# Float on a Note

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**Abstract**

From 1863-1914, banks in the U.S. could issue notes subject to full collateral, a per-period tax on outstanding notes, redemption of notes on demand, and a clearing fee per issued note cleared through the Treasury. The system failed to satisfy a purported arbitrage condition; i.e., the yield on collateral exceeded the tax rate plus the product of the clearing fee and the average clearing rate of notes. The failure is explained by a model in which note issuers choose to issue notes only in trades that both produce a low clearing rate (high float) and are subject to diminishing returns.

## 1 Introduction

Under the U.S. National Banking System (NBS), in effect from 1863–1914, banks with national charters could issue notes under four main restrictions (see, for example, Friedman and Schwartz [4], pp. 20–23): (i) full collateral in the form of government bonds; (ii) a per-period tax on outstanding notes; (iii) redemption of notes into specie (or outside money) on demand; (iv) a clearing fee per issued note that is cleared through the Treasury's clearing system. The simple and predominant view of this system appeals to arbitrage to claim that the system should have produced an upper bound on the yield on collateral; namely, the tax rate plus the product of the clearing fee and the

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average clearing rate of notes. However, as is well known, yields on eligible collateral were generally higher than such a bound (see, [2]).[1]

Because the clearing rate was random, one route to an explanation is risk aversion regarding clearing fees. We rule out such risk aversion. Our explanation is a model in which the observed clearing rate is determined by the behavior of note issuers: they choose to issue notes only in situations that give rise to a low clearing rate (high float) and that happen also to give rise to diminishing returns from additional note issue. Those two features of the situations in which notes are issued allow the model to have a steady state with a yield on collateral higher than the presumed bound and not tied closely to it.

The salient features of note issue under the NBS can be described in more familiar terms as a way of operating a central bank discount window. The central bank lends at an interest rate equal to the tax rate in (ii) subject to the collateral requirement in (i). Loans take the form of notes that are identified with the borrower, perhaps by their serial numbers (analogous to notes under the NBS identified by the issuing bank). When notes associated with a given loan are cleared through the central bank's clearing system, the borrowing bank's debt and collateral are debited by the amount cleared, which corresponds to (iii), and a fee, the fee in (iv), is imposed proportional to the amount cleared. Under such a discount window scheme, a borrowing bank would be concerned about the float rate implied by different uses of the notes it receives from the central bank.

Such concern is one crucial ingredient in our explanation. The other is an inverse association between the float rate and the size of placement opportunities for notes.[2] In our model large placement opportunities imply a low float rate, small placement opportunities imply a high float rate, and only small placements occur. Therefore, the average clearing rate implied by the model is as low as it is because the low float-rate placement opportunity

[1] There is data on redemptions that occur through the Treasury's clearing system and on the fee charged for such redemptions. There is no data on redemptions that occur in other ways—over the counter—or on the costs of such redemptions. In [2], the authors infer total redemptions from data on currency clearings for the Federal Reserve System and assume that the fee charged by the Treasury applies to total redemptions. Then, they argue that the implied costs are high enough to bring the presumed bound into equality with the yield on eligible collateral. Others dispute this way of inferring total redemptions and the application of the Treasury fee to total redemptions (see [6] and [7]).

[2] The first ingredient, the idea that float was a relevant concern for an issuer, was set out, but not modeled, in [3]. The second ingredient seems to be new.

is not used. According to the model, the arbitrage claim is not valid because it treats the observed average clearing rate as if it applies to all potential placements of notes, including unlimited opportunities for note placements.

The two main ingredients of our explanation are plausible. The concern about float is dictated by the rules for note issue. The inverse association between the float rate and the size of placement opportunities for notes is consistent with the notion that large placement opportunities would have been available only in organized markets. But notes used in such markets would have very quickly been turned in to other banks, banks which had an incentive to clear them through the Treasury's note-clearing system.[3] Indeed, a high float rate and most conceptions of large markets seem incompatible. In order to achieve high float, notes should be offered in trade to people who are infrequently connected to such markets. And, almost by definition, such placement opportunities are limited in that they give rise to diminishing returns from additional note issue.

The model contains an extreme version of the inverse association between float rate and the size of placement opportunities. There are two kinds of placement opportunities. The large (unlimited) opportunity is the use of notes to purchase bonds offered on tap by the government. However, that use of notes is assumed to give rise to redemption in one period and, therefore, is unprofitable given the parameter values we assume. The other use is in pairwise meetings. That use gives rise to a random and higher float rate and is profitable on average, but not on the margin within meetings.

Although our model is simple and extreme, it suffices to illustrate the two main ideas. Moreover, it seems to be the first model in which float of any sort arises from decisions. While float on banknotes seems not to be an issue in modern economies, float on other financial instruments is. Our model provides hints about how to model float in other contexts.

# 2 The Model

Much of the background environment is the same as that of the Trejos-Wright [10] and Shi [8] models. There are $N \geq 3$ perishable types of goods at each date and a $[0, 1]$ continuum of each of $N$ specialization types of people. For

[3] Friedman and Schwartz hint at this when they say "An issuing bank ... had no way of identifying banks that returned its notes to the Treasury for redemption; hence its New York City correspondents could do so with impunity." ([4], footnote 8, page 21.)

$n \in \{1, 2, ..., N\}$, a type $n$ person consumes only good $n$ and is able to produce only good $n + 1$ (modulo $N$). Each person maximizes expected discounted utility.

Here, following Cavalcanti and Wallace [1], we divide the unit interval of each specialization type into two parts: the interval $[0, \alpha]$ are "bankers," while the remainder $(\alpha, 1]$ are "nonbankers," where $\alpha \in (0, 1/2)$ and is best thought of as being small.[4] We also divide each discrete date into two stages, called the "morning" and the "afternoon," with no discounting between the stages. The morning is reserved for the random pairwise meetings of the Trejos-Wright and Shi models, although, to simplify the model, we assume that bankers do not meet other bankers. The afternoon is reserved for bankers meeting the government and is used solely for note-clearing, bond purchases, and the levying of taxes and fees. Nonbankers do nothing in the afternoon.

