The Markovian Dynamics of "Smart Money"

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Abstract

I develop a Markov model of smart money chasing past winning funds while taking into account associated costs. The model also allows market capital entry and exit. The steady-state capital allocations are derived using constant transition probabilities. The results suggest that downside risk is significantly attributed to investor overreaction, even though a small degree of investment movement as opposed to capital immobility can in fact stabilize the market. Furthermore, performance sensitivity makes it possible that two much-debated fund styles, passive indexing and active management, are simultaneously profitable. If money is insensitive, the model becomes a zero-sum game where one strategy's profitability is always at the cost of the other.

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1. Introduction

The term “smart money” has been widely used in finance to refer to the investments following the winning funds or being able to identify superior fund styles. The movements of smart money are mostly driven by fund performance and investor sentiments. The market condition is also believed to influence fund selection. There are many observations of investment movements, for instance, a shift from active investment management to passive indexing in recent years. In the US equity markets, 14.8% of actively managed domestic equity funds liquidated or merged during 2000-2002, including 6.6% in 2002 alone. At the state level, Connecticut State Trust Fund, in charge of the state’s $12.7 billion pooled pension fund, bumped up their indexed portion from $3.74 billion to $3.99 billion in just two months from July to September 2002, with the long-term goal for the indexed proportion of equities set to be 50%. At the firm’s level, in July 2002 the trustees of Intel’s profit-sharing and pension plans fired their 10 external money managers and decided to switch an additional $300 million of equities to in-house passive management. All of the above are examples of a common phenomenon of moving from active to passive investment management.

It is natural to expect that overall capital flows driven by fund selections bring about a considerable effect on financial markets. The literature however has not seen much investigation on this regard. Instead, there have been abundant but diverse stories on the performance of mutual funds and management styles. Research along this line studies the cost-benefit comparison across funds and debates on whether active fund management adds value (Grinblatt and Titman 1989, 1993; Brown and Goetzmann 1995; Carhart 1997; Wermers 1997, 2000). Attempts have been made to answer whether actively managed funds outperform the market, and if so, whether it is due to pure luck or they are "hot hands" with stock-picking talents. Controversy still remains. Another line of research studies how fund flows are related to past performance (Grinblatt, Titman and Wermers 1995; Gruber 1996; Chevalier and Ellison 1997, 1999; Sirri and Tufano 1998; Zheng 1999). These studies essentially examine whether money is "smart" based upon the performance-flow relationship and the effectiveness of betting on past winners.

This paper studies the market impact arising from dynamic capital movements. The dynamics of capital flows among fund managements is modelled using the Markov chain. The transition probabilities of the Markov chain are modelled as the functions of the cost-adjusted fund performance. The focus here is on the investment risk induced by capital movements. It is in fact not difficult to imagine the link between investment risk (reflected by price behaviour) and capital movements. This can be simply understood from the process where prices determine fund performance that in turn influences the movements of smart money, whose dynamics then shapes new prices.

In order to understand the long-term behaviour of investment flows, the steady-state closed-form solution is obtained for the Markovian dynamics conditional on the simplified assumptions of constant transition probabilities. A change in transition probabilities implies a change in the popularity of different fund styles. The current study applies the analysis of comparative statics to study the impact on prices due to changes in various aspects of transition probabilities.

Using simulation experiments, it is also investigated the dynamic impact of fund selections and capital flows. Simulation of the pricing process is carried out each trading period. The movements of smart money not only depend on fund performance but also investor sentiment. The study thus also applies varying levels of investor sentiment in response to fund profitability.

The resulting investment risk is assessed by the market drawdown which quantifies uninterrupted falls of prices[1]. Drawdown as a risk measure has become increasingly popular among researchers and investors largely because of the recent crash of equity markets around the world. Drawdown provides a downside approach to risk as opposed to conventional risk measures, such as standard deviation, that do not differentiate deviations above and below the mean. Tolerance of downside risk cannot be easily compensated for by the long-term validity of the employed strategy or the attractive expected return characteristic. For example, regardless of the expected future abnormal returns it is unlikely for a consumer/investor to tolerate a drawdown of more than 50% of his account.

Another feature of drawdown is that it concerns the duration of loss periods so that consecutive losses are distinguished from intermittent losses. It is highly uncommon that a fund manager can hold a client whose account is in a drawdown for a lengthy period of time even if the drawdown size is small. In this study, the degree of drawdown is examined in three aspects: the number, the duration, and the depth/size of drawdown. These various aspects of drawdown results are compared in different models of dynamic capital movements.

2. The Pricing Model

Consider a market of S stocks. Let $\mathbf{P}_t$ and $\mathbf{S}_t$ denote the S-dimensional vectors of the prices and the outstanding stock shares. Prices are determined by market equilibrium. Demand is regarded as the time-varying flow of capital into different investment portfolios. Let K be the total size of capital in the economy and for simplicity it is assumed to be fixed. This assumption does not preclude the market capital size from varying; investment entry and exit are modelled as discussed later.

Let θ_t^i denote the capital ratio invested in strategy (or fund) i, and θ_t^X denote the capital ratio staying out of the market, i.e. non-investment. There are N different investment strategies including non-investing, and their capital ratios sum to one, $\sum_{i=1}^{N}\theta_t^i = 1$. $K\theta_t^i$ is the size of the capital flowing into fund i. Let $\mathbf{w}_t^i$ denote the $S\times 1$ vector of the portfolio weights on S stocks by fund type i, and $\mathbf{w}_t^{i}{}'\mathbf{1}_S = 1$ except that $\mathbf{w}_t^X = \mathbf{0}_S$. $\mathbf{1}_S$ is an $S\times 1$ vector of ones and $\mathbf{0}_S$ is an $S\times 1$ vector of zeros.

Market equilibrium at time t requires

$$\mathbf{S}_t \odot \mathbf{P}_t = K\sum_{i=1}^{N}\theta_t^i \mathbf{w}_t^i, \tag{1}$$

where $\odot$ is the element-by-element multiplication. Notice that although non-market participants, represented by $i = X$, do not invest, i.e. $\mathbf{w}_t^X = \mathbf{0}_S$, they still affect price formation through the constraint $\sum_{i=1}^{N}\theta_t^i = 1$; that is, investment entry and exit can change the capital ratios of different investment strategies and also the total market capital size.

Denote by Θ_t the $N \times 1$ vector of θ_t^i for $i = 1,..,N$. Denote by $\mathbf{W}_t$ the vector of fund i's portfolio weights $\mathbf{w}_t^i$ for $i = 1,..,N$. $\mathbf{W}_t$ thus has a dimension of $NS \times 1$. The equilibrium condition (1) can also be as

$$\mathbf{S}_t \odot \mathbf{P}_t = K\, \Theta_t^{'}\, \mathbf{W}_t. \tag{2}$$

Market equilibrium (2) yields the $S \times 1$ vector of stock prices at time *t* as

$$\mathbf{P}_t = K\, \Theta_t^{'}\, \mathbf{W}_t \odot \mathbf{S}_t^{-1}. \tag{3}$$

Equation (3) states that the dynamic equilibrium process is intrinsically determined by first, the investment movements among funds (Θ_t), and second, the portfolio allocations of different fund managements ($\mathbf{W}_t$). The dynamics of capital movements is modelled in the next section. We now turn to the discussion of management types and portfolio choices.

