Stabilizing, Pareto Improving Policies in an OLG model with Incomplete Markets: The Rational Expectations and Rational Beliefs Case[1].

by

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Abstract

One way to interpret the current policies of many central banks is that they seek to stabilize economic activity. One possible justification for such a policy is that there is volatility in macro variables that individual agents cannot insure against. We study the simplest possible extension of the stochastic 2-period, one agent and one commodity OLG model, where we have added 1 more period, with only one potential activity, namely trading of contingent commodities. We assume, however, that markets are incomplete. In this case the monetary equilibrium is not Pareto Optimal and for an open set of economies an equilibrium where fluctuations in realized savings are removed Pareto dominates the monetary equilibrium. A combination of fiscal and monetary policy may achieve this equilibrium . The policy considered has a simple rationale, namely that it removes some of the uncertainty that agents face by reducing price or interest rate volatility.

We consider two fundamental sources of such volatility, namely respectively an objective and a subjective signal about the distribution of future endowments. The first case is when agents have Rational Expectations while the second case is studied in the context of agents having Rational Beliefs, beliefs which are consistent with empirical observations but not (necessarily) correct.

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1 Introduction

In a finite world with complete markets (and the usual asumptions), the observed price volalitily, although having a negative impact on risk averse agents, is simply reflecting the randomness of underlying fundamentals. A policy that seeks to curb the price volatility will only inhibit the well functioning of the market economy and is thus mistaken. None the less one can interpret the contemporay policies of many central banks as trying to control the level of economic activity using the interest rate . If one believes that the rationale of such policies is that they are Pareto improving, then some of the assumptions of the classical General Equilibrium model has to be reconsidered. In this paper we consider a simple 3-period OLG model with incomplete markets and show how economic stabilization may be walfare improving. Volatility in economic activity is created by volatility in demand and supply. Thus curbing the volatility is done by controlling (directly or indirectly) the actions of agents. There is then clearly a trade-off, since from the individual agent's viewpoint, the ideal policy would control the actions of all other agents, and thus control the (price) volatility he is facing, but would allow himself to act freely. In this paper this trade-off is exposed and some conditions for when a stabilizing policy is considered beneficial for the individual (representative) agents are provided.

The role of the OLG model is twofold. Firstly, in it an explanation for existence of money is derived. However, this is not the the fundamental reason for using the OLG model here. It is conjectured that also if the existence of money was modeled by assuming cash-in-advance constraints the argument for price-stabilization would be valid. The more fundamental reason for using the OLG model is then that it is a realistic infinte horizon model. The infinite horizon is an informal justification for assuming that agents' beliefs have settled down to being consistent with empirical regularities, that is, depending on the choice of auxilliary assumptions, that they are either Rational Expectations or Rational Beliefs.

Sunspots in the OLG model have been used to explain why unwarranted economic volatility unrelated to fundamentals may be present in the economy (se for instance Azariadis(1981)). But there are some open questions regarding this explanation. Firstly, removing the price instability caused by sunspot is not improving when one uses Contingent Pareto Optimality as welfare concept (as was alredy noted by Cass and Shell(1983) - see Nielsen(2001) for further details). If on the other hand one wishes to use Equal Treatment Optimality as a criterion then the stationary distribution is no longer optimal in a model with stochastic fundamentals. Stabilization is then no longer an obvious choice and just describing the optimal allocations becomes difficult even in the two-period model that is usually being used (see Nielsen, 2001). Thirdly, one may doubt whether a realistic version of the

OLG model (where agents live for many periods) can explain the magnitude and form of real and nominal fluctuations observed in the economy. The sunspots effects in the OLG model are the result of random reallocations from/to newborn generations, via an inflationary tax/subsidy. Apart from the issue whether this is a realistic description of what happens during the business cycle is the question, whether the magnitude of such transfers in a monetary equilibrium are realistic. In the case where agents live for 2 periods a relatively high proportion of the economy's ressources can be shifted from one generation to another in equilibrium. But how much can be shifted between the first generation and the 77 remaining in a monetary equilibrium? This is an issue that seems to call for further investigations.

Another explanation provided for "excess" price volatility is incompleteness of markets[2] as for instance studied in Calvet(1998) . The New Keynesian macroeconomics litterature in assuming that individual suppliers whose prices are fixed in the short run cannot insure against shocks that would make them want to change prices, see for instance Woodford(2001). Both in Calvet, Woodford and here there is thus some incompleteness of markets that is not explained from fundamental principles. This seems to be the case for most, if not all, studies of the explicit welfare effects of monetary policies (see also, Lorenzoni, 2000).

The standard two-period one commodity OLG model with a single agent trivially has (sequentially) complete markets, in the sense that agents who live at the same time can freely trade in all spot and contingent commodities, but will not do so. It is only by assuming that (representative) agents live for more than two periods, that there are more than one commodity or that there are hetereogenous agents that the issue of completeness becomes nontrivial. Except for the last possibility such models are unfortunately in general difficult to study if one are interested in equilibria which are ergodic and stationary or, more generally, from which an empirical distribution of prices can be extracted (see Duffie, Geanakoplos, Mas-Colell, and McLennan(1994) and Gottardi(1996)). Therefore, we have chosen to study only the simplest possible extension of the representative agent model, namely where agents live for 3 periods and their potential activity in the first period is only to trade contingent commodities. Markets are however assumed to be incomplete and such trade cannot take place. It should be noted, that as is often the case for incomplete markets models, no explanations for why markets are incomplete is offered here. We show how stabilizing economic activity may improve welfare. Of course this means that active monetary/fiscal policies matter, something which has already been shown in many contexts (see for instance Gottardi(1995) for the case of an OLG model, with

[2]Note that when markets are incomplete another type of sunspot induced volatility may be present

heterogeneous agents who live for two periods). But the explicit study of the effects of economic stabilization found in this paper appears to be new.

The paper is organized as follows. In the next section a simple version of the OLG model is described. Furthermore we introduce two interpretations of the model, one assuming that agents hold Rational Expectations the other assuming that they hold Rational Beliefs. We also consider two possible concepts of Pareto Optimality known from the literature and argue in favour of one of them. In the following section we study the features of the monetary equilibria and provide the policy results. The appendices deals with more technical issues. In the first we show some of the results for a more general version of the model. In the other appendix we provide a brief introduction to the theory of Rational Beliefs.

2 The one-commodity OLG model; Rational Expectations and Rationale Beliefs

2.1 Model and Monetary Equilibrium

We consider an overlapping generations model with one commodity where agents are born in the first period , receive endowments e_a in the second period and a random endowment, e_t either e_b or e_c in the third and final period of their lives. Furthermore, in the second period of their lives the agents receive a signal, $z_t \in \{1, 2\}$ about the (objective or perceived) distribution, $\pi^z = (\pi_b^z, \pi_c^z)$ (with $\pi^1 \neq \pi^2$) of the endowments in the last period of their lives[3]. The stochastic sequence $\{z_t\}$ is i.i.d. and independent of past e_t with probability vector (q_1, q_2).

Agents, which are all ex-ante identical, only have utility, u (defined on $\Re^2_{++}$, C^2, strictly increasing, strictly concave, and with indifference curves whose closures are contained in $\Re^2_{++}$) over consumption in the second and third period of their lifes[4]. So we essentially have a classical OLG model, except that agents are born before they know what "type" they are, i.e. before they know the signal about the distribution of the endowments in the last period of their lifes. We consider a monetary equilibrium for this economy, assuming that the amount of outside fiat money is 1 unit. In such an equilibrium there will be two possible prices (of money in terms of the commodity good),p_1 and p_2, at each date t, depending on signal of the then middle-aged.

DEFINITION 1 *Monetary Equilibrium. Price vector* $(p_1, p_2) \in \Re^2_{++}$ *such that when an agent with*

[3]In appendix 4.1 we study the case where the number of signals and second period endowments is arbitrary but finite.