The common period utility function is

$$U(c_1, p_1, c_2, p_2) = u(c_1) - p_1 + c_2 - p_2, \tag{1}$$

where $c_1$ is consumption in the morning, $p_1$ is production in the morning, $c_2$ is consumption in the afternoon, and $p_2$ is production in the afternoon and where all of these goods are the type-specific goods of the Trejos-Wright and Shi models. Goods are perishable at each stage. Of course, $c_2 = p_2 = 0$ for each nonbanker. There is a common discount factor $\beta \in (0, 1)$. As usual, we assume that $u'(0)$ is sufficiently large. The assumption of linearity for afternoon consumption and production greatly simplifies the analysis in ways we will explain.

There are three kinds of assets: outside money, bonds, and notes issued by bankers. The afternoon and the nature of bonds is modeled to achieve simplicity. We assume that assets are indivisible and that individual wealth holdings are bounded, but otherwise general. Interest, taxes, and fees are real and only bankers hold bonds and pay taxes and fees.[5] The government makes available to bankers one-period bonds: each bond costs a unit of outside money (or a note) and is a title to one unit of outside money *and* $r$ amount of the good at the next afternoon. (The bonds are registered, as opposed to being payable-to-the-bearer, and, therefore, cannot be traded

[4] There were thousands of National Banks under the NBS.

[5] The assumption that interest, taxes, and fees are real is an adaptation of an analogous assumption in [5]. There, storage costs for assets appear as a utility cost that does not affect the state of the economy. Here, interest, taxes, and fees do not affect the state of the economy.

in pairwise meetings in the morning.) We consider only equilibria in which nonbankers view all notes and outside money as perfect substitutes.

*The morning.* Because note-clearing occurs in the afternoon, each banker starts a morning with the following balance sheet. The assets are outside money, $y_m$, and bonds, $y_b$; the liabilities are own-notes held by the government, $y_g$, and own-notes held by nonbankers, $y_n$. All four items are measured in units of outside money.

| Banker Balance Sheet | |
|---|---|
| Assets | Liabilities |
| $y_m$ (outside money) | $y_g$ (own-notes held by the government) |
| $y_b$ (bonds) | $y_n$ (own-notes held by nonbankers) |

We distinguish between the two kinds of notes because they have different consequences for float. A banker's wealth is $y_m+y_b-y_g-y_n$. Each nonbanker holds some outside money and some notes. Letting $z$ denote the wealth of a person, either a banker or a nonbanker, we impose the restriction $z \in \{0, 1, ..., Z\} \equiv \mathbf{Z}$. The upper bound is imposed to achieve compactness. The lower bound for bankers mimics the full-backing feature of NBS notes.

We exclude pairwise meetings between bankers because such meetings would not add much to the model. In particular, we assume that each banker always meets a nonbanker, that a nonbanker meets a banker with probability $\lambda = \alpha/(1-\alpha)$, and that a nonbanker meets another nonbanker with probability $1-\lambda$. In other respects, meetings are random.

In meetings between nonbankers, we assume take-it-or-leave-it offers by buyers (consumers). In meetings between bankers and nonbankers, we assume that the banker always makes a take-it-or-leave-it offer—whether the banker is a buyer (consumer), a seller (producer), or is neither. Among other things, this permits the banker to exchange own-notes for the notes of other bankers or outside money on a one-for-one basis. Such swaps of monies, among which the nonbanker is indifferent, generate one source of note clearings for the afternoon.[6] Throughout, we allow randomization, lotteries, in trades.

[6]The assumption that bankers get to make take-it-or-leave-it offers in all meetings should not be regarded as an essential assumption. It is a simplifying assumption that implies that there is no wealth transfer in no-coincidence meetings. For our purposes, the main conclusion we want is that a nonbanker's wealth when leaving a meeting with a banker is entirely in the form of that banker's notes. Given that the nonbanker is indifferent about whose notes are held and that the banker is not, that conclusion will

*The afternoon.* First, bonds mature (turn into outside money and pay a real coupon), a tax is paid on own-notes outstanding (where the tax base is liabilities at the start of the morning plus any notes issued in the morning pairwise meeting), notes are cleared (which matters because each banker is charged a fee, lost to nature, proportional to the amount of its notes that are cleared), and a lump-sum tax is levied (which in equilibrium balances the within-period budget of the government). We assume that all notes held by the government are cleared and that all notes held by bankers are cleared.[7] Thus, after clearing, no banker holds other bankers' notes. Then bankers buy new bonds (or outside money) from the government, either with outside money or own-notes.

Although all our claims for this model are about steady states, a few remarks are in order about how the economy evolves from a somewhat special initial condition. Consider an initial condition at the beginning of a morning in which each person, banker or nonbanker, holds only some outside money, except that each banker has, in addition, $Z$ own-notes. (Under the NBS, the government printed all the notes, which, however, were identified by the issuing bank.) As is standard, own-notes held by a banker are neither an asset nor a liability. We assume that the banker starts with $Z$ notes because that is the maximum feasible quantity that might be traded in a morning meeting. The total amount of outside money, expressed as the average amount per unit interval of each specialization type, is $\bar{z}$. As the economy evolves, average wealth per specialization type remains $\bar{z}$; that is, in our model wealth gets passed around among people, but is not created.

At the beginning of the afternoon, each banker shows the government how many own-notes are still in the banker's possession and the tax is levied on the difference between the amount so far created and that amount. In addition, the banker must demonstrate satisfaction of the non negative wealth constraint. Because liabilities consist of notes outstanding, either held by the government or held by nonbankers, the liabilities can be identified. Therefore, satisfaction of the non-negative wealth constraint is demonstrated by the banker showing enough assets. To assure that each banker has enough own-notes at the end of each afternoon, we can suppose that each banker's

follow from any bargaining outcome that produces an outcome in the pairwise core for the meeting.

[7] We could makes notes submitted for clearing by bankers a choice variable, but, if we did, then one equilibrium would have all notes submitted. We simply assume that that is the action taken.

stock of own-notes is augmented to bring it up to $Z$. We assume that unlimited own-notes are available for the purchase of bonds or additional outside money in the afternoon.

We do not make explicit the penalty for failing to meet the non negative wealth constraint. We simply assume that it is large enough to prevent the constraint from being violated. Nor do we make explicit the penalty for failing to pay the lump-sum tax that is levied to balance the government's budget.

