General Rationalizability and its Robustness for Strategic Form Games with Incomplete Information

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Abstract

We extend the Δ-rationalizability (see Battigalli and Siniscalchi (2003)) to infinite strategic form games with incomplete information. The most important feature of the Δ-rationalizability is that there is no specified epistemic type space à la Harsanyi. However, we can impose a collection of exogenous restrictions on first order beliefs over payoff types and strategies represented by a collection of correspondences Δ. When Δ represents only restrictions on beliefs over payoff types, we show that the Δ-rationalizable sets are nonempty under general topological conditions. Robustness with respect to almost common belief for rationality of Δ-rationalizability is established under general conditions by two alternative approaches. We can approximate common belief by finite order of mutual beliefs; we can approximate common belief by common p-belief. One important feature of our analysis in the robustness is that in the second approach, different level of belief is allowed for every order of mutual belief among players.

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1 Introduction

One of the most important solution concept in game theory is the rationalizability. In contrast to the equilibrium approach, the rationalizable set is defined to be the logical consequence of the following two promises: (O1) every player is rational in the sense that each one forms a subjective probability distribution over his opponents' choice of strategy and maximizes his expected utility relative to the prior, and (O2) the fact (O1) and the structure of the game are common knowledge among the players. One advantage of the definition is that it clarifies what is assumed and what is implied. In their seminal papers, Pearce (1984) and Bernheim (1984) first formalize these ideas and provide many interesting properties of the rationalizable sets.

In practice, however, the assumptions required for the rationalizability rarely hold. In his seminal paper, Bernheim argues that:

"Practically speaking, we might not expect agents to check the consistency of their beliefs for more than a finite number of levels. This leads us to ask whether the theory of strategic behavior developed above is robust to deviations form perfect rationality." Bernheim (1984), p.1017.

Following this line of analysis, this paper develops a general rationalizability for strategic form games with incomplete information and investigate the robustness of the solution concept. The rationalizability we provide are defined to be the combinations of payoff types and strategies that are implied by the following two assumptions: (R1) each player is rational and his first order belief satisfies some exogenous restrictions and (R2) the structure of the game and the fact (R1) is common belief among players. The two assumptions are essentially the same as the two promises mentioned above.

However, two important features of our analysis are worthy discussed. Rather than modelling asymmetric information on payoff relevant parameters by means of the Harsanyi types(Harsanyi (1967)), we use the payoff types only to denote the possible values of the payoff relevant parameters. The payoff type for each player can be interpreted as the private information that he have about which game being played and the expected utility of other players. We can accommodate exogenous restrictions on players' first order beliefs in stead of imposing a specific type space. Thus, the structure of the game is then defined to be the spaces of the payoff types, the strategy spaces, and the payoff mappings of each player. In such a framework we might embed the set of payoff types into a *universal type space*[1] and this approach would avoid untended restrictions on beliefs due to the adoption of a "small" type space, as noted by Battigalli and Siniscalchi (2003). The analysis here can be viewed as a coherent theory in the sense of Gul (1996) since we explicitly express what is assumed and what is implied.

The imposed exogenous restrictions on each player's first order beliefs over the payoff types and the strategies of his opponents are represented by a set of correspondences. Each of them maps the payoff types for each player into a subset of all his possible first order beliefs — all probability distributions over the strategy spaces times the space of the payoff types of his opponents. These correspondences are generally denoted Δ. Note that the restrictions may depend on the occurrence of payoff types.

The Δ-rationalizable sets, as mentioned above, are thus defined to be the combinations of payoff types and strategies which are consistent with the assumptions (R1) and (R2). This definition coincides with the Δ-rationalizable sets defined in Battigalli and Siniscalchi (2003) for finite strategic form games with incomplete information. Thus, our work can be viewed as a generalization of theirs. However, when the space of payoff types and the strategy space become infinite, the existence and the "best response property" become nontrivial. These results are proved when the underlying payoff type spaces and strategy spaces are compact metric, the correspondences in Δ are closed and all the payoff functions are continuous.

The robustness is then defined to be that when the common certainty assumption in (R2) is replaced by some appropriate almost common belief assumption, the implication of the strategic behavior of these assumptions should be close to the Δ-rationalizability. Since the solution concept we consider is set-valued, we define the closeness in terms of

[1]For more discussion on this point, see Battigalli and Siniscalchi (2003) and Battigalli and Siniscalchi (1999).

sets rather than points. However, one can still ask whether each point in the set implied by the almost common belief is close to some point in the set implied by the common certainty assumption as in Bernheim (1984).

We define the almost common belief in two alternative ways. We can approximate by means of finite order of mutual belief, as in Bernheim (1984). We can use common p-belief formalized by Monderer and Samet (1989) as a approximation of common certainty. The robustness in the first approach is easily established by the mathematical properties of the Hausdorff distance topology. In the second approach, an important departure from the literature in our analysis is that we allow different level of belief in every order of mutual belief in defining common p-belief.

Thus, the $\mathbf{p}$ in the common p-belief becomes a sequence in $(0, 1]$ and an event $\mathcal{E}$ is common $\mathbf{p}$-belief if every player assigns probability at least p_1 on the event $\mathcal{E}$(this event is denoted $\mathcal{E}_1$), every player assigns probability at least p_2 on the event $\mathcal{E}_1$(this event is denoted $\mathcal{E}_2$), every player assigns probability at least p_3 on the event $\mathcal{E}_2$(this event is denoted $\mathcal{E}_3$), etc. One of our main contribution is to establish the robustness of the Δ-rationalizability to such a definition of almost common knowledge under the same topological conditions mentioned above.

At first glace, it seems that these topological conditions are hard to understand. However, it is well-known that compactness and continuity are important conditions for the existence of the best responses. Since we are interesting in the robustness of a set-valued solution concept, it seems difficult to describe the closeness of subsets when the underlying space is not metrizable. Indeed, there are many interesting topologies that can be endowed in the space of all subsets of a topological space(see, for example, Aliprantis and Border (1999)) in terms of the open sets rather than metric. Yet, the topology on the space of all nonempty compact subsets induced from the Hausdorff distance coincides with all these topologies in metric spaces. So it seems a good start point to begin with metric spaces.

Gul (1996) has provided a similar robustness result by the second approach for finite games with complete information. However, in finite games, the implication of common

p-belief of rationality becomes the same for sufficiently large p while this is not true in general for games with infinite strategy spaces. Another important difference between his analysis and ours is that his use of common p-belief can accommodate lexicographic preferences, while we focus on preferences consistent with the subjective expected utility models. The techniques needed for the lexicographic preferences with infinite state spaces are beyond the scope of the current paper. However, most results in Gul (1996) can be generalized to finite strategic form games with incomplete information by our framework.

