Unraveling of Dynamic Sorting

ETTORE DAMIANO
University of Toronto

LI, HAO
University of Toronto

WING SUEN
The University of Hong Kong

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ABSTRACT: We consider a two-sided, finite-horizon search and matching model with heterogeneous types and complementarity between types. The quality of the pool of potential partners deteriorates as agents who have found mutually agreeable matches exit the market. When search is costless and all agents participate in each matching round, the market performs a sorting function in that high types of agents have multiple chances to match with their peers. However, this sorting function is lost if agents incur an arbitrarily small cost in order to participate in each round. With a sufficiently rich type space, the market unravels as almost all agents rush to participate in the first round and match and exit with anyone they meet.

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CORRESPONDING AUTHOR: Ettore Damiano. Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada M5S 3G7. Phone: (416) 946-5821. Fax: (416) 978-6713. E-mail: ettore.damiano@utoronto.ca.

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1. Introduction

Many entry-level markets for professionals (e.g., academic economists, hospital interns, and federal law clerks) are organized around annual recruitment cycles. Some markets use centralized matching procedures, such as the celebrated Gale-Shapley deferred acceptance algorithm used to match interns to hospitals in both the U.K. and North America (e.g., Roth, 1984; Roth and Xing, 1994). In these markets, participants use interviews to gather information about each other before submitting their preferences to a central clearinghouse, which makes all matches according to a pre-specified algorithm. In contrast, matches are typically formed sequentially in a decentralized market. For example, in the North American market for academic economists, information about candidates and academic positions is gathered from applications, interviews and campus visits, and matches are made in sequence throughout the recruitment cycle. Matching opportunities change over time as participants exit the market after successful searches. The non-stationarity of the search process and its implications for search and matching efficiency have received some recent attention from economists interested in comparing centralized and decentralized match-making. For example, in their study of the market for clinical psychologists, Roth and Xing (1997) describe how market participants sometimes choose to match with less desirable partners lest the pool of acceptable matching partners dries up quickly. Since market participants cannot consider more than a few choices simultaneously, the frenzy in the early stages of the market results in reductions in market scope and sorting efficiency.[1] In a similar vein, Niederle and Roth (2003) use data from the entry-level market for American gastroenterologists to show that after the market was decentralized, gastroenterologists are more likely to be employed at the same hospital in which they were residents.

The relationship between search and evolving matching opportunities introduces interesting considerations in search dynamics and sorting efficiency. These considerations have not been adequately analyzed in the existing theoretical literature, which focuses on

[1] A different type of sorting inefficiency involves mismatches because information about quality of applicants and about positions is not yet available when participants sign early contracts (Li and Rosen, 1998; Li and Suen, 2000; Suen, 2000). Early contracting occurs in this type of models because it provides insurance benefits to risk-averse participants.

steady state analysis (Burdett and Coles, 1997; Shimer and Smith, 2000a). One exception is Jackson and Palfrey (1998), who study how bargaining procedures affect the search outcome in a model where heterogeneous buyers and sellers meet randomly and leave the market after successful trade. We are instead interested in the search process in the context of matching markets, where participants are heterogeneous and matching efficiency arises from complementarity.[2] In our stylized model, a job market operates in two rounds. Applicants differ in a one-dimensional, continuous quality, called "type," and so do firms. We assume the match value function exhibits complementarity between worker type and firm type, so that in a frictionless matching environment, the perfect sorting that matches the highest quality worker to the highest quality firm and so on, maximizes the total match output. In our matching market, search frictions exist and meetings are random. In the first round market, participants decide whether or not to form a match upon meeting each other. If they do, they get their match payoffs and withdraw from the market. Otherwise, they proceed to the second round, where all remaining agents again meet randomly. Since this is the last round, they match with whomever they meet. We investigate whether there will be excessive search and matching in the first round.