[4]When making genericity statements the topology of C^2 uniform convergence on compacta (MasColell(1985),p.50) is used.

signal $z = k$ solves:

$$Max_{M\geq 0}\sum_{i=2}^{2}\sum_{s\in\{b,c\}} u(e_a - p_k M, e_s + p_i M)q_i\pi_s^k \tag{1}$$

the solution is $M = 1$

Using the First Order Conditions we then have that an equilibrium is uniquely characterized by

$$\sum_{i=2}^{2}\sum_{s\in\{a,b\}}[-\frac{\partial u}{\partial x_1}(e_a - p_k, e_s + p_i)p_k + \frac{\partial u}{\partial x_2}(e_a - p_k, e_s + p_i)p_i]q_i\pi_s^k = 0, k = 1, 2 \tag{2}$$

Such an equilibrium (where money is valued) may or may not exist, depending on preferences (and beliefs). In appendix 4.1 a sufficient condition for existence is provided.

REMARK 1

We will sometimes refer to the two-period version of the model considered here. In this version agents live for two periods corresponding to the two last periods of the three-period version of the model, i.e. when they are born the signal about the distribution of endowments in the last period of their life has already been realized. Despite this difference, for the two-period version of the model the definition of monetary equilibrium is exactly as above. ■

2.2 Interpretation of z_t

We provide two different not necessarily exclusive explanations for the presence of the signal z_t. According to the first, z_t is an objective signal about the distribution of the endowments the next period, and as such can be considered to be a supply shock, about which agents hold Rational Expectations.

According to the other explanation, z_t is a signal which coordinates the subjective expectations of the agents, expectations which may not necessarily be correct. That subjective beliefs are indeed present and significant even among major actors on the financial markets is convincingly demonstrated in Kurz(2002). When the signals are guiding subjective beliefs they can better be interpreted as a demand shock in that it effects the beliefs i.e. preferences of agents. In the context of this second explanation agents are supposed to hold Rational Beliefs about the distribution of the endowments[5]. The empirical distribution of the endowments is assumed to be known[6]. The Rational Beliefs story postulates that agents may think that more can be known than just this empirical distribution. Specifically, agents may form statistical models or theories according to which the endowment process,

[5]See Appendix 4.2 for a brief introduction to the theory of Rational Beliefs, and Kurz(1994a) and Kurz(1994b) for more comprehensive introductions.

[6]We note in passing that this assumption together with the assumption that the true distribution is known to be stationary would lead to the conclusion that agents have Rational Expectations about the distribution of the endowments.

$\{e_t\}$ is correlated with a process of signals $\{z_t\}$. We do not assume that the agents know the empirical distribution of the joint process $\{e_t, z_t\}$, only that they know they know the two marginals of the empirical distribution, each assumed to be i.i.d.. We denote the empirical distribution of $\{e_t\}$ by $\overline{B} = (\overline{B}_1, \overline{B}_2)$. The type of statistical model we consider in the context of this paper is as follows. When agents at date t observe $z_t = i, i = 1, 2$ they pick the belief $B^i = (\pi_b^i, \pi_c^i)$, which they hold to be the distribution of the the endowments in the following period. To simplify and unify the exposition of the model z_t is assumed to be known to be i.i.d. Assuming, as we did, that both the empirical distribution of $\{e_t\}$ and of $\{z_t\}$ is known, rational agents can only adopt a statistical model, or belief if it generates the same empirical distribution of the endowments as the known one. The model generates the empirical distribution $q_1 B^1 + q_2 B^2$ and consequently, what we call the rationality requirement is

$$q_1 B^1 + q_2 B^2 = \overline{B} \tag{3}$$

It is interesting to note, that endowing agents with rational Beliefs is one way to formulate over-confidence on their part. When agents have (non-stationary) Rational Beliefs they use a model for the savings decission which is more informative than the data itself, namely the two empirical distributions, i.e. they are confident that they know more than what the data provides them with. This puts the theory on line with several psychological studies of human behavior, cited in Odean(1998) and Daniel, Hirschleifer, and Subrahmanyam(1998).

Looking forward the statistical models or beliefs that agents are employing are rational. Looking backwards agents may discover that the model did not work well, i.e. that z_t was on average not a good predictor of e_{t+1}. The gambler in the casino or the investor on the stock market may look back and realize that his past strategy was not performing well. He may never the less believe that his model will perform better in the future. Or he may choose another model/beliefs. From our viewpoint this amounts to the same, the last just being a renaming of the signal. What matters is that agents continues to use subjective rational beliefs in a stable way, i.e. such that not only exogenous but also endogenous variables are having an empirical distribution.

REMARK 2

Let us note that there is a problem of interpretation of the model of the beliefs as presented here. Presumably agents would know the emprical distribution of the joint process of prices and endowments. But since prices are determined by beliefs, i.e. z_t, implicitly they know the empirical distribution of the joint process $\{z_t, e_t\}$, something we assumed that they did not realize. In fact this problem cen be

remedied, but at the expensive of introducing a much more complicated model, see Nielsen (1998b) for details. ■

One approach to understanding the Rational Beliefs story is to think of it as a logically consistent way to formalize the idea that the priors may have a permanent effect on the beliefs of agents. With only a minor departure from the Rational Expectations framework this is achieved. As a result agents may have diverse subjective beliefs, even when confronted with the same statistical data, something we, however, do not emphasize in this context. Another consequence is that volatility in endogenous variables may not have as the only source, volatilty in "fundamentals", but may be generated by subjective changes in the myopic beliefs of agents. This story may help to explain the apparant observation, for instance in the stock markets, that endogenous variables like prices tend to be more volatile than exogenous variables like technological shocks[7]. This observation is one motivation for many recent theoretical investigations of the effects of stabilizing policies, based on indeterminacy, i.e. sunspot equilibria (see f.i. Christiano and Harrison, 1996). However, in this new strand of macroeconomic literature the assumption that agents have Rational Expectations is maintained.

In the context of the present model, we do not need to distinguish between the two types of sources of volatility, signals about endowments (technology) or subjective signals. This is due to two factors. Firstly, for the Rational Beliefs version of the model we have assumed perfect correlation between beliefs. Some correlation is needed for subjective changes in expectations to show up in aggregated like prices, but the correlation need certaintly not be perfect[8]. As the heterogeneity of beliefs increases the analysis below becomes more complicated and this is one reason why we have chosen the simpler case. Incidently, one way to emprically separate the Rational Expectations case from the Rational Beliefs case is by looking for heterogeneity of beliefs among equally informed market participants. Such heterogeneity has been observed in the markets for foreign exchange (see Taylor, 1995) and in financial markets (see Odean, 1998 and Daniel, Hirschleifer, and Subrahmanyam, 1998) for reviews of the literature and further references. See also Kurz(1997) for a view on endogenous fluctuations and diversity of beliefs from the viewpoint of the theory of Rational Beliefs). The fact that there are diversity of opinions about the lifespan of bubbles or whether increasing prices on stocks or land constitute a bubble at all may for then cast doubt on the assertation that such bubbles are rational.

[7]This kind of volatility which becomes endogenous uncertainty (Defined in Kurz, 1974) when agents do not know the future beliefs of others has in the context of an OLG model previously been studied in Kurz and Wu(1996) and Motolese(1998)

[8]In Kurz(1998) it is demonstrated that even without correlation between beliefs excess volatility may be generated.

2.3 Some concepts of Optimality

Several concepts of Pareto Optimality have been studied in the literature, two notable being that of Conditional Pareto Optimality (also called Dynamical Pareto Optimality in Cass and Shell(1983) and that of Equal Treatment-Pareto Optimality (defined in Muench(1977)). In Peled(1982) these two notions are compared and it is argued that Conditional Pareto Optimality is the right criterion. When using the Conditional Pareto Optimality criterion the definition of an agent includes the stochastic state in which he is born. Thus Pareto improving transfers cannot make an agent born in some state worse off, even though the "same" agent born in another state might be made better off. The notion of Conditional Pareto Optimality is then weaker than the notion of Equal Treatment-Pareto Optimality, according to which such transfers may be considered Pareto improving if they make all agents better off in an expected sense. While Conditional Pareto Optimality seems to be natural it also leads to some apparantly strange conclusions. For instance, it is easy to show that all sun-spot equilibria in the model of Azariadis(1981) are conditionally Pareto Optimal (this observation is the same as Proposition 6 of Cass and Shell(1983)). This is the motivation for studying a three period model, where we have added a period, right after birth, to the life of (representative) agents. The only effect of this period is that is allows for an addition of utility across states of beliefs, as it would allow for an addition of utility across sun-spot states in the Azariadis model. In this sense we move closer to the concept of Equal Treatment-Pareto Optimality and the fact that this is possible with a minor change of the model shows that the two concepts of optimality are not as disparate as they may seem.