The rest of this paper is organized as follows. Section 2 provides the definitions and important properties of our main solution concept. An existence theorem and the "best response property" are proved there. Section 3 focuses on the robustness of the solution concept. Two alternative approaches are discussed and the robustness are established there for both ways. The Appendix consists of some mathematical preliminaries needed for our analysis and the omitted proofs.

2 Rationalizable Sets

In this section, the Δ-rationalizable sets are defined formally for strategic form games with incomplete information via a procedure of iterative deletion of combinations of the payoff types and the strategies. The exogenous restrictions are explicitly expressed in terms of correspondences. The procedure will find out all combinations of payoff types and strategies implied by assumptions (R1) and (R2).

For those correspondences that represent only the restrictions on each player's beliefs over the payoff types of his opponents, the existence of the Δ-rationalizable sets is proved under general topological conditions. A corresponding "best response property" is also proved for the Δ-rationalizable sets. We also define a notion of consistency for the Δ-rationalizable sets.

2.1 Definitions and Notations

Consider a strategic form game with incomplete information:

$$G \equiv (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}),$$

where N denotes a finite set of players, Θ_i is the set of all possible payoff types, S_i is the strategy space, and $u_i : S \times \Theta \to \Re$ denotes the payoff function for player i. Throughout this paper, $\Theta_i \times S_i$ is assumed to be a compact metric space and u_i is a continuous function for all $i \in N$.

As mentioned in the introduction, we use exogenous restrictions by meas of correspondences from the space of the payoff types for each player into all his possible first order beliefs. Formally, for each player i, consider the correspondence $\Delta_i : \Theta_i \twoheadrightarrow \Delta(\Theta_{-i} \times S_{-i})$ for each player i and let $\Delta \equiv (\Delta_i)_{i\in N}$. To avoid confusion, we will use Δ^{θ^i} in place of $\Delta_i(\theta^i)$. Then the set Δ^{θ^i} is interpreted as the exogenous restriction of the beliefs for player i when θ^i occurs. Let Σ be the collection of all such correspondences.

The Δ-rationalizable sets are defined to be the combinations of payoff types and strategies that are implied by the assumptions (R1) and (R2). The following definition formalizes this idea and provide a procedure of iterative deletion of combinations of the payoff types and the strategies.

Definition 2.1. Let $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a strategic form game with incomplete information, and let $\Delta \in \Sigma$. Let $T_i \subset \Theta_i \times S_i$ and let $T \equiv \prod_{i\in N} T_i$. Define $\Lambda_i(T, \Delta) \equiv \{(\theta^i, s^i) \in \Theta_i \times S_i$: there exists $\mu \in \Delta^{\theta^i}$ such that $\mu(T_{-i}) = 1$ and $s^i \in F^i[\mu, \theta^i]\}$, where $F^i[\mu, \theta^i] \equiv \arg\max_{s^i\in S_i} \int_{\Theta_{-i}\times S_{-i}} u_i(s, \theta)\, d\mu(s^{-i}, \theta^{-i})$. When $T = \emptyset$, let $\Lambda_i(T, \Delta) = \emptyset$.

Then the set of Δ-rationalizable strategies is defined to be $R(G, \Delta) \equiv \bigcap_{n=1}^{\infty} \Lambda^n(\Theta \times S, \Delta)$, where $\Lambda(\Theta \times S, \Delta) \equiv \prod_{i\in N} \Lambda_i(\Theta \times S, \Delta)$ and $\Lambda_i^n(\Theta \times S, \Delta) \equiv \Lambda_i(\Lambda^{n-1}(\Theta \times S, \Delta), \Delta)$.

Battigalli and Siniscalchi (2003) provides a corresponding definition for finite strategic form games with incomplete information. We extend the work to games with infinite strategy spaces and infinite payoff types spaces. As in the finite case, the restrictions on beliefs represented by Δ, in general, may be inconsistent with the two assumptions (R1) and (R2). In this case, the set of Δ-rationalizable strategies will be empty.

The restrictions represented by the correspondences are general to enough to incorporate many interesting cases. On one extreme, let Δ be defined by $\Delta^{\theta^i} \equiv \Delta(\Theta_{-i} \times S_{-i})$ for all $i \in N$ and $\theta^i \in \Theta_i$, that is, there is no restriction on players' beliefs. On the other extreme, let $\eta \in \Delta(\Theta)$ and define Δ by $\Delta^{\theta^i} \equiv \{\mu \in \Delta(\Theta_{-i} \times S_{-i}) : \text{marg}_{\Theta_{-i}} = \text{marg}_{\Theta_{-i}} \eta\}$ for all $i \in N$ and $\theta^i \in \Theta_i$. Then Δ represents a common prior over the payoff types.

2.2 Existence and Other Properties

For the existence and robustness of the Δ-rationalizable sets, a sub-collection of Σ is introduced here before the discussion of our main results in this section. Let Σ_0 denotes the sub-collection consisted of closed correspondences[2]. Closed correspondences in this place are exactly those with closed graphs and satisfying a minimal continuity assumption (more precisely, upper hemi-continuity)[3]. These properties are important to ensure the existence and robustness.

One useful result for the sub-collection Σ_0 is the following lemma:

Lemma 2.1. *Suppose that T_i is a closed subset of $\Theta_i \times S_i$ for each $i \in N$ and let $T = \prod_{i \in N} T_i$. If $\Delta \in \Sigma_0$, then $\Lambda_i(T, \Delta)$ is closed and hence compact.*

2.2.1 Existence

Another important sub-collection of Σ are those only representing restrictions on each player's beliefs about the payoff types of his opponents. More specifically, Δ belongs to this subset if for each $i \in N$ and for each $\theta^i \in \Theta_i$, there exists a nonempty subset P of $\Delta(\Theta_{-i})$ such that $\Delta^{\theta^i} = \{\mu \in \Delta(\Theta_{-i} \times S_{-i}) : \text{marg}_{\Theta_{-i}} \mu \in P\}$. We denote by Σ_1 for the set of all such correspondences which are closed. Clearly, the two extreme cases mentioned in the last subsection lie in this sub-collection.

[2] For formal definition of closed correspondence, see Appendix.

[3] For more extensive and rigorous discussion of closed correspondences, see Appendix

The following theorem shows that all elements in Σ_1 are consistent with (R1) and (R2).

Theorem 2.1. *Let $G = (N, \{\Theta_i\}_{i \in N}, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game with incomplete information. If $\Delta \in \Sigma_1$, then for all $\theta_0 \in \Theta$, the set $(\{\theta_0\} \times S) \cap R(G, \Delta)$ is not empty.*

The compactness of the strategy spaces is also important for games with complete information to ensure the existence of the rationalizable sets, and Lipman (1994) provides a example where common belief of rationality is not the intersection of any finite order of mutual belief of rationality when the underlying strategy spaces are not compact. It should be noted that the metrizable assumption is not essential for Lemma 2.1 and Theorem 2.1, but it is important for our analysis of the robustness.

