.
2. *Clayton copula:* $C(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}$.
3. *Frank copula:* $C(u, v; \theta) = -\theta^{-1} \log\left([1 + (e^{-\theta u} - 1)(e^{-\theta v} - 1)]/(e^{-\theta} - 1)\right)$
4. *Gumbel copula:* $C(u, v; \theta) = \exp -\left((-\log u)^{\theta} + (-\log v)^{\theta}\right)^{\frac{1}{\theta}}$
5. *Joe copula:* $C(u, v; \theta) = 1 - \left((1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right)^{\frac{1}{\theta}}$
6. *Plackett copula:*

   $C(u, v; \theta) = [1 + (\theta - 1)(u + v) - \{\{(1 + (\theta - 1)(u + v)\}^2 - 4\theta(\theta - 1)uv\}^{\frac{1}{2}}]/\{2(\theta - 1)\}$.
7. *Joe–Clayton copula.*

$$C(u, v; \theta, \delta) = 1 - \left(1 - \left(\left(1 - (1-u)^{\theta}\right)^{-\delta} + \left(1 - (1-v)^{\theta}\right)^{-\delta} - 1\right)^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}}$$

These copulas cover a very wide range of distributional shapes. The ML and IF estimators are those that correspond to the case when, the marginal distributions are assumed to be normal. To evaluate the robustness properties, three other sets of marginal distributions are considered; the cases studied are:

(N-N): X_1 and X_2 are normally distributed.

(T-T): $X_1 \sim t_3$ and $X_2 \sim t_3$.

(T-ST): $X_1 \sim t_3$ and $X_2 \sim$ skewed t-distribution with df=3 and skewness = 0.5.

(T-C): $X_1 \sim t_3$ and $X_2 \sim \chi^2_2$.

Since the maximum likelihood estimation method turned to be extremely time consuming, we needed to restrict the number of samples to a manageable proportions.

3.2 Results

The results are presented in Tables 1 - 5. The main observations are summarized below:

The two marginal distributions are correctly specified as Normal-Normal: The results are given in Tables 1 and 2. Since the marginal distributions and the copula are correctly specified, there is no mis-specification and hence all the estimators are consistent. The MLE and IFM are based on correct specification of the marginal distribution and the likelihood function. Therefore, one would expect that these estimators would have good properties. As expected, they perform better than the semiparametric estimator. However, the difference is small. The bias is small for each of the four estimators.

The two marginal distributions are incorrectly specified as Normal-Normal: The results in Tables 3, 4, and 5 are for the case when the parametric methods ML and IF incorrectly assume that each of the marginal distributions is Normal; hence these estimators may not be even consistent. The semiparametric method assumes that the marginal distributions are continuous, but apart from that it does not assume any functional form for these distributions. Thus, in contrast to the ML and IF methods, the semiparametric method is not based on incorrect assumption for the marginal distributions. It is known that the semiparametric estimator is consistent and asymptotically normal; the rate of convergence is the usual $n^{1/2}$. Tables 3-5 show that, as expected, the semiparametric method performs considerably better than IFM and MLE.

Note that the marginal distributions for the cases in Tables 3-5, have long tails. In all these cases, we restricted each parameter to an interval of the form (a, b) where a and b are finite, even when the true parameter range was unbounded; this was necessary to avoid overflow/underflow. However, when the size of interval is too small or too large, in many of these cases, the MLE was on the boundary. Consequently, the ML iteration failed to converge. For some copulas, this occurred quite frequently. For example, for Frank copula with parameter interval is $(0, 200)$ and the true copula parameter $\theta = 3.5$, the MLE failed to converge in about 80% of the cases. Consequently, the standard deviations of the estimators were quite large. The reason for this failure appears to be the following, although this may not the only possible explanation.

Since the MLE and IFM assume that the marginal distribution is normal when in fact this is not the case. Since the marginal distribution has long tails, some observations tended to be in the extreme tails of the relevant normal distribution. Consequently, whatever be the value of (μ_1, σ_1), the value of $\phi\{(X_{i1} - \mu_1)/\sigma_1\}$ was close to zero for and hence, the absolute value of $\log \phi\{(X_{i1} - \mu_1)/\sigma_1\}$ was very large for some i. Consequently, the observations in the extreme tails tend to be highly influential. This caused computational and other difficulties.

For Frank copula, the MLE-iteration did not converge for 80% of the cases indicating that the MLE was close to the boundary at infinity. Thus, it is clear that if there are extreme observations in the tail then the MLE and IFM estimators are unlikely to be suitable. While, it may be possible to use ideas similar to Winsorization used in the classical robust inference literature to reduce the influence of extreme observations, such a modification is likely to result in an inconsistent estimator.

Very large standard deviations for some of the estimators reflect the fact that the estimators were on the boundary which was chosen to be much larger than the true value.

In summary, Tables 1-5 show that, overall, the semiparametric method is considerably better than the MLE and IFM.

Table 6 shows that an approximate 95% confidence interval based on a normal approximation for the large sample distribution of $\hat{\theta}$ has coverage rates close to 95% for sample size ≥ 50; in some isolated cases, it could drop to a rate in the range 80 - 90 % (see AMH copula with $\theta = 0.4, 0.8$; and Clayton copula with $\theta = 0.221$.).

4 Empirical Example:

In this section, we use a bivariate example to illustrate the estimation methods studied in this paper and to highlight some of the challenges. We consider data from the 1988-89 Household Expenditure Survey conducted by the Australian Bureau of Statistics. For simplicity, we shall restrict to Households consisting of exactly two adults and two children. Let X_1 = proportion of expenditure on housing and X_2 =(1 - proportion of expenditure on food). We wish to estimate the joint distribution of X_1 and X_2.