Let us briefly contrast the present work with that of Nielsen[1998a]. There the emphasis is not on possible Pareto improvements but on the fact that agents are likely to have incorrect beliefs when they hold Rationall Beliefs. It was shown for an OLG model with two countries that a monetary union is better in an ex-post sense (see Hammond(1983)) than letting the exchange rate float. The main point was that, under a floating exchange rates, agents are likely to make forecasts that are wrong and thus to make the wrong decisions about their portfolio of foreign and domestic currencies. Under a monetary union such mistakes are not made, since there is only one currency to hold. Two points of the paper deserve to be mentioned here. One is that it was not assumed that the government knows more than the agents, yet from an ex-post point of view it could deem a monetary union to be superior. Another is that it was only from an ex-post point of view that a monetary union was deemed better. Indeed, if one were using the Pareto criterion the conclusion was likely to be the opposite: Agent who are using non-stationary Rational Beliefs can be viewed as being optimistic about their ability to

forecast - they believe they can read more out of the data than what is in the empirical distribution. This common optimism among agents with rational beliefs means that there are trading possibilities (i.e. possible"bets") that are not being explored under a monetary union. Therefore it is unlikely to lead to a Pareto Optimal allocation. In the present model, we expect that taking into account that agents with Rational Beliefs make mistakes will make stabilizing policies even more attractive.

3 Stabilizing Policies

The following proposition consider price volatility in a monetary equilibrium. It is generalized to generic utility functions in Appendix 4.1.

PROPOSITION 1 *Suppose that* $\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$. *Then* p_1 *and* p_2 *are different in a monetary equilibrium.*

Proof: Suppose not, i.e. $p_1 = p_2 = p$. $\sum_{i=2}^{2} \sum_{s=a}^{b} [-\frac{\partial u}{\partial x_1}(e_a - p, e_s + p)p_k] q_i \pi_s^k$ does not depend on k, but $\sum_{i=2}^{2} \sum_{s=a}^{b} [\frac{\partial u}{\partial x_2}(e_a - p, e_s + p)p] q_i \pi_s^k$ does, since $\frac{\partial u}{\partial x_2}(e_a - p, e_1 + p) \neq \frac{\partial u}{\partial x_2}(e_a - p, e_2 + p)$, a contradiction ■

REMARK 3

Suppose we were only considering a two period model, i.e. where each agent is born with his beliefs and endowment, e_a and have random endowments, e_b or e_c when old. Any (p_1, p_2) which for some weights, (w_1, w_2) solve the problem.

$$Max_{p_1 \geq 0, p_2 \geq 0} \sum_{k=1}^{2} w_k \sum_{i=1}^{2} \sum_{s \in \{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \tag{4}$$

constitutes a stationary conditionally Pareto Optimal allocation(see Appendix 4.2). The First Order Conditions for a solution to this problem are

$$w_k \sum_{i=1}^{2} \sum_{s \in \{b,c\}} -\frac{\partial u}{\partial x_1}(e_a - p_k, e_s + p_i) q_i \pi_s^k + \sum_{j=1}^{2} w_j \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_j, e_s + p_k) q_k \pi_s^j = 0, k = 1, 2 \tag{5}$$

Suppose that the equilibrium conditions (2) for a monetary equilibrium hold. Set $w_1 = p_1 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_2, e_s + p_1) q_1 \pi_s^2$ and $w_2 = p_2 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_1, e_s + p_2) q_2 \pi_s^1$ and the conditions (5) will also hold so that the monetary equilibrium is indeed Conditionally Pareto optimal. This remark (which is generalized in Appendix 4.1) serves as a motivation for considering a 3-period model. ■

3.1 Pareto improving policies

In a monetary equilibrium, (p_1, p_2), the expected utility of a young agent is

$$\sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \tag{6}$$

It is intuitively clear from this expression that the agent would like to be able to react on his own belief (i.e. have his savings depend on the signal he receives) but would prefer that others were not able to (i.e. would prefer that the return on his savings does not depend on other agents' signals).

In a monetary equilibrium the prices are transfers between generations. To see if we can improve on the allocation associated with a monetary equilibrium it is natural to consider the following problem:

$$Max_{p_1\geq 0, p_2\geq 0} \sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p_k, e_s + p_i) q_i \pi_s^k \tag{7}$$

The First Order Conditions for an interior solution to this problem are

$$\sum_{i=1}^{2} \sum_{s\in\{b,c\}} -q_1 \frac{\partial u}{\partial x_1}(e_a - p_1, e_s + p_i) q_i \pi_s^1 + \sum_{j=1}^{2} q_j \sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_k, e_s + p_1) q_1 \pi_s^j = 0 \tag{8}$$

$$\sum_{i=1}^{2} \sum_{s\in\{b,c\}} -q_2 \frac{\partial u}{\partial x_1}(e_a - p_2, e_s + p_i) q_i \pi_s^2 + \sum_{k=1}^{2} q_k \sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_k, e_s + p_2) q_2 \pi_s^k = 0 \tag{9}$$

PROPOSITION 2 *The Monetary Equilibrium is generically not Pareto Optimal*

Proof: If we compare with the conditions (2) which hold in a monetary equilibrium it is easy to see that for a monetary equilibrium to solve the problem (7) the following condition needs to hold:

$$p_1 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_2, e_s + p_1) \pi_s^2 = p_2 \sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_1, e_s + p_2) \pi_s^1 \tag{10}$$

Again, without making the statement precise it is clear, that generically (8), (9), and (10) (3 equations with 2 unknowns) can generically not hold, meaning that a monetary equilibrium cannot be conditionally Pareto Optimal. This is formally shown in Appendix 4.1. ■

The transfers in the solution to (7) are in general still random. Let us now consider another problem. Suppose we restrict the young agents to making a transfer between the date when they are middle aged and the date where they are old, independently of the beliefs they have, when middle aged, i .e. suppose we consider the following problem:

$$Max_{p\geq 0} \sum_{k=1}^{2} q_k \sum_{i=1}^{2} \sum_{s\in\{b,c\}} u(e_a - p, e_s + p) q_i \pi_s^k \tag{11}$$

A solution to this problem does not constitute a Conditionally Pareto Optimal allocation (see, Nielsen,2001 for definitions) - in particular there is a Tradeable Contingent Improvement to it . The reason is that

with the rigid transfer scheme imposed by the solution to (11) there is no room for reaction to changes in second period endowments, e_t.

It is now easy to see, that if preferences are time separable,then a solution to (7) has $p_1 = p_2$, i.e. is a solution to (11).

PROPOSITION 3 *Suppose* $\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$. *Then a solution to (7) has* $p_1 = p_2$ *and is a solution to (11).*

Proof: The assumption implies that for any p $\frac{\partial u}{\partial x_1}(e_a - p, e_s + p_i) = \frac{\partial u}{\partial x_1}(e_a - p, e_{s'} + p_j), \forall i, j, s, s'$. If we let $\hat{p}$ be the solution to (11) is then easy to see that (8) and (9) hold with $p_1 = p_2 = \hat{p}$ ■

Under the stated conditions the result implies that in the Rational Expectations monetary equilibria studied here, if it is possible for the government to pursue a policy which results in the fixed price $\hat{p}$ then such a policy is better than the laisez faire policy. In other words it is desirable to remove all volatility in economic activity related to the signal. Note that for utility functions in an open neighbourhood of time separable utility functions the solution to (11) will still Pareto dominate the Monetary Equilibrium. Before turning to how to implement the constant price let us briefly consider local changes in the two prices.