2.2.2 Best Response Property

When the strategy spaces are finite, it is clear that only finite rounds of elimination is needed, as noted by Battigalli and Siniscalchi (2003). However, in games with infinite available strategies, it may be insufficient to achieve common belief of rationality among players even after a infinitely countable many rounds of iterative elimination, as shown by Lipman (1994).

An important aspect mentioned above is that common belief of an event can be defined by two alternative approaches — that one can define common belief by the intersection of all finite order of mutual belief and that one can define common belief via self-evident events. The point is that the definition by the two approaches should be equivalent. It should be noted that both Pearce (1984) and Bernheim (1984) define the rationalizable sets via the "best response property" and show that this definition is equivalent to that from a iterative elimination procedure. The 'best response property" is very closed to a self-evident event since a set of strategy combinations satisfies this property if all strategies in it can be rationalized by the same set. The theorem stated below is an analogy to the "best response property" for Δ-rationalizable sets.

Theorem 2.2. *Let $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a normal form game, and let $\Delta \in \Sigma_0$. Then we have $\Lambda(R(G,\Delta),\Delta) = R(G,\Delta)$.*

2.2.3 Consistency

Since the restrictions we consider is quite general, it is important to demonstrate the consistency between different games with essentially the same structures.

To illustrate the idea, consider two strategic form games with incomplete information

$$G_0 = (N, \{\Theta_i^0\}_{i\in N}, \{S_i\}_{i\in N}, \{v_i\}_{i\in N})$$

and

$$G_1 = \left(N, \{\Theta_i^1\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right),$$

and suppose that $\Theta^0 \subseteq \Theta^1$ and $v_i(s,\theta) = u_i(s,\theta)$ for all $\theta \in \Theta^0$ and $s \in S$. Intuitively, if two restrictions associated with G_0 and G_1 coincide on Θ^0, then the Δ-rationalizable sets associated with these restrictions should be the same on Θ^0. The following proposition formally demonstrates this intuition.

Proposition 2.1. *Let G_0 and G_1 be given as above. Suppose that Δ_0 and Δ_1 are correspondences associated with G_0 and G_1, respectively, and satisfy that for all $i \in N$ and $\theta \in \Theta^0$, $\Delta_0^{\theta^i} = \Delta_1^{\theta^i}$. Then we have that $R(G_0,\Delta_0) = R(G_1,\Delta_1) \cap (\Theta^0 \times S)$.*

Proof. Note that since $\Delta_0^{\theta^i} = \Delta_1^{\theta^i}$ for all $i \in N$ and $\theta^i \in \Theta_i^0$, it follows that for all $T = \prod_{i\in N} T_i \subseteq \Theta^0 \times S$, $\Lambda_i(T,\Delta_0) = \Lambda_i(T \cup ((\Theta \setminus \Theta^0) \times S), \Delta_1) \cap (\Theta^0 \times S)$. □

Remark 2.1. If Θ^0 is a singleton set, then G_0 becomes a game with complete information. In this case, Proposition 2.1 shows that Δ-rationalizable set is a generalization of that for games with complete information.

3 Robustness

In this section, we investigate the robustness of the Δ-rationalizable sets. Recall that two fundamental assumptions of the Δ-rationalizable sets are (R1) and (R2). We approximate the common belief assumptions in (R2) by two alternative ways — via finite order of mutual belief as in Bernheim (1984) and via common p-beliefs formalized in Monderer and Samet (1989).

Remark 3.1. Borgers (1994) and Gul (1996) use common p-belief of rationality and exogenous restrictions to refine the rationalizable sets. The purpose of common p-belief there is to capture "cautious behavior" and thus the usage has to be compatible with non-Archimedian probabilities. However, we use the common p-belief here to capture the approximate common belief only, and we do not discuss preferences other than those consistent with the subjective expected utility models.

3.1 Robustness to Finite Orders

It might be hard to achieve infinite order of mutual beliefs, as noted in Bernheim (1984). Thus, it is important to investigate the implications of finite order of mutual beliefs.

Remark 3.2. To describe the nearness of sets, let $\mathcal{K}(\Theta \times S)$ denote the collection of all nonempty compact subsets of $\Theta \times S$. Since $\Theta \times S$ is assumed to be compact metric, a nature metric on $\mathcal{K}(\Theta \times S)$ is the Hausdorff distance[4], denoted by $d_{\mathcal{H}}$.

The theorem stated below shows that, for Δ-rationalizable sets, we can approximate the implications of the common belief assumption by finite order of mutual beliefs.

Theorem 3.1. *Let $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a strategic form game with incomplete information, and let $\Delta \in \Sigma_0$. Moreover, suppose that $R(G, \Delta) \neq \emptyset$. Then $d_{\mathcal{H}}(\Lambda^n(\Theta \times S, \Delta), R(G, \Delta)) \to 0$ in $\mathcal{K}(\Theta \times S)$ as $n \to \infty$.*

Remark 3.3. Note that the Hausdorff distance between two sets A and B is defined by $d_{\mathcal{H}}(A, B) \equiv \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$ for any metric space (X, d). Thus, the assertion in the theorem above can be replaced by the claim that for any any given $\epsilon \geq 0$, there is some $M \in \mathbb{N}$ such that $n \geq M$ implies that for any $(\theta^i, s^i) \in \Lambda^n_i(\Theta \times S, \Delta)$, there exists $(\theta^i_0, s^i_0) \in R(G, \Delta)$ such that $d((\theta^i_0, s^i_0), (\theta^i, s^i)) \leq \epsilon$.

[4]For a formal definition, see Appendix

3.2 Robustness to Common p-belief

3.2.1 Definitions and Some Properties

To take the second approach, some definitions and notations are needed.

Definition 3.1. Let $G = (N, \{\Theta_i\}_{i \in N}, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game with incomplete information, and let $\Delta \in \Sigma_0$. For any $T = \prod_{i \in N} T_i \subseteq \Theta \times S$ and any $p \in (0, 1]$, define $\Lambda_i^p(T, \Delta) \equiv \{(\theta^i, s^i) \in \Theta_i \times S_i :$ there is some $\mu \in \Delta^{\theta^i}$ so that $\mu(T_{-i}) \geq p$ and $s^i \in F^i[\mu, \theta^i]\}$.