3. Portfolio Managements and Fund Styles

In the literature of fund managements, debate has centred on two distinct approaches, namely, passive indexing and active portfolio managements. Indexing refers to passive investments that follow market indexes to form investment portfolios. The portfolios are therefore designed based upon the index weights. Each stock's index weight is a measure of its relative market capitalization, and is calculated as the multiplication of the stock price and the number of shares outstanding, normalized by the market capital size. Therefore, the $S \times 1$ vector of the indexing portfolio weights is given by

$$\mathbf{w}_t^P = \left(\mathbf{S}'_{t-1}\mathbf{P}_{t-1}\right)^{-1}\left(\mathbf{S}_{t-1} \odot \mathbf{P}_{t-1}\right). \tag{4}$$

The logic behind indexing is that since each stock's index weight measures its relative market capitalization, the index weight actually reflects an estimate of the 'relative value' of the company. Indexing is therefore believed to track the 'relative values' of stocks while at the same time to benefit from diversification.

Conversely, instead of passively following market indexes, some fund managers trade actively and strategically. This type of investment management is often referred to as active portfolio management. Active portfolio management can have a wide variety of styles. Fund managers may apply systematic trading rules ranging from simple pattern recognition, such as head-and-shoulders, to sophisticated genetic algorithm. Or they may select a particular class of stocks due to their high expectations on a certain stock attribute such as growth, small cap, global, or emerging markets.

Modern portfolio theory established by the pioneering work of Markowitz (1959) provides a cornerstone in building active portfolios. The key idea of the theory is to maximize the expected reward consistent with the willingness to bear risk, i.e. the mean-variance efficient frontier. A basic form in line with the mean-variance analysis can be given by the $S \times 1$ vector $\mathbf{\Omega}_t^{-1}\mathbf{E}_t$, where $\mathbf{E}_t$ is the $S \times 1$ vector of the expected returns on S stocks, and $\mathbf{\Omega}_t$ is the $S \times S$ covariance matrix at time t. The vector of active portfolio weights that satisfies $\mathbf{w}_t^{i}{}' \mathbf{1}_S = 1$ is therefore given by

$$\mathbf{w}_t^A = \left[(\mathbf{\Omega}_t^{-1}\mathbf{E}_t)'\mathbf{1}_S\right]^{-1}\left[\mathbf{\Omega}_t^{-1}\mathbf{E}_t\right]. \tag{5}$$

Active portfolios allocated according to (5) place more weights on the stocks that are expected to yield higher returns per unit risk. In order to maintain optimality, this form of investment managements involves frequent portfolio revision in response to the information affecting prices. An example of how an active manager forms return conjectures based on the simple moving-average trading rule is given below. $\mathbf{E}_t$ is set as some monotonically increasing function of the moving-average price difference, and is defined as[2]

$$\mathbf{E}_t = f(\mathbf{P}_{t-1} - \frac{1}{m}\sum_{i=1}^{m}\mathbf{P}_{t-i}),\ f \in \Re^+ .$$

m is the moving average length. The expected return is thus based on the comparison of the latest available price and the moving-average price of a chosen length of history. The function f is not required to retain a certain range expect $\Re^+$, since it will undergo normalization, as shown by (5), before the portfolio weights are formed.

Throughout this study, we will consider $\mathbf{W}_t$ as a $3S \times 1$ vector given by

$$\mathbf{W}_t = \begin{pmatrix} \mathbf{w}_t^P \\ \mathbf{w}_t^A \\ \mathbf{w}_t^X \end{pmatrix},$$

where $\mathbf{w}_t^P$ is given by (4), $\mathbf{w}_t^A$ by (5), and $\mathbf{w}_t^X = \mathbf{0}_S$.

The profitability of fund management style i is assessed by the returns generated by its portfolio choices. Let π_t^i denote the profitability of fund style i for time period t. π_t^i is a scalar defined by

$$\pi_t^i = (\mathbf{w}_{t-1}^i)'\mathbf{R}_t, \tag{6}$$

where $\mathbf{R}_t = \mathbf{P}_t \odot \mathbf{P}_{t-1}^{-1}$ is the $S \times 1$ vector of stock returns from time t-1 to t.

4. The Markov Model of "Smart Money"

As equation (3) shows, one crucial factor in the dynamic equilibrium process is the time-varying capital flows among different investment portfolios. We model the dynamics of investment flows using a Markov chain. In a Markov process, the distribution of next states depends on the transition probabilities and the distribution of current states. Transition probabilities govern the probability of moving from one state to another, which in this study is considered to be time-varying and a function of some explanatory variables. This dependency property of a Markov chain makes it a natural and appealing choice for dynamic modelling.

Denote by θ_t^P and θ_t^A the capital ratios of passive and active portfolio investments, and as before, θ_t^X is the capital ratio staying out of the market. The ecology of capital ratios is thus a 3×1 vector given by

$$\Theta_t = \begin{pmatrix} \theta_t^P \\ \theta_t^A \\ \theta_t^X \end{pmatrix}, \text{ and } \Theta_t^{'} \mathbf{1}_3 = 1. \tag{7}$$

The Markovian dynamics is characterized by

$$\Theta_{t+1}^{'} = \Theta_t^{'} \mathbf{M}_t, \tag{8}$$

where $\mathbf{M}_t$ is the transition matrix, and is defined as

$$\mathbf{M}_t = \begin{bmatrix} \mathrm{Pr}_t^{PP} & \mathrm{Pr}_t^{PA} & \mathrm{Pr}_t^{PX} \\ \mathrm{Pr}_t^{AP} & \mathrm{Pr}_t^{AA} & \mathrm{Pr}_t^{AX} \\ \mathrm{Pr}_t^{XP} & \mathrm{Pr}_t^{XA} & \mathrm{Pr}_t^{XX} \end{bmatrix}. \tag{9}$$

Pr_t^{ij} denotes the transition probability of capital moving from fund style (strategy) *i* to *j*. For example, Pr_t^{XX} denotes the probability of remaining out of market, and Pr_t^{AX} measures the probability that a client closes his account with active portfolio management and exits the market. The transition matrix is subject to the constraint $\mathbf{M}_t \mathbf{1}_3 = \mathbf{1}_3$, i.e.

$$\sum_j \mathrm{Pr}_t^{ij} = 1. \tag{10}$$

The off-diagonal transition probabilities in (9) can be further expressed in terms of the probability of staying with the original fund and the conditional probability on leaving. Denote by λ_t^{ij} the probability of moving to fund style *j* conditional on a definite departure from fund i, where $i \neq j$. The off-diagonal transition probabilities are given by

$$\Pr_t^{ij} = \lambda_t^{ij}(1 - \Pr_t^{ii}), \text{ for } i \neq j, \text{ and } \sum_j \lambda_t^{ij} = 1. \tag{11}$$

The transition matrix (9) now becomes

$$\mathbf{M}_t = \begin{bmatrix} \Pr_t^{PP} & (1-\lambda_t^{PX})(1-\Pr_t^{PP}) & \lambda_t^{PX}(1-\Pr_t^{PP}) \\ (1-\lambda_t^{AX})(1-\Pr_t^{AA}) & \Pr_t^{AA} & \lambda_t^{AX}(1-\Pr_t^{AA}) \\ (1-\lambda_t^{XA})(1-\Pr_t^{XX}) & \lambda_t^{XA}(1-\Pr_t^{XX}) & \Pr_t^{XX} \end{bmatrix}. \tag{12}$$

The use of conditional probabilities λ_t^{ij} helps to capture the idea of transition from one state to another in a more hierarchical fashion. Notice that the use of conditional probabilities does not simplify estimation as it involves no parameter reduction; we have six free parameters in (9) and also six in (12). This holds true even when the number of states increases.

