Asymptotically Well-behaved Demand Functions for Normal Goods

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Abstract: Marshallian demand functions are well-behaved (downward sloping and consistent to the consumer surplus) only if they are defined for neutral goods, i.e., the case of quasi-linear utility functions. This paper considers a possibility that Marshallian demand functions for normal goods become well-behaved when the initial income is sufficiently large. As a main result, this paper provides necessary and sufficient conditions for a standard utility function under which the derived Marshallian demand function becomes well-behaved for sufficiently large income levels. Moreover, a formula is provided to compute the well-behaved demand function directly from the utility function.

Key words: Marshallian demand function, Consumer Surplus, Normal Good, Income effect.

JEL classification: D11, D63

1. Introduction

The well-behaved (downward sloping and consistent to the consumer surplus) Marshallian demand function forms a basis of the partial equilibrium analysis both in positive and normative perspectives, since the downward slopness and some other regularity conditions imply that there exists a competitive equilibrium uniquely in the partial equilibrium market and the consistency to the consumer surplus enable us to evaluate alternative policies in the market by means of the consumer surplus measure.

It is well-known that Marshallian demand functions are well-behaved only if they are defined for neutral goods, i.e., the case of quasi-linear utility functions. This paper considers a possibility that Marshallian demand functions for normal goods become well-behaved when the initial income is sufficiently large. As a main result, this paper provides *necessary* and *sufficient* conditions for a standard utility function under which the derived Marshallian demand function becomes well-behaved for sufficiently large income levels. Moreover, a formula is provided to compute the well-behaved demand function directly from the utility function.

In the next section, some basic concepts such as utility function, Marshallian demand function, equivalent variation and compensating variation are introduced in a simple two-good setting where one good is a specific good and the other good is money (numeraire), and the well-behavedness of Marshallian demand function is specified by two conditions: (i) Downward slopness ; (ii) Consistency to the consumer surplus, and a well-known fact for the well-behavedness is stated as Proposition 1 that the well-behaved Marshallian demand function derived only from a quasi-linear utility function, i.e., the specific good is a neutral good.

In section 3, a weaker concept of well-behavedness of Marshallian demand function is proposed. Specifically, if a Marshallian demand function has a limit function when initial income is sufficiently large and if the limit function is well-behaved, then we call the Marshallian demand function "asymptotically well-behaved demand function". If a demand function is asymptotically well-behaved, then income effects are very small when the initial income is sufficiently large, which can be recognized as a formalization and proof of Marshall's assertion that income effects are very small when its budget share is sufficiently small.[1]

As a main result of this paper, necessary and sufficient conditions are provided for a standard utility function under which the derived Marshallian demand function is asymptotically well-behaved. The necessary and sufficient conditions are the replaceability and the regularity at the limit. The replaceability condition for utility function is that an amount of the specific good is replaceable by some fixed amount of numeraire, independent from the initial consumption level of numeraire. This condition implies that the marginal rate of substitution between the two goods are bounded when the consumption level of numeraire is sufficiently large. The regularity condition is that the limit marginal rate of substitution has smoothness properties, which is a technical condition. Under the two conditions, our main result implies that we can justify the well-behaved Marshallian demand function even for strictly normal goods if initial income is sufficiently large. Moreover, our main result also implies that we can not drop these conditions for the well-behavedness.

In section 4, a formula is provided to compute the limit demand function directly from the utility function. Moreover, a numerical example is presented.

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[1] For Marshall's original arguments on the smallness of income effects, see Chipman (1990). Vives (1987 and 1999, Chapter 3) has shown the smallness of income effects if the number of commodities are sufficiently large, but his setting is essentially different from ours.

2. Globally well-behaved demand functions

There are two types of consumption goods, x-good and y-good: *x* is a specific good and *y* is numerare. Letting $X = Y = {}_+$, the consumption set is given by $X \times Y = {}^2_+$. Let us consider a consumer whose initial endowment of y-good is $I > 0$ and the preferences are represented by a smooth (twice continuously differentiable) utility function $U(x, y)$. We assume the following standard conditions:

Monotonicity: $U_x(x, y) > 0$ and $U_y(x, y) > 0$ for all $(x, y) \geq (0, 0)$.[1]

Strict quasi-concavity: $2 \cdot U_x(x,y) \cdot U_y(x,y) \cdot U_{xy}(x,y) - [U_x(x,y)]^2 \cdot U_{yy}(x,y) - [U_y(x,y)]^2 \cdot U_{xx}(x,y) > 0$ for all $(x, y) \ll (0, 0)$.

Normalize the price of numerare as 1, and denote the price of x-good by $p > 0$. For each $p, I > 0$, Marshallian demand function $D(p, I)$ is defined by

$$D(p, I) = \operatorname*{\mathbf{argmax}}_{x \quad B(p, I)} U(x, I - px), \text{ where } B(p, I) = \{ (x, y) \quad X \times Y : p \cdot x + y \leq I \}.$$

For price $p > 0$ and utility level $u \quad U(X \times Y)$,[2] the expenditure function $e(p, u)$ is defined by

$$e(p, u) = \min_{(z, w) \quad F(u)} pz + w, \quad \text{where } F(u) = \{ (z, w) \quad X \times Y : U(z, w) \geq u \}.$$

For a pair of prices p^0, p^1 with $p^0 > p^1$ and initial income $I > 0$, the *equivalent variation* $EV(p^0, p^1, I)$ and *compensating variation* $CV(p^0, p^1, I)$ are defined by

$$EV(p^0, p^1, I) = e(p^0, u^1) - I \quad \text{and} \quad CV(p^0, p^1, I) = I - e(p^1, u^0),$$

where $u^i = U(D(p^i, I), I - p^i D(p^i, I))$ for $i = 0, 1$. The pair of prices (p^0, p^1) is called *regular* if not only $(D(p^0, I), I - p^0 D(p^0, I))$ and $(D(p^1, I), I - p^1 D(p^1, I))$, but also the minimizers for $e(p^0, u^1)$ and $e(p^1, u^0)$ are strictly positive vectors.