To illustrate the definition, let us say that an event $\mathcal{E}$ is p-believed by player i if he assigns probability at least p to the event $\mathcal{E}$. There is mutual p-belief in $\mathcal{E}$ if it is p-believed by every player. For any given Δ, consider the following set of assumptions:

$\mathcal{A}^0$: every player i is rational and his beliefs satisfy the set of restrictions Δ_i,

$\mathcal{A}^1$: there is mutual p_1-belief in $\mathcal{A}^0$,

$\mathcal{A}^2$: there is mutual p_2-belief in $\mathcal{A}^0 \cap \mathcal{A}^1$,

....

$\mathcal{A}^{n+1}$: there is mutual p_{n+1}-belief in $\mathcal{A}^0 \cap \mathcal{A}^1 \cap ... \cap \mathcal{A}^n$,

....

Let us denote $\Lambda^{(p_1,\ldots,p_n)}(\Theta \times S) \equiv \Lambda^{p_n}(\Lambda^{(p_1,\ldots,p_{n-1})}(\Theta \times S))$ for all $n \in \mathbb{N}$. Then the set $\bigcap_{n \in \mathbb{N}} \Lambda^{(p_1,\ldots,p_n)}(\Theta \times S)$ consists of combinations of payoff types and strategies that are consistent with the event $\bigcap_{n \in \mathbb{N}} \mathcal{A}^n$. Note that we allow different levels of belief in each order of mutual belief among players. Clearly, the event $\mathcal{A}^0$ is the same as the assumption (R1). But we relax the common belief assumption in (R2) to common **p**-belief defined in the introduction.

When $p = p_1 = ... = p_n = ...$, we denote $\Lambda^{(p_1,\ldots,p_n)}(\Theta \times S)$ by $\Lambda^{p,n}(\Theta \times S)$. Note that in this case, the set $R^p(G, \Delta) \equiv \bigcap_{n \in \mathbb{N}} \Lambda^{p,n}(\Theta \times S)$ corresponds to the event that $\mathcal{A}^0$ is common p-belief among the players in the sense of Monderer and Samet (1989).

Some important properties of the operator Λ^p are summarized in the following lemma:

Lemma 3.1. *Suppose that $\Theta \times S$ is a compact metric space and u_i is continuous for each $i \in N$. Then:*

(a) For any $T = \prod_{i \in N} T_i$ closed in $\Theta \times S$ and for any $p \in (0,1)$, $\Lambda_i^p(T, \Delta)$ is closed.

(b) For any $p \in (0,1)$ and for any $T_1 = \prod_{i \in N} T_{1,i} \subseteq T_2 = \prod_{i \in N} T_{2,i}$, we have that $\Lambda_i^p(T_1, \Delta) \subseteq \Lambda_i^p(T_2, \Delta)$. Moreover, for all $n \in \mathbb{N}$, $\Lambda^{p,n}(\Theta \times S, \Delta) \subseteq \Lambda^{p,n-1}(\Theta \times S, \Delta)$.

(c) For any p, $p' \in (0,1)$ with $p \leq p'$ and for any $T = \prod_{i \in N} T_i$ closed in $\Theta \times S$, we have that $\Lambda_i^{p'}(T, \Delta) \subseteq \Lambda_i^p(T, \Delta)$.

(d) For any sequence $\{p_n\} \subset (0,1)$, we have the following inclusions:

$$\bigcap_{m=1}^{\infty} \Lambda^{\sup\{p_n\},m}(\Theta \times S, \Delta) \subseteq \bigcap_{m=1}^{\infty} \Lambda^{(p_1,..,p_m)}(\Theta \times S, \Delta) \subseteq \bigcap_{m=1}^{\infty} \Lambda^{\inf\{p_n\},m}(\Theta \times S, \Delta)$$

.

Proof. We omit the proofs of (b) and (c).

(a) Note that if $\{\mu_n^i\}_{n \in \mathbb{N}} \subseteq \Delta_i(\theta_n^i)$ is a sequence satisfying that $\mu_n^i(T_{-i}) \geq p$ and $\mu_n^i \to \mu^i$, then we have $\mu^i(T_{-i}) \geq \limsup_{n \to \infty} \mu_n^i(T_{-i}) \geq p$(see, for example, Aliprantis and Border (1999), Chapter 14). Then all other arguments are similar to the proofs in Lemma 2.1.

(d) By (b) and (c), it is easy to check that

$$\Lambda^{\sup\{p_n\},m}(\Theta \times S, \Delta) \subseteq \Lambda^{(p_1,..,p_m)}(\Theta \times S, \Delta) \subseteq \Lambda^{\inf\{p_n\},m}(\Theta \times S, \Delta)$$

for all $m \in \mathbb{N}$. □

As mentioned in section 2.2.2, it is important to note that the rationalizable sets can also be defined via the "best response property". We argued there that the equivalence of the two alternative ways comes from the equivalence of two approaches in defining common belief. An analogous equivalence theorem holds for common p-beliefs and the p-evident events as demonstrated by Kajii and Morris (1997). The theorem stated below shows that an analogous "best response property" for common p-beliefs.

Theorem 3.2. *For any $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ and $\Delta \in \Sigma_0$, we have that for all $p \in (0,1)$, $R^p(G,\Delta) = \Lambda^p(R^p(G,\Delta),\Delta)$.*

3.2.2 The Robustness Result

To show the robustness with respect to approximate common belief via common p-beliefs, some notions on the convergence are needed. Since we allow different levels of belief in each order of mutual belief, the parameter representing the approximation becomes a sequence in $(0,1]$ rather than a number in $(0,1]$. To avoid the non-convergent behavior of coordinates with large subindex, we impose the sup-norm topology on the set of parameters.

Formally, let $\Omega \equiv (0,1]^{\mathbb{N}}$, and let $d_\infty : \Omega \times \Omega \to \Re$ be a metric on Ω defined by $d_\infty(\mathbf{p},\mathbf{q}) \equiv \sup_{n\in\mathbb{N}} |p_n - q_n|$. Let $\mathbf{1} \equiv (1,...,1,...) \in \Omega$.

With these notations, the theorem stated below shows that every Δ-rationalizable set is robust with respect to common $\mathbf{p}$-belief.

Theorem 3.3. *Let $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a strategic form game with incomplete information, and let $\Delta \in \Sigma_0$. Suppose that $\boldsymbol{p}^n \to \boldsymbol{1}$ in (Ω, d_∞). Then either*

(1) there exists some $M \in \mathbb{N}$ such that $n \geq M$ implies that $\bigcap_{k\in\mathbb{N}} \Lambda^{(p_1^n,\dots,p_k^n)}(\Theta\times S,\Delta) = \emptyset = R(G,\Delta)$, or

(2)$\bigcap_{k\in\mathbb{N}} \Lambda^{(p_1^n,\dots,p_k^n)}(\Theta\times S,\Delta) \to R(G,\Delta)$ in $(\mathcal{K}(\Theta\times S), d_{\mathcal{H}})$.