In our benchmark model, there is no participation cost, and all agents participate in the first round market. Equilibrium involves a uniform threshold such that an applicant accepts an offer from a firm if the latter's type exceeds the threshold, and waits for the second round otherwise. If all applicants and firms follow this strategy, types lower than the threshold will not find a match and will participate in the second round market. The presence of these low types in the first round market imposes a negative search externality on the higher types, so that some of the latter will not be lucky enough to find an acceptable match and will also participate in the second round market. In equilibrium the negative search externality is such that the expected type in the second round market equals the acceptance threshold. In this equilibrium, the job market performs a "dynamic sorting" function by giving higher types a better chance to match with their peers and realize their

[2] Smith (1995) first studies an infinite-horizon matching model with no entry, where non-steady state dynamics is driven by temporary matches that are formed because finding acceptable partners takes time and waiting is costly in terms of foregone production. Shimer and Smith (2000b) examine the possibility that efficient search and matching requires non-stationarity.

higher match values. It turns out that the equilibrium level of search externality is optimal in the sense that the total match value cannot be increased by changing the threshold.

The dynamic sorting function of the job market is robust to alternative modeling assumptions, including asymmetric type distributions and match value functions, more than two matching rounds, and discrete types. However, this dynamic sorting unravels if applicants and firms have to incur a small cost in order to participate in each round of the market. This is because the negative search externality is crucial to dynamic sorting, and it is destroyed by the participation cost. To begin, agents of type lower than the acceptance threshold have no reason to pay the cost to be in the first round market since they face zero probability of forming a match. As these types withdraw, the quality of the first round pool improves, so higher types now have greater chances of meeting their peers and they exit the market in greater numbers. As a result, the quality of the second round pool worsens. But this division into a high quality market in the first round and a low quality market in the second round cannot be an equilibrium. The best types in the low quality second round pool would be acceptable in the first round, and would therefore have incentives to join the first round. As more of the best types from the second round market join the first round market, the pool in the second round worsens further, which lowers the acceptance threshold in the first round still further. When the participation cost is arbitrarily small, the market loses its sorting function as almost all agents rush to participate in the first round and match with just about anyone they meet. The second round market collapses. Needless to say, such unraveling outcome is very inefficient, even though the participation cost is arbitrarily small.

Our result that almost no sorting can be achieved with an arbitrarily small participation cost depends on the assumption that types are continuously distributed. When the type space is discrete, with costless, sequential participation, a different kind of sorting emerges as equilibrium which does not rely on the negative search externality responsible for the sorting equilibrium in the continuous type case. For example, when there are two types on both sides and two rounds of search, it is an equilibrium that high type agents participate in the first round and accept only high type agents while low type agents wait and participate in the second round. Since the two types are segregated, sorting does not rely on the search externality, and our previous unraveling argument does not apply. In fact, for small participation costs, there is a mixed strategy equilibrium (in participation and acceptance decisions) whose outcome is close to the perfect sorting outcome. More generally, when there are at least as many rounds of matching as the number of types, almost perfect sorting is an equilibrium outcome for small participation costs. However, if there are more types than the number of matching rounds, sorting inefficiency becomes significant. For any fixed number of matching rounds, as the type space becomes richer, the types that can be almost perfectly sorted are increasingly concentrated at the bottom of the type distributions. Our unraveling result obtains again in the sense that almost all types randomly match and exit in the first round with no sorting.

2. A Non-stationary Matching Model

To analyze how the search process interacts with matching opportunities over time, we consider a finite-horizon, two-sided matching market where there is no infusion of new agents in the relevant horizon. Matching can occur in any of the several matching rounds, but agents leave the market once they form a match. The distribution of agents changes endogenously over time. Agents decide whether to search and whether to form a match based on their expectations about future matching opportunities. The flavor of our main results can be conveniently conveyed in a model with two rounds. The extension to multiple matching rounds will be discussed later.

Agents on each side of the market differ in a one-dimensional productive characteristic, called "type." Types of agents on the two sides of the market are distributed continuously and symmetrically on the support $[a, b] \subset (0, \infty)$, with density function f and distribution function F. Our results will be extended to asymmetric type distributions later. Throughout the paper, the two sides of the market are assumed to have the same size. Continuous type space is a simple representation of matching environments where the number of interview rounds is limited relative to the number of types, because the needs and matching characteristics of participants are diverse and activities such as application, interviewing, and decision-making take time. In other markets, relevant information about qualities of participants may not be so refined due to difficulties in observing match characteristics or idiosyncrasies in evaluating potential matches. These markets are better represented by a model with a discrete type space, and the implications will be addressed later.