The transition probabilities characterize the Markovian dynamics of investment flows. We consider that the probability of capital flowing from one fund management to another is not exogenously prearranged, but instead it depends on the relative fund performance that is regularly updated with new stock prices. That is, the present study endogenizes the transition probabilities to capture how smart money follows the winning fund. Endogenizing transition probabilities in fact completes the investment cycle by linking stock prices, which are shaped by investment flows, with the probabilities that determine the dynamics of investment flows. The following presents how the transition probabilities are endogenized.

We consider that the probability of staying with the original fund style i, $\Pr_t^{ii}$, reflects a measure of self efficiency, and that the conditional probability of moving to management style j on abandoning i, λ_t^{ij}, reflects a comparison across new fund styles other than i. An illustrative example is given below. Suppose $\Pr_t^{ii}$ is a logistic function. The transition probabilities are given by

$$\Pr_t^{ii} = \frac{\exp(\alpha\, \pi_t^i)}{1+\exp(\alpha\, \pi_t^i)} = 1 - \frac{1}{1+\exp(\alpha\, \pi_t^i)}; \tag{13}$$

$$\Pr_t^{ij} = \lambda_t^{ij}\left(\frac{1}{1+\exp(\alpha\, \pi_t^i)}\right), \text{ where } \sum_j \lambda_t^{ij} = 1 \text{ and } i \neq j.$$

Fund profitability π_t^i, defined by (6), is chosen to be the explanatory variable but with a slight modification. Here we use log return instead of simple return for $\mathbf{R}_t$. There are two main reasons. First, the sign of π_t^i will now clearly indicate whether or not a loss has occurred. Second, this has the benefit of making the symmetric logistic transition function centre at 0.5 when the profitability is neutral. According to (13), clearly higher profitability leads to a higher probability of staying.

Profitability is one most straightforward measure of investment performance. Other choices include risk-adjusted measures such as efficiency by the Sharpe ratio. Further, the choice of explanatory variables in transition probabilities can go beyond the performance measures to include factors such as market conditions. Although market conditions are not modelled here, the coefficient α is related to investor sentiments that may to some extent reflect market conditions.

We assume $\alpha \geq 0$. The coefficient on profitability, α, measures the smart money's responsiveness to a change in fund profitability. This can be seen by rearranging (13),

$$\alpha = \frac{\partial \ln\left(\frac{\Pr^{ii}}{1-\Pr^{ii}}\right)}{\partial \pi^{i}}.$$

$\frac{\Pr^{ii}}{1-\Pr^{ii}}$ is sometimes called the *odds* of staying with fund style *i*. α is the multiplier on the explanatory variable of the logarithm of the odds of staying. A high α leads to a high probability of staying if the fund management makes positive profits, but also a high probability of changing if a loss occurs. Given a change in the fund profitability, a high α implies a dramatic change in transition probabilities. Therefore, a large α characterizes the "overreacting" smart money. For example, nervous investors change their fund styles or fund managers after one single bad moment. A counterexample is given by pension funds. Pension Funds tend to have a relatively lower α and be sticky to their fund managers.

The conditional probability λ_t^{ij} reflects, given a sure change in the fund management, how smart money picks up a new fund style *j*. λ_t^{ij} can be viewed as a function that compares both the benefits and the costs of all fund styles excluding *i*, since it is conditional on a sure leave from *i*. Let f be a monotonically increasing function that maps $\Re \to \Re^{+}$. We define conditional probabilities consistent with the requirement (11) by

$$\lambda_t^{ij} = \frac{f(\pi_t^j - c_t^j)}{\sum_{k \neq i} f(\pi_t^k - c_t^k)}, \tag{14}$$

where $\pi - c$ represents the cost-adjusted profits, and $\pi, c \in \Re$. Costs may include transaction costs and management fees, and are defined as a constant fraction of portfolio returns.

$$c_t^j = \begin{cases} \overline{c}\pi_t^j & when \;\; j = A. \\ \underline{c}\pi_t^j & when \;\; j = P. \\ 0 & when \;\; j = X. \end{cases} \tag{15}$$

We further impose $\overline{c} > \underline{c} > 0$ to indicate that low-cost indexing still incurs some transaction expenses, and that investors pay higher management fees and commissions to invest in actively managed funds than indexing funds. When active portfolio management no longer outperforms others, this high entry cost encourages a shift to lower-cost passive management or even a market exit.

5. Steady State

Steady state concerns the long-term behaviour of a dynamic system. This section solves the steady-state solution for the Markovian dynamics (8) with the transition matrix given by (12), under the simplifying assumption of constant transition probabilities. That is, the tendency of capital moving from one investment style to another is assumed fixed over time. Although unrealistic, this assumption simplifies the calculation to a great extent.

In steady state, the Markov chain reaches a stationary distribution and Θ has the ergodic[3] property $\Theta^{'}\mathbf{M} = \Theta^{'}$. In addition, since Θ represents the capital ratios, it must satisfy $\Theta^{'}\mathbf{1}_3 = 1$ as given by (7). Thus, the steady-state solution is in fact the normalized left[4] eigenvector of M, corresponding to the eigenvalue unity. We solve for the steady-state solution and it is given by:

$$\theta^P = \frac{1}{\kappa}(1-\Pr^{XX})(1-\Pr^{AA})\big((1-\lambda^{XA})\lambda^{AX} + (1-\lambda^{AX})\big),$$
$$\theta^A = \frac{1}{\kappa}(1-\Pr^{XX})(1-\Pr^{PP})\big(1-(1-\lambda^{XA})\lambda^{PX}\big), \text{ and} \tag{16}$$
$$\theta^X = \frac{1}{\kappa}(1-\Pr^{AA})(1-\Pr^{PP})\big(1-(1-\lambda^{PX})(1-\lambda^{AX})\big), \text{ where}$$
$$\kappa = (1-\Pr^{XX})(1-\Pr^{AA})\big((1-\lambda^{XA})(\lambda^{AX}) + (1-\lambda^{AX})\big) + (1-\Pr^{XX})(1-\Pr^{PP})\big(1-(1-\lambda^{XA})\lambda^{PX}\big) + (1-\Pr^{AA})(1-\Pr^{PP})\big(1-(1-\lambda^{PX})(1-\lambda^{AX})\big)$$

and $\Pr^{ii} < 1$, $i = P, A, X$.