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1 The partial derivatives $U_x(0, y)$ and $U_y(x, 0)$ at the boundary should be regarded as the *right* partial derivatives $U_x^+(0, y)$ and $U_y^+(x, 0)$, respectively.

2 $U(X \times Y)$ is the range of U, i.e., $U(X \times Y) = \{ u \quad : u = U(x, y) \text{ for some } (x, y) \quad X \times Y \}$.

A standard definition of well-behaved Marshallian demand function is given as follows:

Definition: A Marshallian demand function $D(p, I)$ is called *well-behaved* at $I > 0$ if $D(p, I)$ has the following properties:

(Downward slopeness): $D(p, I) > D(p+\delta, I)$ for all p with $D(p, I) > 0$ and all $\delta > 0$.

(Consistency to the consumer surplus): $\int_p^{+\infty} D(p, I)dp < +\infty$ for all $p > 0$,[3] and

$$EV(p^0, p^1, I) = \int_{p1}^{p0} D(p, I)dp = CV(p^0, p^1, I) \quad \text{for all regular } (p^0, p^1) \text{ with } p^0 > p^1.$$

A Marshallian demand function $D(p, I)$ is called *globally well-behaved* if $D(p, I)$ is well-behaved at all $I > 0$.

The downward slopeness means that $D(p, I)$ is a decreasing function, which is an important condition when one proves the existence and uniqueness of a competitive equilibrium in a partial equilibrium market model. The consistency to the consumer surplus implies that the consumer surplus is well-defined and it coincides with the finite integral of demand function. The following proposition is well-known:

Proposition 1: Suppose that a utility function $U(x, y)$ is monotone and strictly quasi-concave. Then the Marshallian demand function $D(p, I)$ is globally well-behaved if and only if x-good is a neutral good, i.e., $U(x, y)$ satisfies that

$$\partial MRS(x, y)/\partial y = U_y(x,y)\cdot U_{xy}(x,y) - U_x(x,y)\cdot U_{yy}(x,y) = 0 \quad \text{for all } (x, y) \ll (0, 0),$$

where $MRS(x, y)$ is the marginal rate of substitution of x-good for y-good at (x, y) defined by $MRS(x, y) = U_x(x,y)/U_y(x,y)$.

[3] $\int_p^{+\infty} D(p, I)dp = \lim_{q \to +\infty} \int_p^q D(p, I)dp$.

3. Asymptotically well-behaved demand functions for normal goods

In the previous section, we observe that the well-behaved demand functions are derived only from a class of quasi-linear utility functions and then the well-behaved demand functions can be applicable only for neutral goods, which is too restrictive.

In this section, we will show that the (almost) well-behaved demand functions can be derived even under the condition that x-good is a normal good. At first, we additionally assume the following condition:

Normal good: $\partial MRS(x, y)/\partial y = U_y(x,y) \cdot U_{xy}(x,y) - U_x(x,y) \cdot U_{yy}(x,y) > 0$ for all $(x, y) \ll (0, 0)$.

This condition is equivalent to $\partial D(p, I)/\partial I > 0$ (whenever $D(p, I) > 0$). Then the asymptotically well-behaved demand function is defined as follows:

Definition: A Marshallian demand function $D(p, I)$ is called *asymptotically well-behaved* if there exists a real valued function $d(p)$ on $P \equiv \ _{++}$ such that:

(Uniform convergence): $D(p, I)$ converges to $d(p)$ uniformly on compacta in P as $I \to +\infty$.

(Downward slopeness): $d(p) > d(p+\delta)$ for all p with $d(p) > 0$ and all $\delta > 0$.

(Consistency to the limit consumer surplus): $\int_p^{+\infty} d(p)dp < +\infty$ for all $p > 0$ and

$$\lim_{I \to +\infty} EV(p^0, p^1, I) = \int_{p1}^{p0} d(p)\, dp = \lim_{I \to +\infty} CV(p^0, p^1, I) \quad \text{for all } p^0 > p^1 > 0.$$

In other words, a Marshallian demand function $D(p, I)$ is called asymptotically well-behaved if $D(p, I)$ has the well-behaved limit function $d(p)$. The uniform convergence implies that

$$\lim_{I \to +\infty}[D(p, I+\delta) - D(p, I)] = 0 \quad \text{and} \quad \lim_{I \to +\infty} p \cdot D(p, I)/I = 0 \quad \text{for any } p, \delta > 0.$$

Hence income effects on the Marshallian demand function converge to 0 as $I \to +\infty$ in the marginal and average senses.

In order to characterize the asymptotically well-behaved demand function, we introduce two conditions. The first one is the following condition:

Replaceability: For any $x \quad X$, there exists $K_x \geq 0$ such that

$$U(0, y+K_x) \geq U(x, y) \quad \text{for all } y \quad Y.$$

This condition means that an additional amount of x-good is (uniformly) replaceable for some amount of y-good, independent of the consumption level of y-good. Then we have the following lemma:

Lemma 1: Suppose that $U(x, y)$ is monotone and strictly quasi-concave, and that x-good is a normal good. Moreover, suppose that $U(x, y)$ satisfies the replaceability. Then

(**i**) $\lim_{y \to +\infty} MRS(x, y) = \lim_{y \to +\infty} U_x(x,y)/U_y(x,y) < +\infty$ for all $x > 0$, and

(**ii**) $\sigma(x) \equiv \lim_{y \to +\infty} MRS(x, y)$ is continuous for almost all $x > 0$ and $\sigma(x)$ is weakly decreasing on $x > 0$.