We assume complementarity between agents' types. In particular, match value to a type x agent, if matched with a type y agent on the other side of the market, is xy. In our symmetric model, complementarity implies that the total match value is maximized by the "perfect sorting," where each type x agent is paired with a type $y = x$ agent on the other side of the market. All our results extend to the class of more general match value functions that are multiplicatively or additively separable, and monotone in types. This class includes, for example, the match value function used by Burdett and Coles (1997). However, since additively separable match value functions do not exhibit complementarity between types, how types are matched does not affect the total match value and therefore sorting efficiency is not an issue.[3] Given our focus on the sorting efficiency in a non-stationary environment, we need a match value function that exhibits complementarity and choose xy for simplicity.

We adopt a simple search technology in our model: if the type distribution function is G, then the probability that any type x agent meets an agent of type y or lower from the other side of the market is $G(y)$. Later on we modify this random meeting technology to accommodate different distributions of types and masses of participants on the two sides of the market. If the market operates for only one round, all types are randomly matched and there is no sorting. We refer to this outcome as the "random matching," which represents the opposite extreme of the perfect sorting in our model in terms of total match value. Our objective is to investigate whether better sorting can be achieved by multiple search rounds in a non-stationary environment. While more realistic representations of search frictions have been considered in the literature (Montgomery, 1991; Lagos, 2000; Shimer, 2001), we choose the simple random meeting technology because it makes the evolution

[3] In our model all agents are matched with probability 1 and there is no discounting. Without these assumptions, the total match value may depend on the search and matching decisions even when the match value function is additively separable. This type of inefficiency is outside the focus of the present paper.

of the distribution of types analytically more tractable.[4] Moreover, the random meeting technology does not exhibit any scale effect, and this allows us to focus on efficiency gains that arise solely from better sorting.

A few additional assumptions are in order. First, agents who fail to find a match at the end of all matching rounds suffer a large cost, which we normalize by assuming an outside option value of 0 for all types. Since all matches have strictly positive values, every agent prefers any match to the outside option. Later we extend our results to situations where some agents face binding outside options. Next, we assume that agents are risk-neutral, and do not discount. Adding a discount factor does not change our conclusions qualitatively. Further, it is reasonable to assume no-discounting in a setup where production takes place only after the conclusion of the job market regardless of when matches are formed. Finally, we assume that there are no side payments.[5] This assumption is appropriate in matching markets where wage bargaining plays a minor role in match formation (e.g., dating and marriage, tenure track academic positions, and federal law clerks).

The remainder of the paper is organized as follows. In Section 3 we consider the case of no participation cost, and demonstrate gains in sorting efficiency achieved by dynamic sorting in multiple search rounds. Section 4 shows that an arbitrary small participation cost causes unraveling of dynamic sorting and reduces it to the random matching. In Section 5 we investigate how the unraveling result depends on the richness of the type space. Section 6 concludes the paper with a brief summary and some final remarks.

3. Full Participation and Dynamic Sorting

Since an unmatched agent gets a payoff of 0, agents accept anyone they meet in the second (and last) round of the market. Anticipating this, an agent of type x agrees to match with

[4] In the search and matching literature, random meeting technology is sometimes referred to as "linear," as opposed to "quadratic" (e.g. Smith, 1995). With a quadratic search technology, the matching payoff of any agent is unaffected by the matching decision of agents with whom he is not willing to match. This rules out negative search externality that is crucial for our results.