We now apply the steady-state results to illustrate the limiting cases of the functional forms[5] given by (13). We consider both the cases when $\alpha = 0$ and when $\alpha \to \infty$. First, $\alpha = 0$ leads to $\Pr^{ii} = \frac{1}{2}$ and $\Pr^{ij} = \frac{\lambda^{ij}}{2}$, where $\sum_j \lambda^{ij} = 1$ for $i \neq j$. The steady-state capital ratios now become

$$\theta^P = \frac{1}{\kappa}\big[(1-\lambda^{XA})\lambda^{AX} + (1-\lambda^{AX})\big],$$
$$\theta^A = \frac{1}{\kappa}\big[1-(1-\lambda^{XA})\lambda^{PX}\big], \text{ and} \tag{17}$$
$$\theta^X = \frac{1}{\kappa}\big[1-(1-\lambda^{PX})(1-\lambda^{AX})\big], \text{ where}$$
$$\kappa = (1-\lambda^{XA})\lambda^{AX} + (1-\lambda^{AX}) + 1 - (1-\lambda^{XA})\lambda^{PX} + 1 - (1-\lambda^{PX})(1-\lambda^{AX}).$$

Moreover, it is interesting to observe that if all the conditional probabilities λ^{ij} are set to be $\frac{1}{2}$, the steady-state results (17) even reduce to $\theta^P = \theta^A = \theta^X = \frac{1}{3}$. Thus, when $\alpha = 0$ and $\lambda^{ij} = \frac{1}{2}$, we will have fixed and equal capital ratios among different fund styles over time. We shall refer to this case as the static benchmark model. This extreme case of a small α is consistent with the discussion before, regarding the stickiness (or under-reaction) of investment when α is low.

On the other hand, when $\alpha \to \infty$, two situations arise. If $\pi^i > 0$, then $\lim_{\alpha \to \infty} \Pr^{ii} = 1$ and $\lim_{\alpha \to \infty} \Pr^{ij} = 0$ for $i \neq j$. If instead $\pi^i < 0$, then $\lim_{\alpha \to \infty} \Pr^{ii} = 0$ and $\lim_{\alpha \to \infty} \Pr^{ij} = \lambda^{ij}$, where $\sum_j \lambda^{ij} = 1$ for $i \neq j$. However a non-degenerate steady-state solution fails to exist in either of these situations, as the limiting transition matrices in both examples are reducible and do not satisfy the properties of ergodicity for the existence of a steady state of a Markov chain.

6. Comparative Statics

Steady state describes the long-term behaviour of capital movements, but it does not tell us how prices respond to capital movements caused by a change in transition probabilities. Price formation reflects the dynamics of investment flows characterized by transition probabilities. The impact on prices of a change in transition probabilities can be understood analytically by comparative statics and numerically by simulation. Simulation experiments are carried out in the next section. This section applies the analysis of comparative statics to examine the impact on steady-state prices due to the following three causes ranging from general to specific: first, a change in transition probabilities, second, a change in conditional probabilities on leaving the current state, and third, a change in the featuring factor of transition probabilities such as the responsiveness to profitability.

From market equilibrium condition (1), the price vector can be rewritten as

$$\mathbf{P}_t = K\sum_{i=1}^{N}\theta_t^i\,(\mathbf{w}_t^i \odot \mathbf{S}_t^{-1}). \tag{18}$$

It is convenient to define an $S \times 1$ vector $\mathbf{H}_t^i = \mathbf{w}_t^i \odot \mathbf{S}_t^{-1}$, and express the price vector as

$$\mathbf{P}_t = K\sum_{i=1}^{N}\theta_t^i\,\mathbf{H}_t^i. \tag{19}$$

Since $\mathbf{w}_t^i$ represents the portfolio weights on S stocks by strategy i, $\mathbf{H}_t^i$ can be simply understood as *demands per share*, which mainly reflects the strategy's expectation on future returns of different stocks.

Suppose prices are in steady state denoted by $\mathbf{P}^*$. We obtain the following results of comparative statics. Their proofs are given in Appendix A.

$$\frac{\partial \mathbf{P}^*}{\partial \mathrm{Pr}^{sk}} = K\theta^s\left(\mathbf{H}^k + \sum_{j\neq k}\frac{\partial \mathrm{Pr}^{sj}}{\partial \mathrm{Pr}^{sk}}\mathbf{H}^j\right), \text{ where } \sum_{j\neq k}\frac{\partial \mathrm{Pr}^{sj}}{\partial \mathrm{Pr}^{sk}} = -1. \tag{20}$$

$$\frac{\partial \mathbf{P}^*}{\partial \lambda^{sk}} = K\theta^s\left(1-\mathrm{Pr}^{ss}\right)\left(\mathbf{H}^k + \sum_{j\neq k}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}}\mathbf{H}^j\right), \text{ where } \sum_{j\neq k}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}} = -1. \tag{21}$$

$$\frac{\partial \mathbf{P}^*}{\partial \alpha} = K\sum_i \left(\theta^i \sum_{j \neq i} \frac{\partial \Pr^{ij}}{\partial \alpha} (\mathbf{H}^j - \mathbf{H}^i) \right) \tag{22}$$

Suppose now α is fund-specific so it can be written as α^i. (22) now becomes

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K\theta^s \sum_{j \neq s} \frac{\partial \Pr^{sj}}{\partial \alpha^s} \left(\mathbf{H}^j - \mathbf{H}^s\right). \tag{23}$$

In the case of logistic transition probabilities (12), we obtain

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K\theta^s \Pr^{ss}(1-\Pr^{ss})\pi^s \left(\mathbf{H}^s - \sum_{j \neq s} \lambda^{sj} \mathbf{H}^j \right), \text{ where } \sum_{j \neq s} \lambda_t^{sj} = 1. \tag{24}$$

An application of (20) is given as follows. Suppose now passive indexing becomes a popular investment approach, i.e. $k = passive\ indexing = P$ in equation (20). What is the resulting impact on stock prices? The sign will depend on the expectation of the indexed fund on future stock returns. More precisely, its impact on the price of a particular stock is positive, if the demand per share by indexers on the stock is greater the normalized sum of the demand per share by others, i.e. the second term on the RHS of equation (20) is greater than zero. An important implication of (20) is that a strategy's optimism on a particular stock can have a positive impact on the price of the stock if the strategy becomes popular.

Furthermore, since $\mathbf{H}^X = \mathbf{0}_S$, an application of (20) suggests that the tendency in staying out of the market (i.e. $k = non-investing = X$ in equation 20) always has a negative impact on prices. The resulting price falls can be easily understood as a consequence of a lack of investments.

Equation (21) tells us how prices are affected by a change in conditional probabilities. Its implication is similar to that of (20) by the same reasoning. (22) and (23) are better understood by their application (24) with logistic transition probabilities. As discussed before, α measures the responsiveness to profitability. Let us consider the case of active fund management, i.e. $s = active\ fund\ management = A$ in equation (24). Overreacting smart money or a higher α implies a significant increase in capital inflow when actively managed fund makes profits, i.e. $\pi^A > 0$. By equation (24), this results in a positive impact on the price of a particular stock if active management has a high expectation on the stock's returns (i.e. the last term on the RHS of equation 24 is greater than zero).