Suppose we are in a monetary equilibrium, so that (2) holds but that (10) does not hold, i.e. it is not Conditionally Pareto Optimal. If in stead

$$p_1 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_2, e_s + p_1)\pi_s^2 > p_2 \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_1, e_s + p_2)\pi_s^1 \tag{12}$$

then the LHS of (8) is > 0, while the LHS of (9) is < 0. This means that if we slightly increase the price associated with signal 1 and slightly decrease the other price then we increase utility for all agents. To interprete this observation note the following result:

PROPOSITION 4 *Assume* $e_c < e_b$ *and* $\pi_c^1 < \pi_c^2$. *If* $\frac{\partial^2 u}{\partial x_1 \partial x_2} \geq 0$ *then* $p_1 < p_2$.

Proof: We assume the opposite, $p_1 \geq p_2$ and obtain a contradiction. We then have

$$(a) \quad \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_1}(e_a - p_1, e_s + p_i) q_i \pi_s^1 > \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_1}(e_a - p_1, e_s + p_i) q_i \pi_s^2$$

$$\geq \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_1}(e_a - p_2, e_s + p_i) q_i \pi_s^2$$

and

$$(a) \quad \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_1, e_s + p_i) q_i \pi_s^1 < \sum_{i=1}^{2} \sum_{s \in \{b,c\}} \frac{\partial u}{\partial x_2}(e_a - p_1, e_s + p_i) q_i \pi_s^2$$

$$\leq \sum_{i=1}^{2} \sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_1}(e_a - p_2, e_s + p_i) q_i \pi_s^2$$

which implies that the condition, (2), for a monetary equilibrium cannot hold ■

Thus if the assumptions of the result hold as well as (12) then decreasing the "price volatility" slightly, by increasing the smaller price and decreasing the larger price increases welfare. Of course if the assumptions of the result still hold but instead (12) holds with reverse inequality, then increasing "price volatility" increases welfare.

3.2 Implementing the Pareto-improvement

Let $\hat{p}$ be the solution to (11). We find a combination of fiscal and monetary policies that results in the utility associated with (11) being obtained. Let M_t and p_t be given s.t. $M_t p_t = \hat{p}$. Suppose the current signal has the value k. Then define next period's price to be

$$p_{t+1}^k = \frac{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_1}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k}{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k} p_t$$

Then let next period's taxes be

$$\tau_{t+1}^k = p_{t+1}^k M_t - \hat{p}$$

They then solve the problem:

$$Max_{M\geq 0} \sum_{s\in\{b,c\}} u(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)\pi_s^k$$

with First Order Conditions

$$\sum_{s\in\{b,c\}} [-\frac{\partial u}{\partial x_1}(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)p_t + \frac{\partial u}{\partial x_2}(e_a - p_t M, e_s + p_{t+1}^k M - \tau_{t+1}^k)p_{t+1}^k]\pi_s^k = 0 \quad (13)$$

Using the definitions of prices and taxes the LHS of (13) becomes

$$\sum_{s\in\{b,c\}} [-\frac{\partial u}{\partial x_1}(e_a - \hat{p}, e_s + \hat{p})p_t +$$

$$\sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - \hat{p}, e_s + \hat{p})]\pi_s^k \frac{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_1}(e_a - \hat{p}, e_s + \hat{p})p_t\pi_s^k}{\sum_{s\in\{b,c\}} \frac{\partial u}{\partial x_2}(e_a - \hat{p}, e_s + \hat{p})\pi_s^k} = 0, k = 1, 2$$

Thus $M = M_t$ solves the problem of the agents with the future prices as defined. Finally we define M_{t+1} via the equation

$$p_{t+1}^k M_{t+1} = p_{t+1}^k M_t - \tau_{t+1}^k = \hat{p}$$

So the government is selling money to the middle aged, if $\tau_{t+1}^k < 0$, else it is buying money from the old. With this combination of monetary and fiscal policies all uncertainty about the signal is removed,

and the only uncertainty left is the fundamental uncertainty about the resources of the economy, when the agents are old. It should be noted though, that the implicit assumptions about the information available to the governments are quite strong. Most importantly, it is assumed that at any date, the government is able to find out what signal agents have received.

Suppose for simplicity that u is separable, $= u_1 + u_2$. We then have $p^k_{t+1} = \frac{u_1'(e-\hat{p})}{\sum_s u_2'(e_s-\hat{p})\pi^k_s} p_t$. When the signal is positive, assigning high probability to high endowments in the next period, the interest rate, $\frac{p^k_{t+1}}{p_t}$ is large, inducing agents to transfer more. Also the growth in the money supply is smaller than if the signal is bad. This is how the government in this simple model counteract tendencies to decreased savings, and increased current consumption. Note also, that with this policy, there is no price uncertainty for any agent. At date t the middle aged know what price will prevail in the following period, $t+1$.

REMARK 4

Suppose that preferences are time separable, i.e. $\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$. Let $\bar{\pi}_s = q_1\pi^1_s + q_2\pi^2_s$. Then we can define prices in a different way:

$$p^{k_t}_{s_{t+1},t+1} = \frac{\overline{\pi}_{s_{t+1}}}{\pi^{k_{t+1}}_{s_{t+1}}} p_t, s = b, c$$

Inserting in (13) since $\sum_{s\in\{b,c\}}[-\frac{\partial u}{\partial x_1}(e_a - \hat{p}, e_s + \hat{p})\pi^k_s$ is independent of k, we now get:

$$p_t \sum_{s\in\{b,c\}} [-\frac{\partial u}{\partial x_1}(e_a - \hat{p}, e_s + \hat{p}) + \frac{\partial u}{\partial x_2}(e_a - \hat{p}, e_s + \hat{p})\overline{\pi}_s] = 0$$

since this is the First Order Conditions for solving (11). In this case, since $p_t = \frac{\hat{p}}{M_t}$ and $p^k_{s,t+1}M_{t+1} = \hat{p}$ we get

$$M_{t+1} = \frac{\pi^{k_t}_{s_{t+1}}}{\overline{\pi}_{s_t}} M_t$$

In other words,

$$M_T = \times^T_{t=1} \frac{\pi^{k_t}_{s_{t+1}}}{\overline{\pi}_{s_t}} M_1$$

or

$$\frac{1}{T}\ln M_T = \frac{1}{T}\sum_{t=1}^{T} \ln\left[\frac{\pi^{k_t}_{s_{t+1}}}{\overline{\pi}_{s_{t+1}}}\right] \tag{14}$$

■

REMARK 5

What happens to the money stock under the monetary policy considered here? Under the assumption of structural independence (defined in Nielsen(1994)) the RHS of (14) tends to

$$\sum_{k=1}^{2}\sum_{s\in\{b,c\}} q_k\overline{\pi}_s \ln\left[\frac{\pi_s^k}{\overline{\pi}_s}\right] < 0$$

as $T \to \infty$. This means that $M_t \to 0$ as $T \to \infty$. ■

3.3 Concluding Remarks

The two versions of the model presented above, with respectively RE and RB are not exclusive. However, it is an imperical matter to investigate, whether the main driving force behind fluctuations in prices is real informational shocks (or shocks to technology and preferences) or whether it is rather fluctuations in (subjective) beliefs.

Both the policies of the European Central Bank and the American Federal Reserve Board have, wholy are partially, been directed towards stabilizing inflation, that is to keep inflation inside a target zone. To achieve this goal interest rates are being adjusted in reaction to developments in the real economy. Stabilizing prices (or price movements) will reduce the macroeconomic part of the price uncertainty that agents face and as long as this price uncertainty is not already completely insurable may have a positive impact on economic welfare. This was what the present study seeked to formalize. However, if we take into account that agents typically have diverse and in particular wrong expectations such stabilization has another beneficial effect (as was demonstrated, in another context in Nielsen, 1998a) namely in reducing the mistakes agents make in forecasting prices. That such an effect is intended may be seen from the Federal Reserve Board who's policy has been to some extent geared towards the American stock markets and in particular to preventing (probably unsuccessfully) a build-up of a bubble and its consecutive burst. This two beneficial effects of an active stabilization policy are not exclusive and may, for clarity best be studied separately.