One of the challenges that we face is the specification of a suitable copula. Since there are a large number of copulas, specifying one that would suit a particular example is not easy at all. Even if one has some idea of the shape of the joint density function of (X_1, X_2), it is not easy to deduce the shape of the copula, which is the shape of the density function of $(F_1(X_1), F_2(X_2))$ where F_1 and F_2 are the *cdf* s of X_1 and X_2 respectively. Therefore, what we can do is to consider different copulas and evaluate their goodness of fit.

An Archimedean copula is defined to be the one that is of the form

$$C(u, v) = \phi^{-1}\{\phi(u) + \phi(v)\}$$

where ϕ is a strictly decreasing smooth convex function. This family is known to capture a range of functional forms; see Genest and Rivest (1993) for a graphical way to guide the choice of a suitable member of this family. Let $\lambda(t) = \phi(t)/\{(d/dt)\phi(t)\}$. There is a one-one correspondence between the functions λ and ϕ; hence choosing the particular Archimedean copula is equivalent to choosing the function λ. A nonparametric estimator of λ is given by $\hat{\lambda}(t) = t - n^{-1} \sum I(t - V_i)$ where I is the indicator function and

$$V_i = \text{Number of } \{(X_{1j}, X_{2j}) : X_{1j} < X_{1i}, X_{2j} < X_{2i}\}/(n-1).$$

Now, a graphical way of choosing an Archimedean copula is to draw $\hat{\lambda}(t)$ and the λ functions for different copulas and see which of the λ functions is 'close' to $\hat{\lambda}$. Figure 5 shows the λ functions for Clayton, Joe and Frank copulas; the λ functions for the other Archimedean copulas considered in simulation are not shown because they were not close to λ.

We fitted the Clayton, Joe and Frank copulas by the semiparametric methods. To assess the goodness of fit, the domain $[0, 1] \times [0, 1]$ of the copula was first partitioned into 25 squares; then the squares with small frequencies were amalgamated and chi-square test of fit was applied. The chi-square statistics and the p-values are given in Table 11. Note that the p-values for Clayton and Joe copulas are considerably smaller than that for Frank copula. Figure 5 also shows that the λ functions for Clayton and Joe copulas are not as close as that for the Frank copula.

A plot of (X_1, X_2) is given in Figure 3. While this plot provides an overview of the joint distribution of (X_1, X_2), it is not that helpful in suggesting a suitable function form for the copula. To this end, we need to plot $\{(F_{1n}(X_{1i}), F_{2n}(X_{2i})) : i = 1, \ldots, n\}$, where F_{1n} and F_{2n} are the empirical cdf s of the two variables. Figure 4 shows that the points tend to concentrate near (0,0) and (1, 1). The shapes of the density functions of the estimated copulas are shown in Figures 6,8 and 10. The single peaks for Clayton and Joe copulas do not appear to be consistent with the nature of the scatter in peaks in $\{(F_{1n}(X_{1i}), F_{2n}(X_{2i})) : i = 1, \ldots, n\}$.

Thus among the Archimedean copulas that we considered, Frank copula appears to fit the data best.

The estimates of the copula parameters corresponding to semiparameteric method and the parametric method with normal margins turned out to be close (see Tables 10 and 11). If we use a t-distribution for the margins with the degrees of freedom as an unknown parameter, then the MLE and IFM-estimate of the copula parameters change substantially. Strictly speaking, one needs to consider the standard errors of the estimates and goodness of fit statistics to quantify this - these are not necessarily easy tasks. This raises the question which distribution should we use for the margin ? The fact that the semiparametric estimate of the copula parameter is consistent and the fact the MLE and IFM estimates are close to the semiparametric estimates suggest that if we were to choose a suitable distributions for the margin, then the normal distribution appears better than a t-distribution for this example. The more important point is that this example illustrates the advantages of using the semiparametric method as opposed to the fully parametric ML and IF methods in dealing the unknown marginal distributions.

5 Conclusion

In this paper, we evaluated a semiparametric method of estimating the copula. A simulation study showed that the semiparametric method, which estimates the marginal distributions nonparametrically, is more robust than the fully parametric ML and IF methods. A data example involving the household expenditure survey data, compared and contrasted the three methods. The example illustrated the difficulties in choosing the correct marginal distributions to implement fully parametric methods. By contrast, the semiparametric method estimates the marginal distributions nonparametrically by the empirical distribution function and hence the difficult task of choosing the correct form for the marginal distribution does not arise. The simulation study also highlighted the difficulties that arise due to non-convergence of the computational iterations for MLE and IFM; the semiparametric method did not exhibit such difficulties.

We recognize that in the copula approach, the marginal distribution is treated as a nuisance function while the copula is the function of interest. By contrast, for the ML approach, the marginal distribution and copula are treated as equally important. Therefore, a direct comparison of the two methods may not be completely justified. However, if copula is the basic function of interest which captures the features of dependence between X and Y that are invariant under monotonic transformation of the marginal distribution, and hence treats the marginal distribution itself as a nuisance function, then the fully parametric methods, ML and IFM, do not appear to be as good as the semiparametric method.

An attractive feature of the semiparametric method studied here is that it lends itself to the more general setting where the observations may not be *iid*. For example, the marginal variables may have a regression or a time-series structure. In such general settings, choosing the correct joint distribution for the error term and using it in inference are more difficult tasks. The semiparametric method considered here appears to provide a reasonably flexible way of approaching this; we are currently working on this and hope to report the results elsewhere.