7. Simulation Experiments and Results Discussion

Computational simulation of market history provides a dynamic perspective on the impact of smart money movements. In this section, the pricing process is simulated for each trading period. The process can be understood as follows: stock prices determine fund profitability that in turn influences the flows of smart money; the dynamics of smart money then determines stock prices through market equilibrium, and the whole process repeats.

The simulation is based upon the equilibrium price equation (3). The portfolio allocations of passive and active fund managements are given by (4) and (5) respectively, with m set to be 10. Besides, f is set as an adjusted hyperbolic tangent function, $f(x) = \tanh(x/2) + 1$. To avoid the problem of non-existence of the inverse covariance matrix and for the sake of simplicity, we assume an identity matrix for $\mathbf{\Omega}_t$. The dynamics of smart money Θ_t is modelled by the Markov chain (8) with the transition matrix illustrated by (12). Further, transition probabilities are endogenized as functions of fund profitability and associated expenses, as given by (13), (14) and (15). The costs of active and passive fund managements as a fraction of their profitability, $\bar{c}$ and $\underline{c}$, are set to be 0.2 and 0.02 respectively. The pricing process is simulated for 2000 trading periods.

We apply different levels of investor sentiment in response to fund profitability. The results are compared with the static benchmark model ($\alpha = 0, \lambda^{ij} = \frac{1}{2}$), with no capital movements and an allocation of fixed and equal capital ratios among non-investment, indexing and actively managed funds. Results are assessed in terms of "drawdown". Drawdown is a risk measure that quantifies uninterrupted falls in security prices. The study considers three aspects of drawdowns, namely the duration, the depth, and the number of drawdowns, in both indexing and actively managed accounts.

For a price series p_t, a drawdown is simply defined as a sequence of $\{p_t\}_{t=d_0}^{d_0+d}$ for $d > 1$, where all the prices of the sequence fall down while there exists a price rise both immediately before and after the sequence; d is the duration or the length of the drawdown. In terms of a returns series $r_t = \frac{p_t}{p_{t-1}}$, a drawdown can be equivalently defined as a sequence of $\{r_t\}_{t=d_0+1}^{d_0+d}$ for d > 1, where $r_t < 1$ for $t = d_0 + 1,\ d_0 + 2, ..,$ $d_0 + d$, while $r_{d_0} > 1$ and $r_{d_0+d+1} > 1$. Without loss of generality, we assume $d_0 = 0$.

The absolute depth of a drawdown is calculated by the difference between p_0 and p_d, i.e. $p_0 - p_d$, and the relative depth by the ratio $D_d = \frac{p_0 - p_d}{p_0}$. In this study, we compute the relative depth, D_d, that relates to the beginning position of a drawdown. D_d is calculated conditional on the preceding local peak, so that a drawdown is considered less severe if its preceding local peak is comparatively high. The relative depth of a drawdown can also be expressed in terms of returns by $1 - \prod_{t=1}^{d} r_t$. Finally, let N_d denote the number of drawdowns in a series. Note that N_d is bounded above by $(T+1)/2$ for a series of T trading periods.

Table 1 reports the results of drawdown[6] calculated from index (or market) returns and also active investment returns. Figure 1 provides the distributions of the sizes of the drawdown.

Table 1 Drawdown results of the investment returns from the market index fund and actively managed fund.

		Number of Period Loss (Frequency)	Number of Drawdown (Frequency)	Average Duration of Drawdown	Average Depth of Drawdown	Maximum Depth of Drawdown
Fixed Θ, $\theta^P = \theta^A = \theta^X = 1/3$	Index	806 (0.3224)	615 (0.246)	1.31057	0.0390024	0.18136
	Active	1694 (0.6776)	614 (0.2456)	2.75896	0.133933	0.677253
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 1)$	Index	791 (0.3164)	622 (0.2488)	1.2717	0.0289949	0.404647
	Active	1709 (0.6836)	608 (0.2432)	2.81086	0.11926	0.618233
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 7)$	Index	1139 (0.4556)	395 (0.158)	2.88354	0.187774	0.381794
	Active	1351 (0.5404)	388 (0.1552)	3.48196	0.352543	0.648755
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 11)$	Index	1083 (0.4332)	363 (0.1452)	2.98347	0.463669	0.556648
	Active	1222 (0.4888)	359 (0.1436)	3.4039	0.573109	0.692364
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 100)$	Index	826 (0.3304)	329 (0.1316)	2.51064	0.556938	0.653068
	Active	1110 (0.444)	326 (0.1304)	3.40491	0.598585	0.777556