The only assumptions we make about $\bar{z}$ and $Z/\bar{z}$ is that both are sufficiently large. The main assumption we make is

$$r - \rho < \tau < \beta(r - \lambda\rho), \tag{2}$$

where $r$ is the bond coupon payment, $\tau$ is the tax per outstanding note, and $\rho$ is the fee per note cleared. The first inequality is consistent with most of the data on interest rates and the clearing fee for the NBS episode if a period in the model is something like a week or less.[8] As for the second inequality, we show that there is a steady state in which $\lambda$, the probability that a note held by a nonbanker "meets" a banker, is the average clearing rate. (It is also an upper bound on the average clearing rate.) With the average clearing rate equal to $\lambda$, the second inequality is equivalent to failure of the purported arbitrage condition if $\beta$ is sufficiently close to one. We further assume that $r, \tau$, and $\rho$ are sufficiently small, which, as we explain later, should be regarded as an implication of a sufficiently large $\bar{z}$—which, in turn, should be interpreted as a low degree of asset indivisibility.

Our non negative wealth constraint on bankers is a bit weaker than the NBS constraint that a bank have bonds deposited that match all the own-notes that it receives from the Treasury. And, although the model's bankers are most straightforwardly viewed as issuers of transferable trade-credit IOUs under some constraints, there is an interpretation that makes them more like NBS banks. If each NBS bank issued notes in the form of a loan to a single borrower, then each banker in the model can be viewed as a consolidation of the NBS bank and its borrower. Consideration of such a consolidated entity makes sense because an NBS bank had to be concerned about the float rate on its notes and, therefore, had to be concerned about how its customers who borrowed in the form of notes used those notes.

[8] As noted in [3, p.356], the Treasury's clearing fee was "sufficient to offset more than two weeks of interest at 2% per year."

In our model, the classification of agents between bankers and nonbankers is exogenous. We will, however, make some comments later about the welfare of bankers and nonbankers that are pertinent for thinking about entry into and exit from banking.

# 3 Definition of a Steady State

We show that there exists a steady state with several features. First, $y_m = y_g = 0$ for every banker. That is, at the start of each morning, each banker has a single asset, bonds, and a single liability, notes held by nonbankers. Second, when a banker meets a nonbanker holding wealth $z$, which with probability one is in the form of other bankers' notes, the banker replaces that wealth with own-notes. That, in turn, implies that those notes of other bankers get cleared in the afternoon. Such behavior implies that the expected number of clearings of own-notes is $\lambda y_n$. However, to demonstrate that there is such a steady state, we have to begin with a somewhat general definition, one that does not impose that behavior.

The steady state consists of $(w, v, \pi)$, where, $w : \mathbf{Z} \to \mathbb{R}_+$ is a value function defined on individual nonbanker's wealth and $v : \mathbf{Y} \to \mathbb{R}$, is a value function defined on individual banker balance sheets, where

$$\mathbf{Y} = \{y = (y_m, y_b, y_g, y_n) \in \mathbb{Z}_+^4 : 0 \leq y_m + y_b - y_g - y_n \leq Z\}, \tag{3}$$

and $\pi = (\pi_n, \pi_b)$ is a pair of probability measures over wealth where $\pi_i : \mathbf{Z} \to \mathbb{R}_+$ and $\pi_n(z)$ $[\pi_b(z)]$ is the measure of nonbankers (bankers) of each specialization type who have wealth $z$. All of these pertain to the start of a date prior to pairwise meetings. Notice that we are not making the distribution of bankers over elements of $\mathbf{Y}$ a part of the description of a steady state. We can get by without describing that distribution because the behavior that affects future wealth turns out to depend only on current wealth, not on its composition.[9]

Now we describe choices and payoffs in pairwise meetings, starting with the meetings between nonbankers. Let $W$ be an upper bound on output that we describe in the existence proof. For $(z, z') \in \mathbf{Z}^2$ (where $z$ is the wealth of the buyer and $z'$ is that of the seller), we let $\Gamma(z, z', w)$ be a set of probability

[9] Our definition of a steady state is standard, but lengthy. Some readers may wish to skip it on a first reading and go directly to the existence result in the next section.

measures on $[0, W] \times \{\max\{0, z + z' - Z\}, ..., z\}$, given by

$$\Gamma(z, z', w) = \{\sigma : E_\sigma[-q + \beta w(z + z' - z'')] \geq \beta w(z')\}. \tag{4}$$

Here, $E_\sigma$ denotes the expectation with respect to $\sigma$, the arguments of which are $(q, z'')$—$q$ being output and $z''$ being the post-trade wealth of the buyer. Then,

$$f(z, z', w) = \max_{\sigma \in \Gamma(z,z',w)} E_\sigma[u(q) + \beta w(z'')] \tag{5}$$

is the payoff for a nonbanker buyer with $z$ who meets a nonbanker seller with $z'$.

Now we turn to meetings between bankers and nonbankers, the new part of the model. For $(z, y) \in \mathbf{Z} \times \mathbf{Y}$, where $z$ is the wealth of the nonbanker and $y$ is the portfolio of the banker, we let

$$\begin{aligned} d(z, y) &= \{(d_a, d_l) \in \mathbb{Z} \times \mathbb{Z}_+ : -y_m \leq d_a \leq z, \\ \tilde{y} - Z &\leq d_l - d_a \leq \min\{Z - z, \tilde{y}\}, \end{aligned} \tag{6}$$

where $d_a$ is the addition to the banker's assets and $d_l$ is the addition to the banker's liabilities in the meeting, and where $\tilde{y} = y_m + y_b - y_g - y_n$, the wealth implied by $y$. (The second two inequalities impose the bounds on wealth.) Notice that if $d_a$ is never negative, which turns out to be the case, then the restriction $-y_m \leq d_a$ can be omitted and $d(z, y)$ depends on $y$ only by way of the wealth implied by $y$.

We start with afternoon decisions. A banker with $y$ at the start of the morning, with final morning trade $(q, d_a, d_l)$, and with afternoon redemptions from notes previously held by nonbankers $x$ chooses $(y'_m, y'_b, y'_g)$ to maximize

$$v(y'_m, y'_b, y'_g, y_n + d_l - x) \tag{7}$$

subject to

$$y'_m + y'_b \leq y'_g + y_m + y_b + d_a - x - y_g. \tag{8}$$

(Notice that if $y$ implies non negative wealth and $(d_a, d_l)$ satisfies (6), then any $(y'_m, y'_b, y'_g)$ that satisfies (8) gives rise to a $y'$ that implies non negative wealth.) The payoff for such a banker is

$$\begin{aligned} g(d_a, d_l, y, x, v) &= r y_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota + \\ &\quad \beta \max v(y'_m, y'_b, y'_g, y_n + d_l - x), \end{aligned} \tag{9}$$

where $\iota$ is the lump-sum tax and where the maximization is subject to (8).