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Table 1: Estimated means and variances when both marginal distributions are correctly specified as normal: Number of samples=250, Number of observation in each sample=100

	Mean				Variance (×100)			
θ	BM	MLE	IFM	SP	BM	MLE	IFM	SP
				AMH copula				
-0.9	-0.81	-0.80	-0.81	-0.81	6.13	6.49	6.20	6.20
0.1	0.09	0.09	0.09	0.09	8.24	8.24	8.15	8.41
0.5	0.47	0.46	0.46	0.47	5.51	5.46	5.41	5.65
0.9	0.87	0.87	0.87	0.88	0.93	1.06	1.07	1.10
				Clayton copula				
0.11	0.12	0.11	0.11	0.13	1.21	1.10	1.08	1.34
2.00	2.01	2.04	2.01	2.08	7.55	15.0	13.9	16.3
3.95	3.97	4.04	3.96	3.99	17.2	38.9	37.6	42.4
6.39	6.40	6.43	6.20	6.14	44.4	88.8	84.0	89.0
				Frank Copula				
-5.0	-4.98	-4.99	-4.96	-5.01	49.7	54.9	54.8	57.1
0.5	0.53	0.54	0.54	0.55	22.8	24.0	23.8	25.3
2.5	2.50	2.48	2.47	2.51	45.8	49.3	48.5	50.8
5.0	5.1	5.1	5.1	5.1	58.8	68.0	67.4	70.4
				Gumbel Copula				
1.5	1.51	1.51	1.51	1.54	1.35	1.97	1.97	2.29
6.5	6.6	6.6	6.6	6.4	36.4	68.4	67.3	64.0
9.0	9.0	9.1	9.0	8.6	46.6	95.3	97.7	99.2
11.0	11.1	11.2	11.1	10.3	83.9	182	184	175
				Joe Copula				
1.2	1.23	1.23	1.23	1.25	1.79	1.89	1.79	2.13
3.0	3.01	3.04	3.00	3.07	9.4	16.9	16.5	20.3
4.0	4.09	4.09	4.01	4.06	17.9	40.7	39.1	39.2
5.0	5.05	5.04	4.92	4.96	21.2	45.7	43.8	49.4
				Plackett Copula				
0.5	0.52	0.52	0.52	0.52	2.69	2.75	2.74	2.87
5.5	5.8	5.8	5.7	5.8	216	239	230	255
8.0	8.1	8.2	8.0	8.2	426	456	420	487
10.0	10.2	10.4	10.2	10.3	744	897	857	880

Table 2: Estimated means and variances when both marginal distributions are correctly specified as normal and the copula is the Joe-Clayton family

First Parameter								
	Mean				Variance (×100)			
θ_1, θ_2	BM	MLE	IFM	SP	BM	MLE	IFM	SP
1.5,0.5	1.52	1.52	1.51	1.56	3.81	4.15	4.12	5.43
1.5,2.5	1.53	1.53	1.54	1.63	8.69	9.86	9.75	12.5
3.5,0.5	3.48	3.50	3.47	3.50	12.6	21.4	21.8	27.5
3.5,2.5	3.54	3.51	3.51	3.56	17.4	26.3	29.7	41.1
Second parameter								
1.5,0.5	0.51	0.51	0.51	0.56	4.85	5.93	5.42	7.25
1.5,2.5	2.56	2.58	2.54	2.59	15.0	26.9	25.7	32.0
3.5,0.5	0.52	0.53	0.54	0.66	10.3	11.7	11.6	20.2
3.5,2.5	2.59	2.58	2.55	2.63	29.2	50.6	46.8	71.9

Table 3: Estimated means and variances when both marginal distributions are t_3, but incorrectly specified as normal distribution: Number of samples=250, Number of observation in each sample=1000

	Mean			Variance ($\times$100)		
True value	MLE	IFM	Semiparametric	MLE	IFM	SP
			AMH copula			
0.3	0.49	0.47	0.30	1.89	1.63	0.66
0.5	0.69	0.67	0.50	0.75	0.67	0.45
0.7	0.82	0.82	0.70	0.18	0.17	0.22
			Clayton copula			
0.105	0.04	0.13	0.11	0.10	0.35	0.16
1.718	2.70	2.19	1.73	3372	10.1	1.07
6.389	14.6	7.35	6.32	56286	109	8.37
			Frank copula			
0.5	0.94	0.92	0.51	15.4	13.2	3.28
3.5	5.17	4.85	3.50	34.4	19.1	3.81
5	6.96	6.53	5.03	48.0	22.0	5.85
			Gumbel copula			
1.5	1.95	1.69	1.51	137	1.93	0.20
6.5	9.2	7.5	6.5	1182	45.5	4.73
11	12.9	12.6	10.9	558	167	14.7
			Joe copula			
1.2	1.51	1.25	1.20	265	0.85	0.16
2.4	5.67	2.87	2.42	2765	9.05	1.17
5	12.0	5.93	5.00	2298	43.7	4.25
			Plackett copula			
0.5	0.31	0.33	0.50	0.58	0.45	0.22
5.0	8.99	7.87	5.07	362	119	18.6
10.0	16.7	14.8	10.1	483	300	68.5

Table 4: Estimated means and variances when marginal distributions are t_3, and $skew-t(3, 0.5)$ but incorrectly specified as normal distribution: Number of samples=250, Number of observation in each sample=1000