The second condition is the following condition:

Regularity at the limit: $\sigma(x)$ is continuous for all $x > 0$ and decreasing on $x > 0$.

The main result of this paper is the following proposition:

Proposition 2: Suppose that a utility function $U(x, y)$ is monotone and strictly quasi-concave, and that x-good is a normal good, i.e., $U(x, y)$ satisfies the normal good condition. Then the Marshallian demand function $D(p, I)$ is asymptotically well-behaved if and only if $U(x, y)$ satisfies the replaceability and the regularity at the limit.

4. Direct derivation of the limit demand function

This section provides a formula to compute the limit demand function directly from a utility function U. The main result of this section is the following proposition:

Proposition 3: Suppose that a utility function U(x, y) is monotone and strictly quasi-concave, and that x-good is a normal good. Moreover, suppose that U(x, y) satisfies the following two conditions:

(i) $\lim_{y \to +\infty} [\, U(x, y) - U(0, y)] < +\infty$ and $0 < \lim_{y \to +\infty} U_y(0, y) < +\infty$.

(ii) $h(x) \equiv \dfrac{\lim_{y \to +\infty} [\, U(x, y) - U(0, y)\,]}{\lim_{y \to +\infty} U_y(0, y)}$ is twice differentiable and $h''(x) < 0$ on $x > 0$.

Then the Marshallian demand function D(p, I) is asymptotically well-behaved, and the limit demand function d(p) is given by

$$d(p) = f^{-1}(p) \quad \text{if } p \quad (0, p^*);$$
$$= 0 \quad \text{otherwise,}$$

where $f = h'$ and $p^* \equiv \lim_{x \to +0} f(x)$.

Let us consider the utility function $U(x, y) = \log(x+1) + \log(y+1) + y$. Since U satisfies (i) in Proposition 3 and $h(x) = \log(x+1)$, $h'(x) = 1/(x+1)$ and $h''(x) = -1/(x+1)^2 < 0$. Hence it holds by Proposition 3 that

$$d(p) = 1/p - 1 \quad \text{if } p \quad (0, 1];$$
$$= 0 \quad \text{otherwise.}$$

If $p = 0.5$, then $d(p) = d(0.5) = 1$ and the *income expansion path* is given by

$$x = \frac{y}{y+2} \quad \text{or} \quad y = \frac{2x}{1-x}$$

For given initial income level $I > 0$, the original Marshallian demand of x-good is determined by the intersection of the income expansion path and the budget line, $0.5x + y = I$. Hence we have that

$$\lim_{I \to +\infty} D(0.5, I) = \lim_{I \to +\infty} \frac{I - 0.5x}{(I - 0.5x)+2} = \lim_{y \to +\infty} \frac{y}{y+2} = 1,$$

which implies that $d(0.5) = 1$ is the *vertical asymptote* of the income expansion path. See the following figure:

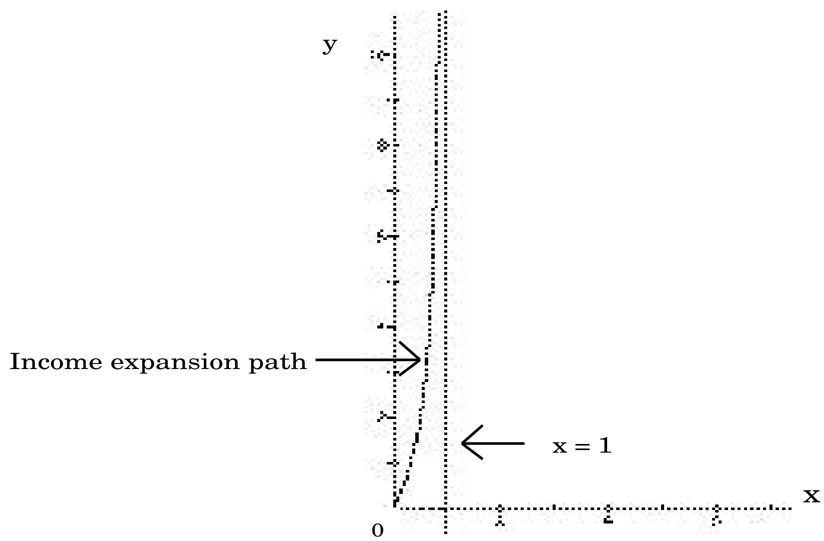

Figure 1

The above example tells us that there exists a utility function which satisfies all conditions for the asymptotically well-behaved demand function.

5. Proofs

Proof of Proposition 1: Suppose that U is monotone and strictly quasi-concave. It is well-known that if $\partial MRS(x, y)/\partial y = 0$ for all $(x, y) \ll (0, 0)$, then $D(p, I)$ is globally well-behaved. For example, see Mas-Colell, *et. al.* (1995, Section 3.I, page 83). Conversely, if $D(p, I)$ is globally well-behaved, we will prove that $\partial MRS(x, y)/\partial y = 0$ for all $(x, y) \ll (0, 0)$. Suppose that $\partial MRS(x^*, y^*)/\partial y > 0$ for some $(x^*, y^*) \ll (0, 0)$. Set $p^* = MRS(x^*, y^*) > 0$, $u^* = U(x^*, y^*)$ and $I^* = e(p^*, u^*) > 0$. Then it holds that $\partial D(p^*, I^*)/\partial I > 0$. Hence there is an open rectangular Q in $^2_{++}$ such that (p^*, I^*) Q and $\partial D(p, I)/\partial I > 0$ for all (p, I) Q. Moreover it holds that

$$\partial^2 e(p^*, u^*)/\partial p \partial u = [\, \partial D(p^*, I^*)/\partial I \,] \cdot [\, \partial e(p^*, u^*)/\partial u].$$

Since $\partial e(p^*, u^*)/\partial u > 0$, there exists an open rectangular Q^* in $^2_{++}$ such that

(p^*, u^*) Q and $\partial^2 e(p, u)/\partial p \partial u > 0$ for all (p, u) Q^*.