[5] It is straightforward to define full participation equilibria with side payments under some bargaining rule, say Nash bargaining. With specific assumptions on the type distribution (uniform) and the match value function (symmetric power functions), we are able to show that if complementarity between types is strong enough, costless search leads to dynamic sorting and unraveling occurs with costly participation. Whether these results are general is subject of future research.

y in the first round if and only if $xy \geq xm$, where m is the symmetric expected type in the second round. This implies a uniform acceptance threshold m for all types of agents. Note that types lower than m are rejected by all types in the first round. Nevertheless, since participation is free, these low types have no reason to skip the first round search. Indeed, any robustness criterion that allows for a small chance that agents make mistakes in acceptance decisions would ensure full participation in the first round market.

Given any first round acceptance threshold k, the expected type m in the second round market is determined by the distribution of types that remain unmatched after the first round. Since two agents match and leave the market only when each agent's type is greater than k, the relative size of the second round market is $R(k) = 1 - (1 - F(k))^2$. Then, m is determined by k according to:

$$m(k) = \int_a^b x \, \mathrm{d}G(x; k),$$

where $G(x; k)$ is the distribution of types in the second round, given by

$$G(x; k)R(k) = \begin{cases} F(x), & \text{if } x \leq k; \\ F(k) + (F(x) - F(k))F(k), & \text{if } x > k. \end{cases}$$

Since $R(a) = 0$, the above does not define $m(a)$. Let us define $m(a)$ by continuity:

$$m(a) = \lim_{k \to a} m(k). \tag{3.1}$$

We can verify that $G(x; k)$ stochastically dominates $G(x; k')$ if $k > k'$. It follows that $m'(k) > 0$ for any $k \in (a, b)$.

Definition 3.1. *A threshold type k^e is a full participation equilibrium if $k^e = m(k^e)$.*

An equilibrium in our model occurs when the expected type $m(k)$ that results from an acceptance threshold k precisely justifies k. The above reference to full participation is to distinguish the equilibrium defined here from later definitions of equilibrium when search is costly and participation is endogenous. Our first result characterizes the existence and uniqueness of an equilibrium with an interior threshold k^e.

Proposition 3.2. *(i) A full participation equilibrium $k^e \in (a, b)$ exists; and (ii) it is unique if the type distribution $F(x)$ is log-concave.*

PROOF. (i) By definition, we have:

$$m(b) = \int_a^b x \, \mathrm{d}F(x) < b;$$

$$m(a) = \lim_{k \to a} \left(\int_a^k \frac{xf(x)}{R(k)} \, \mathrm{d}x + \int_k^b \frac{xf(x)}{2 - F(k)} \, \mathrm{d}x \right) = \frac{1}{2}a + \frac{1}{2} \int_a^b xf(x) \, \mathrm{d}x > a.$$

Since $m(k)$ is a continuous function, by the Intermediate Value Theorem, an equilibrium $k^e \in (a, b)$ exists.

(ii) Write $m(k)$ as:

$$m(k) = w(k)q(k) + (1 - w(k))Q(k), \tag{3.2}$$

where $q(k) = \mathrm{E}[x \mid x < k]$, $Q(k) = \mathrm{E}[x \mid x \geq k]$, and $w(k) = F(k)/R(k)$. Take derivative of equation (3.2), we get

$$m'(k) = w(k)q'(k) + (1 - w(k))Q'(k) + w'(k)(q(k) - Q(k)).$$

If F is log-concave, then $q'(k) < 1$ and $Q'(k) < 1$ (An, 1998). Further, $q(k) < Q(k)$ and $w'(k) > 0$. Thus, $m'(k) < 1$, implying a unique equilibrium. *Q.E.D.*

It is evident from the above proof that the existence of an equilibrium with an interior threshold k^e does not depend on the definition of $m(a)$. On the other hand, our definition of $m(a)$ (equation 3.1) rules out $k = a$ as an equilibrium. Letting $m(a) = a$ makes $k = a$ an equilibrium according to Definition 3.1, but it would not be robust. For example, if agents who are indifferent between accepting their match and waiting for the second round "tremble" with an arbitrarily small probability and reject their match, the second round mean would be strictly greater than a, making it non-optimal to accept type a. The uniqueness of equilibrium depends on a characterization of the slope of $m(k)$. Since $m'(k) > 0$, in general expectations about the prospects in the second round market can be self-fulfilling and multiple equilibria may occur.[6] Proposition 3.2 uses a log-concavity