	Mean			Variance (×100)		
True value	MLE	IFM	Semiparametric	MLE	IFM	SP
			AMH copula			
0.3	0.64	0.60	0.300	4.64	3.89	0.58
0.5	0.82	0.79	0.50	0.81	0.82	0.39
0.7	0.91	0.90	0.70	0.11	0.12	0.22
			Clayton copula			
0.105	0.08	0.23	0.11	1.01	1.76	0.15
1.718	3.59	1.96	1.73	8056	11.1	1.34
6.389	6.18	3.97	6.34	3671	41.4	9.40
			Frank copula			
0.5	1.01	0.93	0.49	45.5	29.8	4.01
3.5	5.36	4.84	3.52	74.3	29.4	5.95
5	7.06	6.37	5.00	64.6	28.6	6.10
			Gumbel copula			
1.5	1.90	1.63	1.51	49.0	1.68	0.17
6.5	6.30	5.19	6.46	649	23.9	4.71
11	7.99	6.14	10.89	973	59.0	16.9
			Joe copula			
1.2	1.47	1.21	1.21	240	7.61	0.17
2.4	6.3	2.61	2.40	3408	27.7	0.98
5	12.4	5.05	4.99	2615	52.0	4.28
			Plackett copula			
0.5	0.30	0.34	0.50	0.51	0.40	0.25
5.0	9.64	7.77	5.05	646	175	20.6
10.0	16.6	13.4	9.90	491	268	63.2

Table 5: Estimated means and variances when marginal distributions are t_3, and $\chi_2(2)$ but incorrectly specified as normal distribution: Number of samples=250, Number of observation in each sample=1000

	Mean			Variance (×100)		
True value	MLE	IFM	Semiparametric	MLE	IFM	SP
			AMH copula			
0.3	0.54	0.48	0.30	5.38	3.60	0.56
0.5	0.83	0.78	0.49	1.29	1.48	0.39
0.7	0.93	0.91	0.70	0.09	0.11	0.25
			Clayton copula			
0.105	0.06	0.20	0.11	0.71	0.90	0.14
1.718	1.63	1.50	1.73	21.8	3.98	1.08
6.389	4.05	2.63	6.31	3684	63.7	8.06
			Frank copula			
0.5	0.71	0.67	0.49	8.73	7.75	3.61
3.5	4.68	4.20	3.54	26.4	9.00	4.41
5.0	6.33	5.60	5.00	29.8	11.9	7.19
			Gumbel copula			
1.5	1.58	1.48	1.50	32.2	0.50	0.21
6.5	4.96	3.54	6.46	255	7.51	4.69
11.0	5.51	3.82	10.9	243	9.92	14.6
			Joe copula			
1.2	1.31	1.14	1.21	344	0.16	0.19
2.4	3.97	2.11	2.41	2149	1.57	1.17
5.0	7.72	3.51	4.99	2632	10.3	3.95
			Plackett copula			
0.5	0.37	0.40	0.50	0.46	0.33	0.25
5.0	7.39	6.12	5.01	157	38.5	17.4
10.0	13.3	10.5	10.0	303	85.2	64.9

Table 6: **95% coverage of copulas with Normal margins: Number of samples=250**

	Sample sizes		
true value	50 samples	100 samples	250 samples
	AMH copula		
-0.8	95.1	96.4	97.4
-0.4	94.9	95.9	94.3
0.4	88.7	92.4	94.7
0.8	81.6	88.3	92.8
	Clayton copula		
0.221	84.3	92.9	95.3
0.822	94.7	93.7	94.2
3.055	94.4	94.0	95.6
5.050	94.3	95.1	94.9
	Frank copula		
-4.5	94.6	94.4	95.5
-2.5	94.4	94.7	95.7
1.5	97.1	94.0	95.7
4.5	95.6	96.1	94.4
	Gumbel copula		
2	94.3	94.8	93.6
4	95.8	95.9	96.1
8	95.1	95.3	95.5
10	94.8	93.1	96.4
	Joe copula		
1.4	93.2	93.2	92.9
2.2	94.5	93.2	94.1
3.8	93.7	93.6	95.2
4.6	95.3	94.2	95.3
	Plackett copula		
1	90.8	93.9	96.4
3	93.7	96.0	97.8
7	95.9	98.2	99.1
9	96.6	98.1	99.7

Table 7: **95% coverage of copulas with t-distribution margins: Number of samples=250**

	Sample sizes		
true value	50 samples	100 samples	250 samples
	AMH copula		
-0.8	94.1	97.1	97.6
-0.4	93.9	94.8	93.3
0.4	89.1	91.2	95.3
0.8	81.6	90.3	92.7
	Clayton copula		
0.221	85.0	92.2	94.5
0.822	94.3	94.9	94.2
3.055	92.9	94.7	95.7
5.050	93.6	94.8	95.4
	Frank copula		
-4.5	95.5	95.4	95.4
-2.5	94.9	93.6	95.8
1.5	96.8	95.3	94.8
4.5	95.5	93.6	95.0
	Gumbel copula		
2	93.2	94.3	94.3
4	95.2	94.9	95.4
8	93.9	95.4	95.8
10	93.4	94.5	95.0
	Joe copula		
1.4	93.3	94.1	95.0
2.2	93.7	93.8	95.0
3.8	94.4	94.1	94.0
4.6	95.7	94.0	94.5
	Plackett copula		
1	90.9	95.1	96.5
3	93.1	94.1	96.8
7	95.2	97.8	99.2
9	97.8	98.6	99.7

Table 8: **95% coverage of copulas with t-distribution and skew-t distribution margins: Number of samples=250**