Now, we are ready to describe banker choices in the morning. We define $\Gamma_i(z, y, w)$, $i = 1, 2, 3$, sets of probability measures on $[0, W] \times d(z, y)$, by

$$\Gamma_1(z, y, w) = \{\sigma : E_\sigma[-q + \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{10}$$

$$\Gamma_2(z, y, w) = \{\sigma : E_\sigma[u(q) + \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{11}$$

and

$$\Gamma_3(z, y, w) = \{\sigma : E_\sigma \beta w(z - d_a + d_l)] \geq \beta w(z)\}, \tag{12}$$

where $E_\sigma$ again denotes the expectation with respect to $\sigma$ and where the arguments of $\sigma$ are now $(q, d_a, d_l)$. (Notice that the $\Gamma_i$ depend on $y$ only by way of the dependence of the set $d(z, y)$ on $y$.) Then, the banker payoffs are

$$f_1(z, y, w, v) = \max_{\sigma \in \Gamma_1(z,y,w)} E_\sigma[u(q) + E_x g(d_a, d_l, y, x, v)] \tag{13}$$

$$f_2(z, y, w, v) = \max_{\sigma \in \Gamma_2(z,y,w)} E_\sigma[-q + E_x g(d_a, d_l, y, x, v)] \tag{14}$$

$$f_3(z, y, w, v) = \max_{\sigma \in \Gamma_3(z,y,w)} E_\sigma[E_x g(d_a, d_l, y, x, v)], \tag{15}$$

where $E_x$ denotes expectation w.r.t the distribution of $x$, and where $g$ is given by (9). Here, $f_1(z, y, w, v)$ is the payoff for a banker buyer with $y$ who meets a nonbanker seller with $z$, $f_2(z, y, w, v)$ is the payoff for a banker seller with $y$ who meets a nonbanker buyer with $z$, and $f_3(z, y, w, v)$ is the payoff for a banker with $y$ who meets a nonbanker with $z$ in a no-coincidence meeting. As we will see, the form of $g$ will be such that the only aspect of the distribution of $x$ that is needed to compute the expectation w.r.t. $x$ is the mean of that distribution. That is why we can get by without including the distribution of clearings as part of the steady state.

We denote the set of maximizers in (5) by $\widetilde{\Delta}(z, z', w)$ and those in (13), (14), and (15) by $\widetilde{\Delta}_1(z, y, w, v)$, $\widetilde{\Delta}_2(z, y, w, v)$, and $\widetilde{\Delta}_3(z, y, w, v)$, respectively. Because it can be shown that all the maximizers are degenerate in $q$, in what follows, for a maximizer, we denote the maximizing $q$ by $\hat{q}$.

Now, using $\widetilde{\Delta}$ and the $\widetilde{\Delta}_i$, we define sets of optimal end-of-trade wealth that are needed to define the steady state conditions for $\pi$. For a meeting between a nonbanker buyer with $z$ and a nonbanker seller with $z'$, we define $\Delta(z, z', w)$, a set of probability measures on $\mathbf{Z}$, by

$$\Delta(z, z', w) = \{\delta : \delta(.) = \widetilde{\delta}(\hat{q}, .),\ \widetilde{\delta} \in \widetilde{\Delta}(z, z', w)\}.$$

Then we define $\;(w,\pi)$, a set of probability measures on $\mathbf{Z}$, by

$$\begin{aligned}(w,\pi) &= \{\omega : \omega(z) = \textstyle\sum_{(z',z'')} \pi_n(z')\pi_n(z'')[\delta(z) + \delta(z' - z + z'')], \\ & \quad \text{for } \delta \in \Delta(z', z'', w)\}, \end{aligned} \tag{16}$$

where, as a convention, $\delta(x) = 0$ if $x \notin \mathbf{Z}$.

For a banker with $y$ who meets a nonbanker with $z'$, for $i = 1, 2, 3$, and for $(z, z', y) \in \mathbf{Z}^2 \times \mathbf{Y}$, we let $z^{-1}(z', y) = \{(d_a, d_l) : z' - d_a + d_l = z\}$. (That is, $z^{-1}(z', y)$ is the set of asset trades that leave the nonbanker with wealth $z$.) Then we define $\Delta_i(z', y, w, v)$, a subset of probability measures on $\mathbf{Z}$ by

$$\begin{aligned}\Delta_i(z', y, w, v) &= \{\delta : \delta(z) = \textstyle\sum_{z^{-1}(z',y)} \widetilde{\delta}(\hat{q}, d_a, d_l), \\ \text{for } \widetilde{\delta} &\in \widetilde{\Delta}_i(z', y, w, v)\}.\end{aligned}$$

Let $\mu$ be an arbitrary measure on $\mathbf{Y}$ that is consistent with $\pi$. We define $\Theta(w, v, \pi)$, a set of probability measures on $\mathbf{Z} \times \mathbf{Z}$ by

$$\begin{aligned}\Theta(w, v, \pi) &= \{(\theta_n, \theta_b) : \\ \theta_n(z) &= \frac{1}{N} \textstyle\sum_{(z',y)} \pi_n(z')\mu(y)(\delta_1 + \delta_2 + (N-2)\delta_3)(z), \\ \theta_b(z) &= \frac{1}{N} \textstyle\sum_{(z',y)} \pi_n(z')\mu(y)(\delta_1 + \delta_2 + (N-2)\delta_3)(z' - z + \tilde{y}), \\ \text{for } \delta_i &\in \Delta_i(z', y, w, v)\},\end{aligned} \tag{17}$$

where $\tilde{y} = y_m + y_b - y_g - y_n$ and where, as a convention, $\delta_i(x) = 0$ if $x \notin \mathbf{Z}$.

