A conditional distribution model for limited stock index returns

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Abstract

When a price limit regime exists for all of the stocks involved in an index, the index return is an aggregate of limited variables and thereby it is restricted to the same limits. We argue that neither a censored nor a truncated distribution model is appropriate for the aggregate return. The proposed mixed beta distribution allows for varying conditional mean and volatility, and with increasing volatility it changes from leptokurtic to platykurtic densities. The model is illustrated and statistically evaluated with an empirical application to the Shanghai stock market index returns under a 10 % price change limit regime.

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1 Introduction

Daily price limits for all traded stocks are implemented in many stock markets, for example, in France, Italy, Japan, and especially in emerging markets, for example, China, Korea, Taiwan, and Thailand. Within a trading day, the price for a single stock cannot move outside the limits, and the daily return is restricted to an interval $[a, b]$, $a < 0 < b$. As a consequence, the daily return on the stock market *index* is restricted to the same interval. The focus of this paper is on modelling the conditional distribution of stock index returns, when a price change limit applies to the individual stocks. Generally, the adequate specification of the conditional distribution of index returns is an important issue for the assessment of the expected volatility and market risk, for portfolio selection as well as for the pricing of derivatives.[1]

The intention of price limits is to decrease volatility. In fact, setting a floor a and a ceiling b for the stock price returns within a day, the variance of the return on individual stocks as well as on the stock market index is restricted to be less than or equal to $\frac{(b-a)^2}{4}$, where the maximum variance is reached when the probability mass is equally distributed to the extreme returns a and b. The variance bound, however, does not guarantee that the price limits effectively decrease volatility.

How should the statistical analysis of index returns account for the price limits? Common practice in the analysis of limited variables suggests to check whether the censored or the truncated distribution model is appropriate. For a sample of individual stock returns that includes those daily returns which hit the limits, the doubly censored distribution appears to be suitable. For at least two reasons, however, the censored distribution model does not match with the index return data. First and primarily, even when the index movements are affected quite frequently by limit hits of individual stocks, the index return itself will rarely, if at all, hit the limit. Due to its aggregate nature the index return data usually will not even reveal on which days it is affected by limit hits of stocks. Second, not

[1] For an assessment of market risk under a price limit regime see Friedmann and Sanddorf-Köhle (2001).

only the index return but also the individual stocks may be influenced by the price limit regime already when the price approaches the limit. Several authors suggest a *magnet effect* of price limits, i.e. the asset price accelerates towards the limits as it gets closer to the limits, see Aran and Cook (1997), Lehmann (1989), and Subrahmanyam (1994), and for empirical support Cho et al. (2002). The reasons proposed for the magnet effect are mainly the fear for illiquidity, causing traders to sell which pulls the price closer to the floor, and behavioral investors who believe in price trends and, anticipating that the ceiling will be reached, contribute to accelerate price changes as price gets closer to the ceiling.

Because practically no observed index returns hit the limits, a doubly truncated distribution model with support on the finite interval $[a, b]$ may be considered to be more promising. The natural candidate for truncation is the normal distribution or a more general family of bell shaped distributions. Any family of doubly truncated bell shaped density functions, however, is fairly restrictive with respect to the analysis of price limit effects. For example, it does not allow for U-shaped densities, which concentrate the probability mass close to the floor and to the ceiling of the admissible range of index returns. For periods of extreme volatility, with an increasing proportion of stocks which hit the limits, U-shaped densities should not be excluded a priori. More formally, the class of doubly truncated bell shaped distributions restricts the possible variances to the interval $[0, \frac{(b-a)^2}{12}]$. The maximum variance applies to the continuous uniform distribution over $[a, b]$ as a limiting case of doubly truncating a bell shaped density whose variance tends to infinity. Thus, to use a truncated normal model for the index returns under a price limit regime would artificially restrict the volatility and thereby exaggerate any potential dampening effect of the price limit on the volatility.

Another drawback of using a truncated or censored normal distribution for approximating the conditional return distribution is the implied reduction of the kurtosis. Empirical evidence provided by index return data under price limit regimes indicates that the conditional distribution model should be flexible enough to allow for leptokurtosis, see for example Su and Fleisher (1998) and Friedmann and Sanddorf-Köhle (2002).

Summarizing so far, it is not surprising that the common approach to the analysis of stock index returns is to ignore any price limits imposed on the traded stocks. Instead, in this paper we propose a flexible statistical model to account for the price limits in the specification of the conditional mean and volatility dynamics as well as in the conditional distribution model with support on $[a, b]$.

The paper is organized as follows. Section 2 considers the price limit implications for an MA(1)-GJR-GARCH(1,1) specification of the conditional mean and volatility dynamics in a parametric conditional distribution framework. Section 3 develops a mixed beta distribution model with a time varying $\mu - \sigma-$parameterization. The distribution model is based on two beta distributions, and, contrary to the truncated normal model, it allows for the whole range of admissible volatilities. Depending on the time varying conditional mean and volatility the kurtosis and skewness also vary over time, where the shape of the density changes with increasing volatility from a leptokurtic bell shape over a hat shape (with the probability concentrated around the center and close to the limits) to a platykurtic U-shape. The distribution depends, apart from the time varying mean and volatility, on two additional time invariant parameters which determine the weighting between the two beta distributions and the volatility spread between them.

Section 4 provides an empirical application of the model to Chinese stock indices subject to a 10% price limit. We consider the daily returns on the Shanghai A index (domestic investors) and the Shanghai B index (foreign investors) for the time period from December 1996 until September 1999. Based on the maximum likelihood estimates we apply various graphical evaluations and formal tests of the model specification, as suggested by Diebold, Gunther, and Tay (1998). The applied test procedures are based on the Rosenblatt transformation of the return series, which should provide identically, uniformly distributed random variables with support on the unit interval, if the specification is correct. After a subsequent transformation, tests for Gaussian white noise are used to evaluate the model specification. Overall the results confirm that the model provides a close approximation to the conditional distribution of the Shanghai A and B index returns subject to a 10 % price limit regime. We conclude with section 5.