Recall that in our analysis, one can avoid unintended restrictions on beliefs due to the adaption of "small" type spaces. In fact, we do not impose any restrictions other than those represented by Δ. Thus, the robustness we proved here is more general than the models defined on a specific type space.

Remark 3.4. In our two robustness theorems, closeness are defined on the space $\Theta \times S$ rather than $\{\theta^i\} \times S_i$ for each θ^i. However, a slight modification in our proofs can provide the stronger version of convergence. In deed, in both cases, we show that as n increase, both $R^{p_n}(G,\Delta)$ and $\Lambda^n(\Theta\times S)$ decrease if $p_n \uparrow 1$. So clearly for any $\theta^i \in \Theta_i$, both

$R_i^{p_n}(G, \Delta) \cap (\{\theta^i\} \times S_i)$ and $\Lambda^n(\Theta \times S) \cap (\{\theta^i\} \times S_i)$ decrease. Note that all these sets are also compact, so the intersection is empty if and only if for some n, $R_i^{p_n}(G, \Delta) \cap (\{\theta^i\} \times S_i)$ or $\Lambda^n(\Theta \times S) \cap (\{\theta^i\} \times S_i)$ is empty. So all arguments apply to these sets too.

4 Concluding Remarks

This paper provides a general definition for Δ-rationalizable sets and proves its robustness with respect to two alternative definitions of almost common belief. Within this framework, one can easily extend the analysis of Gul (1996) to strategic form games with incomplete information. However, all analysis provided here focuses on strategic form games. So it is interesting to investigate the robustness of the extensive-form rationalizable sets and the Δ-rationalizable sets for extensive form games with incomplete information defined by Battigalli and Siniscalchi (2003).

Nevertheless, one should note that the common belief assumption here is replaced by common "strong belief" defined by Battigalli and Siniscalchi (2003) in defining Δ-rationalizable sets for extensive form games with incomplete information. Thus, it is not clear how to define the robustness in analyzing Δ-rationalizable sets for extensive form games with incomplete information.

Another interesting extension of the current paper is to investigate the robustness of the assumption that the preference consistent with the subjective expected utility model is common belief among players. Indeed, in the framework of Epstein and Wang (1996), this is a sensible question. However, it is not clear how to define a corresponding notion of common p-belief there.

5 Appendix

The appendix consists of two parts. Some necessary mathematical preliminaries are presented in the first part. The second part contains the proofs of the theorems stated in this paper.

5.1 Mathematical Preliminaries

5.1.1 Correspondences

Fist, we list some definitions and properties of correspondences between topological spaces that are needed for the following analysis.

Definition 5.1. Let X and Y be two topological spaces. A correspondence $\varphi : X \twoheadrightarrow Y$ is **upper hemicontinuous** at point $x \in X$ if for every open neighborhood U of $\varphi(x)$, the upper inverse $\varphi^u(U) \equiv \{x \in X : \varphi(x) \subseteq U\}$ is a neighborhood of x in X. We say that φ is upper hemicontinuous on X if it is upper hemicontinuous at every point of X.

Definition 5.2. A correspondence $\varphi : X \twoheadrightarrow Y$ between topological spaces is **closed** if its graph

$$\mathrm{Gr}\varphi \equiv \{(x, y) \in X \times Y : y \in \varphi(x)\}$$

is closed.

The following two theorems are very useful for our analysis. The proofs can be found in, for example, Aliprantis and Border (1999), Chapter 16.

Lemma 5.1. *A closed correspondence is closed-valued.*

Theorem 5.1. *A closed-valued correspondence with compact Hausdorff range space is closed if and only if it is upper hemicontinuous.*

Theorem 5.2. *Let $\varphi : X \twoheadrightarrow Y$ be a closed-valued correspondence between topological spaces. Suppose that Y is a compact Hausdorff space. Then the two statements below are equivalent:*

(a) For any $x \in X$, if $x_\alpha \to x$ in X and $y_\alpha \in \varphi(x_\alpha)$ for each α, then the net $\{y_\alpha\}$ has a limit point in $\varphi(x)$.

(b) φ is upper hemicontinuous.

By these theorems, we have the following immediate corollary:

Corollary 5.1. *Let $\varphi : X \twoheadrightarrow Y$ be a correspondence between metric spaces and suppose that Y is compact. Then the two statements below are equivalent:*

(a) For any $x \in X$, if $x_n \to x$ in X and $y_n \in \varphi(x_n)$ for each α, then the sequence $\{y_n\}$ has a limit point in $\varphi(x)$ and φ is closed-valued.

(b) φ is closed.

5.1.2 Hausdorff Distance

We begin with the definition of the Hausdorff distance between subsets of a metric space.

Definition 5.3. Let (X, d) be a metric space. For each pair of nonempty subsets A and B of X, define

$$d_{\mathcal{H}}(A, B) \equiv \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

The following notion of convergence, however, is defined solely in terms of the topology of X.

Definition 5.4. Let $\{E_n\}$ be a sequence of subsets of a topological space X. Then:

(a) A point $x \in X$ belongs to the **topological lim sup**, denoted $\operatorname{Ls} E_n$, if for every neighborhood V of x there are infinitely many n with $V \cap E_n \neq \emptyset$.

(b) A point $x \in X$ belongs to the **topological lim inf**, denoted LiE_n, if for every neighborhood V of x, we have $V \cap E_n \neq \emptyset$ for all but finitely many n.

Two useful lemmas are provided below:

Lemma 5.2. *Let E_n be a sequence of nonempty compact subsets of a Hausdorff space X. Suppose that $E_{n+1} \subseteq E_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} E_n = \operatorname{Ls} E_n = \operatorname{Li} E_n$.*

Proof. Let $x \in \bigcup_{n \in \mathbb{N}} (X \setminus E_n)$. Then there is some $n \in \mathbb{N}$ such that $x \in (X \setminus E_n)$. Since E_n is compact and hence closed in X, there is some neighborhood V of x such that

$V \subseteq (X \setminus E_n)$. Considering that $E_{n+1} \subseteq E_n$ for all $n \in \mathbb{N}$, $V \subseteq (X \setminus E_m)$ for all $m \geq n$. Therefore, x does not belong to $\operatorname{Ls} E_n \cup \operatorname{Li} E_n$.