	Sample sizes		
true value	50 samples	100 samples	250 samples
	AMH copula		
-0.8	95.5	96.5	96.8
-0.4	94.1	95.9	93.2
0.4	89.2	93.7	93.1
0.8	81.1	89.9	94.1
	Clayton copula		
0.221	84.5	92.8	95.4
0.822	94.0	93.9	94.6
3.055	94.3	94.2	95.6
5.050	92.9	95.1	94.9
	Frank copula		
-4.5	95.8	95.6	94.4
-2.5	94.9	94.8	95.8
1.5	96.2	94.5	96.3
4.5	95.2	94.0	95.0
	Gumbel copula		
2	95.7	93.7	94.4
4	96.3	95.7	93.8
8	94.2	95.9	95.4
10	92.7	96.7	95.5
	Joe copula		
1.4	94.1	92.0	94.9
2.2	93.4	94.1	94.9
3.8	95.8	95.5	95.1
4.6	94.8	94.4	93.0
	Plackett copula		
1	92.9	92.9	96.6
3	93.7	94.5	97.9
7	97.1	98.0	99.3
9	96.9	98.7	99.7

Table 9: **95% coverage of copulas with t-distribution and χ_2 distribution margins: Number of samples=250**

	Sample sizes		
true value	50 samples	100 samples	250 samples
	AMH copula		
-0.8	94.9	96.7	97.8
-0.4	92.9	94.9	93.7
0.4	89.2	90.6	94.1
0.8	79.9	89.9	92.8
	Clayton copula		
0.221	83.7	92.8	95.0
0.822	93.4	94.7	95.0
3.055	94.5	95.2	95.5
5.050	93.2	95.2	94.8
	Frank copula		
-4.5	95.6	96.2	94.1
-2.5	95.1	95.2	94.9
1.5	97.9	94.4	96.0
4.5	93.9	93.3	95.1
	Gumbel copula		
2	95.5	93.2	91.6
4	95.4	94.9	93.8
8	93.3	95.7	95.6
10	94.6	95.0	96.0
	Joe copula		
1.4	92.8	93.6	94.1
2.2	96.1	93.9	94.8
3.8	94.3	93.2	93.4
4.6	94.0	94.3	94.9
	Plackett copula		
1	92.8	94.7	96.9
3	93.9	94.2	97.0
7	96.0	98.1	98.9
9	96.4	98.1	99.7

Table 10: **Parametric Estimations under different marginal assumptions**

	normal distributions		t distributions		skew-t distributions
Copula model	IFM	MLE	IFM	MLE	IFM
Clayton	0.446	0.419	14.361	14.399	7.707
Frank	2.470	2.505	29.189	29.266	9.511
Joe	1.394	1.417	14.517	16.148	1.133

Table 11: Goodness of Fit tests

Copula	Parameter estimate	Chi-square statistic	P-value
	Semiparametric Method		
Clayton	0.396	39.895	0.011
Frank	2.384	22.990	0.402
Joe	1.469	31.172	0.093
	IFM		
Clayton	0.446	38.202	0.004
Frank	2.470	23.862	0.160
Joe	1.394	32.0117	0.022
	MLE		
Clayton	0.419	38.9825	0.003
Frank	2.505	24.2927	0.146
Joe	1.417	31.4839	0.025