2 Modelling the conditional distribution of limited index returns

Suppose that the range of values of daily index returns is restricted to the interval $[a, b]$. For the distribution of the index return r_t conditional on the past of the process up to $t-1$ we assume a parametric distribution model with support $[a, b]$, which specifies the conditional density f_t for a given conditional mean $\mu_t = \mathrm{E}_{t-1}(r_t)$, given conditional variance $\sigma_t^2 = \mathrm{Var}_{t-1}(r_t)$, and additional time invariant parameters $\boldsymbol{\nu}$,

$$r_t | r_{t-1}, r_{t-2}, \ldots \sim f_t(r) = f(r; \mu_t, \sigma_t^2, \boldsymbol{\nu}), \quad r \in [a, b]. \tag{1}$$

With regard to the conditional mean return we will check the index return data for linear dependencies. A priori, there is some theoretical and empirical evidence that price limits cause linear dependencies in the individual stock return data, see Shen and Wang (1998). Fama (1989) supposed that price discovery is delayed, when price constraints prevent price from reacting fully to news ε, that means when price constraints prevent price from reaching its new equilibrium. Accounting for this potentially delayed price response to news, we assume a first order moving average process,

$$\mu_t = \mu + \psi \varepsilon_{t-1}, \quad |\psi| < 1, \tag{2}$$

where the news variable ε_t is represented by the unpredictable index return, i.e.

$$\varepsilon_t \equiv r_t - \mu_t.$$

Apart from the commonly observed nonlinear dependencies in daily stock market returns the case of a price limit regime provides an additional argument for volatility spillovers due to the shift of trading activities until the subsequent trading days, see Fama (1989), and Kim and Rhee (1997) for empirical support of the volatility spillover hypothesis. For the volatility dynamics we propose to use the GARCH approach of Glosten, Jagannathan and Runkle (1993), GJR for short, providing a flexible and parsimonious approximation of conditional variance dynamics. Its special characteristic is to allow for asymmetric volatility effects:

$$\sigma_t^2 = \omega + (\alpha + \gamma I)\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{3}$$

Here, $I = 1$ if $\varepsilon_{t-1} < 0$ and $I = 0$ if $\varepsilon_{t-1} \geq 0$. For $\sigma_t^2 > 0 \; \forall \; t$ the conditions $\omega > 0$, $\alpha \geq 0$, $\alpha + \gamma \geq 0$, and $\beta \geq 0$ are required.

For the process to be well-defined in the framework of a price limit regime with $a < r_t < b$, the conditional variance has to fulfil the restriction

$$\sigma_t^2 \leq (b - \mu_t)(\mu_t - a), \tag{4}$$

where $(b - \mu_t)(\mu_t - a)$ is the maximum variance over all probability distributions with support $[a, b]$ and mean μ_t.[2] The upper bound for the variance is even lower, if the Bernoulli distribution is not a special case of the assumed probability model. For example, for the doubly truncated normal distribution the upper bound for the variance is equal to $(b - a)^2/12$.

Due to the restricted range of ε_t, implied by the price change limit, the conditional variance of the MA(1)-GJR GARCH(1,1) specification with $\gamma \geq 0$ is bounded by

$$\frac{\omega}{1-\beta} \leq \sigma_t^2 \leq \frac{\omega + \max[(\alpha+\gamma)\varepsilon_{\min}^2 \,,\, \alpha\varepsilon_{\max}^2]}{1-\beta}, \tag{5}$$

where

$$\varepsilon_{\min} = \begin{cases} \dfrac{(a-\mu)+\psi(\mu-b)}{1-\psi^2} & \text{for} \quad 0 \leq \psi < 1 \\ \dfrac{a-\mu}{1+\psi} & \text{for} \quad -1 < \psi \leq 0 \end{cases}$$

$$\varepsilon_{\max} = \begin{cases} \dfrac{(b-\mu)+\psi(\mu-a)}{1-\psi^2} & \text{for} \quad 0 \leq \psi < 1 \\ \dfrac{b-\mu}{1+\psi} & \text{for} \quad -1 < \psi \leq 0 \end{cases} .$$

Therefore, a sufficient condition for the predicted volatility to be well-defined is

$$\frac{\omega + \max[(\alpha+\gamma)\varepsilon_{\min}^2 \,,\, \alpha\varepsilon_{\max}^2]}{1-\beta} \leq \min_{\mu_t}[(b-\mu_t)(\mu_t - a)], \tag{6}$$

[2]The maximum variance is reached with distributing the probability mass to the interval limits, according to

$$p := \text{Prob}(r_t = b) = \frac{\mu_t - a}{b - a}, \quad \text{Prob}(r_t = a) = 1 - p.$$

where the conditional mean μ_t is restricted by

$$\mu + \min[\psi\varepsilon_{\min}\,,\,\psi\varepsilon_{\max}] \leq \mu_t \leq \mu + \max[\psi\varepsilon_{\min}\,,\,\psi\varepsilon_{\max}]. \tag{7}$$

3 Beta distribution and mixed beta distribution

The beta distribution model is known to comprise a large variety of different shapes of a distribution over a finite range. After briefly reviewing the beta distribution, we will generalize it to a mixture of beta distributions in the sense of section 3. We will demonstrate that the mixed beta distribution captures very well the special features of the index return distribution when a price limit regime applies to the individual stocks.

We have assumed that the range of values of the index returns is restricted to the interval $[a, b]$. Modelling the conditional distribution of the index return r with a beta distribution is equivalent to assuming that the conditional distribution of the transformed return x,

$$x = \frac{r - a}{b - a} \tag{8}$$

is given by the standard form of the beta distribution with the probability density function

$$f(x) = \frac{1}{B(c, d)} x^{c-1}(1 - x)^{d-1}, \quad 0 < x < 1, \tag{9}$$

with real parameters $c > 0$, $d > 0$, and B denoting the beta function. The mean and variance of the standard beta distribution are given by

$$\mu_x = \frac{c}{c + d}, \quad \sigma_x^2 = \frac{\mu_x(1 - \mu_x)}{c + d + 1}. \tag{10}$$

For our purpose it is convenient to reparameterize the density in terms of μ_x and σ_x^2 as the given parameters, with

$$0 < \mu_x < 1, \quad \text{and} \quad 0 < \sigma_x^2 < \mu_x(1 - \mu_x) \leq \frac{1}{4},$$

where the upper bound of the variance applies to the limiting case of a Bernoulli distribution. Obviously the range of values for the mean and for the variance is not subject to any restriction other than the finite support of the distribution. Mean and variance of the transformed return x are related to the mean μ and variance σ^2 of the index return r by $\mu_x = \frac{\mu-a}{b-a}$ and $\sigma_x^2 = \frac{\sigma^2}{(b-a)^2}$. Then the parameters c and d can be expressed by

$$c = \mu_x \theta, \quad \text{and} \quad d = (1-\mu_x)\theta, \quad \text{with} \quad \theta = \frac{\mu_x(1-\mu_x)}{\sigma_x^2} - 1. \tag{11}$$

Notice that the parameter $\theta = c+d$ indicates the precision of the distribution, with $\theta = 0$, if the variance takes its maximum value, and θ going to infinity, if the variance tends to zero.