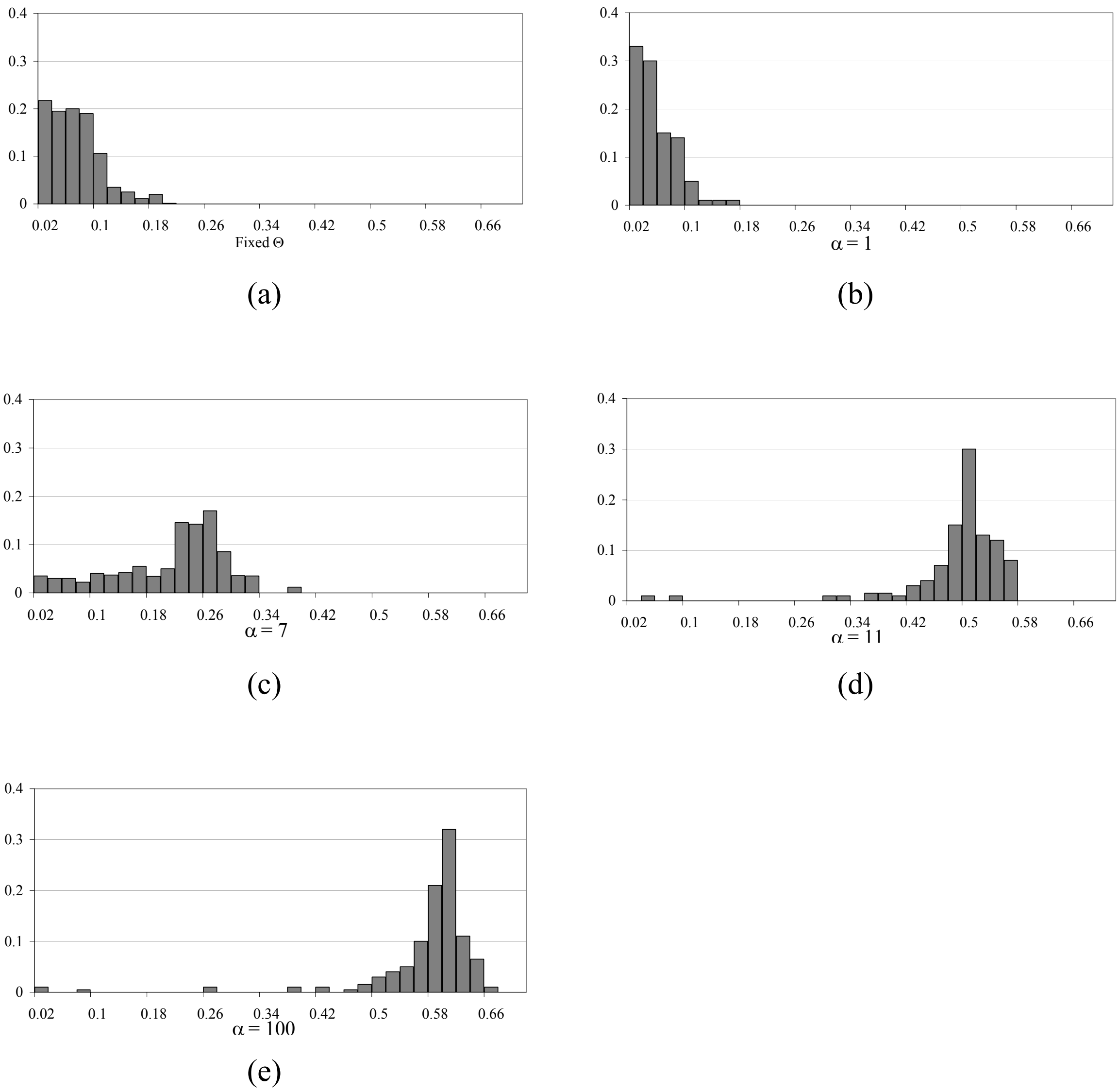

Figure 1 The distributions of the relative sizes of drawdown.

A number of patterns can be found from the results. These patterns can lead to rather different implications on investor sentiment. First, comparing only the results of varying levels of α excluding the static benchmark model, we find that overall the number of drawdown decreases but the average duration of drawdown increases as α

gets larger. Furthermore, the average depth of drawdown also increases with α. The results suggest that overreaction aggravates both the duration and the size of drawdown in investment returns in the market index and active managed accounts.

We may then tend to think that the static benchmark model with no investment movements would have a lower downside risk. Surprisingly, the static model in fact leads to a larger drawdown size in average than the dynamic model with a small α. This observation implies the existence of a stabilizing force when there is a limited degree of capital movements.

There is an interesting observation that the number of period loss of actively managed fund always exceeds its counterpart of market index. This outcome is considered to be attributed to active management being inherently more risky.

Table 2 reports the mean and variance of both market returns and active portfolio returns, together with the correlation coefficient between these two return series. Although overall the mean returns of both management strategies[7] slightly increase with α, their variances exhibit an unproportionally large increase at the same time. Overreaction adversely induces a volatile market. Higher return volatility in the case of overreacting smart money is consistent with the drawdown results.

Table 2 Mean, variance, and correlation of index returns and active investment returns.

		Mean	Variance	Correlation Coefficient
Fixed Θ, $\theta^P = \theta^A = \theta^X = 1/3$	Index	1.02608	0.00282958	-1
	Active	0.973918	0.00282958	
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 1)$	Index	1.01962	0.00179315	-0.990231
	Active	0.976985	0.00241346	
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 7)$	Index	1.02767	0.0114034	0.594797
	Active	0.975898	0.0145883	
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 11)$	Index	1.04795	0.058714	0.921535
	Active	1.00201	0.0591932	
$\Theta'_{t+1} = \Theta'_t M_t$ $\Pr_t(\alpha = 100)$	Index	1.06738	0.0890357	0.888454
	Active	1.04008	0.092746	

Now we turn to the discussion of the correlation outcomes. The profitability of these two investment strategies moves in the opposite directions in the benchmark model and also when α is low, i.e. when clients' money is sticky to the original investments. The reason to the observed negative return correlation can be grasped intuitively that when there is no source of investment inflow and the market capital size remains fixed, one strategy can only be profitable at the cost of the other. In the case of the static benchmark model, there is no investment entry or exit and the market capital ratio always remains $2/3$. Appendix B provides a proof that index returns and active investment returns sum to a constant, by imposing the constraint of fixed and equal capital ratios among different investment strategies. Since the sum of these two returns at each time period is fixed under certain assumptions, it becomes clear that one strategy is profitable at the expense of the other, and so their profitability moves in the reverse directions as shown by their correlation coefficient. The sum of these two returns remaining fixed also explains why their variances are virtually the same by a straightforward proof.

The return correlation however becomes positive when α increases. This observation implies a counter-intuitive situation when active and passive fund managements can be simultaneously profitable. Considering the price mechanism, this is not as surprising as it seems. In the presence of overreacting smart money, a profitable active fund management quickly attracts a vast investment inflow from not only index believers but also non-market participants. This pushes up the prices of the stocks on which active fund management puts more weights, and hence the overall market index price. Here the investment inflow from out of market ($\Pr^{XA}$) is crucial. If the investment inflow into active fund management comes merely from passive management, the weakened passive investment is likely to offset the push-up effect on the market index by the strong active investment.

The most straightforward example is found in bull markets, where various active fund managements can be profitable at the same time when the market index is soaring[8]. This can be grasped by that market conditions influence investor sentiments, and in particular, bull markets trigger massive new investment inflows that boost profitability. Even in the rare case when passive indexing has no investment inflow, the market price can still go up due to the push-up effect of a strong active investment.

8. Concluding Remarks

Since the birth of mutual funds and the emergence of diversified fund styles, there have been numerous observations on significant investment movements among different fund managements, e.g. a recent trend towards passive indexing. These phenomena have drawn my interest into studying their impact on financial markets. This paper develops a Markovian model of smart money chasing past winning funds while taking into account the associated costs. This study also allows market capital entry and exit by making non-participation as one investment choice. In seeking the long-term behaviour, the steady-state capital allocations among different investment strategies are derived under some simplified assumption on transition probabilities. It then studies the resulting investment risk induced by the Markovian dynamics of smart money, using both the analyses of comparative statics and computational simulation.

One major finding of the results suggests that downside risk can be significantly attributed to overreacting smart money drastically moving from one fund style to another, even though a small degree of investment movement as opposed to capital immobility can in fact stabilize the market. Here downside risk is measured in terms of both the duration and the depth of drawdown. Moreover, the results using the conventional risk measure are consistent with the drawdown results.

This paper also finds that when money is sensitive to fund performance, profitable active fund management is likely to trigger vast capital inflow that pushes up the asset prices of active portfolios and hence the overall index prices, given that there is no offsetting effect from possibly weakened passive investment. Therefore, performance sensitivity and new capital inflow make it possible that two much debated portfolio management styles, passive indexing and active fund management, can be simultaneously profitable. By the same token, a rapid investment withdraw triggered by overreacting investors in response to either bad news or underperformance can lead to active fund management being just as devastating as market index. On the contrary, if money is insensitive and investment capital remains immobile, the model is in fact a typical zero-sum game where one strategy will be profitable at the cost of the other.

Is the returns-chasing behaviour enabled by market liquidity socially desirable? Several implications can be drawn from the results. The returns-chasing behaviour induces a natural selection of investment funds, so that ill-performing funds are liquidated or merged while outperforming ones accumulate even more capital. From the viewpoint of seeking a valid investment tool, the increased competition level may enhance the effectiveness of asset managements. Besides, some degree of liquidity is desirable since it may work as a stabilizing force to the market as the results suggest. However, there are tradeoffs as well as benefits. The market with overreacting smart money chasing past winners and abandoning poor performing funds implies a higher downside risk. Particularly in bear markets, vast investment withdraws in a hasty fashion can exacerbate the already worsening market condition.