Then there is $p^1 > p^*$ such that:

(p^*, p^1) is regular;

(p^1, u^1) Q^*, where $u^1 = U(D(p^1, I),\ I - p^1 D(p^1, I)) < u^*$.

Then it holds that $\partial e(p, u^*)/\partial p > \partial e(p, u^1)/\partial p$ for all p (p^*, p^1), which implies

$$EV(p^*, p^1, I) - CV(p^*, p^1, I) = \int_{p^*}^{p1} [\, \partial e(p, u^*)/\partial p - \partial e(p, u^1)/\partial p \,]\, dp > 0.$$

This is a contradiction. Similarly we can derive a contradiction in case of $\partial MRS(x^*, y^*)/\partial y < 0$. **QED**

In order to prove Lemma 1, Propositions 2 and 3, we need a concept and a claim: Suppose that U is monotone and strictly quasi-concave. For each $x \geq 0$, there exists unique $I_x \geq 0$ such that $U(0, I_x) = U(x, 0)$. For given $I \geq I_x$, the *Hicksian surplus* for adding $x \geq 0$ units of x-good in terms of y-good (numerare) is defined by the amount of y-good c such that $U(0, I) = U(x, I - c)$. We denote the surplus by $S(x; I)$. Mathematically, $S(x; I)$ is a parametric representation of the indifference curve starting from $(0, I)$. If U is monotone and strictly quasi-concave, $S(x; I)$ is well-defined for all $x \geq 0, I \geq I_x$.

Claim 1: Suppose that U is monotone and strictly quasi-concave, and that x-good is a normal good. Then the following statements hold:

(0) For any $x > 0, I > I_x$, $S(x; I)$ is twice continuously differentiable for x and it holds that $S'(x; I) > 0$ and $S''(x; I) < 0$.

(**i**) If $U(x^*, y^*) = U(x, y)$ with $x^* > x$, then $U(x^*, y^*+\delta) > U(x, y+\delta)$ for all $\delta > 0$.

(**ii**) $S(x; I+\delta) > S(x; I)$ for all $x > 0, I \geq I_x$ and all $\delta > 0$.

(**iii**) $S'(x; I+\delta) > S'(x; I)$ for all $x > 0, I \geq I_x$ and all $\delta > 0$.

Proof of Claim 1: (0) This is a direct consequence of the Implicit Function Theorem. (**i**) Suppose $U(x^*, y^*) = U(x, y)$ with $x^* > x$, and fix any $\delta > 0$. If $U(x^*, y^*+\delta) > U(x, y+\alpha)$ for all $\alpha > 0$, then, setting $\alpha = \delta$, it holds that $U(x^*, y^*+\delta) > U(x, y+\delta)$, which implies (**i**) holds. If $U(x^*, y^*+\delta) = U(x, y+\alpha)$ for some $\alpha > 0$, there remains to show $\alpha > \delta$. By the strictly quasi-concavity of U, there is a concave indifference path $I: [x, x^*] \to Y$ connecting (x, y) and (x^*, y^*), i.e., $I(x) = y$ and $I(x^*) = y^*$. Similarly, there is a concave indifference path $J: [x, x^*] \to Y$ connecting $(x, y+\alpha)$ and $(x^*, y^*+\delta)$. It holds by the monotonicity of U that $y > y^*$ and $y+\alpha > (y^*+\delta)$. Then we have that

$$y - y^* = \int_x^{x^*} | I(x)/dx | \, dx = \int_x^{x^*} MRS(x, I(x)) \, dx \; ; \qquad (1)$$

$$y+\alpha - (y^*+\delta) = \int_x^{x^*} |J(x)/dx| \, dx = \int_x^{x^*} MRS(x, J(x)) \, dx. \qquad (2)$$

Since $\partial MRS(r, s)/\partial y > 0$ for all $(r, s) \ll (0, 0)$, and since $I(x) < J(x)$ by the monotonicity and $\delta > 0$, we have that $MRS(x, I(x)) < MRS(x, J(x))$ for all $x \quad (x, x^*)$, which implies that $\int_x^{x^*} MRS(x, I(x)) \, dx < \int_x^{x^*} MRS(x, J(x)) \, dx$. Hence we have by (1) and (2) that $y - y^* < y+\alpha - (y^*+\delta)$ and $\alpha > \delta$.

(**ii**) Fix any $x > 0, I \geq I_x$ and $\delta > 0$. It holds by the definition of S that $U(x, I - S(x; I)) = U(0, I)$. Then it holds by Claim 1(i) that $U(x, I - S(x; I)+\delta) > U(0, I+\delta)$. Moreover it holds by the definition of S that $U(x, I+\delta - S(x; I+\delta)) = U(0, I+\delta)$. Hence we have that $U(x, I+\delta - S(x; I)) > U(x, I+\delta - S(x; I+\delta))$, which implies $I+\delta - S(x; I) > I+\delta - S(x; I+\delta)$ and $S(x; I+\delta) > S(x; I)$.