Figure 1: Histogram of Housing Expenditure

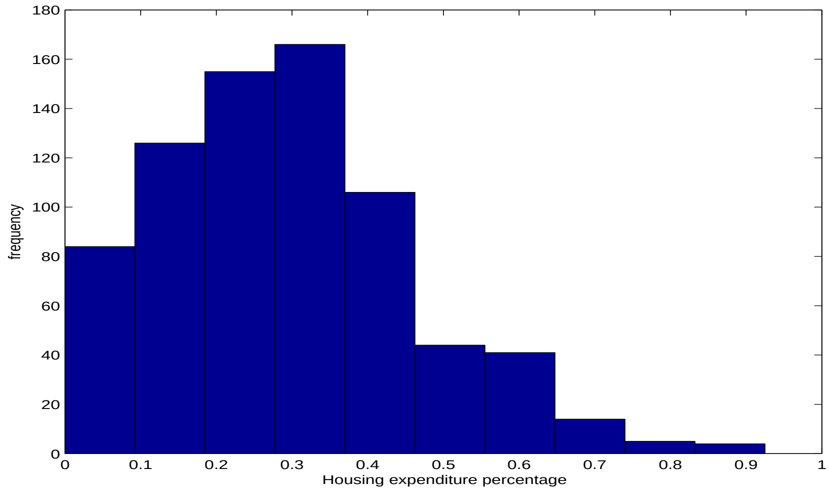

Figure 2: Histogram of 1-Food Expenditure

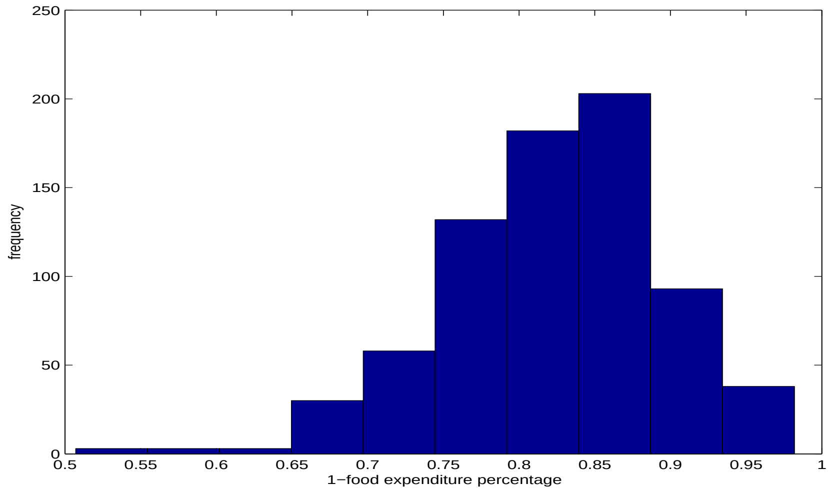

Figure 3: Scatter diagram of Housing and 1-Food expenditure

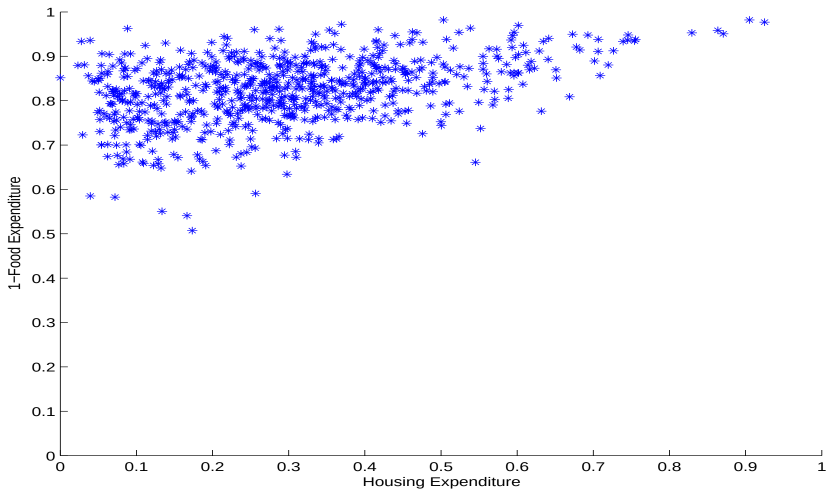

Figure 4: Scatter diagram of Empirical distributions of Housing and 1-Food expenditure