4 Appendices

4.1 Generalization of the Results

The Generalized Model For the generalized model we assume that there are $K \geq 2$ signals and thus K conditional distributions $\pi^k, k = 1, ..., K$, Furthermore, that there are $S \geq 2$ states, $e_1 < e_2 < ... < e_s < ...e_S$ for the endowments in the last period of the life of the agents, such that $\sum_{s=1}^{S} e_s\pi_s^j < \sum_{s=1}^{S} e_s\pi_s^{j+1}, \forall j$. Let $\bar{e} = max\{e, e_1, \ldots, e_S\}$. The assumptions from 2.2 about u are maintained.

A Monetary Equilibrium is then a vector $(p_1, ..., p_K)$ s.t. for each k the solution to the problem

$$Max_M \sum_{i=1}^{K} \sum_{s=1}^{S} u(e - p_k M, e_s + p_i M) q_i \pi_s^k \tag{15}$$

is $M = 1$. Note that an equilibrium price will be in $(0, \bar{e})^K$

The proof and the formulation of the following proposition follow Peled(1982).

PROPOSITION 5 *Suppose that* $\frac{\frac{\partial u}{\partial C_1}}{\frac{\partial u}{\partial C_2}}(e, e_s) < 1, \forall s$. *Then there exists a monetary equilibrium for the economy.*

Proof: In step 1 of the proof existence of an equilibrium is shown and in step 2 it is shown that in this equilibrium money has value. Consider normaliced prices for the consumptio and money, $p = (p^c, p^m) \in \Delta^{2K}$. For $p \in int\Delta^{2K}$ let $M_k(p)$ be the solution to

$$Max_{M \geq 0} \sum_{s=1}^{S} \sum_{i=1}^{K} u(e - M\frac{p_k^m}{p_k^c}, e_s + M\frac{p_i^m}{p_i^c}) q_i \pi_s^k.$$

Let $C_k(p) = e - M\frac{p_k^m}{p_k^c}$, $(C, M)(p) = [C_1(p), ..., C_K(p), M_1(p), ..., M_K(p)]$, and $E(p) = (E^c(p), E^m(p)) = (C, M)(p) - (e, e, .., e, 1, ..1)$ - the excess demand at price p. We have

$$E(p)p = \sum_{k=1}^{K} p_k^c \left[-M_k(p)\frac{p_k^m}{p_k^c} + \frac{p_k^m}{p_k^c} \right] + \sum_{k=1}^{K} p_k^m \left[M_k(p) - 1 \right] = 0.$$

Consider a sequence $\Delta^n \subset int\Delta^{2K}$ s.t. $\Delta^n \uparrow \Delta^{2K}$, where Δ^n is non-empty, compact and convex. From Debreu's theorem it follows that for each n there is a $(\hat{p}^n, \hat{E}^n)$ s.t.

(i) $\hat{p}^n \in \Delta^n$

(ii) $\hat{E}^n = E(\hat{p}^n)$

(iii) $\hat{E}^n p \leq 0, \forall p \in \Delta^n$

$\hat{E}^n$ is bounded from below by $(e, e, .., e, 1, .., 1)$. It follows from (iii) that $\hat{E}^n$ is also bounded from above. We conclude that $(\hat{p}^n, \hat{E}^n)$ has a convergent subsequence $(\hat{p}^{n_q}, \hat{E}^{n_q}) \to (\bar{p}, \bar{E}) \in \Delta^{2K} \times \Re^{2K}$. We have $\bar{p}\bar{E} = 0$ and $\bar{E}p \leq 0, \forall p \in \Delta^{2K}$. Notice that if $M_k(\bar{p}) = 0$ i.e. $E_k^m(\bar{p}) = 0$ then $E_k^c(\bar{p}) = 0$. By (iii) we have that $\bar{E} \leq 0$ (for if for some j, $\bar{E}_j > 0$, choose $p_j = 1$, $p_r = 0, r \neq j$). Since $\bar{E}\bar{p} = 0$, if $\bar{E}_k^m < 0$, $\bar{P}_k^m = 0$. Clearly, $\bar{p} \in int\Delta^{2K}$ it is an equilibrium price. Suppose that $\bar{p} \in \partial\Delta^{2K}$. Clearly, if $\bar{p}^c \gg 0$. Furthermore, we must have that $\bar{p}_k^m = 0, \forall k$. For else, if $\bar{p}_{k'}^m > 0$ and $\bar{p}_k^m = 0$ we must have that $\hat{E}_k^{nm} = E_k(p^{n_q}) \to \infty$,which can be seen from the FOC to the problem of the consumer:

$$\sum_{s=1}^{S} \sum_{i=1}^{K} \left[\frac{\partial u}{\partial C_1} \left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n_q}} \right) \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}} - \frac{\partial u}{\partial C_2} \left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n}} \right) \frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}} \right] q_i \pi_s^k = 0 \tag{16}$$

In the second step we rule out that $\hat{p}^{n_q} \to (\bar{p}^c, 0 \dots 0)$ where $\bar{P}^c \gg 0$. Rewrite the FOC as

$$\sum_{s=1}^{S}\sum_{i=1}^{K} \frac{\partial u}{\partial C_1}\left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n_q}}\right) \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}} \left[\frac{\frac{\partial u}{\partial C_2}\left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n}}\right)}{\frac{\partial u}{\partial C_1}\left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n_q}}\right)} \frac{\hat{p}_i^{m,n}}{\hat{p}_i^{c,n}} \frac{\hat{p}_k^{c,n}}{\hat{p}_k^{m,n}} - 1\right] q_i \pi_s^k = 0$$

Since M_k^n is bounded we have that

$$\frac{\frac{\partial u}{\partial C_2}\left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n}}\right)}{\frac{\partial u}{\partial C_1}\left(e - M_k^n \frac{\hat{p}_k^{m,n}}{\hat{p}_k^{c,n}}, e_s + M_k^n \frac{\hat{p}_i^{m,n}}{\hat{p}_k^{c,n_q}}\right)} \to \frac{\frac{\partial u}{\partial C_2}(e, e_s)}{\frac{\partial u}{\partial C_1}(e, e_s)} > 1 \text{ as } n \to \infty$$

There is also some k' s.t.. for some sequence n_q

$$\frac{\hat{p}_i^{m,n_q}}{\hat{p}_i^{c,n_q}} \geq \frac{\hat{p}_{k'}^{m,n_q}}{\hat{p}_{k'}^{c,n_q}}, \forall q, \forall i$$

If we consider what is inside the bracket in the reformulated FOC, be see that for this subsequence and for $k = k'$ is is $> 0, \forall i$, which means that the equality cannot hold for all n for this k' ■

Regularity of the Monetary Equilibrium Let $\mathcal{U}$ be the set of C^2 utility functions defined on $\Re^2_{++}$ which are strictly increasing, strictly concave, and with indifference curves who's closure is contained in $\Re^2_{++}$. Let $\mathcal{U}^*$ be the subset of $\mathcal{U}$ for which a monetary equilibrium exists - it has non-empty interior. Define F on $\mathcal{U} \times \Re^K_{++}$ with values in $\Re^K$. $F_k(u,p)$ is equal to the left hand side of (16) above (ignoring the subscripts). Thus $p > 0$ is a Monetary Equilibrium price for the economy u if and only if $F(u,p) = 0$. F is continuous.

PROPOSITION 6 *For an open and dense set of utility functions u we have that if $F(u,p) = 0$ for some p then there is an open ball, B_u containing u and an $\underline{p} \in \Re^K_{++}$ s.t. for all $u' \in B_u$ we have that if $F(u',p) = 0$ then $p \geq \underline{p}$.*

Proof: Suppose we have for some u that there is a sequence $p^n \to 0$ s.t. $F(u,p^n) = 0, \forall n$. We then have $F(u,0) = 0$ something which obviously only hold for a closed no-where dense set of utility function. Suppose then that there is $\underline{p} \in \Re^K_{++}$ s.t. $F(u,p) = 0 \Rightarrow p > \underline{p}$. We can then not have a sequence $u^n \to u$ s.t. for each n there is $p^n \leq \underline{p}$ with $F(u^n,p^n) = 0$ ■

Let $\mathcal{V} = \{u \in \mathcal{U} : F(u,0) \neq 0\}$. Regularity for an economy u means that $F(u,p) = 0, p > 0 \Rightarrow \partial_p F(u,p)$ has full rank i.e. rank K. The set of regular u contains an open set. For suppose that u is regular and in $\mathcal{V}$. If there were a sequence $u^n \to u$ where u^n were not regular there would be a sequence p^n s.t. $F(u^n,p^n) = 0$ and $p^n > 0$ but $|\partial_p F(u^n,p^n)| = 0, \forall n$ (where $|\cdot|$ means determinant).