(**iii**) For given any $x > 0$, $I \geq I_x$ and $\delta > 0$, set $y = I - S(x; I)$. Since $S'(x; I) = MRS(x, y)$ and $S'(x; I+\delta) = MRS(x, y+\beta)$ for some positive $\beta < \delta$ by Claim 1(i). Since $\partial MRS(x, y)/\partial y > 0$ by the normal good condition, it holds that $S'(x; I+\delta) = MRS(x, y+\beta) > MRS(x, y) = S'(x; I)$. **QED**

Proof of Lemma 1: (**i**) Suppose $\lim_{I\to+\infty}S'(x, I) = +\infty$ for some $x > 0$. Since $S'(x; I)$ is increasing for I by Claim 1(iii), we can set $I^* \geq I_x$ such that

$$I > I^* \qquad S'(x; I) > (2/x)K_x ,$$

where K_x is the real number in the condition of replaceability. Then it holds that

$$I > I^* \qquad (x/2)\, S'(x; I) > K_x \qquad S(x; I) > K_x$$

$$U(x , I - K_x) > U(x, I - S(x; I)) = U(0, I).$$

Hence $U(x, I) > U(0, I+K_x)$ for all $I > \max(I^*, K_x)$, which contradicts with the selection K_x. Thus $\lim_{I\to+\infty}S'(x; I) < +\infty$ for all $x > 0$. Setting $\sigma(x) = \lim_{I\to+\infty}S'(x; I)$ for all $x > 0$, we have that $\lim_{y\to+\infty}MRS(x, y) = \lim_{y\to+\infty} S'(x; y+S(x; I)) = \sigma(x)$ for all $x > 0$.

(**ii**) Since $S'(x; I)$ is decreasing with respect to x for given I by Claim 1(0), we have that $\sigma(x) = \lim_{I\to+\infty}S'(x; I)$ is weakly decreasing with respect to x. Then it holds by Royden (1988, Theorem 3, Ch. 5, page 100) that $\sigma(x)$ is differentiable for almost all $x > 0$, and then $\sigma(x)$ is continuous for almost all $x > 0$. **QED**

Proof of Proposition 2: We need the following lemma proved in Appendix:

Lemma 2: Suppose that U is monotone and strictly quasi-concave, and that x-good is a normal good.

(**a**) If U satisfies the replaceability, then $\lim_{I\to+\infty}S(x; I) < +\infty$ for all $x \geq 0$.

Denote $s(x) = \lim_{I\to+\infty}S(x; I)$ for all $x \geq 0$. If U satisfies the replaceability and the regularity at the limit, then it holds that:

(**b**) $\sigma(x) = s'(x)$ for all $x > 0$.

(**c**) For any $p > 0$, there exists $x^*(p) > 0$ uniquely such that

$$s(x^*(p)) - px^*(p) \;>\; s(x) - px \quad \text{for all } x \neq x^*(p) .$$

(**d**) Let $p^* = \lim_{x \to +0} s'(x) > 0$. (It is possible $p^* = +\infty$). Then it holds that

$$x^*(p) = \sigma^{-1}(p) \quad \text{for all } p < p^*.$$

(**e**) $\lim_{I \to +\infty} D(p, I) \equiv d(p) < +\infty$ and $d(p) = x^*(p)$ for any $p > 0$.

(**f**) $d(p)$ is downward slope and consistent to the limit consumer surplus.

Let us start the proof of Proposition 2. Suppose that U is monotone and strictly quasi-concave, and that x-good is a normal good. If U satisfies the two additional conditions, then it follows from Lemma 2(e, f) that the limit demand function $d(p)$ is well-defined and well-behaved. The uniformity of convergence holds by Dini's Theorem (Lang, 1969, Theorem 2, page 325). Conversely, suppose that $D(p, I)$ is asymptotically well-behaved. Let $d(p)$ be the limit demand function. We need a claim:

Claim 2 : $\lim_{p \to +0} d(p) = +\infty$.

Proof : By the normality, it suffices to prove $\lim_{p \to +0} D(p, I) = +\infty$ for all $I > 0$. Suppose $\lim_{p \to +0} D(p, I) = a < +\infty$ for some $I > 0$. Set $b = MRS(a+1, 0) > 0$. Since $D(p, I)$ is increasing as $p \to +0$, it holds that $\lim_{p \to +0} MRS(D(p, I), I - pD(p, I)) = 0$, which implies $MRS(D(p^*, I), I - p^*D(p^*, I)) < b$ for some $p^* > 0$. Then it holds by the quasi-concavity that $MRS(a+1, c) < MRS(D(p^*, I), I - p^*D(p^*, I))$ for some $c > 0$ such that $U(a+1, c) = U(D(p^*, I), I - p^*D(p^*, I))$. Hence $MRS(a+1, c) < b = MRS(a+1, 0)$, which contradicts with the normality. **QED**

Since $d(p)$ is decreasing, it holds by Claim 2 that the inverse $d^{-1}(x)$ is well-defined for all $x > 0$. We need a claim:

Claim 3 : $\lim_{I \to +\infty} S'(x; I) \equiv \sigma(x) < +\infty$ and $\sigma(x) = d^{-1}(x)$ for all $x > 0$.

Proof : Set $p = d^{-1}(x)$. Suppose $\sigma(x) > d^{-1}(x)$ or $\sigma(x) = +\infty$ for some $x > 0$. Since $\lim_{I \to +\infty} S'(x; I) > p$, it holds that $D(p, I) > x$ for some I and $\lim_{I \to +\infty} D(p, I) = d(p) > x$, which contradicts with $p = d^{-1}(x)$.