Conversely, suppose that $x \in \bigcap_{n\in\mathbb{N}} E_n$. Then for any neighborhood V of x, it is clear that $V \cap E_n \neq \emptyset$. Thus, $x \in \operatorname{Ls} E_n \cap \operatorname{Li} E_n$. □

Lemma 5.3. *Suppose that E_n,E'_n and E''_n are three sequences of subsets of a topological space X. Suppose that $E_n \subseteq E'_n \subseteq E''_n$ for all $n \in \mathbb{N}$, then $\operatorname{Ls} E_n \subseteq \operatorname{Ls} E'_n \subseteq \operatorname{Ls} E''_n$ and $\operatorname{Li} E_n \subseteq \operatorname{Li} E'_n \subseteq \operatorname{Li} E''_n$.*

Proof. Directly from the definition. □

Fix a compact metric space (X, d) and let $\mathcal{K}$ denotes the collection of all nonempty compact subsets of X. Then $(\mathcal{K}, d_{\mathcal{H}})$ is a metric space. The following theorem states that two notions of convergence coincide on $\mathcal{K}$. The proof can also be found in Aliprantis and Border (1999), Chapter 2.

Theorem 5.3. *If X is a compact metric space, then for any sequence of nonempty compact subsets K_n and K a compact subset, $\lim d_{\mathcal{H}}(K_n, K) = 0$ if and only if $\operatorname{Ls} E_n = \operatorname{Li} E_n = K$.*

5.2 Proofs

Let $G = (N, \{\Theta_i\}_{i\in N}, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ be a strategic form game with incomplete information. The following lemma is trivial but is important for the following analysis:

Lemma 5.4. *For any $T_1 = \prod_{i\in N} T_{1,i} \subseteq T_2 = \prod_{i\in N} T_{2,i}$, we have $\Lambda_i(T_1, \Delta) \subseteq \Lambda_i(T_2, \Delta)$. Moreover, for all $n \in \mathbb{N}$, $\Lambda^n(\Theta \times S, \Delta) \subseteq \Lambda^{n-1}(\Theta \times S, \Delta)$.*

Proof of Lemma 2.1:

Proof. Let $\{(\theta^i_n, s^i_n)\}_{n\in\mathbb{N}}$ be a sequence in $\Lambda_i(T, \Delta)$ with limit (θ^i, s^i). It suffices to show that $(\theta^i, s^i) \in \Lambda_i(T, \Delta)$.

For each $n \in \mathbb{N}$, there exists $\mu_n^i \in \Delta^{\theta_n^i}$ such that $\mu_n^i(T_{-i}) = 1$ and $s_n^i \in F^i[\mu_n^i, \theta_n^i]$. Since $\Theta_{-i} \times S_{-i}$ is a compact metric space, so is $\Delta(\Theta_{-i} \times S_{-i})$(see, for example, Aliprantis and Border (1999),Chapter 14).

Since Δ_i is a closed correspondence, by Corollary 5.1 there is a convergent subsequence $\{\mu_m^i\}_{m\in\mathbb{N}}$ of $\{\mu_n^i\}_{n\in\mathbb{N}}$ with limit $\mu^i \in \Delta^{\theta^i}$. Moreover, for any $s_0^i \in S_i$, we have $\int_{\Theta_{-i}\times S_{-i}}[u_i(s_n^i, s^{-i}, \theta_n^i, \theta^{-i}) - u_i(s_0^i, s^{-i}, \theta_n^i, \theta^{-i})]d\mu_n^i(s^{-i}, \theta^{-i}) \geq 0$.

Sice $\mu_n^i \to \mu^i$, for any given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $n \geq N_1$ implies that

$$\Big|\int_{\Theta_{-i}\times S_{-i}} \left[u_i(s^i, s^{-i}, \theta^i, \theta^{-i}) \;-\; u_i(s_0^i, s^{-i}, \theta^i, \theta^{-i})\right]\, d\mu_n^i(s^{-i}, \theta^{-i})$$
$$-\int_{\Theta_{-i}\times S_{-i}} [u_i(s^i, s^{-i}, \theta^i, \theta^{-i}) - u_i(s_0^i, s^{-i}, \theta^i, \theta^{-i})]\, d\mu^i(s^{-i}, \theta^{-i})\Big| \leq \epsilon/2.$$

Moreover, Since $\Theta \times S$ is compact metric, u_i is uniformly continuous for each $i \in N$. So for some $N_2 \in \mathbb{N}$, we have that

$$\big|[u_i(s_n^i, s^{-i}, \theta_n^i, \theta^{-i}) - u_i(s_0^i, s^{-i}, \theta_n^i, \theta^{-i})] - [u_i(s^i, s^{-i}, \theta^i, \theta^{-i}) - u_i(s_0^i, s^{-i}, \theta^i, \theta^{-i})]\big| \leq \epsilon/2$$

for all $(\theta^{-i}, s^{-i}) \in \Theta_{-i} \times S_{-i}$ and $n \geq N_2$.

Thus, for all $n \geq N_0 = \max\{N_1, N_2\}$,

$$\Big|\int_{\Theta_{-i}\times S_{-i}} \left[u_i(s_n^i, s^{-i}, \theta_n^i, \theta^{-i}) \;-\; u_i(s_0^i, s^{-i}, \theta_n^i, \theta^{-i})\right]\, d\mu_n^i(s^{-i}, \theta^{-i})$$
$$-\int_{\Theta_{-i}\times S_{-i}} [u_i(s^i, s^{-i}, \theta^i, \theta^{-i}) - u_i(s_0^i, s^{-i}, \theta^i, \theta^{-i})]\, d\mu^i(s^{-i}, \theta^{-i})\Big| \leq \epsilon.$$

Then,

$$\int_{\Theta_{-i}\times S_{-i}} [u_i(s^i, s^{-i}, \theta^i, \theta^{-i}) \;-\; u_i(s_0^i, s^{-i}, \theta^i, \theta^{-i})]\, d\mu^i(s^{-i}, \theta^{-i})$$
$$= \lim_{n\to\infty} \int_{\Theta_{-i}\times S_{-i}} [u_i(s_n^i, s^{-i}, \theta_n^i, \theta^{-i}) \;-\; u_i(s_0^i, s^{-i}, \theta_n^i, \theta^{-i})]\, d\mu_n^i(s^{-i}, \theta^{-i}) \geq 0.$$

Therefore, $s^i \in F^i[\mu^i, \theta^i]$ and hence $(\theta^i, s^i) \in \Lambda_i(T, \Delta)$. $\square$

Proof of Theorem 2.1:

Proof. By Lemma 5.4, for each $n \in \mathbb{N}$, we have that $\Lambda_i^{n+1}(\Theta \times S, \Delta) \subset \Lambda_i^n(\Theta \times S, \Delta)$. Note that for any family of closed subsets in a compact space, finite intersection property implies that this family has a nonempty intersection. By Lemma 2.1, it suffices to show that for all $n \in \mathbb{N}$ and$\theta^i \in \Theta_i$, $\Lambda_i^n(\Theta \times S, \Delta) \cap (\{\theta^i\} \times S_i) \neq \emptyset$. We proceed our proof by induction in n.