It is well known that the family of beta distributions includes unimodal distributions for $c > 1$, $d > 1$ (i.e. low volatility), U-shaped distributions for $c < 1$, $d < 1$, and J-shaped distributions for $(c-1)(d-1) < 0$.[3]

The skewness α_3 and the kurtosis α_4 of the beta distribution can be expressed in terms of μ_x and θ as

$$\alpha_3 = \frac{4(0.5-\mu_x)}{\sqrt{\mu_x(1-\mu_x)}} \cdot \frac{\sqrt{1+\theta}}{2+\theta}, \tag{12}$$

$$\alpha_4 = 3 - \frac{6}{3+\theta}\left(1 - k\left(\frac{1+\theta}{2+\theta}\right)\right), \tag{13}$$

with

$$k = \frac{1}{\mu_x(1-\mu_x)} - 4 \geq 0.$$

From (13) it follows immediately, that a necessary condition for the beta distribution to be leptokurtic is $k > 1$, i.e. $\mu_x(1-\mu_x) < 0.2$. When applying the beta distribution model to index return series under a symmetric percentage price change limit, the transformed conditional mean return will typically be close to 0.5. Thus, the beta distribution would be restricted to have a negative excess kurtosis. In order to obtain a more flexible family of distributions we propose to generalize the simple beta distribution to a mixture of two beta distributions with parameters c_1, d_1, and c_2, d_2, which have the same mean μ_x, but differ in their respective precision θ_1 and θ_2, say $\theta_2 < \theta_1$, or equivalently $\sigma_2^2 > \sigma_1^2$. Generally, the excess kurtosis of the mixture of two distributions with the same mean and

[3]See Johnson, Kotz, Balakrishnan (1996), chapter 25.

existing moments up to the fourth order can be arbitrary large, depending on the mixing parameter and the second moments of the component distributions, see appendix A.1.

Thus we propose the following probability density function:

$$f(x) = \pi f_1(x) + (1-\pi) f_2(x), \qquad 0 < x < 1 \tag{14}$$

with

$$\begin{aligned} f_1(x) &= \frac{1}{B(c_1, d_1)} x^{c_1 - 1}(1-x)^{d_1 - 1} \\ f_2(x) &= \frac{1}{B(c_2, d_2)} x^{c_2 - 1}(1-x)^{d_2 - 1}, \end{aligned}$$

with

$$\begin{aligned} c_1 = \mu_x \theta_1 \quad &\text{and} \quad d_1 = (1-\mu_x)\theta_1, \\ c_2 = \mu_x \theta_2 \quad &\text{and} \quad d_2 = (1-\mu_x)\theta_2, \end{aligned}$$

The variance of the mixed beta distribution (14) is given by

$$\sigma_x^2 = \pi \sigma_1^2 + (1-\pi)\sigma_2^2 = \mu_x(1-\mu_x)\left(\frac{\pi}{1+\theta_1} + \frac{(1-\pi)}{1+\theta_2}\right) =: \frac{\mu_x(1-\mu_x)}{1+\theta}, \quad \theta > 0, \tag{15}$$

where the precision θ of the mixed distribution is related to the precision of the component distributions by

$$\frac{1}{1+\theta} = \frac{\pi}{1+\theta_1} + \frac{1-\pi}{1+\theta_2}. \tag{16}$$

Defining $\eta := \theta_2/\theta_1$, $0 < \eta < 1$, equation (16) can be solved for θ_1 and $\theta_2 = \eta\theta_1$, as a function of any given precision $\theta \geq 0$, that means, as a function of any given variance $\sigma_x^2 \leq \mu_x(1-\mu_x)$:

$$\theta_1 = g(\theta) + \sqrt{g^2(\theta) + \theta/\eta}, \quad \text{and} \quad \theta_2 = \eta\, \theta_1, \tag{17}$$

with

$$g(\theta) = \frac{1}{2}\left(\left(\pi + \frac{1-\pi}{\eta}\right)\theta - \left(\frac{\pi}{\eta} + 1 - \pi\right)\right). \tag{18}$$

In the following we consider the mixed beta distribution parameterized with $\mu_x, \sigma_x^2, \pi, \eta$ for modelling the conditional distribution of stock index returns. Using the mixed beta

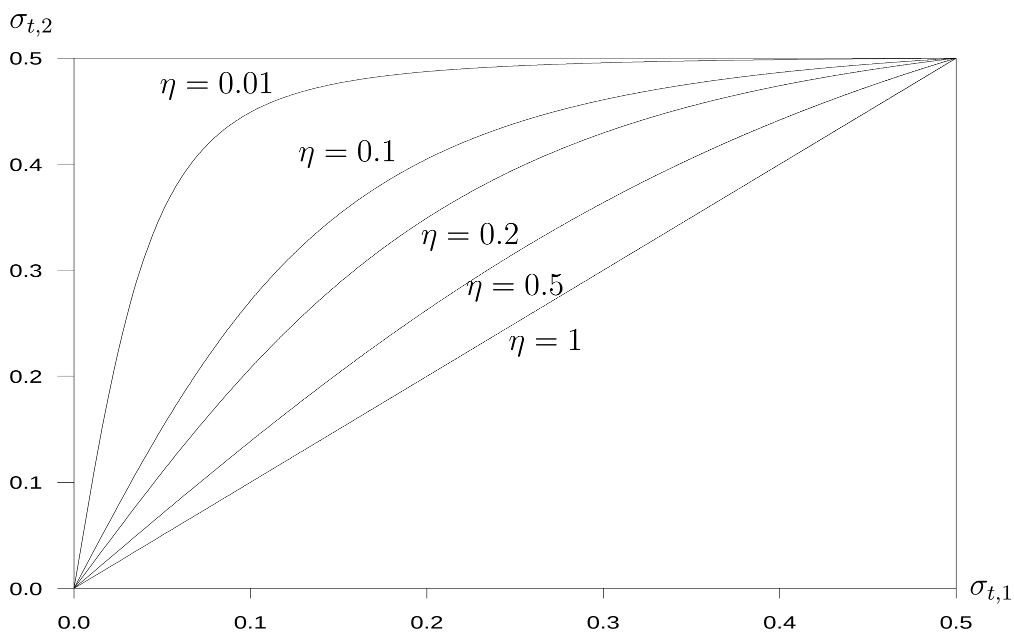

Figure 1:

Component volatilities $(\sigma_{t,1}, \sigma_{t,2})$ for different values of η with $\mu_x = 0.5$

distribution for modelling the time varying conditional distribution of index returns under a price limit regime, we allow for time varying conditional moments μ_t and σ_t^2 of the index returns, which transform according to section 2 into the conditional moments $\mu_{x_t} = \frac{\mu_t - a}{b-a}$ and $\sigma_{x_t}^2 = \frac{\sigma_t^2}{(b-a)^2}$. Then the precision θ_t is implied by (15). We assume time invariant weights $\pi, 1-\pi$ of the component distributions, as well as a time invariant ratio $\eta = \theta_{t,2}/\theta_{t,1}$. This assumption means that the dynamics of $(\theta_{t,1}, \theta_{t,2})$ are implied by the dynamics of $(\mu_{x_t}, \sigma_{x_t}^2)$ according to (17) and (18).