Now we can complete the conditions for a steady state. The value function $w$ must satisfy

$$w(z) = [\lambda + \frac{(1-\lambda)(N-1)}{N}]\beta w(z) + \frac{(1-\lambda)}{N} \sum \pi_n(z') f(z, z', w), \tag{18}$$

and the value function $v$ must satisfy

$$v(y) = \frac{1}{N} \sum \pi_n(z)[f_1(z, y, w, v) + f_2(z, y, w, v) + (N-2) f_3(z, y, w, v)]. \tag{19}$$

The measures $\pi = (\pi_n, \pi_b)$ must satisfy

$$\sum_z z[\alpha\pi_b(z) + (1-\alpha)\pi_n(z)] = \bar{z}, \tag{20}$$

and

$$\pi = (\lambda\theta_n + (1-\lambda)\omega, \theta_b) \text{ for some } (\omega, (\theta_n, \theta_b)) \in \quad (w,\pi) \times \Theta(w, v, \pi). \quad (21)$$

**Definition 1** *A steady state is* $(w, v, \pi)$ *that satisfies* $(18) - (21)$.

We have chosen to omit from the definition the condition that the lump-sum tax, $\iota$, is such as to achieve government budget balance; namely,

$$\tau(1-\alpha)\sum_z z\pi_n(z) + \iota\alpha = r\bar{z}. \quad (22)$$

Because $\iota$ does not affect behavior, we can simply insert into (9) the $\iota$ that satisfies (22) for a $(w, v, \pi)$ that satisfies definition 1. This is written under the assumption that the clearing fee covers a real resource cost and, therefore, is not a source of net revenue for the government. If it were, $\tau$ in this expression would be replaced by $\tau + \lambda\rho$ and nothing else would be affected.

# 4 Existence of a Steady State

We have the following result on existence.

**Proposition 1** *There exists a steady state* $(w, v, \pi)$ *in which* $w$ *is strictly increasing and strictly concave,* $h(\cdot) \equiv v(0, \cdot, 0, 0)$ *is strictly concave and satisfies* $h(z+1) - h(z) \geq \frac{r}{1-\beta}$. *Also,* $v$ *satisfies*

$$\begin{aligned} v(y_m, y_b, y_g, y_n) &= v(0, y_m + y_b, y_g, y_n) - ry_m, & (23)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b - y_g, 0, y_n) - (\rho + \tau - r)y_g, & (24)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b + s, y_g, y_n + s) - \eta s, & (25) \end{aligned}$$

*where* $\eta = \frac{r - \lambda\rho - \tau}{1 - \beta + \lambda\beta}$. *Finally,* $\pi$ *satisfies*

$$0 < (1-\alpha)\sum_z \pi_n(z)z < \bar{z}. \quad (26)$$

Condition (23) says that the only consequence for a banker of having some money at the start of a morning rather than having all assets in the form of bonds is the implied loss of interest payments. Condition (24) says that the only consequence of having issued notes to buy bonds is the loss implied by the first inequality in (2). Condition (25) says that the only consequence of $s$ additional bonds and $s$ additional notes held by nonbankers is the gain implied by the second inequality in (2). Inequality (26) says that average wealth for both bankers and nonbankers is positive. If nonbankers have no wealth, then they are in autarky. If bankers have no wealth, then there is no goods trade between bankers and nonbankers.

Our existence proof has several steps. The proof is in the Appendix and it is separated into lemmas. The general idea is to show that the mapping implicit in the definition of a steady state has a fixed point that satisfies the properties in proposition 1. Therefore, some of the proof involves showing that the mapping preserves those properties. Relative to the arguments in Zhu [11], the new parts of the argument involve showing that the mapping preserves (23)-(25), which is established by Lemma 2, and that both bankers and nonbankers have positive average wealth. As the proof of Lemma 2 shows, in a steady state with a $v$ that satisfies the properties in the proposition, no banker holds outside money or uses own-notes to buy outside money or bonds from the government ($y_m = y_g = 0$). Also, each banker replaces all the notes held by the nonbanker who is met with own-notes, from which we conclude that a banker with $y_n$ outstanding at the start of a morning experiences average clearings equal to $\lambda y_n$.

As regards the distribution of individual bankers by size in terms of gross assets, our steady state determines only averages: from the above properties and (20), average bond holding per banker is $\bar{z}/\alpha$ and average banker liability in the form of notes held by nonbankers is $\sum_z \pi_n(z)z/\alpha$.

We also know the average afternoon pay-off (to bankers). It is negative and is equal to the average of clearing fees, because the clearing fee is lost to nature. If the clearing fee were a source of net revenue for the government, then the average afternoon pay-off would be zero. Obviously, these conclusions depend on the assumption that a lump-sum tax is levied only on bankers. Under our assumptions, the lump-sum tax is positive. If the lump-sum tax is imposed on the entire population and if the measure of bankers is small, then the average afternoon pay-off to bankers is positive. After all, bankers hold their wealth in the form of bonds, while nonbankers hold non-interest bearing notes. If everyone pays the lump-sum tax that finances interest, then the bankers gain.

One should not, however, conclude that bankers are on average worse off than nonbankers under our scheme. After all, bankers get to make take-it-or-leave-it offers in all meetings. Average nonbanker welfare is

$$W = \sum_z \pi_n(z) w(z). \tag{27}$$

Average banker welfare can be expressed in terms of the objects in our steady state by making use of (25). By (25),

$$v(0, y_a, 0, y_n) = v(0, z, 0, 0) + \eta y_n, \tag{28}$$

where $z = y_a - y_n$, the banker's wealth. It follows that average banker welfare is

$$V = \sum_z [\pi_b(z) v(0, z, 0, 0) + (\frac{1-\alpha}{\alpha}) \eta \pi_n(z) z], \tag{29}$$

where the second term within the summation comes from market-clearing; namely, equating the average note liability of bankers to the average notes held by nonbankers. (The distribution we use to compute average banker liability is described below in footnote 10.)

If we view entry as a once-for-all decision (before portfolios are assigned) and if we are permitted to assume any distribution of one-time utility costs of entry over the entire population, then we can construct one that makes $\alpha$ the fraction who choose to be bankers. Thus, for $x \in \mathbb{R}$, let $H(x)$ denote the fraction of each specialization type whose utility cost of entry into being a banker does not exceed $x$. If $H$ is strictly increasing and $H(V - W) = \alpha$, then the fraction $\alpha$ choose to be bankers.[10]

Previous investigators were not concerned about entry. Given that they were not able to reconcile the second inequality in (2) with finite profits for an existing bank, it was premature to consider entry. Because we reconcile

[10] The point of view behind this one-time entry decision is the following. Aside from one-time utility costs of entry, people are identical prior to entry and prior to being assigned portfolios. A person who chooses to be a nonbanker gets expected utility $W$: the person is assigned wealth $z$ with probability $\pi_n(z)$. A person who chooses to be a banker gets $V$ via the following construction of initial banker portfolios.