Suppose $\sigma(x) < d^{-1}(x)$ for some $x > 0$. Since $\sigma(x) < p$ implies $\lim_{I \to +\infty} S'(x; I) < p$, there exists ε $(0, p)$ such that $S'(x; I) < p - \varepsilon$ for all I. Hence $D(p - \varepsilon, I) \le x$ for all I and $d(p - \varepsilon) \le x = d(p)$, which contradicts with the downward slopeness. **QED**

Since $d(p)$ is continuous by the continuity of $D(p, I)$ and uniform convergence, and since $d(p)$ is decreasing by the downward slopeness, it follows from Claim 3 that $\sigma(x)$ is continuous and decreasing on $x > 0$, and then U satisfies the regularity. There remains to prove that U satisfies the replaceability. Fix any $x^* > 0$. It holds by the Fundamental Theorem of Calculas that

$$\int_{1/m}^{x^*} S'(x; I_{x^*}+n)\, dx = S(x^*; I_{x^*}+n) - S(1/m; I_{x^*}+n) \text{ for all } m = 1, 2 \cdots \text{ and all } n = 1, 2 \cdots.$$

By the definition of (improper) Riemann integral, continuity of $S(x; I_{x^*}+n)$ on $[0, x^*]$ and $S(0; I_{x^*}+n) = 0$, we have that

$$\int_{0}^{x^*} S'(x; I_{x^*}+n)\, dx = \lim_{m \to +\infty} \int_{1/m}^{x^*} S'(x; I_{x^*}+n)\, dx = S(x^*; I_{x^*}+n) \quad \text{for all } n.$$

Since $\{ S'(x; I_{x^*}+n) \}_n$ is monotone on $(0, x^*]$ by Claim 1(iii), and since the limit function $\sigma(x) = d^{-1}(x)$ is Riemann integrable on $[0, x^*]$, it holds that:

$\{ S(x^*; n) \}_n$ is monotone;

$$S(x^*; I_{x^*}+n) = \int_{0}^{x^*} S'(x; I_{x^*}+n)\, dx \le \int_{0}^{x^*} d^{-1}(x)\, dx < +\infty \quad \text{for all } n.$$

Setting $\int_{0}^{x^*} d^{-1}(x)\, dx + I_{x^*} = K_{x^*}$, we have that

$$I_{x^*} \le K_{x^*} \quad \text{and} \quad S(x^*; I_{x^*}+w) \le K_{x^*} \quad \text{for all } w \ge 0. \qquad (3)$$

It holds by the definition $S(x^*; z)$ that

$$U(0, z) = U(x^*, z - S(x^*; z)) \quad \text{for all } z \ge I_{x^*}. \qquad (4)$$

Thus, for any $y \ge 0$, setting $z = y+K_{x^*}$ and $w = K_{x^*}+ y - I_{x^*}$, it holds by (4) and (3) that

$$U(0, y+K_{x^*}) = U(x^*, y+K_{x^*} - S(x^*; y+K_{x^*})) \ge U(x^*, y+K_{x^*} - K_{x^*}) = U(x^*, y).$$ **QED**

Proof of Proposition 3: Suppose that U satisfies all the condition in Proposition 3. Then the following lemma holds:

Lemma 3: (**i**) U satisfies the replaceability. (**ii**) Set $Y(x, I) \equiv I - S(x; I)$, then it holds that

$$S(x; I) = f^{-1}[U(x, Y(x, I))] - Y(x, I),$$

where f is a function defined by $f(y) = U(x, y)$ for all $y \geq 0$.

Lemma 3 is proved in Appendix . By Lemma 3(i) and Lemma 2(a), it holds that

$$\lim_{I \to +\infty} S(x; I) \equiv s(x) < +\infty \quad \text{for all } x \geq 0,$$

which implies $\lim_{I \to +\infty} Y(x, I) = \lim_{I \to +\infty} [I - S(x; I)] = +\infty$ for all $x \geq 0$. Then it holds by this and Lemma 3(ii) that

$$\lim_{I \to +\infty} S(x; I) = \lim_{y \to +\infty} [f^{-1}(U(x, y)) - y]. \tag{5}$$

Since $\lim_{y \to +\infty} f'(y) < +\infty$ by $\lim_{y \to +\infty} U_y(0, y) < +\infty$, it holds by Mean Value Theorem that

$$\lim_{y \to +\infty} f''(y) = 0. \tag{6}$$

Hence it follows from Taylor's Formula and Inverse Function Theorem that

$$\begin{aligned} f^{-1}(U(x, y)) - y &= f^{-1}([U(x, y) - U(0, y)] + U(0, y)) - y \\ &= f^{-1}(U(0, y)) + [U(x, y) - U(0, y)]/f'(y) \\ &\qquad + [U(x, y) - U(0, y)]^2 f''(y+\xi)/2[f'(y)]^3 - y \\ &= [U(x, y) - U(0, y)]/f'(y) + [U(x, y) - U(0, y)]^2 f''(y+\xi)/2[f'(y)]^3 \end{aligned}$$

where $\xi \quad [0, U(x, y) - U(0, y)]$. Thus we have by (5), (6) and the condition (i) in Proposition 3 that

$$\begin{aligned} s(x) &= \lim_{I \to +\infty} S(x; I) \\ &= \lim_{y \to +\infty} [f^{-1}(U(x, y)) - y] = \lim_{y \to +\infty} [U(x, y) - U(0, y)]/f'(y) \\ &= \frac{\lim_{y \to +\infty} [U(x, y) - U(0, y)]}{\lim_{y \to +\infty} U_y(0, y)} . \end{aligned}$$

Since $h'(x) = s'(x)$ is continuous and decreasing on $x > 0$ by condition (ii) in Proposition 3, U satisfies the regularity at the limit and d(p) is given by $h'(x) = s'(x) = p$. **QED**

Appendix

Proof of Lemma 2: (**a**) Fix any $x \geq 0$. It holds by the replaceability that there exists $K_x \geq 0$ such that

$$U(0,\ I - S(x; I) + K_x) \geq U(x,\ I - S(x; I)) \quad \text{for all } I > I_x.$$

Since $U(x,\ I - S(x; I)) = U(0, I)$, we have that $U(0,\ I - S(x; I) + K_x) \geq U(0, I)$ and

$$I - S(x; I) + K_x \geq I \quad \text{for all } I > I_x.$$

Hence it holds that $K_x \geq S(x; I)$ for all $I > \lambda(x)$. Since $S(x; I)$ is increasing with respect to I by Claim 1(ii), it holds that $\lim_{I \to +\infty} S(x; I) < +\infty$.