The assumption of a constant η implies a nonlinear relation between the volatilities $\sigma_{t,1}$ and $\sigma_{t,2}$, with $0 < \sigma_{t,1}^2 < \sigma_{t,2}^2 \leq \mu_{x_t}(1 - \mu_{x_t})$, of the mixed distribution's components. Figure 1 displays this relation, where $\mu_{x_t} = 0.5$, for various values of η. For tranquile periods the high volatility $\sigma_{t,2}$ is allowed to be considerably higher than $\sigma_{t,1}$, whereas in extremely volatile periods both standard deviations are enforced to approach the upper bound. Obviously our specification covers a wide range of admissible relations between the component volatilities. Notwithstanding the parsimonious specification with a constant η the shape of the proposed density model adapts to different volatilities in an extremely

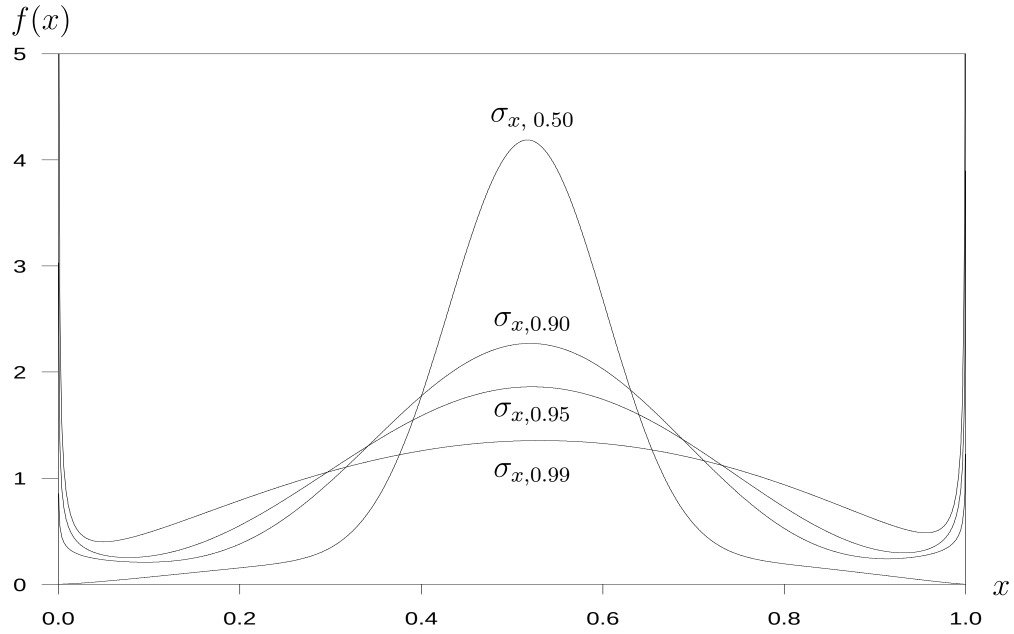

Figure 2:

Conditional densities of Shanghai B index returns for

$\sigma_{x,0.50} = 0.12, \sigma_{x,0.90} = 0.19, \sigma_{x,0.95} = 0.22, \sigma_{x,0.99} = 0.26$

(Mixed beta distribution with $\pi = 0.811, \eta = 0.127, \mu_x = 0.516$**)**

flexible way. As an example, consider the different shapes of the mixed beta density for alternative variances from an application to the conditional distribution of the daily returns on the Shanghai B-share index, when it was affected by a 10%-price change limit imposed on the individual shares.[4] Figure 2 displays the conditional densities of the transformed returns over the unit interval for the median volatility $\sigma_{x,0.50} = 0.12$, and for the volatility percentiles $\sigma_{x,0.90} = 0.19$, $\sigma_{x,0.95} = 0.22$, $\sigma_{x,0.99} = 0.26$. The parameters of the density are $\pi = 0.811$, $\eta = 0.127$, with the unconditional mean $\mu_x = 0.516$. The shape of the distribution also easily adapts to a time varying conditional mean μ_{x_t}. The conditional distribution of the transformed returns in turn determines the conditional distribution of the index return $r_t = a + (b - a)x_t$. Notice that in contrast to common

[4]Our empirical analysis relates to the dynamics of the daily stock index returns on the Shanghai A-share index (domestic investors) and on the B-share index (foreign investors) when a 10%-price change limit was imposed on the individual shares (for details of the specification of the volatility dynamics and for a statistical evaluation of the model specification see section 4).

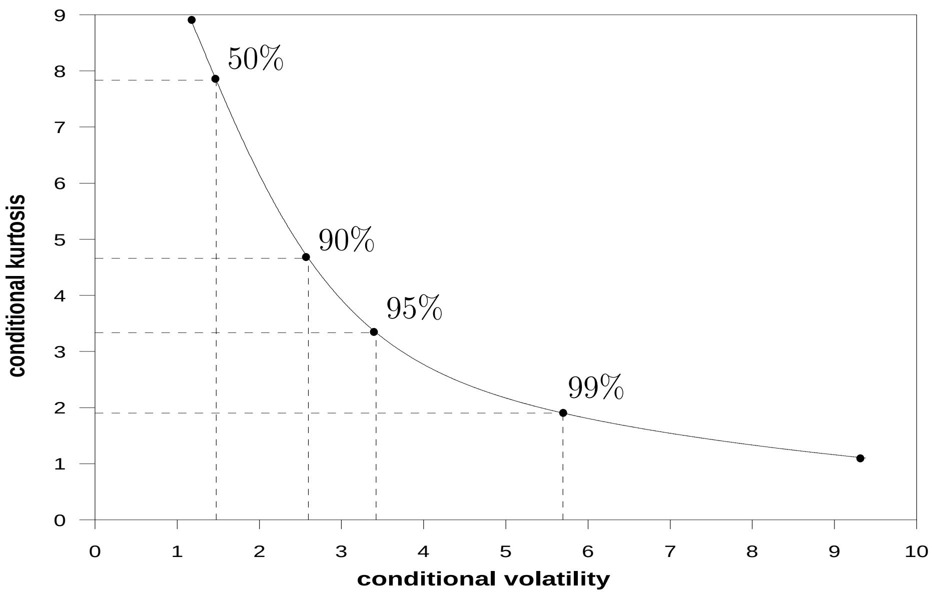

Figure 3: Kurtosis as a function of volatility (Shanghai A index)

$\pi = 0.942$, $\eta = 0.075$, $\mu = 0.075\%$

volatility modelling with an infinite range of the returns, in our case the conditional standard deviation σ_t cannot be considered as a scaling factor applied to an invariant distribution of standardized returns. Using the mixed beta distribution with a limited range according to the price change limits, the shape of the distribution changes with the standard deviation σ_t. In particular the time varying volatility implies that the kurtosis also varies over time, and, if $\mu_{x_t} \neq 0.5$, the skewness as well. The range of possible values for the kurtosis and skewness depends on the parameters π, η. It is getting larger with an increasing difference in the variance of the two component distributions (η decreasing) and an increasing weight π of the low volatility component.