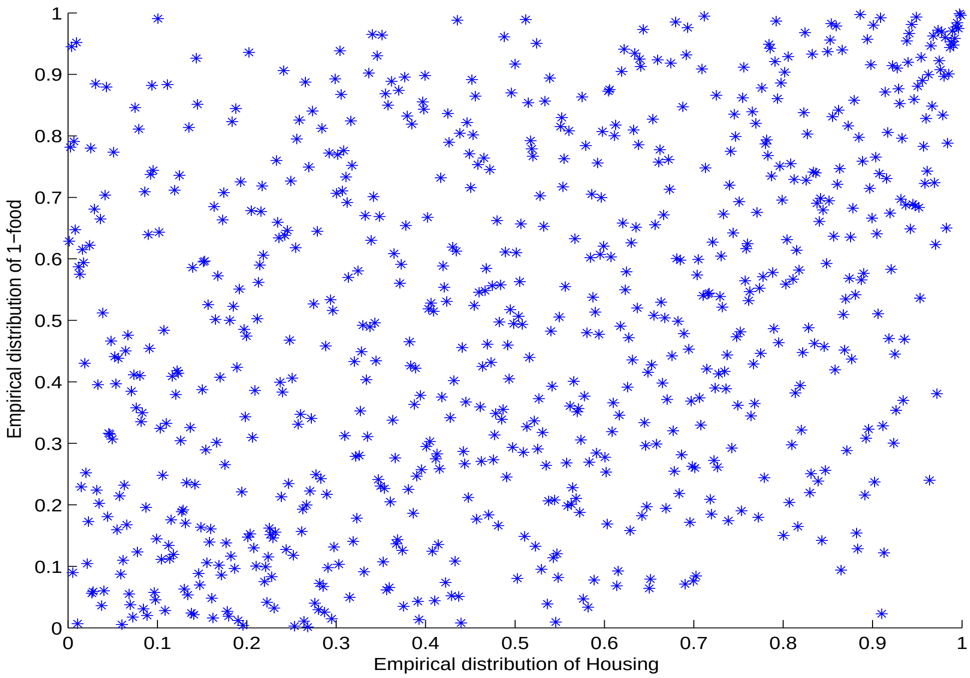

Figure 5: λ functions

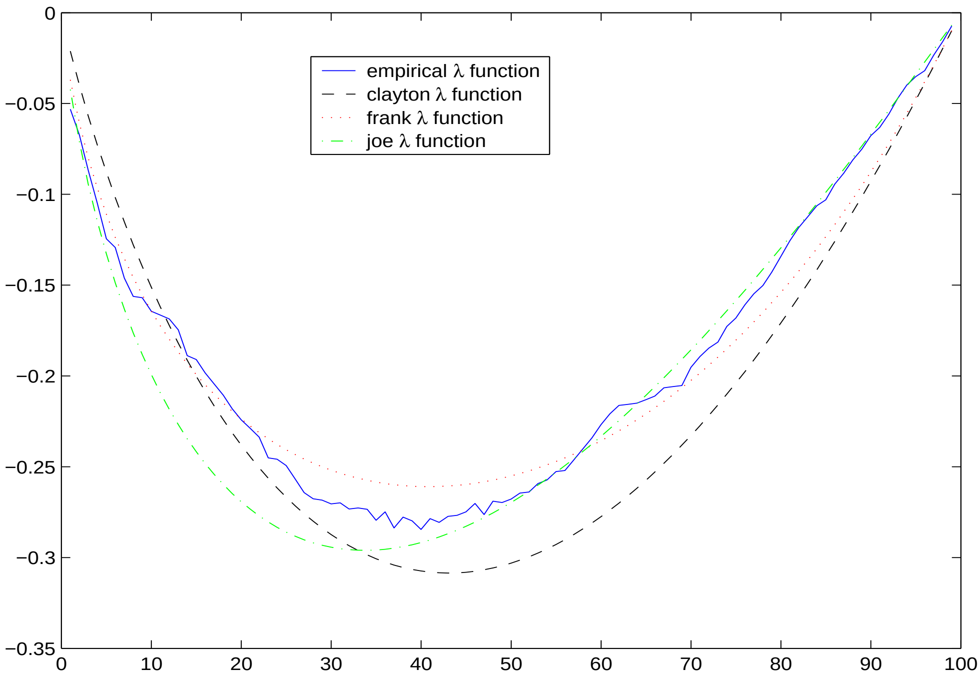

Figure 6: Clayton copula pdf surface when θ=0.396

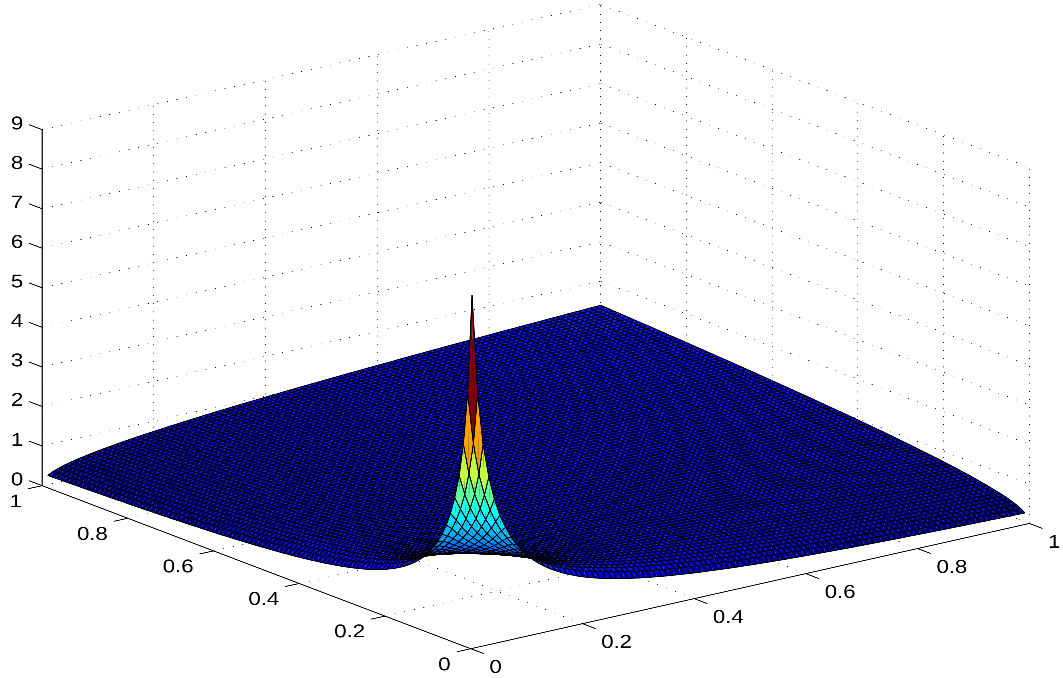

Figure 7: Clayton copula cdf surface with empirical cdf when θ=0.396

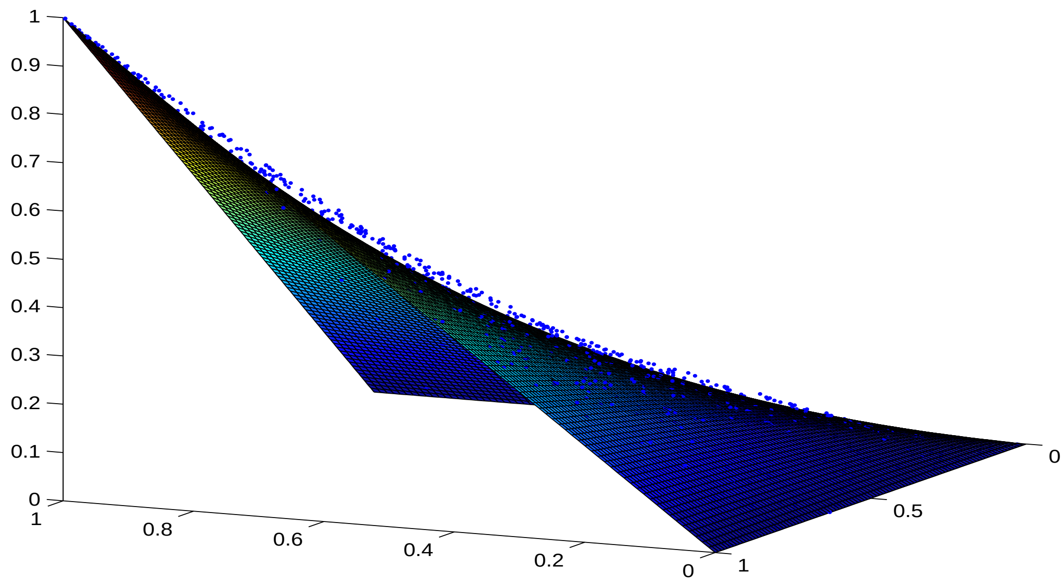