Let $x = (1/\alpha - 1) \sum \pi_n(z) z$. Notice that $x > 0$. Now let $\theta$ satisfy $\theta[x] + (1-\theta)([x]+1)$, where $[x]$ is the largest integer that does not exceed $x$. Then a person who chooses to be a banker is assigned $(y_b, y_n) = ([x] + z, [x])$ with probability $\theta \pi_b(z)$ and $(y_b, y_n) = ([x]+1+z, [x]+1)$ with probability $(1-\theta)\pi_b(z)$. Of course, this distribution of portfolios does not persist, but we do not care. All that matters is that it is consistent with the wealth measure $\pi_b$ and with market clearing. And, of course, this is just one way to assign initial banker portfolios.

(2) with finite profits for an existing bank, our model opens the way for more elaborate models of entry and exit.[11]

In our model, $r$, $\tau$, and $\rho$ are measured in goods per note. In the data for the NBS episode, the paradox is stated in terms of interest rates. To convert the inequalities (2) into interest-rate units, we need only divide both sides of both inequalities by the average goods value per note. Obviously, such a restatement leaves the inequalities intact. Moreover, because the trades of goods for notes is determined by the objects in our steady state, both total output (real GDP) and the nominal value of total output (nominal GDP) are implied by a steady state. Therefore, a total output deflator is determined and that along with the average note holding implies an average goods value per note.

In addition, we have the freedom to determine realistic magnitudes for interest rates computed as just described. Consider different magnitudes for $(r, \tau, \rho)$ in $\mathbb{R}^3_+$ along a ray through the origin determined so that inequalities (2) hold. As $(r, \tau, \rho)$ gets large, the average goods value per note does not get proportionally large because the value function $v$ is bounded independently of $(r, \tau, \rho)$. It follows that the interest rate computed as just described gets large as $(r, \tau, \rho)$ gets large. As $(r, \tau, \rho)$ gets small, the value of a note in a proposition 1 steady state is bounded away from zero. It follows that the interest rate goes to zero as $(r, \tau, \rho)$ gets small. Those two features suggest that we are free to choose $(r, \tau, \rho)$ in such a way as to get any magnitude for the steady-state interest rate. Moreover, suppose we consider economies with different magnitudes for $\bar{z}$ for fixed $Z/\bar{z}$. These economies, in effect, have different degrees of asset indivisibility: the larger is $\bar{z}$, the greater the degree of divisibility. Because $r$, $\tau$, and $\rho$ are in goods per unit of money, it does not make sense to hold them fixed for economies with different $\bar{z}$'s. Therefore, when we assume, as we do in a part of the existence proof, that $(r, \tau, \rho)$ is sufficiently small, we are, in effect, assuming that assets are sufficiently divisible.

Finally, we want to compare our model to closely related models. First, although the division of agents into bankers and nonbankers is borrowed from Cavalcanti and Wallace [1], the models are very different. In [1] bankers have

[11] If the lump-sum tax were levied on everyone, then no banker would subsequently choose to switch to being a nonbanker if that choice were offered. Bankers face the same probabilities of being consumers and producers as do nonbankers and have more freedom to choose trades; in particualr, they can emulate the actions of nonbankers. Of course, nonbankers would then want to switch to being bankers if that were permitted.

known histories. Those histories, by way of threatened punishment, permit the mechanism designer to partially control banker behavior. In particular, that allows outcomes in which the possibility of issuing notes frees banker spending for consumption from the banker's recent acquisition of financial wealth. That can happen because there is no non negativity restriction on financial wealth in [1]. If there were such a constraint, then the main result in [1]—namely, that note-issue implements strictly more outcomes than does a fixed stock of outside money—would not hold. In the current model, the non negative wealth constraint rules out any such role for note issue. Indeed, it is hard to avoid the conclusion that the full-backing requirement under the NBS ruled out any such role of notes in the actual NBS economy.

In a complicated way, the role of note issue in the current model is to pay interest on wealth (or money) held by bankers—interest financed in part by lump-sum taxes. However, the model is not equivalent to a model without note issue in which bankers simply get an additive period-utility of $r$ per unit of money held. In such a model, $r$ would be an opportunity cost for a banker who surrenders some wealth for consumption in a pairwise meeting. In our model, a banker who surrenders the same wealth does not sacrifice $r$. The banker issues notes and those notes are not redeemed immediately. Therefore, the banker surrenders less than $r$ per note issued. In effect, the banker uses the opportunity to consume to engage in some additional arbitrage. That being the case, the goods trade is not the same as in a model in which bankers get period-utility of $r$ per unit of money held.

# 5 Concluding Remarks

As we noted at the outset, there seem to be two possible explanations of the paradox. One appeals to risk aversion regarding clearing costs, while the other explains how the observed average float arises from the behavior of note issuers. These are not mutually exclusive explanations. We omitted the first and showed that the second is sufficient to explain the paradox. Adding risk aversion would presumably only reinforce the result. However, adding risk aversion is difficult.

If there is risk aversion concerning clearing fees, then moments of the distribution of clearings higher than the mean would matter for bankers. And those higher moments depend not only on the total stock of own-notes outstanding, but also on how that stock is distributed among nonbankers: widely dispersed holdings of a given stock imply a smaller variance of clearings than bunched holdings. But that, in turn, implies that the history of a banker's note issues matters because that history influences how widely dispersed is a given stock of outstanding notes. At a minimum, therefore, dealing with risk aversion greatly increases the dimensionality of the state space.

With risk neutrality regarding clearing costs, bankers would like to issue an unlimited quantity of notes that are subject to the steady-state average clearing rate, $\lambda$. The model would not have an equilibrium if a banker could do that. Therefore, a critical feature of the model is that the clearing rate $\lambda$ is available only for notes issued in pairwise meetings, meetings in which the opportunity to arbitrage is limited by the wealth held by the nonbanker in the meeting. That is the model's representation of the general idea that opportunities to place notes that generate high float are limited.

Finally, it is worth recalling that a major defect of the NBS was considered to be its failure to produce an elastic currency—its failure, that is, to limit fluctuations in nominal interest rates. As emphasized by Friedman and Schwartz [4, page 189], elasticity of the currency was considered so important that the title of the act that created the Federal Reserve System included the phrase, "to furnish an elastic currency." Although our steady state has no fluctuations, it is consistent with the inelasticity concern about the NBS because there is a steady state for all bond coupons satisfying (2). In the model, an elastic currency could be achieved by the simple step of eliminating the clearing fee $\rho$. Given the risk neutrality in the model, if the clearing cost is financed by lump-sum taxes rather than by the user fee $\rho$, then $r \leq \tau$ is necessary for equilibrium. (If $r > \tau$, then there is no solution to the afternoon problem of bankers; they want to buy an unlimited quantity of bonds with own-notes.)