As an illustration consider figure 3, which displays the kurtosis as a function of the volatility of the daily returns on the Shanghai A-share index, for which we have specified a time invariant conditional mean return. In the graphical display of the functional relation between kurtosis and volatility we have marked the volatility/kurtosis percentiles for 50%, 90%, 95%, and 99%. One observes that the conditional distribution is leptokurtic during the periods with low volatility, that means for more than 95% of the daily returns on the Shanghai-A index. Only when the predicted volatility is very high, the limited range of the returns induces a negative excess kurtosis. Generally the range for the volatility and for the kutosis of the limited index returns is restricted by the parameters for the volatility dynamics as indicated.

Negative excess kurtosis is an inherent implication of extremely high volatility under a price limit regime. The market volatility arrives at its upper bound when the probability mass is equally distributed to the price limits, with the excess kurtosis equal to -2. Our distribution model includes this implication of price limits as a special case. Using a more restrictive model such as the truncated normal distribution, the excess kurtosis is also negative, arriving at the value of -1.2 in the limiting case of a uniform distribution.

Whether the excess kurtosis is positive in tranquile periods is an empirical question. In the framework of our model specification it is related to the empirical relevance of using a mixture of different beta distributions instead of a simple beta distribution. Figure 3 already indicates the necessity of mixing, because the excess kurtosis is positive for more than 95 % of our predicted volatilities.

4 An empirical application

Daily price limits are currently in place in many international stock markets, e.g. in France, Italy, Japan, and especially in emerging markets, e.g. China, Korea, Taiwan, and Thailand. In China, after a period of more than four years without any price regulation, a price change limit was introduced on December 16, 1996 (Chinese Securities News, December 1996).

In this section we will apply the proposed framework for modelling the conditional distribution under a price change limit regime to Chinese stock market index returns, for the period from December 1996 until September 1999. Apart from daily price limits one main characteristic of Chinese stock markets is market segmentation. There are two types of shares traded at the Shanghai stock exchange. So called A-shares, traded in domestic currency, are designated only for private Chinese citizens and domestic institutions. B-shares, introduced in Shanghai in February 1992, are designed to attract foreign capital. B-shares can be owned and traded by foreign investors only, and are held mainly by international institutional investors.

The data base for our analysis consists of the value-weighted daily closing price indices for the Shanghai A-shares and B-shares.[5]

4.1 Basic Statistics

Table 1 presents basic statistics of continuous daily market return rates. The second column, giving the mean, standard deviation, and the respective $t-$value, shows that for all of the index return series the mean daily return is not significantly different from zero. The third column presents the skewness and kurtosis together with the Jarque-Bera statistic. According to the kurtosis of the return series we conclude that the distributions are clearly nonnormal. This conclusion is strongly supported by the Jarque-Bera statistic.

With respect to the minimum and maximum returns, notice that the price change limit refers to the discrete rate of return, i.e. $0.9p_{t-1} \leq p_t \leq 1.1p_{t-1}$, while the analyzed data represents continuous return rates $r_t = \ln(p_t) - \ln(p_{t-1})$. Thus under the price limit regime the observed return rates should satisfy the restriction $\ln(0.9) = -10.54\% \leq r_t \leq 9.53\% = \ln(1.1)$.[6]

The Ljung-Box statistics in the last column of table 1 were used to test for serial correlation. With respect to the significance level notice that the Ljung-Box statistics have been corrected to allow for ARCH effects.[7] In all cases the null hypothesis of white noise cannot be rejected. Due to the correction for general ARCH effects, however, the power of the modified Ljung-Box test decreases.

[5] The data is from Global Financial Data.

[6] For computational reasons we have extended the range slightly to $[1.01 \cdot \ln(0.9), 1.01 \cdot \ln(1.1)]$ in order to avoid realizations at the limits.

[7] See Diebold (1995) pp. 444-445.

Table 1:

Daily Market Returns from 1992 to 1999: Test Statistics

Index	Mean StdDev $t(\mu = 0)$		Skewn. Kurt. JB		Min Med. Max	LB*(6) LB*(12) LB*(18)	
Sha.A	0.055		−1.06		−10.80	6.07	(0.416)
$T = 677$	2.008		9.27		0.089	12.78	(0.385)
	0.708	(0.479)	1207.3	(0.000)	7.45	19.60	(0.356)
Sha.B	−0.061		0.26		−10.09	10.32	(0.112)
$T = 678$	2.829		5.07		−0.145	17.73	(0.124)
	−0.557	(0.578)	123.9	(0.000)	9.41	20.48	(0.306)

p-values in parentheses, JB: Jarque-Bera-Statistic, LB*(m): under ARCH corrected Ljung-Box-Test.

4.2 Estimation results

For the conditional mean and volatility dynamics we estimate the model

$$\begin{aligned} \mu_t &= \mu + \psi \varepsilon_{t-1}, \quad |\psi| < 1, \\ \sigma_t^2 &= \omega + (\alpha + \gamma I)\varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

Let $\boldsymbol{\phi} := (\mu, \psi, \omega, \alpha, \beta, \gamma)'$ and $\boldsymbol{\nu} := (\pi, \eta)'$. The log-likelihood function is specified using the mixed beta distribution model according to section 4:

$$\ln L(\boldsymbol{\phi}, \boldsymbol{\nu} | r_T, r_{T-1}, \ldots; a, b) = \sum_t \ln \left[\pi \ f_1(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b) + (1 - \pi) \ f_2(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b) \right].$$

The recursive formula for the log-likelihood of the observation r_t is given in the appendix A.2. Obviously the log-likelihood function is highly nonlinear in the parameters. The maximization of the log-likelihood function has been performed with the BHHH algorithm. The application of the BHHH algorithm to various sets of initial values has converged to the same maximum. The parameter estimates are presented in table 2. For the original model specification the estimate of ψ is insignificant in case of the A-share index, while the estimate of γ is insignificant for the B-share index. In table 2 we report only the estimation results for the final specifications. The estimates of the GJR parameters α and β are highly significant for both segments, confirming the presence of autoregressive

Table 2:

Parameter Estimates

	Mean		GJR-Parameter				Mixed Beta		
	μ	ψ	ω	α	β	γ	$1-\pi$	η	LogL
Sha.A	0.075		0.637	0.196	0.534	0.162	0.059	0.078	-1265.5
	(1.46)		(4.60)	(3.53)	(8.36)	(2.03)	(3.37)	(4.16)	
Sha.B	-0.184	0.134	0.893	0.193	0.690		0.189	0.127	-1558.9
	(-2.08)	(3.55)	(3.56)	(5.26)	(12.8)		(4.64)	(7.33)	

t-values in parentheses

conditional heteroskedasticity in the series. With respect to the asymmetric reaction of the predicted volatility to good and bad news, the expected pattern of a stronger volatility response to bad news is significant only in case of the A-share index.