Figure 8: Frank copula pdf surface when θ=2.384

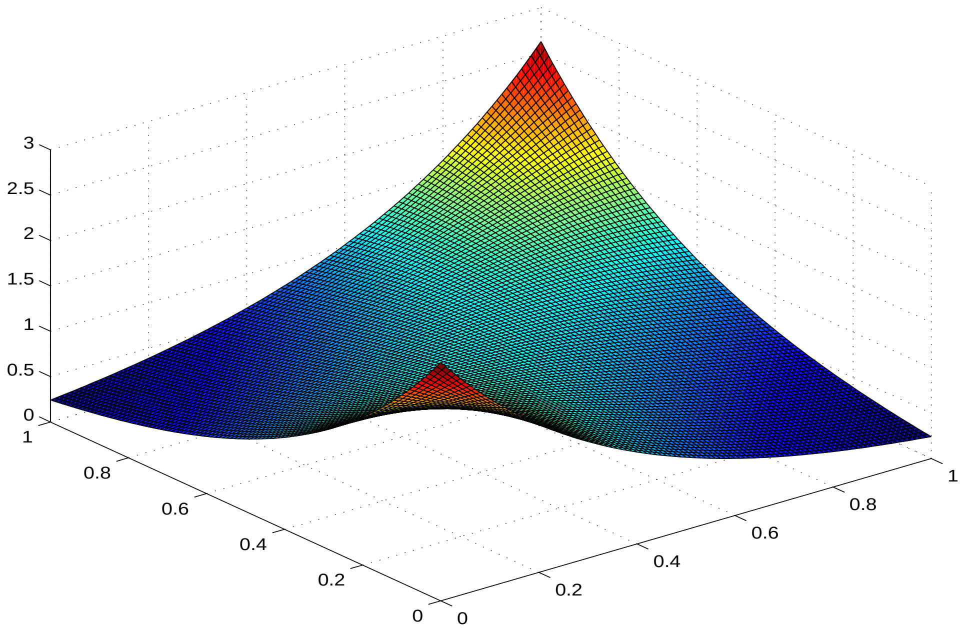

Figure 9: Frank copula cdf surface with empirical cdf when θ=2.384

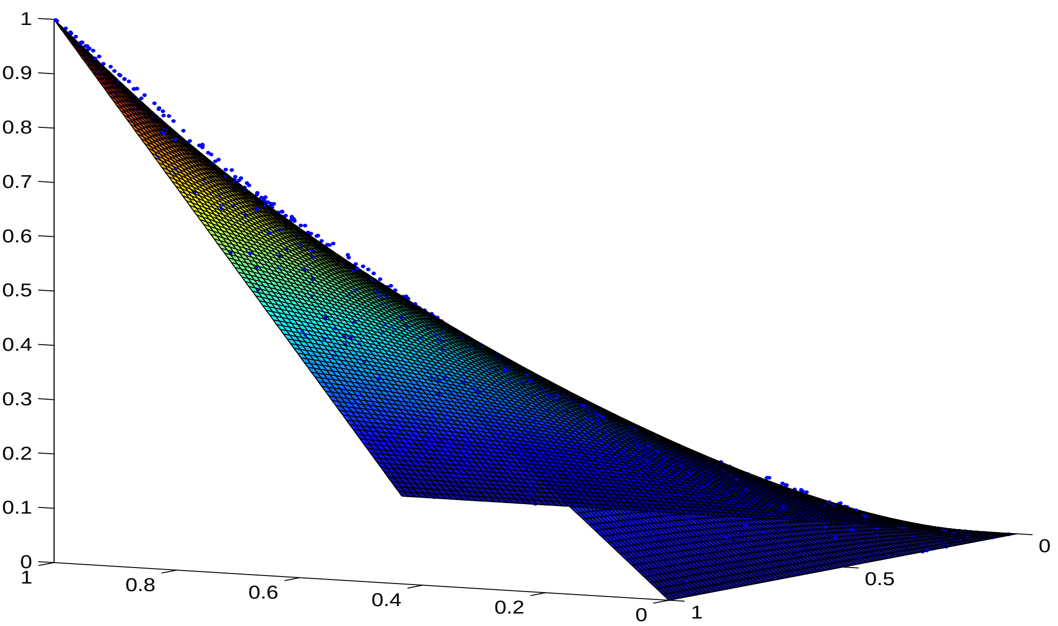

Figure 10: Joe copula pdf surface when θ=1.469

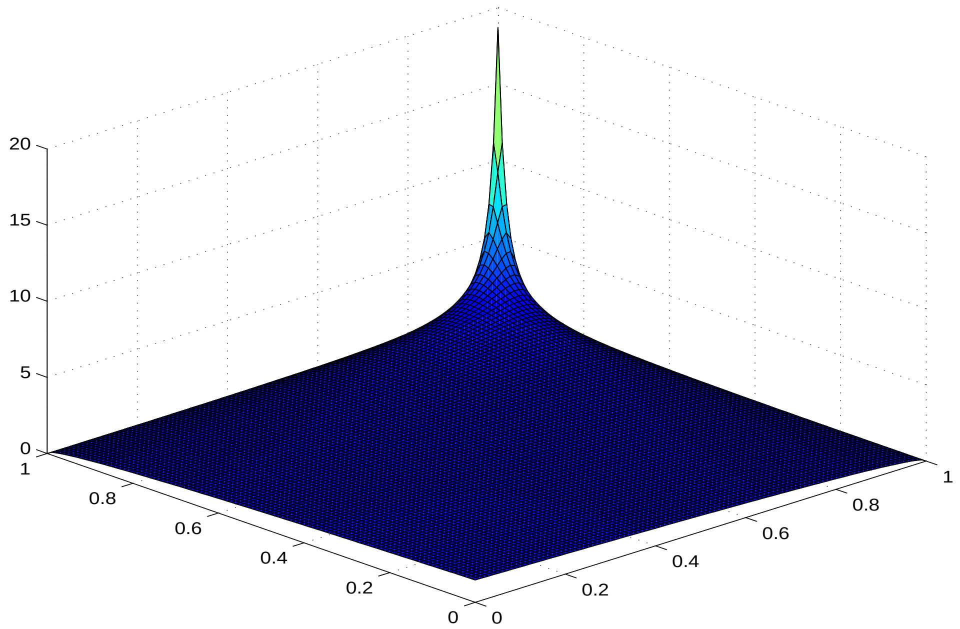

Figure 11: Joe copula cdf surface with empirical cdf when θ=1.469