Since p^n is bounded (u^n, p^n) has a cluster point $(u, \bar{p})$, implying by the continuity of F and $|\partial_p F|$ that $F(u, \bar{p}) = 0$ (so that $\bar{p} > 0$ and $|\partial_p F(u, \bar{p}| = 0$, a contradiction.

PROPOSITION 7 *Suppose that the rank of the matrix*

$$\Pi = \begin{pmatrix} \pi_1^1 & \pi_2^1 & \cdot & \cdot & \cdot & \pi_S^1 \\ \pi_1^2 & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \pi_1^K & \cdot & \cdot & & & \pi_K^K \end{pmatrix}$$

is K. Then the set of regular u is dense in $\mathcal{U}$.

Proof: Pick any $u \in \mathcal{V}$. Then show that there is a sequence $\{u^n\} \to u$ s.t. $\forall n :$ "$F(u^n, p) = 0, p > 0 \Rightarrow |\partial_p F(u^n, p)| \neq 0$. First we find an open ball, B, around u and contained in $\mathcal{U}^*$ s.t. (using Proposition 6) there is a lower bound $\underline{p}$ such that any equilibrium price p for $u' \in B$ is $\geq (\underline{p}, \ldots, \underline{p})$. We consider the following form of parametrization:

$$u_{(\epsilon, r_1, \ldots, r_L)} = u + \epsilon \sum_{j=1}^{L} u_j$$

for $(\epsilon, r_1, \ldots, r_L) \in [0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j$ where the R_js are open intervals and $u_j(C_1, C_2) = -e^{-r_j C_2}$. In the proof we choose L and $R_1, \ldots, R_L$ s.t. if we let $\hat{F}(\epsilon, r_1, \ldots, r_L, p) = F(u_{(\epsilon, r_1, \ldots, r_L)}, p)$ then $\partial \hat{F}(\epsilon, r_1, \ldots, r_L, p)$ has rank K for all $(\epsilon, r_1, \ldots, r_L, p) \in [0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j \times [\underline{p}, \bar{e})^K$. It follows from the transversality theorem that the set of $(\epsilon, r_1, \ldots, r_L)$ for which $F(u_{(\epsilon, r_1, \ldots, r_L)}, p) = 0 \Rightarrow |\partial_p F(u_{(\epsilon, r_1, \ldots, r_L)}| \neq 0$ has full Lebesgue measure in $[0, \bar{\epsilon}) \times \prod_{j=1}^{L} R_j$ which implies the result.

We have

$$\hat{F}(\epsilon, p)_k = F(u, p) + \epsilon \sum_{j=1}^{L} r_j \sum_i \sum_s e^{-r_j[e_s + p_i]} p_i q_i \pi_s^k = F(u, p) + \epsilon \sum_{j=1}^{L} r_j \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right]$$

Then

$$\partial \hat{F}(\epsilon, p) = \begin{pmatrix} & \partial_p \hat{F} & \partial_\epsilon \hat{F} & A \\ & & & \end{pmatrix}$$

where $A(p)$ is an K by L matrix with elements

$$a_{kj}(r_j, p) = \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right] -$$

$$r_j \left\{ \left[\sum_i e^{-r_j p_i} p_i^2 q_i \right] \left[\sum_s e^{-r_j e_s} \pi_s^k \right] + \left[\sum_i e^{-r_j p_i} p_i q_i \right] \left[\sum_s e^{-r_j e_s} e_s \pi_s^k \right] \right\}$$

We now let p be given and show in the two lemmas below that there are $r_1(p), \ldots, r_K(p)$ such that the K by K matrix $A(r_1, \ldots, r_L, p)$ with elements $a_{kj}(r_j(p), p)$ has full rank. This implies that there are open intervals, $R_j(p)$ around r_j and an open ball, $B(p)$ around p such that for any $(p, r_1, \ldots, r_K) \in B(p) \times R_1(p) \times \cdots \times R_K(p)$ we have full rank of $A(r_1, \ldots, r_L, p)$. Now $\{B(p) : p \in [\underline{p}, \bar{e}]^K\}$ forms an open covering of $[\underline{p}, \bar{e}]^K$ and consequently there is a finite subcover $\{B(p_h)\}_{h=1}^H$. We then let $A(p, r_1, \ldots, r_L)$ be the K by $KH = L$ matrix defined on $[\underline{p}, \bar{e}]^K \times \prod_{h=1}^H \prod_{k=1}^K R_k(p_h)$ and we see that $A(p, r_1, \ldots, r_L)$ has full rank. This completes the proof ■

LEMMA. 1 *There are $r_1(p), \ldots, r_K(p)$ such that the K by K matrix $A(r_1, \ldots, r_L, p)$ with elements $a_{kj}(r_j(p), p)$ has full rank*

Proof: We start with some $r_K > 0$ and get the K'th row of the matrix:

$$A_K(r_K) = \begin{pmatrix} a_{1K}(r_K) \\ \cdot \\ \cdot \\ \cdot \\ a_{KK}(r_K) \end{pmatrix}$$

We then find $0 < r_{K-1} < r_K$ s.t. $A_{K-1}(r_{K-1})$ is not parallel to $A_K(r_K)$. In the n'th step we find $0 < r_{K-n} < r_{K-n+1}$ s.t. $A_{K-n}(r_{K-n})$ is not in the subspace spanned by $[A_K(r_K), \cdots, A_{K-n+1}(r_{K-n+1})]$. For this procedure we need to show the following: Let $\bar{r} > 0$. Then for all sets of K-vectors $[V_1, \ldots, V_{K-1}]$ there is $0 < r < \bar{r}$ s.t. $A_k(r) \notin span[V_1, \ldots, V_{K-1}]$. Letting N be a normal to $span[V_1, \ldots, V_K]$ it is enough to show that for all $N \in \Re^K$ there is $0 < r < \bar{r}$ s.t. $A_k(r) \cdot N \neq 0$.

If $A_k(0) \cdot N \neq 0$ then, by continuity, this also holds for some $0 < r < \bar{r}$. Consider then the case where $A_k(0) \cdot N = 0$. If for all $r \in (0, \bar{r})$, $A_k(r) \cdot N = 0$ then, in particular, $\partial^l A_k(0) \cdot N = 0$ for $l = 1, 2, ...$ So the result is proven by showing that the matrix

$$\left[\partial^{l_1} A_k(0) \middle| \partial^{l_2} A_k(0) \middle| \cdots \middle| \partial^{l_K} A_k(0)\right]$$

- a K by K matrix has full rank for some $l_1, \ldots, l_K$. The derivatives in this matrix are K-vectors with elements $(\sum_{i=1}^l a^{li} \sum_s e_s^i \pi_s^k$. Because of this form of the derivatives and the lemma below, we need only prove the folowing: There are $l_1, \ldots, l_K$ such that

$$\begin{pmatrix} \sum_s e_s^{l_1} \pi_s^1 & \cdot & \cdot & \cdot & \cdot \sum_s e_s^{l_K} \pi_s^1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sum_s e_s^{l_1} \pi_s^K & \cdot & \cdot & \cdot & \cdot \sum_s e_s^{l_K} \pi_s^K \end{pmatrix}$$

has full rank. Suppose that

$$\begin{pmatrix} \sum_s e_s^{l_1}\pi_s^1 & \cdot & \cdot & \cdot & \cdot\sum_s e_s^{l_k}\pi_s^1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sum_s e_s^{l_1}\pi_s^K & \cdot & \cdot & \cdot & \cdot\sum_s e_s^{l_k}\pi_s^K \end{pmatrix}$$

has maximal rank and let $N \in \Re^K$ be a normal to the space spanned by the k columns in this matrix. By the assumption that Π has rank K we have $N \cdot \Pi \neq 0$. Consider

$$N \cdot \begin{bmatrix} \sum_s e_s^l \pi_s^1 \\ \cdot \\ \cdot \\ \cdot\ \sum_s e_s^l \pi_s^K \end{bmatrix} = \sum_{k=1}^{K} N_k \sum_s e_s^l \pi_s^k = \sum_s e_s^l \sum_k N_k \pi_s^k.$$