Inserting the parameter estimates in (7) and (5), we get the following bounds for the conditional mean and volatility for the two index returns:

$$\begin{array}{lcc} \text{A index:} & \mu_t = 0.075\ \% & 1.176\ \% \le \sigma_t \le 9.386\ \% \\ \text{B index:} & -1.774\ \% \le \mu_t \le 1.331\ \% & 1.697\ \% \le \sigma_t \le 9.516\ \%. \end{array}$$

For both of the index returns the model implied range of volatility is consistent with the admissible range (6) due to the price limit, because

$$\begin{aligned} \sqrt{\min_{\mu_t}((b-\mu_t)(\mu_t-a))} &= 10.017\% \quad \text{for the A index,} \\ \sqrt{\min_{\mu_t}((b-\mu_t)(\mu_t-a))} &= 9.865\% \quad \text{for the B index.} \end{aligned}$$

For comparison, using a doubly truncated bell shaped density for the conditional distribution, the volatility would be restricted to be less than $(b-a)/\sqrt{12} = 5.85\%$, i.e., the volatility in the case of a uniform distribution. Thus, whenever the GARCH model predicts volatilities above this artificial limit a doubly truncated normal distribution model, for example, breaks down.

With respect to a simple beta distribution model as a competing approach, the estimated density parameters η and π clearly indicate the necessity of using a mixture of two beta distributions as low and high volatility components. For an illustration of the wide range of different shapes of the estimated densities, see figure 2. The estimates of $1-\pi$, giving the weight of the high volatility component, are 0.058 and 0.189. Both estimates are significant at the 1 % level. The estimated values of η are 0.075 for the domestic index return and 0.127 for the foreign index return, and both are significantly different from 1, displaying a high volatility spread between the two component distributions, compare figure 1. With parameter combinations π close to unity and η near zero, the mixed beta distribution model allows for a wide range of the kurtosis as a function of the volatility, see figure 3, whereas a simple beta distribution model would a priori imply a negative excess kurtosis.

4.3 Density Evaluation

In this section we apply several specification tests to the estimated model. The applied testing procedure basically relies on the research of Rosenblatt (1952), who shows that the transformation of a random variable with its own cumulative distribution function gives a uniformly distributed random variable.

Let $\{f_t(r_t|r_{t-1}, r_{t-2}\ldots)\}_{t=1}^T$ be the sequence of the true conditional densities governing a return series r_t, and let $\{\hat{f}_t(r_t|r_{t-1}, r_{t-2}\ldots)\}_{t=1}^T$ be a corresponding series of specified densities with the estimates inserted for the parameters. Finally, let $\{r_t\}_{t=1}^T$ denote the corresponding series of realizations. Diebold, Gunther, and Tay (1998) show, that the transformed random variables

$$u_t = \int_{-\infty}^{r_t} \hat{f}_t(w)\, dw = \hat{F}_t(r_t), \quad t = 1, \ldots, T$$

have support on the unit interval and are i.i.d. and uniformly distributed, if the specified density coincides with the true density, i.e. $\hat{f}_t(r_t|r_{t-1}, r_{t-2}, \ldots) = f_t(r_t|r_{t-1}, r_{t-2}, \ldots)$.

A wide variety of tests and graphical tools are then available to check both for independence and uniformity. One simple graphical display to evaluate the appropriateness of a probability model is the so-called P-P plot.[8] The P-P plot is based on the ordered ran-

[8] See e.g. Spanos (1999) pp. 232

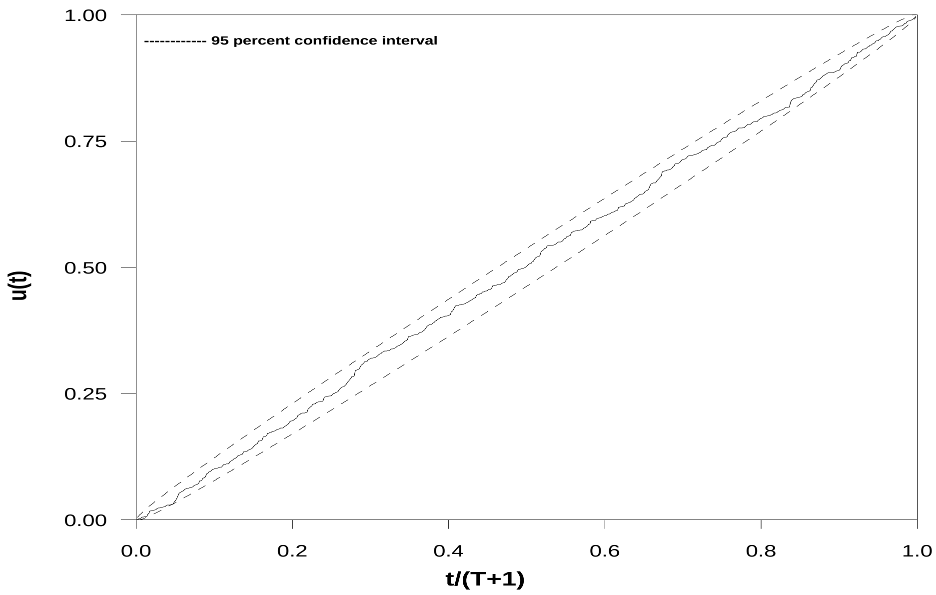

Figure 4:

P-P plot – Shanghai A index

dom variables $u_{(1)} < \ldots < u_{(t)} < \ldots < u_{(T)}$. In the case where the u_t are i.i.d uniformly distributed, the ordered random variables $u_{(t)}$ are beta distributed with mean $t/(T+1)$, $t = 1, \ldots, T$. This suggests a graphical way to check the distribution assumption using the P-P plot:

$$\left(\frac{t}{T+1}\,,\, u_{(t)}\right) \quad t = 1, \ldots, T.$$

If the distribution model is correctly specified then the transformed variables u_t are uniformly distributed and the plot should roughly look like a straight line through the origin.

For both, the Shanghai A and B index return series, the transformed variables were calculated using the estimated mixed beta distributions.[9] In figure 4 we show the P-P plot together with the 95% confidence band for the transformed Shanghai A return series. In both cases the plot lies within the range of the 95% confidence band. This result indicates that the specified mixed beta distribution provides a good approximation to the

[9]The graphics for the Shanghai A index are presented in the paper; the figures for the Shanghai B index are available from the authors on request. The integration of the mixed beta density function was performed by Monte Carlo simulation with 30,000 replications.