(**b**) By the definitions of $\sigma(x)$ and $s(x)$, it holds that

$$\lim_{n \to +\infty} S'(x; I_x+n) = \sigma(x) \quad \text{and} \quad \lim_{n \to +\infty} S(x; I_x+n) = s(x) \quad \text{for all } x > 0.$$

Fix any x and select a compact interval $[a, b]$ with $a < x < b$. Since $\sigma(x)$ is continuous on $[a, b]$ by the regularity and each $S'(x; I_x+n)$ is continuous for x on $[a, b]$ by Claim 1(0), and since the convergence $\{ S'(x; I_x+n) \}_n$ is monotone on $[a, b]$ by Claim 1(iii), it holds by Dini's Theorem (Lang, 1969, Theorem 2, page 325) that the convergence of $\{ S'(x; I_x+n) \}_n$ is a uniform convergence on $[a, b]$. Hence it holds by Lang (1969, Theorem 12, page 117) that $\sigma(x) = s'(x)$ on (a, b). Thus it holds that $\sigma(x) = s'(x)$ for all $x > 0$.

(**c**) If $\lim_{x \to +0} s'(x) = +\infty$, the assertion (c) holds by the regularity and Lemma 2(b). We consider the case of $\lim_{x \to +0} s'(x) \equiv A < +\infty$. Since $\sigma(x) = s'(x)$ is decreasing for x, and since $\{ S'(x; \lambda(x)+n) \}_n$ monotonically converges to $s'(x)$ for each $x > 0$, it holds that

$$S'(x, I_x+n) \leq A \quad \text{for all } x > 0 \text{ and all } n = 1, 2 \cdots.$$

Since $S(x, I_x+n)$ is continuous on $[0, x]$ and continuously differentaible on $(0, x)$, we have that

$$S(x, I_x+n) = \int_0^x S'(z, I_x+n)\, dz \leq \int_0^x A\, dz = [\, Az \,]_0^x = Ax.$$

For any $\varepsilon > 0$, if $0 < x < \varepsilon/A$, then $S(x, I_x+n) \leq Ax \leq \varepsilon A/A = \varepsilon$ for all $n > 0$. Hence

$$s(x) = \lim_{n \to +\infty} S(x; I_x+n) \leq \varepsilon,$$

which implies that $s(x)$ is continuous at $x = 0$, since $s(0) = 0$. Since $s(x)$ is continuous on $x > 0$ by Lemma 2b, $s(x)$ is a continuous function on $x \geq 0$. Hence the assertion (c) holds by the regularity and Lemma 2b.

(**d**) It holds by Lemma 2b and the regularity that $s(x)$ is monotone and concave on $(0, +\infty)$. Hence for any $p < p^*$, it holds that $s'(x(p)) - p = 0$, which implies that $s'(x^*(p)) = p$ and $\sigma(x^*(p)) = p$. Thus we have that $x^*(p) = \sigma^{-1}(p)$.

(**e**) Set $x^* = x^*(p)$ for $p > 0$. Then it holds that

$$s(x^*) - px^* > s(x) - px \quad \text{for all } x \neq x^*.$$

We need the following claims:

Claim 4: If $s(x^*) - px^* > s(x) - px$, then there exists $I^0 > 0$ such that

$$I > I^0 \qquad U(x^*, I - px^*) > U(x, I - px).$$

Proof: By $s(x^*) - px^* > s(x) - px$, it holds that there exists $I^* > 0$ such that

$$I > I^* \qquad S(x^*; I) - px^* > S(x; I) - px. \tag{7}$$

It holds by $\mathbf{C_1}$ that there are two real numbers a and b such that

$$U(x^*, I+a - [a+px^*]) = U(x^*, I - px^*) = U(0, I+a) \quad \text{and}$$

$$U(x, I+b - [b+px]) = U(x, I - px) = U(0, I+b),$$

which implies that $a+px^* = S(x^*; I+a)$ and $b+px = S(x; I+b)$. Hence we have by (7) that

$$I > \max(I^*, I+a, I+b) \qquad a > b$$

$$U(x^*, I - px^*) = U(0, I+a) > U(0, I+b) = U(x, I - px). \quad \textbf{QED}$$

Claim 5: $\lim_{I \to +\infty} D(p, I) < +\infty$.

Proof: If $\lim_{I \to +\infty} D(p, I) = +\infty$, then there exists $I^* > p(x^*+1)$ such that

$$I > I^* \qquad D(p, I) > x^*+1 \text{ and } U(D(p, I), I - pD(p, I)) > U(x^*, I - px^*).$$

It holds by the quasi-concavity of U that $U(x^*+1, I - p(x^*+1)) > U(x^*, I - px^*)$ for all $I > I^*$, which contradicts with Claim 4. Hence $\lim_{I \to +\infty} D(p, I) < +\infty$. **QED**

Set $\lim_{I\to+\infty} D(p, I) \equiv b$ in Claim 5, and suppose that $b \neq x^*$. By Claim 4, there exists $I^1 > px^* > 0$ such that

$$I > I^1 \qquad U(x^*, I - px^*) > U(b, I - pb).$$

It holds by the continuity of U and $\lim_{I\to+\infty} D(p, I) \equiv b$ that there exists $I^2 > I^1$ such that

$$I > I^2 \qquad U(x^*, I - px^*) \geq U(D(p, I), I - pD(p, I)),$$

which contradicts with the uniqueness of $D(p, I)$.