Appendix A

Proof A

Suppose prices are in steady state and so is the ecology of capital ratios. Express the ergodic property for a steady-state ecology, $\Theta' = \Theta' \mathbf{M}$, in scalars,

$$\theta^j = \sum_i \theta^i \Pr^{ij} . \tag{A.1}$$

From (19) and (A.1), the steady-state price vector can be written as

$$\mathbf{P}^* = K \sum_j \sum_i \theta^i \Pr^{ij} \mathbf{H}^j . \tag{A.2}$$

By separating whether $i = s$ or $j = k$, (A.2) can be further decomposed into

$$\mathbf{P}^* = K \left\{ \theta^s \Pr^{sk} \mathbf{H}^k + \sum_{i \neq s} \theta^i \Pr^{ik} \mathbf{H}^k + \sum_{j \neq k} \theta^s \Pr^{sj} \mathbf{H}^j + \sum_{j \neq k} \sum_{i \neq s} \theta^i \Pr^{ij} \mathbf{H}^j \right\}. \tag{A.3}$$

From the equivalent of constraint (10), $\sum_j \Pr^{sj} = 1$, it is easy to obtain $\sum_{j \neq k} \dfrac{\partial \Pr^{sj}}{\partial \Pr^{sk}} = -1$.

Also we know that $\left.\dfrac{\partial \Pr^{ij}}{\partial \Pr^{sk}}\right|_{i \neq s} = 0$. Therefore, by partial differentiation of the steady-state price given by (A.3) with respect to the transition probability, we obtain

$$\frac{\partial \mathbf{P}^*}{\partial \Pr^{sk}} = K\theta^s \left(\mathbf{H}^k + \sum_{j \neq k} \frac{\partial \Pr^{sj}}{\partial \Pr^{sk}} \mathbf{H}^j \right), \text{ where } \sum_{j \neq k} \frac{\partial \Pr^{sj}}{\partial \Pr^{sk}} = -1 .$$

Proof B

We first decompose the steady state price (A.2) into four parts.

$$\mathbf{P}^* = K\left\{ \theta^s \Pr^{ss} \mathbf{H}^s + \theta^s \Pr^{sk} \mathbf{H}^k \Big|_{k \neq s} + \sum_{\substack{j \neq s \\ j \neq k}} \theta^s \Pr^{sj} \mathbf{H}^j + \sum_j \sum_{i \neq s} \theta^i \Pr^{ij} \mathbf{H}^j \right\} \qquad \text{(A.4)}$$

Recall that λ^{ij} is not defined when $i = j$, and that the diagonal entries in the transition probability matrix $\mathbf{M}$ have no λ^{ij}, thus

$$\frac{\partial \Pr^{ss}}{\partial \lambda^{sk}} = 0 .$$

Also we know that $\Pr^{ij}$ is independent of λ^{sk} if $i \neq s$, thus

$$\left. \frac{\partial \Pr^{ij}}{\partial \lambda^{sk}} \right|_{i \neq s} = 0 .$$

The partial differentiation of (A.4) with respect to the conditional transition probability hence yields

$$\frac{\partial \mathbf{P}^*}{\partial \lambda^{sk}} = K\left\{ \theta^s \frac{\partial \Pr^{sk}}{\partial \lambda^{sk}} \mathbf{H}^k + \sum_{\substack{j \neq s \\ j \neq k}} \theta^s \frac{\partial \Pr^{sj}}{\partial \lambda^{sk}} \mathbf{H}^j \right\}. \qquad \text{(A.5)}$$

Since the off-diagonal transition probabilities are defined by $\left.\Pr^{sk}\right|_{k\neq s} = \lambda^{sk}(1-\Pr^{ss})$, it is straightforward that

$$\left.\frac{\partial \Pr^{sk}}{\partial \lambda^{sk}}\right|_{s\neq k} = 1-\Pr^{ss}.$$

By chain rule, $\left.\frac{\partial \Pr^{sj}}{\partial \lambda^{sk}}\right|_{\substack{j\neq s \\ j\neq k}} = \frac{\partial \Pr^{sj}}{\partial \lambda^{sj}}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}} = (1-\Pr^{ss})\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}}$.

Also, from (11), $\sum_{j}\lambda^{sj} = 1$, it is easy to show that $\sum_{j\neq k}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}} = -1$.

Therefore, (A.5) reduces to

$$\frac{\partial \mathbf{P}^{*}}{\partial \lambda^{sk}} = K\theta^{s}\left(1-\Pr^{ss}\right)\left(\mathbf{H}^{k} + \sum_{j\neq k}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}}\mathbf{H}^{j}\right), \text{ where } \sum_{j\neq k}\frac{\partial \lambda^{sj}}{\partial \lambda^{sk}} = -1.$$

Proof C1

Suppose $\Pr^{ij} = \Pr^{ij}(\alpha)$.

From the constraint (10), we know that $\sum_j \Pr^{ij}(\alpha) = 1$, and hence $\sum_j \frac{\partial \Pr^{ij}(\alpha)}{\partial \alpha} = 0$.

Separating the diagonal entries from the off-diagonal ones yields

$$\frac{\partial \Pr^{ii}(\alpha)}{\partial \alpha} = -\sum_{j \neq i} \frac{\partial \Pr^{ij}(\alpha)}{\partial \alpha}. \tag{A.6}$$

From (A.2), we can obtain

$$\frac{\partial \mathbf{P}^*}{\partial \alpha} = K \sum_i \sum_j \theta^i \frac{\partial \Pr^{ij}}{\partial \alpha} \mathbf{H}^j = K \sum_i \left(\theta^i \frac{\partial \Pr^{ii}}{\partial \alpha} \mathbf{H}^i + \sum_{j \neq i} \theta^i \frac{\partial \Pr^{ij}}{\partial \alpha} \mathbf{H}^j \right). \tag{A.7}$$

Replacing (A.6) into (A.7) leads to

$$\frac{\partial \mathbf{P}^*}{\partial \alpha} = K \sum_i \left(\theta^i \sum_{j \neq i} \frac{\partial \Pr^{ij}}{\partial \alpha} (\mathbf{H}^j - \mathbf{H}^i) \right).$$

Proof C2

Now, suppose $\Pr^{ij} = \Pr^{ij}(\alpha^i)$, and $\alpha^i \neq \alpha^s$ if $i \neq s$. Thus, $\left.\frac{\partial \Pr^{ij}}{\partial \alpha^s}\right|_{i \neq s} = 0$.

Again from (A.2), we calculate the following.