# 6 Appendix

We begin with some notation. Let $W$ be the unique positive solution to $N(1-\beta)W = u(\beta W)$. Let $\mathbf{W}$ be the set of nondecreasing and concave functions $w : \mathbf{Z} \rightarrow [0, W]$. Let $\mathbf{V}$ be the set of functions $v : \mathbf{Y} \rightarrow [\frac{-\iota}{1-\beta}, W +$

$\frac{rZ}{1-\beta}]$ with $v$ nondecreasing and concave in its second argument and with

$$\begin{aligned} v(y_m, y_b, y_g, y_n) &= v(0, y_m + y_b, y_g, y_n) - r y_m, && (30)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b - y_g, 0, y_n) - (\rho + \tau - r) y_g, && (31)\\ v(y_m, y_b, y_g, y_n) &= v(y_m, y_b + s, y_g, y_n + s) - \eta s, && (32) \end{aligned}$$

where $\eta = \frac{r-\lambda\rho-\tau}{1-\beta+\lambda\beta}$. (That is, $v \in \mathbf{V}$ satisfies most of the conclusions in the proposition except that strict properties are replaced by their weak counterparts to make $\mathbf{V}$ closed.)

Next, we formally define the mapping implied by the definition of a steady state. Let the mapping $\Phi_w$ on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined

$$\Phi_w(w, v, \pi)(z) = [\lambda + \frac{(1-\lambda)(N-1)}{N}]\beta w(z) + \frac{(1-\lambda)}{N} \sum \pi_n(z') f(z, z', w).$$

Let the mapping $\Phi_v$ on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined by

$$\Phi_v(w, v, \pi)(y) = \frac{1}{N} \sum \pi_n(z)[f_1(z, y, w, v) + f_2(z, y, w, v) + (N-2) f_3(z, y, w, v)]. \qquad (33)$$

Let the mapping $\Phi_\pi : \mathbf{W} \times \mathbf{V} \times \mathbf{\Pi} \to \mathbf{\Pi}$ be defined by

$$\begin{aligned} \Phi_\pi(w, v, \pi) &= \{(\lambda\theta_n + (1-\lambda)\omega, \theta_b) \text{ for some} \\ & (\omega, (\theta_n, \theta_b)) \in \quad (w, \pi) \times \Theta(w, v, \pi)\}. \end{aligned}$$

Finally, we let the mapping $\Phi$ on $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ be defined by

$$\Phi(w, v, \pi) = (\Phi_w(w, v, \pi), \Phi_v(w, v, \pi), \Phi_\pi(w, v, \pi)).$$

Next, we show that $\Phi$ maps $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$ into $\mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$.

**Lemma 1** $\Phi_w(w, v, \pi) \in \mathbf{W}$ *and* $\Phi_\pi(w, v, \pi) \in \mathbf{\Pi}$

**Proof.** The proof is standard and is omitted. For reference, see the proof of [9, Proposition 1] ■

**Lemma 2** $\Phi_v(w, v, \pi) \in \mathbf{V}$.

**Proof.** Let $(w, v, \pi) \in \mathbf{W} \times \mathbf{V} \times \mathbf{\Pi}$. Let $h(\cdot) \equiv v(0, \cdot, 0, 0)$. Consider a banker with $y$ who takes $w$ and $v$ as the next date's value functions and $\pi$ as this date's distribution. The afternoon problem of the banker is described in (7) and (8). It follows from (30) that $y'_m = 0$ and from (31) and the first inequality in (2) that $y'_g = 0$. Therefore, the banker must choose $y'_b = y_m + y_b + d_a - x - y_g$. Then using (9), we have

$$\begin{aligned} g(d_a, d_l, y, x, v) &= ry_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota \\ \\ &\quad \beta v(0, y_m + y_b + d_a - x - y_g, 0, y_n + d_l - x) \end{aligned}$$

Next, applying (32) with $s = -y'_n = -(y_n + d_l - x)$ and letting $\tilde{y} = y_m + y_b - y_g - y_n$, we have

$$\begin{aligned} g(d_a, d_l, y, x, v) &= ry_b - \rho(x + y_g) - \tau(y_g + y_n + d_l) - \iota \\ \\ &\quad +\beta h(\tilde{y} + d_a - d_l) + \beta\eta(y_n + d_l - x). \end{aligned}$$

It follows that

$$E_x g(d_a, d_l, y, x, v) = \beta h(\tilde{y} + d_a - d_l) + (\beta\eta - \tau)d_l + A(y), \tag{34}$$

where $A(y) = ry_b - (\rho + \tau)y_g + (\beta\eta - \tau)y_n - (\beta\eta + \rho)E_x x$.

Now, we are ready to consider the maximization in (13)-(15). By concavity of $w$ and $h$, the maximizations over lotteries in (13)-(15) are equivalent to deterministic maximizations with $w$ and $h$ replaced by their extensions to $[0, Z]$ via linear interpolation. Therefore, we can let $(\hat{q}, \hat{d}_a, \hat{d}_l)$ denote a maximizer. If $\hat{d}_a < z$ (the pre-trade wealth of the nonbanker which is an upper bound on $d_a$), then $(\hat{q}, z, \hat{d}_l + z - \hat{d}_a)$ is also feasible. By the second inequality in (2), $\beta\eta - \tau > 0$. Then by (34), $(\hat{q}, z, \hat{d}_l + z - \hat{d}_a)$ produces a higher value of $f_i$. Therefore, a necessary condition for a maximum is $d_a = z$. It follows that the lower bound on $d_a$ in (6) can be omitted. Then the constraint sets in (13)-(15) depend only on $\tilde{y}$, the wealth implied by $y$. Moreover, because $d_a = z$ holds for every banker, it follows that $E_x x = \lambda y_n$. Therefore, we can write

$$f_i(z, y, w, v) = F_i(z, \tilde{y}, w, v) + A^*(y), \tag{35}$$

where $A^*(y) = A(y)$ but with $E_x x = \lambda y_n$. That is,

$$A^*(y) = ry_b - (\rho + \tau)y_g + (\beta\eta - \tau)y_n - (\beta\eta + \rho)\lambda y_n. \tag{36}$$