It is clear that for any $\theta^i \in \Theta_i$ and any $\mu^i \in \Delta^{\theta^i}$, $F^i[\mu^i, \theta^i] \neq \emptyset$ since S_i is compact and $U_i(s^i, \mu^i) \equiv \int_{\Theta_{-i} \times S_{-i}} u_i(s^i, s^{-i}, \theta^i, \theta^{-i})\, d\,\mu^i(s^{-i}, \theta^{-i})$ is continuous in s^i. Thus, $(\{\theta^i\} \times S_i) \cap \Lambda_i(\Theta \times S, \Delta) \neq \emptyset$ for all $i \in N$ and $\theta^i \in \Theta_i$. Suppose that $(\{\theta\} \times S) \cap \Lambda^k(\Theta \times S, \Delta) \neq \emptyset$ for all $k \leq n$ and $\theta \in \Theta$. Let $\theta^i \in \Theta_i$. Considering that $\Delta^{\theta^i} = \{\mu \in \Delta(\Theta_{-i} \times S_{-i}) : \operatorname{marg}_{\Theta_{-i}} \mu \in P\}$ for some $P \subseteq \Delta(\Theta_{-i})$, it follows that there exists $\mu^i \in \Delta^{\theta^i}$ such that $\mu^i(\Lambda_{-i}^n(\Theta \times S, \Delta) = 1$. Then $F^i[\mu^i, \theta^i] \neq \emptyset$ by the compactness of S_i and continuity of U_i. Thus, $\Lambda_i^{n+1}(\Theta \times S, \Delta) \cap (\{\theta^i\} \times S_i) \neq \emptyset$ for all $i \in N$. $\square$

Proof of Theorem 2.2:

Proof. Let $(\theta^i, s^i) \in R_i(G, \Delta)$. Then for each $n \in \mathbb{N}$, there exists $\mu_n^i \in \Delta^{\theta^i}$ such that $\mu_n^i(\Lambda_{-i}^n) = 1$ and $s^i \in F^i[\mu_n^i, \theta^i]$. Since $\Theta \times S$ is compact metric, so is $\Delta(\Theta \times S)$ and hence Δ^{θ^i} is compact. Let $\{\mu_m^i\}_{m \in \mathbb{N}}$ be a convergent subsequence of $\{\mu_n^i\}_{m \in \mathbb{N}}$ such that $\mu_m^i \to \mu^i$ for some $\mu^i \in \Delta^{\theta^i}$.

For each $n \in \mathbb{N}$, $\mu^i(\Lambda_{-i}^n(\Theta \times S)) \geq \limsup_{m \to \infty} \mu_m^i(\Lambda_{-i}^n(\Theta \times S)) = 1$ since $\mu_m^i \to \mu^i$(see, for example, Aliprantis and Border (1999),Chapter 14). From Lemma 5.4 and $R_{-i}(G, \Delta) = \bigcap_{n \in \mathbb{N}} \Lambda_{-i}^n(\Theta \times S)$, we have that $\mu^i(R_{-i}(G, \Delta)) = \lim_{n \to \infty} \mu^i(\Lambda_{-i}^n(\Theta \times S)) = 1$ by the continuity of countably additive measures. By continuity of u_i, it is clear that $s^i \in F^i[\mu^i, \theta^i]$. Thus, $(\theta^i, s^i) \in \Lambda_i(R(G, \Delta), \Delta)$.

For the converse, suppose that $(\theta^i, s^i) \in \Lambda_i(R(G, \Delta), \Delta)$. Then there exists $\mu^i \in \Delta^{\theta^i}$ such that $\mu^i(R(G, \Delta)) = 1$ and $s^i \in F^i[\mu^i, \theta^i]$. Note that $\mu^i(R_{-i}(G, \Delta)) = 1$ implies that $\mu^i(\Lambda_{-i}^n(\Theta \times S)) = 1$ for all $n \in \mathbb{N}$. It follows that $(\theta^i, s^i) \in R_i(G, \Delta)$. $\square$

Proof of Theorem 3.1:

Proof. By Lemma 2.1 and Lemma 5.4, we have that $\Lambda^n(\Theta\times S,\Delta)$ is compact and $\Lambda^{n+1}(\Theta\times S,\Delta)\subset\Lambda^n(\Theta\times S,\Delta)$ for all $n\in\mathbb{N}$. Then, $R(G,\Delta)=\bigcap_{n\in\mathbb{N}}\Lambda^n(\Theta\times S,\Delta)$ implies that $d_{\mathcal{H}}(\Lambda^n(\Theta\times S,\Delta),R(G,\Delta))\to 0$ as $n\to\infty$ by Theorem 5.3 and Lemma 5.2, where $d_{\mathcal{H}}$ is the Hausdorff metric on compact sets. $\square$

Proof of Theorem 3.2:

Proof. Let $(\theta^i,s^i)\in R^p_i(G,\Delta)$. For each $n\in\mathbb{N}$, there exists $\mu^i_n\in\Delta^{\theta^i}$ such that $\mu^i_n(\Lambda^n_{-i})\geq p$ and $s^i\in F^i[\mu^i_n,\theta^i]$. Since $\Theta\times S$ is compact metric, so is $\Delta(\Theta\times S)$ and hence Δ^{θ^i} is compact. Let $\{\mu^i_m\}_{m\in\mathbb{N}}$ be a convergent subsequence of $\{\mu^i_n\}_{m\in\mathbb{N}}$ such that $\mu^i_m\to\mu^i$ for some $\mu^i\in\Delta^{\theta^i}$.

For each $n\in\mathbb{N}$, $\mu^i(\Lambda^n_{-i}(\Theta\times S))\geq\limsup_{m\to\infty}\mu^i_m(\Lambda^n_{-i}(\Theta\times S)\geq p$ since $\mu^i_m\to\mu^i$(see, for example, Aliprantis and Border (1999),Chapter 14). Moreover, since $R^p_{-i}(G,\Delta)=\bigcap_{n\in\mathbb{N}}\Lambda^{p,n}_{-i}(\Theta\times S)$ and $\Lambda^{p,n}_{-i}(\Theta\times S)\subseteq\Lambda^{p,n-1}_{-i}(\Theta\times S)$ by Lemma 3.1, we have that $\mu^i(R^p_{-i}(G,\Delta))=\lim_{n\to\infty}\mu^i(\Lambda^{p,n}_{-i}(\Theta\times S))\geq p$ by continuity of countably additive measures. By continuity of u_i, it is easy to see that $s^i\in F^i[\mu^i,\theta^i]$. Thus, $(\theta^i,s^i)\in\Lambda^p_i(R^p(G,\Delta),\Delta)$.