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K \sum_j \sum_i \theta^i \frac{\partial \Pr^{ij}}{\partial \alpha^s} \mathbf{H}^j$$

$$= K \left\{ \theta^s \frac{\partial \Pr^{ss}}{\partial \alpha^s} \mathbf{H}^s + \sum_{j \neq s} \theta^s \frac{\partial \Pr^{sj}}{\partial \alpha^s} \mathbf{H}^j + \sum_j \sum_{i \neq s} \theta^i \frac{\partial \Pr^{ij}}{\partial \alpha^s} \mathbf{H}^j \right\}$$

$$= K\left\{\theta^s \frac{\partial \Pr^{ss}}{\partial \alpha^s} \mathbf{H}^s + \sum_{j \neq s} \theta^s \frac{\partial \Pr^{sj}}{\partial \alpha^s} \mathbf{H}^j \right\}$$

From the same reasoning as (A.6), we know

$$\frac{\partial \Pr^{ss}}{\partial \alpha^s} = -\sum_{j \neq s} \frac{\partial \Pr^{sj}}{\partial \alpha^s}.$$

It therefore follows that

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K\theta^s \sum_{j \neq s} \frac{\partial \Pr^{sj}}{\partial \alpha^s} \left(\mathbf{H}^j - \mathbf{H}^s\right). \tag{A.8}$$

An example is given below with the logistic transition probabilities (13).

We first obtain

$$\left.\frac{\partial \Pr^{sj}}{\partial \alpha^s}\right|_{j \neq s} = -\lambda^{sj} \Pr^{ss}(1 - \Pr^{ss})\pi^s.$$

From (A.8),

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K\theta^s \sum_{j \neq s} \frac{\partial \Pr^{sj}}{\partial \alpha^s} \left(\mathbf{H}^j - \mathbf{H}^s\right) = K\theta^s \Pr^{ss}(1 - \Pr^{ss})\pi^s \sum_{j \neq s} \lambda^{sj} \left(\mathbf{H}^s - \mathbf{H}^j\right).$$

Since $\sum_{j \neq s} \lambda^{sj} = 1$ as given by (11), it follows that

$$\frac{\partial \mathbf{P}^*}{\partial \alpha^s} = K\theta^s \Pr^{ss}(1 - \Pr^{ss})\pi^s \left(\mathbf{H}^s - \sum_{j \neq s} \lambda^{sj} \mathbf{H}^j\right), \text{ where } \sum_{j \neq s} \lambda_t^{sj} = 1.$$

Appendix B

Recall that market equilibrium at time t is given by (1) as

$$\mathbf{S}_t \odot \mathbf{P}_t = K\sum_{i=1}^{N}\theta_t^i \mathbf{w}_t^i ,$$

where $\mathbf{P}_t$, $\mathbf{S}_t$, and $\mathbf{w}_t^i$ are S-dimensional vectors of prices, outstanding shares, and portfolio weights of strategy i on S stocks. The scalars K and θ_t^i are the total capital size and the capital ratio allocated to investment fund i.

Rearrange the equilibrium condition (1) to obtain the price vector at time t

$$\mathbf{P}_t = K\sum_{i=1}^{N}\theta_t^i \mathbf{w}_t^i \odot \mathbf{S}_t^{-1} . \tag{B.1}$$

The static benchmark model assumes fixed and equal capital ratios among different investment funds. Imposing this assumption together with the simplified assumption of a constant number of outstanding shares, (B.1) becomes

$$\mathbf{P}_t = K\theta\mathbf{S}^{-1} \odot \left(\sum_{i=1}^{N}\mathbf{w}_t^i\right), \text{ and similarly, } \mathbf{P}_{t-1} = K\theta\mathbf{S}^{-1} \odot \left(\sum_{i=1}^{N}\mathbf{w}_{t-1}^i\right).$$

Therefore, returns defined as the price ratios are given by the $S \times 1$ vector

$$\mathbf{R}_t = \mathbf{P}_t \odot \mathbf{P}_{t-1}^{-1} = \left(\sum_{i=1}^{N}\mathbf{w}_t^i\right) \odot \left(\sum_{i=1}^{N}\mathbf{w}_{t-1}^i\right)^{-1} . \tag{B.2}$$

Following the model, consider three investment strategies, i.e. three states, $i = P, A,$ and X. Notice that $\mathbf{w}_t^X = \mathbf{0}_S$ and $\mathbf{w}_t^{i}{}' \mathbf{1}_S = 1$ for $i = P, A$. Let $\mathbf{1}_S$ denote an $S \times 1$ vector of ones. The return vector (B.2) then becomes

$$\mathbf{R}_t = \left(\mathbf{w}_t^P + \mathbf{w}_t^A\right) \odot \left(\mathbf{w}_{t-1}^P + \mathbf{w}_{t-1}^A\right)^{-1}.$$

Fund profitability is measured by its portfolio return. Portfolio return is a scalar defined by (6) and is calculated as the summation of stock returns multiplied by the corresponding portfolio weights. The sum of index return and active investment return is then given by

$$\begin{aligned}\pi_t^P + \pi_t^A &= (\mathbf{w}_{t-1}^P)'\mathbf{R}_t + (\mathbf{w}_{t-1}^A)'\mathbf{R}_t = (\mathbf{w}_{t-1}^P + \mathbf{w}_{t-1}^A)'\mathbf{R}_t \\ &= (\mathbf{w}_{t-1}^P + \mathbf{w}_{t-1}^A)' \left\{ \left(\mathbf{w}_t^P + \mathbf{w}_t^A\right) \odot \left(\mathbf{w}_{t-1}^P + \mathbf{w}_{t-1}^A\right)^{-1} \right\} \\ &= \mathbf{1}_S' \left(\mathbf{w}_t^P + \mathbf{w}_t^A\right).\end{aligned}$$

Since $\mathbf{w}_t^{i}{}' \mathbf{1}_S = 1$ for $i = P, A$, thus

$$\pi_t^P + \pi_t^A = \mathbf{1}_S' \mathbf{w}_t^P + \mathbf{1}_S' \mathbf{w}_t^A = 2.$$

Therefore, the proof has shown that under the restriction of fixed and equal capital ratios among different investment strategies and also the assumption of a constant number of outstanding stock shares, the sum of index return and active investment return is a constant.

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[1] The statistical properties of drawdown have been studies by Sancetta and Satchell (2003).

[2] The definition of E_t essentially follows the feedback function given by Yang and Satchell (2002).

[3] Ergodicity requires the transition matrix to be irreducible and non-periodic. For a more detailed discussion on ergodicity, see, for example, Cox and Miller (1965).

[4] The right eigenvector of M corresponding to the eigenvalue unity is $\mathbf{1}_{3\times1}$, since $M\mathbf{1}_{3\times1} = \mathbf{1}_{3\times1}$ as given by (8).

[5] Notice here we assume constant λ^{ij} since the steady-state results given by (16) are derived when transition probabilities are constant.

[6] The results of drawdown are calculated from portfolio returns π_t instead of simple returns r_t. As π_t given by (6) is just the weighted return in accordance with the underlying portfolio weights, the computation of the number, the duration, and the sizes of drawdown discussed above will still apply.

[7] In the model, the mean index return exceeds the mean active investment return by an insignificant amount. We believe that this observation is highly model-specific and is largely attributed to the employed active trading strategy.

[8] For instance, during the bull market of 1998 – 1999, the Fidelity aggressive growth fund achieved returns of 190%.