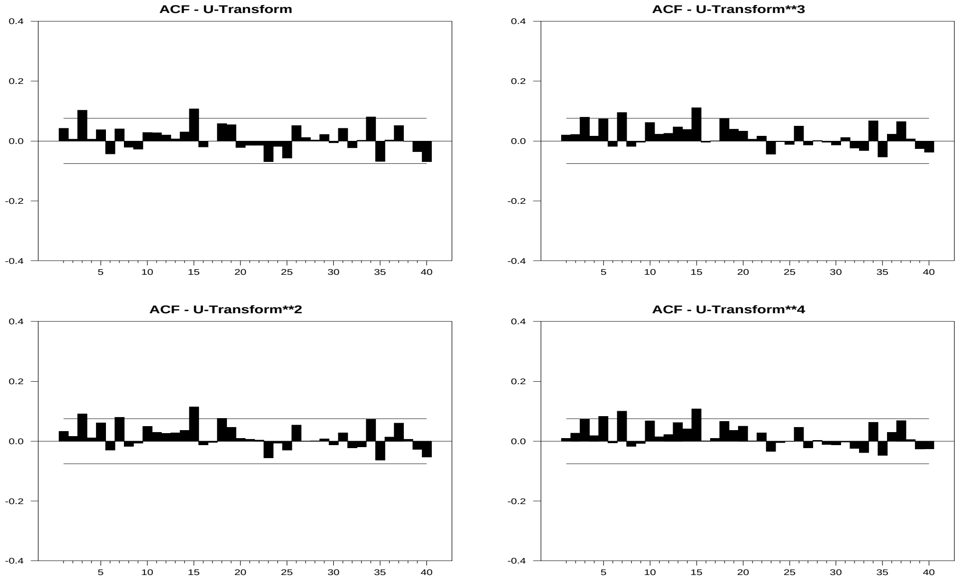

Figure 5:

Histogram of *u* series – Shanghai A index

conditional distribution of the Shanghai A and B-share index returns.

After confirming the appropriateness of the distribution model, we now examine whether the transformed variables u_t are i.i.d. The obvious graphical tool to detect linear dependencies is the correlogram of $(u - \bar{u})$. In order to detect nonlinear dependencies we also examine the correlograms of powers of $(u - \bar{u})$. The correlograms are supplemented with the usual Bartlett 95% confidence intervals. Figure 5 contains the correlogram of $(u - \bar{u})$ and powers of $(u - \bar{u})$ up to order four. For both the A index returns and the B index returns, the correlogram of $(u - \bar{u})$ as well as the correlograms of the powers of $(u - \bar{u})$ display no serious dependencies.

Additional to the graphical assessment of the u-series we apply a nonparametric test for uniformity. Crnkovic and Drachman (1996) suggested to use the Kuiper test, which is based on the distance between the observed density of u and the theoretical density. The Kuiper statistics for the two transformed index return series are given in the sixth column of table 3. At a significance level of 5% the null hypothesis of a uniform distribution cannot be rejected for the u series.

Table 3:

Density Evaluation

Index	Mean $t(\mu_z = 0)$ Variance $t(\sigma_z^2 = 1)$		Skewn. Kurtosis JB		LR KS Kuiper(u)	
Sha.A	0.002		-0.168	(0.075)	0.469	(0.926)
	0.049	(0.961)	3.124	(0.512)	0.024	[0.034]*
	1.014		3.622	(0.164)	1.078	(0.699)
	0.217	(0.828)				
Sha.B	0.020		0.195	(0.039)	3.359	(0.340)
	0.525	(0.599)	3.175	(0.386)	0.028	[0.034]*
	0.997		5.161	(0.076)	1.131	(0.619)
	-0.004	(0.940)				

p-values in parentheses; LR: likelihood-ratio statistic for autocorrelation;

KS: Kolomogorov-Smirnov statistic; *: critical value at the 5% significance level.

The applied tests for uniformity are, as common, nonparametric and notoriously data intensive. Crnkovic and Drachman recommend that for the application of the Kuiper test at least 1000 observations should be available. The poor small sample properties of the test are also confirmed by simulation studies of Berkowitz (2001). On the other hand, it is difficult to develop parametric tests for uniformity, e.g. likelihood ratio tests, because of the discontinuity of the objective function. For this reason, Berkowitz (2001) proposes to consider a further transformation of the data. If the probability model is correctly specified, then the following transformation of the u series

$$z_t = \Phi^{-1}\left(\int_{-\infty}^{r_t} \hat{f}_t(w)\, dw\right) = \Phi^{-1}(u_t) \overset{i.i.d.}{\sim} \mathrm{N}(0,1),$$

where Φ^{-1} denotes the inverse of the standard normal cumulative distribution function, yields an i.i.d. N(0,1) distributed time series.[10] The advantage of this transformation is that under the null the data follow a normal distribution and thus a wide variety of parametric as well as nonparametric tests are available.

[10] See Berkowitz (2001).

If our conditional density specification is appropriate for the Chinese stock index returns, then the corresponding z series should have zero mean and unit variance. The t statistics in table 3 indicate that in all cases the null cannot be rejected. The results of the usual tests for skewness and kurtosis applied to the z series of the Shanghai A returns confirm our density specification. In the case of the Shanghai B returns the skewness test indicates significant positive skewness, while the hypothesis of zero excess kurtosis cannot be rejected at the 5% level. However, when the Jarque-Bera test is applied to the z series, the null of a normal distribution cannot be rejected at the 5% level for both index return series. Using the Kolmogorov-Smirnov test to check for normality of the z series confirms the above conclusion.

Finally, we consider the likelihood ratio test for the z series developed by Berkowitz (2001) to test against a first-order autoregressive alternative with mean and variance possibly different from (0,1). LR in table 3 denotes the likelihood ratio test statistic. For the A index as well as for the B index first-order serial correlation cannot be detected at the 5% significance level.

5 Conclusions

In this paper we propose a new approach for the conditional distribution of daily stock index returns when for all of the involved stocks a price limit is imposed. Obviously the price limits on the individual stocks imply that the index return, and the return on any portfolio as well, is bounded in the same way. Empirically the aggregate return, in contrast to the individual stocks, will rarely, if at all, hit the limit. For this reason a censored distribution model is not suitable for the portfolio return. On the other hand a truncated normal distribution is too restrictive both with regard to the volatility and to the kurtosis of the conditional return distribution. In contrast, the proposed mixed beta distribution model appears to capture the special features of the limited index return very well. In particular, it allows for a wide range of the kurtosis, depending on the time varying volatility.

Our analysis provides some important insights for higher conditional moments of index returns under a price change limit. Not only the volatility is restricted by the limits, but also the kurtosis is typically less than three in periods of extremely high volatility. On the other hand our empirical study of Chinese stock market index returns under price limits indicates that for the major part of the sample, where the volatility is moderate or low, the conditional excess kurtosis is positive and compares with the standard observation for daily return time series. A simple beta distribution model, which is contained in our approach as a special case, would be unable to capture this feature of the data.

The distributional form of asset returns is highly relevant for theoretical and empirical analyses in economics and finance, for example, in asset, portfolio and option pricing. It is beyond the scope of this paper and suggested as a topic of future research to investigate the implications of the mixed beta distribution model, which appears to be appropriate for index returns under price limits.