As noted, we have $\sum_k N_k \pi_s^k \neq 0$. To show that there is l s.t. $\sum_{k=1}^K N_k \sum_s e_s^l \pi_s^k$ we only need to show that for $a \in \Re^S \setminus 0$ there is l such that $\sum_s e_s^l a_s \neq 0$. Let $\bar{s} = max\{s : a_s \neq 0\}$.

$$\sum_s e_s^l a_s = \sum_{s=1}^{\bar{s}} e_s^l a_s = e_{\bar{s}}^l \left[\sum_{s=1}^{\bar{s}-1} \left(\frac{e_s}{e_{\bar{s}}} \right)^l a_s + a_{\bar{s}} \right]$$

and as $l \to \infty$ $\sum_{s=1}^{\bar{s}-1} \left(\frac{e_s}{e_{\bar{s}}}\right)^l a_s \to 0$ and $e_{\bar{s}}^l a_{\bar{s}} \to sign(a_{\bar{s}})\infty$, giving us the result ■

LEMMA. 2 *Let $X_i \in \Re^K, i = 1, 2, \ldots$ Let $L_1 = 1$ and for $j = 2, 3, .., K$ define L_j inductively s.t. L_j is the smallest number s.t. $[X_{L_1}, X_{L_2}, \ldots, X_{L_j}]$ has rank j assuming that such L_j exists. Let $V_i = a_{i1}X_i + a_{i2}X_2 + \cdots + a_{ii}X_i$, with $a_{ij} \neq 0, \forall j$. Then $[V_{L_1}, V_{L_2}, \ldots, V_{L_K}]$ has maximal rank (K).*

Proof: For every j V_{L_j} is linearly independent of $[V_{L_1}, \ldots, V_{L_{j-1}}]$. Suppose not. There would be $\alpha_1, \ldots, \alpha_j$ s.t. $\sum_{i=1}^j \alpha_i V_{L_i} = 0$. But then

$$\sum_{h=1}^{L_j-1} \sum_{i=1}^{j} \alpha_i a^{L_i,h} X_h = \alpha_j a^{L_j L_j} X_{L_j}$$

in contradiction with that X_{L_j} is linearly independent of $[X_1, \ldots, X_{L_j-1}]$ ■

Genericity of Price Volatility

Let $A = \{p \in \Re^K_{++} : p_1 = p_2 = \cdots = p_K\}$ and $B = \{u \in \mathcal{V} : F(u,p) = 0 \Rightarrow p \notin A\}$. We show that B is an open set. So let $u \in B$ and $u^n \to u$. Suppose there were for each n a $p^n \in A \cap [0, \bar{e}]^K$ s.t. $F(u^n, p^n) = 0$. Since $A \cap [0, \bar{e}]^K$ is compact p^n has a cluster point, $\bar{p} \in A$, i.e., since F is continuous, $F(u, \bar{p}) = 0$, a contradiction.

Denseness of price volatility

PROPOSITION 8 *Suppose the set $\{u \in \mathcal{V} : u$ is regular$\}$ is dense. Then B is dense.*

Proof: It is sufficient to show that for every $u \in \{u \in \mathcal{V} : u \text{ is regular}\}$ there is a sequence $u^n \to u$ s.t. $u^n \in B, \forall n$. Such a u has finitely many equilibria, say $p^1, ..., p^M$. Consider the following parametrization:

$$u_\epsilon = u - \epsilon \left[k_1 e^{-r_1 C_2} + k_2 e^{-r_2 C_2} \right]$$

wherek_1, k_2, r_1, and r_2 are to be chosen.

$$F(u_\epsilon, p) = F(u, p) + \epsilon \left\{ k_1 r_1 \sum_i \sum_s e^{-r_1(e_s + p_i)} p_i q_i \pi_s^k + k_2 r_2 \sum_i \sum_s e^{-r_2(e_s+p_i)} p_i q_i \pi_s^k \right\}$$

We have by the implicit function theorem that locally the equilibrium price p^h is a function of ϵ with derivative

$$- \left[\partial_p F(u_0, p^h) \right]^{-1} \partial_\epsilon F(u_0, p^h).$$

$$\partial_\epsilon F(u_0, p^h) = \left\{ k_1 r_1 \sum_i \sum_s e^{-r_1(e_s + p_i^h)} p_i^h q_i \pi_s^h + k_1 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} p_i^h q_i \pi_s^h \right\}_{k=1}^K =$$

$$\left\{ k_1 r_1 \sum_i e^{-r_1 p_i^h} p_i^h q_i \sum_s e^{-r_1 e_s} \pi_s^h + k_2 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} \pi_s^h \right\}_{k=1}^K$$

We can find r_1 and r_2 s.t. $\{\sum_s e^{-r_1 e_s} \pi_s^h\}_{k=1}^K$ and $\{\sum_s e^{-r_2 e_s} \pi_s^h\}_{k=1}^K$ are linearly independent:

$$\partial_r \{ \sum_s e^{-r_1 e_s} \pi_s^h \}_{k=1}^K \neq \lambda \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \equiv \mathbf{1}, \forall \lambda \in \Re$$

Note that we have $\partial_p F(u_\epsilon, p^h)_{|\epsilon=0} = \partial_p F(u, p^h)$ the last having full rank. So there is a unique X^h s.t

$$-[\partial_p F(u, p^h)]^{-1} X^h = \mathbf{1}$$

We can then pick k_1 and k_2 such that

$$\left\{ k_1 r_1 \sum_i e^{-r_1 p_i^h} p_i^h q_i \sum_s e^{-r_1 e_s} \pi_s^h + k_2 r_2 \sum_i e^{-r_2 p_i^h} p_i^h q_i \sum_s e^{-r_2 e_s} \pi_s^h \right\}_{k=1}^K \neq \lambda X^h, \forall h = 1, 2, ..H, \forall \lambda \in \Re$$

. In other words we have $\frac{\partial p^h}{\partial \epsilon}_{|\epsilon=0} \neq \lambda \ \mathbf{1}, \forall h$, so that for ϵ close to 0 we have the equilibrium prices for u_ϵ, $p_\epsilon^h \neq \lambda \ \mathbf{1}, \forall h$ ■

Pareto Optimality

We first confirm that a Monetary Equilibrium is conditionaly Pareto Optimal in the two-period version of the more general model. For this model an allocation, characterized by the transfers/prices $p_1, \ldots, p_K$ is Pareto Optimal if there are $W_1, \ldots, W_K > 0$ s.t.

$$W_k \sum_s \sum_i -\frac{\partial u}{\partial C_1}(e - p_k, e_s + p_i) q_i \pi_s^k + \sum_j W_j \sum_s \frac{\partial u}{\partial C_2}(e - p_j, e_s + p_p) q_k \pi_s^j = 0, \forall k \qquad (17)$$

Let

$$d_k = \sum_s \sum_i \frac{\partial u}{\partial C_1}(e - p_k, e_s + p_i) q_i \pi_s^k$$

and

$$c_k j = \sum_s \frac{\partial u}{\partial C_2}(e - p_k, e_s + p_j) q_j \pi_s^k$$

Then the equilibrium condition can be written $H(u,p) \cdot p = 0$ where

$$H(u,p) = \begin{pmatrix} -d_1 + c_{11} & c_{12} & c_{13} & \cdot & \cdot & \cdot & c_{1K} \\ c_{21} & -d_2 + c_{22} & c_{23} & \cdot & \cdot & \cdot & c_{2K} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ c_{K1} & \cdot & \cdot & \cdot & \cdot & \cdot & -d_K + c_{KK} \end{pmatrix}$$

while (17) can be written $(W_1, \ldots, W_K) \cdot H(p) = 0$. The argument is now the same as in Peled(1984). His Theorem 1 states that if H is an K by K matrix with positive off-diagonal elements and if there is $p \gg 0$ s.t. $H \cdot p = 0$ then there is $W \gg 0$ s.t. $W \cdot H(p) = 0$ ■