For the converse, suppose that $(\theta^i,s^i)\in\Lambda^p_i(R^p(G,\Delta),\Delta)$. Then there exists $\mu^i\in\Delta^{\theta^i}$ such that $\mu^i(R^p(G,\Delta))\geq p$ and $s^i\in F^i[\mu^i,\theta^i]$. Note that $\mu^i(R^p_{-i}(G,\Delta))\geq p$ implies that $\mu^i(\Lambda^{p,n}_{-i}(\Theta\times S))\geq p$ for all $n\in\mathbb{N}$ by Lemma 3.1. It follows that $(\theta^i,s^i)\in R^p_i(G,\Delta)$. $\square$

Proof of Theorem 3.3:

Proof. Note first that for all $p\in(0,1)$, by Lemma 3.1 (a), we have $R^p(G,\Delta)$ is a compact subset of $\Theta\times S$.

Then we claim that $\bigcap_{p\in(0,1)}R^p(G,\Delta)=R(G,\Delta)$.

By Lemma 3.1(c), it is easy to see that $p\leq p'$ implies that $\bigcap_{m=1}^{\infty}\Lambda^{p,m}(S,\Delta)\subseteq\bigcap_{m=1}^{\infty}\Lambda^{p',m}(S,\Delta)$. Thus, $\bigcap_{p\in(0,1)}R^p(G,\Delta)=\bigcap_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\Lambda^{p_k,n}(S,\Delta)$ for any $\{p_k\}_{k\in\mathbb{N}}\subset(0,1)$ with $p_k\uparrow 1$. So $\bigcap_{p\in(0,1)}R^p(G,\Delta)=\bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\Lambda^{p_k,n}(S,\Delta)$.

Then, we show by induction that $\bigcap_{k=1}^{\infty} \Lambda^{p_k,n}(\Theta \times S, \Delta) = \Lambda^n(\Theta \times S, \Delta)$ for all $n \in \mathbb{N}$. Note that it is straightforward to check that $\bigcap_{k=1}^{\infty} \Lambda^{p_k,n}(\Theta \times S, \Delta) \supseteq \Lambda^n(\Theta \times S, \Delta)$

For $n = 1$, note that $\Lambda_i(\Theta \times S, \Delta) = \Lambda_i^{p_k}(\Theta \times S, \Delta) = \bigcap_{k=1}^{\infty} \Lambda_i^{p_k}(\Theta \times S, \Delta)$ for all $i \in N$. Now, fix some $n \geq 2$. If $(\theta^i, s^i) \in \bigcap_{k=1}^{\infty} \Lambda_i^{p_k,n}(\Theta \times S, \Delta)$, then for all $k \in \mathbb{N}$, there exists $\mu_k^i \in \Delta^{\theta^i}$ so that $\mu_k^i(\Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)) \geq p_k$. By taking subsequence, we may assume that $\mu_k^i \to \mu^i$. Then for any $k \in \mathbb{N}$,

$$\mu^i(\Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)) \geq \limsup_{m \to \infty} \mu_m^i(\Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)) \geq \limsup_{m \to \infty} p_m = 1.$$

Moreover, since $\Lambda_{-i}^{p_{k+1},n-1}(\Theta \times S, \Delta) \subseteq \Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)$ for all $k \in \mathbb{N}$, by continuity of probability measure, it follows that $\mu^i(\bigcap_{k=1}^{\infty} \Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)) = \lim_{k \to \infty} \mu^i(\Lambda_{-i}^{p_k,n-1}(\Theta \times S, \Delta)) = 1$. Note also that $\mu^i \in \Delta^{\theta^i}$ since $\Delta\theta^i$ is compact. Therefore, by continuity of u_i and induction hypothesis, $(\theta^i, s^i) \in \Lambda_i^n(S, \Delta)$.

Thus, by Theorem 5.3 and Lemma 5.2, since $R^p(G, \Delta)$ is compact, we have that $d_{\mathcal{H}}(R^p(G, \Delta), R(G, \Delta)) \to 0$ as $p \to 1$. Moreover, note that $R(G, \Delta) = \emptyset$ if and only if for some $\pi \in (0, 1)$, $p \geq \pi$ implies that $R^p(G, \Delta) = \emptyset$. The claim follows directly from that for a family of compact sets, finite intersection property holds if and only if it has a nonempty intersection.

To complete the proof, note that by Lemma 3.1 (d), for any $\mathbf{p} \in \Omega$, we have that

$$\bigcap_{m=1}^{\infty} \Lambda^{\sup\{p_n\},m}(\Theta \times S, \Delta) \subseteq \bigcap_{m=1}^{\infty} \Lambda^{(p_1,..,p_m)}(\Theta \times S, \Delta) \subseteq \bigcap_{m=1}^{\infty} \Lambda^{\inf\{p_n\},m}(\Theta \times S, \Delta).$$

Suppose that $\{\mathbf{p}^n\}$ is a sequence satisfying $\mathbf{p}^n \to \mathbf{1}$ in (Ω, d_∞). Then for each $n \in \mathbb{N}$, it follows that

$$R^{\sup_m\{p_m^n\}}(G, \Delta) \subseteq \bigcap_{k \in \mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta) \subseteq R^{\inf_m\{p_m^n\}}(G, \Delta)$$

.

Thus, by Lemma 5.3,

$$\begin{aligned} R^{\sup_m\{p_m^n\}}(G, \Delta) &\subseteq \mathrm{Li}_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta) \\ &\subseteq \mathrm{Ls}_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta) \subseteq R^{\inf_m\{p_m^n\}}(G, \Delta). \end{aligned}$$

But, $\mathbf{p}^n \to \mathbf{1}$ implies that $\sup_m\{p_m^n\} \to 1$ and $\inf_m\{p_m^n\} \to 1$. By the claim above, it follows that

$$d_{\mathcal{H}}(R^{\sup_m\{p_m^n\}}(G,\Delta), R(G,\Delta)) \to 0$$

and

$$d_{\mathcal{H}}(R^{\inf_m\{p_m^n\}}(G,\Delta), R(G,\Delta)) \to 0$$

as $n \to \infty$ whenever $R(G,\Delta) \neq \emptyset$.

Moreover, if $R(G,\Delta) = \emptyset$, then $\inf_m\{p_m^n\} \geq \pi$ implies that $\bigcap_{k\in\mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta) = \emptyset$. That is, for some $M \in \mathbb{N}$, $n \geq M$ implies that $\bigcap_{k\in\mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta) = \emptyset$.

Thus, we have that

$$d_{\mathcal{H}}(\bigcap_{k\in\mathbb{N}} \Lambda^{(p_1^n,\ldots,p_k^n)}(\Theta \times S, \Delta), R(G,\Delta)) \to 0$$

as $n \to \infty$.

$\square$

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