It follows from (33) and (35) that for $y, y' \in \mathbf{Y}$ with the same implied wealth,

$$\Phi_v(w, v, \pi)(y') - \Phi_v(w, v, \pi)(y) = A^*(y') - A^*(y). \tag{37}$$

Then it follows from (37) and (36) that $\Phi_v(w, v, \pi)$ satisfies (30)-(32). (For (32), let $y' = (y_m, y_b+s, y_g, y_n+s)$, then $A^*(y')-A^*(y) = [r+(\beta\eta-\tau)-(\beta\eta+\rho)\lambda]s = \eta s$.) Finally, it is easy to verify that for all $z$, $F_i(z, z'+1, w, v) > F_i(z, z', w, v)$ if $z' < Z$ and $2F_i(z, z', w, v) \geq F_i(z, z'+1, w, v) + F_i(z, z'-1, w, v)$ if $0 < z' < Z$. It follows that $\Phi_v(w, v, \pi)$ is nondecreasing and concave in its second argument. ■

Now, we equip $\mathbf{V}$ with the topology of pointwise convergence (on $\mathbf{Y}$). We have the following results.

**Lemma 3** $\mathbf{V}$ *is compact.*

**Proof.** Notice that $v \in \mathbf{V}$ is completely determined by $v(0, \cdot, 0, 0)$. Hence, $\mathbf{V}$ is isomorphic to the set of functions $h : \mathbf{Z} \to \mathbb{R}$ with $h(\cdot) = v(0, \cdot, 0, 0)$ for some $v \in \mathbf{V}$. That set is compact. ■

**Lemma 4** $\Phi$ *is u.h.c., compact valued, and convex valued.*

**Proof.** The result follows from the Theorem of Maximum and the convexification by lotteries. ■

We complete the proof of proposition 1 by way of the next lemma. Recall that we have the following assumptions on parameters: $u'(0)$, $\bar{z}$, and $Z/\bar{z}$ are sufficiently large, and $r, \rho$, and $\tau$ are sufficiently small.

**Lemma 5** *There exists* $(w, v, \pi) \in \Phi(w, v, \pi)$ *in which* $w$ *is strictly increasing and strictly concave,* $h(\cdot) \equiv v(0, \cdot, 0, 0)$ *is strictly concave and satisfies* $h(z+1) - h(z) \geq \frac{r}{1-\beta}$*, and* $\pi$ *satisfies* $0 < (1-\alpha)\sum_z \pi_n(z) z < \bar{z}$.

**Proof.** First, we sketch the proof that there is a fixed point of $\Phi$ that satisfies the claim that $w$ is bounded away from 0 at all positive money holdings. (Note that $w(0) = 0$.) Here, we apply arguments used in Zhu [11, Proposition 1]. Indeed, because nonbankers do not profit from their meetings with bankers, aside from somewhat different meeting-rate parameters and a somewhat different law of motion for the evolution of their wealth (the law of motion of their wealth is affected by their meetings with bankers), the nonbankers in this model are exactly like the agents in [11]. For the properties of $w$, the only aspect of the distribution needed is that a sufficient mass of nonbankers are poor enough relative to $\bar{z}$. Hence, as now explained, the arguments in [11, Proposition 1] are easily adapted.

First, we define $\Phi_n(w, v, \pi) = \Phi(w + 1/n, v, \pi)$, where $n$ is a natural number. Next, from Fan's Fixed Point Theorem (a generalized version of Kakutani's) and Lemmas 3 and 4, we conclude that $\Phi_n$ has a fixed point. Next, we prove, exactly as in [11, Lemma 3], that for $(w, v, \pi) \in \Phi_n(w, v, \pi)$, $w(8\bar{z}) \geq D/\beta - 1/n$, where $D$ is the unique solution of $u'(D) = [2/(R\beta)]^2$ with $R = [N(1-\lambda\beta) - (1-\lambda)(N-1)\beta - (1-\lambda)]^{-1}$. In this step, we need the condition that $\sum_{z \leq 4\bar{z}} \pi_n(z) \geq 1/2$. Because $\alpha < 1/2$ and $(1-\alpha)\sum_z \pi_n(z)z \leq \bar{z}$, the condition is satisfied. Next, for a sequence of $(w_n, v_n, \pi_n)$ with $(w_n, v_n, \pi_n) \in \Phi_n(w_n, v_n, \pi_n)$, there exists a limit point, denoted $(w, v, \pi)$. It follows that $(w, v, \pi) \in \Phi(w, v, \pi)$ with $w(8\bar{z}) \geq D/\beta$.

The property that $h(z+1) - h(z) \geq \frac{r}{1-\beta}$ is easily verified (one possible use of an additional unit of banker wealth is to acquire a bond).

Next, we consider the claim that both bankers and nonbankers have positive average wealth. First, assume by contradiction that $\sum_z \pi_n(z)z = 0$. Then the measure of bankers with positive wealth is positive and the measure of nonbankers with positive wealth is zero. Then with positive probability, a banker with positive wealth can issue 1 unit of his own notes to a nonbanker with zero wealth. The payoff is $u[\beta w(1)] - \beta\lambda\rho$. As shown above, $w(8\bar{z}) \geq D/\beta$. This and concavity of $w$ imply that the payoff is positive for sufficiently small $\rho$. (As noted above, small $\rho$ should be interpreted as sufficient asset divisibility.) Hence, such a banker issues a positive amount of notes, which contradicts $\sum_z \pi_n(z)z = 0$.

Next, assume by contradiction that $\sum_z \pi_n(z)z = \bar{z}/(1-\alpha)$, so that bankers have no wealth. Then, there is no trade of goods between bankers and nonbankers. In this case, even the law of motion for nonbanker wealth is unaffected by their meetings with bankers. Then the nonbankers are exactly like the agents in [11], and the full-support result in [11, Lemma 9] holds and implies that $\pi_n$ has full support. Because $w$ is bounded above, for sufficiently large $z$, $w(z+1) - w(z) < \frac{r}{1-\beta}$. As noted above, for all $z$, $h(z+1) - h(z) \geq \frac{r}{1-\beta}$. Hence, all bankers are willing to produce for a nonbanker with sufficiently high $z$ to get at least 1 unit of wealth. But this contradicts $\sum_z \pi_b(z)z = 0$.

Finally, the fact that $w$ is strictly increasing is easily confirmed. (One may apply the corresponding simple argument in [11, Lemma 5].) And strict concavity of $w$ and $h$ can also be easily confirmed. ■

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