We next show that generically the Monetary Equilibrium in the model studied (the 3-period version) is not Pareto Optimal. More precisely we show the following. Let $Q = (q_1, q_2, \ldots, q_K)$

PROPOSITION 9 *Assume that generically u is regular. For generic u we have that whenever $H(u,p) \cdot p = 0$, $Q \cdot H(u,p) \neq 0$*

Proof:

Let $A = \{u \in \mathcal{V} : F(u,p) = 0 \Rightarrow Q \cdot H(u,p) \neq 0\}$. Suppose A were not open, i.e. there were a u in A and a sequence $u^n \to u$ and p^n s.t. $F(u^n, p^n) = 0 = QH(u^n, p^n)$. As usual p^n has a cluster point $\bar{p}$ giving the contradicting consequence, $f(u, \bar{p}) = 0 = QH(u, \bar{p})$ Before proving denseness we prove the following lemma:

LEMMA. 3 *Suppose that p is a particular Monetary Equilibrium for the regular economy u s.t.*

(i) $p_i \neq p_j$ for some i, j

(ii) $Q \cdot H(u,p) = 0$

There is then for every open ball, B around u an $u' \in B$ which is regular s.t. p is still an equilibrium price for u' but $Q \cdot H(u', p) \neq 0$.

Proof: Let $\underline{p} = min\{p_1, \ldots, p_K\} < \bar{p} = max\{p_1, \ldots, p_K\}$. Pick a C^2 function g defined on $\Re^2_{++}$ s.t.

(a) $\frac{\partial g}{\partial C_1}(e - \underline{p}, e_S + \bar{p})\underline{p} = -\frac{\partial g}{\partial C_2}(e - \underline{p}, e_S + \bar{p})\bar{p} \neq 0$

(b) $\frac{\partial g}{\partial C_1}(e-p_k, e_S+p_j) = \frac{\partial g}{\partial C_2}(e-p_k, e_S+p_j) = 0$ for all k, j, s s.t. $p_k \notin \{\underline{p}, \bar{p}\}$ or $p_j \notin \{\underline{p}, \bar{p}\}$ or $s = S$. Note that from (a) it follows that

$$\sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_1}(e-p_k, e_S+\bar{p})\underline{p}q_j\pi_s^k = -\sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_2}(e-p_k, e_S+\bar{p})\bar{p}q_j\pi_s^k, \forall k \text{ s.t. } p_k = \underline{p}$$

Let $u_\epsilon = u + \epsilon g$. There is $\delta > 0$ s.t. for $\epsilon \in [-\delta, \delta]$ we have that u_ϵ fulfils the maintained assumptions about preferences. We also have for all ϵ that $H(u_\epsilon, p) \cdot p = 0$ i.e. that p is still an equilibrium for the perturbed economy. $H(u_\epsilon, p) = H(u, p) + \tilde{H}$ where every row $\tilde{H}^k = 0$ if $p_k \neq \underline{p}$. If $p_k = \underline{p}$ then $\tilde{H}_{kk} = \epsilon \sum_{\{j:p_j=\bar{p}\}} \frac{\partial g}{\partial C_1}(e-p_k, e_S+\bar{p})q_j\pi_s^k$ and $\tilde{H}_{kj} = \frac{\partial g}{\partial C_2}(e-p_k, e_S+\bar{p})q_j\pi_s^k$ for all j s.t. $p_j = \bar{p}$, else $\tilde{H}_{kj} = 0$. It follows that $\tilde{H}_k \cdot p = 0, \forall k$ i.e. that $H(u_\epsilon, p) \cdot p = 0$. Let j be such that $p_j = \bar{p}$. The column $H(u_\epsilon, p)_{\cdot,j}$ has the form $H(u, p)_{\cdot,j} + \tilde{H}_{\cdot,j}$, where $\tilde{H}_{kj} = \frac{\partial g}{\partial C_2}(e-p_k, e_S+\bar{p})q_j\pi_s^k$ whenever $p_k = \underline{p}$, 0 else. It follows that $Q \cdot \tilde{H}_{\cdot,j} \neq 0$, while $Q \cdot H(u, p)_{\cdot,j} = 0$ so that $Q \cdot H(u_\epsilon, p)_{\cdot,j} \neq 0$ ■

We now finish the proof of the Proposition. We have to show that for all u and all open balls, B containing u, $B \cap A \neq \emptyset$. Since the set of u which are regular and have the feature that for all equilibrium prices p, $p_j \neq p_i$ for some i, j is open and dense, we can assume that u has these features. By regularity there are then only finitely many Monetary Equilibria for u, say $p^1, \ldots, p^N$. Let B be an open ball around u so small that the equilibrium prices can be parametrized by N continuous functions, $g_n : B \to (0, \bar{e}]^K, n = 1, \ldots, N$.

Start with p^1. If $Q \cdot H(u, p^1) = 0$ pick according to the lemma u^1 in B s.t $H(u^1, p^1) \cdot p = 0$ but $Q \cdot H(u^1, p^1) \neq 0$. Else let $u^1 = u$. In the j'th step we have u^{j-1} with equilibria $(p^{j-1,1}, \ldots, p^{j-1,N})$ s.t. $Q \cdot H(u^{j-1}, p^{j-1,i}) \neq 0$ for $i \leq j-1$. These inequalities will continue to hold in an open neighborhood, $B^{j-1} \subset B$ of u^{j-1}. Consider $p^{j-1,j}$. If $Q \cdot H(u^{j-1}, p^{j-1,j}) = 0$ pick according to the lemma a u^j in B^{j-1} s.t. $H(u^j, p^{j-1,j}) \cdot p = 0$ but $Q \cdot H(u^j, p^{j-1,j}) \neq 0$. Else let $u^j = u^{j-1}$. u^N then has the desired properties ■

For the more general version of the model is still straigt forward to show that Proposition 3 still holds.

4.2 Brief Introduction to Rational Beliefs

The generic set of variables is denoted $\mathcal{H}$, a subset of $\Re^L$. Depending on the context it can be a set of observable or unobservable variables. For any set Y we will denote by $\mathcal{B}(Y)$ the Borel algebra for Y. Let $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ be the shift transformation i.e. $T(H_1, H_2, ...) = T(H_2, H_3...)$. Let μ be a probability measure on $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty))$ so that $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty), \mu, T)$ is a dynamical system. Finally, let $\mathcal{C}(\mathcal{H}^\infty)$ be the cylinders. The following definitions are taken from Kurz[1994a] :

DEFINITION 2 *Stability: The dynamical system* $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty), \mu, T)$ *as well as the measure* μ *are said to be stable if for all cylinders* $C \in \mathcal{C}(\mathcal{H}^\infty)$*:*

$$lim_{J\to\infty}\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j(h))$$

exists for μ*-a.a.* h

For the case we are studying when the system is stable there is an associated stationary measure, $\bar{\mu}$ s.t. $\bar{\mu}(C)$ is the limit of the sequence in the above definition. This $\bar{\mu}$ is the empirical distribution of the stochastic process and is assumed to be known by all agents.

To know that the true but unknown dynamical system $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty), \mu, T)$ generates $\bar{\mu}$ is not the same as knowing μ . There are many possible stable dynamical systems which will generate the same stationary measure.

DEFINITION 3 *A probability measure* ρ *on* $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty))$ *is said to be a Weakly Rational Belief for the stable dynamical system* $(\mathcal{H}^\infty, \mathcal{B}(\mathcal{H}^\infty), \mu, T)$ *if* $\bar{\rho} = \bar{\mu}$*.*

Thus a belief ρ is rational, if it generates the same empirical distribution as the one being observed. In this paper we use rational beliefs which are generated by a random signal/sunspot, z and two one-period beliefs, $B_i, i = 1, 2$. We assume that the empirical distribution is i.i.d. with one-period distribution $\bar{B}$. Thus we can phrase the rationality conditions in terms of one-period beliefs only as in (3).

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