(**f**) $\lim_{p\to+0} d(p) = +\infty$ holds by $\lim_{x\to+\infty} \sigma(x) = 0$. The proof of downward slopeness is very easy. Let us prove the consistency to the limit consumer surplus. If $\lim_{x\to+0} s'(x) < +\infty$, then $d(p^*) = 0$ for some $p^* > 0$ and $d(p)$ is continuous on $[p, p^*]$ for all $p < p^*$. Hence

$$\int_p^{+\infty} d(p)dp = \int_p^{p^*} d(p)dp < +\infty \quad \text{for all } p < p^*.$$

In case of $\lim_{x\to+0} s'(x) = +\infty$, fix any $p > 0$ and set $x^* > 0$ such that $s'(x^*) = p$. Then it holds by the replaceability that

$$0 \leq \int_{x^*/m}^{x^*} S'(x; I_x+n)\, dx = S(x^*; I_x+n) - S(x^*/m; I_x+n) \leq K_{x^*} \quad \text{for all } m > 1.$$

Since $S'(x, n)$ uniformly converges to $s'(x)$ for all $x \quad [x^*/m, x^*]$, it holds that

$$\lim_{n\to+\infty} \int_{x^*/m}^{x^*} S'(x; I_x+n)\, dx = \int_{x^*/m}^{x^*} s'(x)\, dx \leq K_{x^*} \text{ for all } m > 1,$$

which implies that

$$\lim_{m\to+\infty} \int_{x^*/m}^{x^*} s'(x)\, dx = \int_0^{x^*} s'(x)\, dx \leq K_{x^*}.$$

Thus it holds by Lemma 2 (b, c) that

$$\int_0^{x^*} s'(x)\, dx - px^* = \int_0^{x} d^{-1}(x)\, dx - px^* = \int_p^{+\infty} d(p)dp < +\infty.$$

Finally, we will show that

$$\lim_{I\to+\infty} EV(p^0, p^1, I) = \int_{p1}^{p0} d(p)\, dp = \lim_{I\to+\infty} CV(p^0, p^1, I) \quad \text{for all } p^0 > p^1 > 0.$$

Set $d(p^0) = x^0$ and $d(p^1) = x^1$, then it holds that

$$\int_{p1}^{p0} d(p)\, dp = [\, s(x^1) - px^1] - [\, s(x^0) - px^0]\,. \qquad (8)$$

Let $(x(I), y(I))$ be the expenditure minimizer corresponding to $EV(p^0, p^1, I)$. Then it holds that

$$EV(p^0, p^1, I) = y(I) + p^0[x(I) - D(p^0, I)].$$

As $I \to +\infty$, it holds that $[x(I) - D(p^0, I)] \to 0$ and $y(I) \to A$, where A is given by

$$A = \lim_{I\to+\infty} WTA(D(p^1, I) - D(p^0, I); D(p^0, I), I - p^1 D(p^1, I)) - [p^1 D(p^1, I) - p^0 D(p^0, I)]$$

$$= \lim_{I\to+\infty} WTA(x^1 - x^0; x^0, I) - (p^1 x^1 - p^0 x^0)$$

$$= \lim_{I\to+\infty} [S(x^1; I) - S(x^0; I)] - (p^1 x^1 - p^0 x^0) = [s(x^1) - px^1] - [s(x^0) - px^0].$$

Note that $WTA(\Delta x ; x, y)$ is defined by a real number $e \geq 0$ such that

$$U(x, y+e) = U(x+\Delta x, y).$$

Hence we have by (8) that

$$\lim_{I\to+\infty} EV(p^0, p^1, I) = [s(x^1) - px^1] - [s(x^0) - px^0] = \int_{p1}^{p0} d(p)\, dp.$$

Similarly, we have that

$$\lim_{I\to+\infty} CV(p^0, p^1, I) = [s(x^1) - px^1] - [s(x^0) - px^0] = \int_{p1}^{p0} d(p)\, dp. \qquad \textbf{QED}$$

Proof of Lemma 3: (**i**) By the condition (i) in Proposition 3, there is a real number $y^* > 0$ such that:

$$\sup_{y \in [y^*, +\infty)} [U(x, y) - U(0, y)] < \lim_{y\to+\infty} [U(x, y) - U(0, y)] + 1 \quad \text{and}$$

$$\inf_{y \in [y^*, +\infty)} U_y(0, y) > [\lim_{y\to+\infty} U_y(0, y)]/2 > 0.$$

Since $[0, y^*]$ is compact and U is smooth, we may set:

$$b = \sup_{y \in [0, +\infty)} [U(x, y) - U(0, y)] < +\infty ; \tag{9}$$

$$a = \inf_{y \in [0, +\infty)} U_y(0, y) > 0; \tag{10}$$

Then it holds by (9) that $U(x, y) - U(0, y) \leq b$ for all $y \geq 0$, which implies that

$$U(x, y) \leq U(0, y) + b. \tag{11}$$

Furthermore, it holds by (10) that

$$U(0, y+b/a) - U(0, y) = \int_y^{y+(b/a)} U_t(0, t)\, dt \geq \int_y^{y+(b/a)} a\, dt = a[y+(b/a)] - ay = b,$$

which implies that

$$U(0, y) + b \leq U(0, y+b/a). \tag{12}$$

We have by (11) and (12) that $U(0, y+b/a) \geq U(x, y)$ for all $y \geq 0$.

(**ii**) Since $U(0,I) = U(x, I - S(x; I)) = U(x, Y(x, I))$, it holds that $U(0,I) = U(x, Y(x, I))$, which implies $f^{-1}(U(0, I)) = f^{-1}[U(x, Y(x, I))]$. Since $f^{-1}(U(0, I)) = I$ by $f(I) = U(0, I)$, it holds that $I = f^{-1}[U(x, Y(x, I))]$. Hence we have that

$$S(x; I) = I - I + S(x; I) = I - Y(x, I) = f^{-1}[U(x, Y(x, I))] - Y(x, I).$$ **QED**

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