A Appendix

A.1 Moments of mixed distributions

We derive some simple properties of the central moments of a density function f that is obtained as a mixture of two densities f_1 and f_2 with the same mean μ,

$$f(x) = \pi f_1(x) + (1-\pi) f_2(x), \quad 0 < \pi < 1, \quad x \in \mathbb{R}.$$

The mixed density f has the same mean μ and if for $n \geq 2$ the central moments $\mu_{n,i} = \mathrm{E}(X_i - \mu)^n$ exist for the random variabels X_i with density function $f_i, i = 1, 2$, then the $n-$th central moment of the random variable X with the mixed density f is

$$\mu_n = \mathrm{E}(X-\mu)^n = \pi \mu_{n,1} + (1-\pi)\mu_{n,2}.$$

Denoting the second central moments of f_1, f_2, f also as $\sigma_1^2, \sigma_2^2, \sigma^2$, respectively, the following Lemma holds for the skewness $\alpha_3 = \frac{\mu_3}{\sigma^3}$ and the kurtosis $\alpha_4 = \frac{\mu_4}{\sigma^4}$:

Lemma 1 *The kurtosis (skewness) of the mixed density f can be written as a multiple of an averaged kurtosis (skewness) of the component distributions:*

$$\alpha_n = \frac{\mu_n}{\sigma^n} = M_n \bar{\alpha}_n \quad n = 3, 4,$$

with

$$\begin{aligned} M_n &= \pi \frac{\sigma_1^n}{\sigma^n} + (1-\pi)\frac{\sigma_2^n}{\sigma^n} \geq 1 \\ \bar{\alpha}_n &= \omega_n \alpha_{n,1} + (1-\omega_n)\alpha_{n,2} \\ \omega_n &= \frac{\pi \sigma_1^n}{\pi\sigma_1^n + (1-\pi)\sigma_2^n}. \end{aligned}$$

Proof:

For $n \geq 3$ we have

$$\begin{aligned} \alpha_n = \frac{\mu_n}{\sigma^n} &= \frac{\pi\mu_{n,1} + (1-\pi)\mu_{n,2}}{\sigma^n} \\ &= \frac{\pi\sigma_1^n\alpha_{n,1} + (1-\pi)\sigma_2^n\alpha_{n,2}}{\sigma^n} \\ &= \frac{(\pi\sigma_1^n + (1-\pi)\sigma_2^n)}{\sigma^n} \cdot \frac{(\pi\sigma_1^n\alpha_{n,1} + (1-\pi)\sigma_2^n\alpha_{n,2})}{(\pi\sigma_1^n + (1-\pi)\sigma_2^n)} \\ &= M_n \cdot \bar{\alpha}_n. \end{aligned}$$

With regard to the size of M_n we get

$$M_n = \frac{\pi\sigma_1^n + (1-\pi)\sigma_2^n}{\sigma^n} = \frac{\pi(\sigma_1^2)^{\frac{n}{2}} + (1-\pi)(\sigma_2^2)^{\frac{n}{2}}}{(\pi\sigma_1^2 + (1-\pi)\sigma_2^2)^{\frac{n}{2}}} = \frac{\mathrm{E}[(\tilde{\sigma}^2)^{\frac{n}{2}}]}{(\mathrm{E}[\tilde{\sigma}^2])^{\frac{n}{2}}} \geq 1$$

by Jensen's inequality, where $\tilde{\sigma}^2$ takes the values σ_1^2, σ_2^2 with probabilities $\pi, 1-\pi$, respectively. ■

Corollary 1 *The kurtosis of the mixed distribution has no upper bound.*

Proof:

Let the variance of one of the component distributions tend to zero and its respective weight tend to one, say $\sigma_1^2 = \delta(1-\pi), \pi \to 1$, for some $\delta > 0$, whereas $\sigma_2^2 > 0$ is fixed, then $M_4 \approx (1-\pi)^{-1}$ grows to infinity. The unboundedness of M_4 implies that the kurtosis α_4 of the mixed density has no upper bound. ■

The following corollary gives another instructive representation of the factor M_4 for the kurtosis of the mixed density.

Corollary 2 *Equivalent to the formula for α_4 in Lemma 1 the kurtosis of the mixed density f is given by*

$$\alpha_4 = (1 + \mathrm{Var}\left(\frac{\tilde{\sigma}^2}{\sigma^2}\right)) \cdot \bar{\alpha}_4.$$

Proof:

With $\mathrm{E}(\tilde{\sigma}^2) = \pi\sigma_1^2 + (1-\pi)\sigma_2^2 = \sigma^2$ we have

$$\mathrm{Var}\left(\frac{\tilde{\sigma}^2}{\sigma^2}\right) = \mathrm{E}\left(\frac{\tilde{\sigma}^4}{\sigma^4}\right) - \left(\frac{\mathrm{E}(\tilde{\sigma}^2)}{\sigma^2}\right)^2 = \mathrm{E}\left(\frac{\tilde{\sigma}^4}{\sigma^4}\right) - 1,$$

thus $M_4 = \mathrm{E}\left(\frac{\tilde{\sigma}^4}{\sigma^4}\right) = 1 + \mathrm{Var}\left(\frac{\tilde{\sigma}^2}{\sigma^2}\right)$. ■

A.2 Recursive formula for the log-likelihood

Information up to $t-1$ implies $\sigma_{t-1}^2, \varepsilon_{t-1}$, hence the parameters of the conditional density are:

$$\begin{aligned}
\mu_t &= \mu + \psi\varepsilon_{t-1} && \text{and} \quad \mu_{x_t} = \frac{\mu_t - a}{b-a} \\
\sigma_t^2 &= \omega + (\alpha + \gamma I)\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 && \text{and} \quad \sigma_{x_t}^2 = \frac{\sigma_t^2}{(b-a)^2} \\
\theta_t &= \frac{\mu_{x_t}(1-\mu_{x_t})}{\sigma_{x_t}^2} - 1 = \frac{(\mu_t - a)(b-\mu_t)}{\sigma_t^2} - 1 \\
g(\theta_t) &= \frac{1}{2}\left[\left(\pi + \frac{1-\pi}{\eta}\right)\theta_t - \left(\frac{\pi}{\eta} + 1 - \pi\right)\right] \\
\theta_{t,1} &= g(\theta_t) + \sqrt{g^2(\theta_t) + \frac{\theta_t}{\eta}} \\
\theta_{t,2} &= \eta\,\theta_{t,1} \\
c_t &= \mu_{x_t}\,\theta_{t,1} \\
d_t &= (1-\mu_{x_t})\,\theta_{t,1}.
\end{aligned}$$

Given the observation r_t, its contribution to the log-likelihood is

$$\ln\left[\pi\ f_1(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b) + (1-\pi)\ f_2(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b)\right]$$

with

$$\begin{aligned}
f_1(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b) &= \frac{1}{(b-a)^{c_t+d_t-1}B(c_t, d_t)}\,(r_t - a)^{c_t-1}(b-r_t)^{d_t-1} \\
f_2(r_t; \boldsymbol{\phi}, \boldsymbol{\nu}, a, b) &= \frac{1}{(b-a)^{\eta(c_t+d_t)-1}B(\eta c_t, \eta d_t)}\,(r_t - a)^{\eta c_t-1}(b-r_t)^{\eta d_t-1}
\end{aligned}$$

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