[6] The issue of multiple equilibria is certainly interesting, but is orthogonal to the purpose of the present paper. Li and Suen (2004) deal with the issue of multiple equilibria in an early contracting model based on the trade-off between insurance benefits and sorting inefficiency.

condition on the type distribution to rule out multiple equilibria.[7]

In an equilibrium with an interior first round threshold k^e, the market performs a sorting function by giving types higher than k^e a better chance to match with their peers and realize their higher match values. How large is the efficiency gain from dynamic sorting relative to the random matching? For a numerical example, consider the uniform type distribution F on $[1, 2]$, which is log-concave. The unique equilibrium is given by $k^e = 1.38$, with a total match value of $V^* = 2.272$, compared to a total match value of $V^0 = 2.25$ under the random matching. The percentage gain from dynamic sorting seems small, less than 1%, but it would be significantly greater if either the support of the types is wider ($[1, 10]$ instead of $[1, 2]$), or the match value function exhibits stronger complementarity (x^2y^2 instead of xy). To isolate the sorting gains from any effect that may arise from rescaling the types, we need a more accurate measure. In our present example with uniform type distribution on $[1, 2]$ and match value function xy, the total match value from the perfect sorting is only $V^\infty = 2.333$. This suggests that we measure the efficiency gain by $(V^* - V^0)/(V^\infty - V^0)$, which implies a relative gain of 26.5% from dynamic sorting.

Dynamic sorting through selective first round acceptance is imperfect due to the kind of search frictions we have imposed. An interesting question is whether it can be improved without changing the search technology. We ask: are agents in the market too selective, or do they rush to match in the first round? To answer this question, consider the problem of choosing a threshold type k to maximize the total match value

$$V(k) = (1 - R(k))Q^2(k) + R(k)m^2(k).$$

The next result shows that the sorting efficiency of dynamic sorting cannot be improved.

Proposition 3.3. *If k^* maximizes the total match value, then k^* is a full participation equilibrium.*

[7] Unlike in Burdett and Coles (1997), in our model log-concavity is not required for the existence of a non-stationary equilibrium, and is instead used to ensure uniqueness of equilibrium. In fact, log-concavity of the function $\int_a^x F(t)\, dt$ suffices to guarantee uniqueness of equilibrium. We use a stronger condition, namely log-concavity of $F(x)$, in order to simplify the proof.

PROOF. The derivative $V'(k)$ of $V(k)$ with respect to k is given by

$$2(1 - R(k))Q(k)Q'(k) + 2R(k)m(k)m'(k) - R'(k)(Q(k) - m(k))(m(k) + Q(k)). \quad (3.3)$$

Note that for all values of k, $m(k)$ and $Q(k)$ also satisfy the relationship,

$$(1 - R(k))Q(k) + R(k)m(k) = m^u,$$

where m^u is the unconditional mean of the distribution F of types. Differentiating the above identity with respect to k, we have

$$(1 - R(k))Q'(k) + R(k)m'(k) - R'(k)(Q(k) - m(k)) = 0.$$

Substituting the above into equation (3.3), with a few steps of manipulations we get

$$V'(k) = 2(Q(k) - m(k))f(k)(1 - F(k))(m(k) - k).$$

Since $V'(a) > 0$ and $V'(b) < 0$, the optimal threshold k^* is interior and satisfies $V'(k^*) = 0$. Thus, $m(k^*) = k^*$ and k^* is an equilibrium threshold. *Q.E.D.*

Equation (3.3) in the proof of Proposition 3.3 shows that raising the first round acceptance threshold k has two opposite effects on the total match value. On one hand, since $Q'(k) > 0$ and $m'(k) > 0$, an increase in the acceptance threshold from its equilibrium value improves the quality of matches realized in both the first round and the second round. This suggests that agents may not be selective enough in their choice of matching partners in the first round. On the other hand, since $R'(k) > 0$, raising the first round acceptance threshold increases the size of the second round market, which has a lower match quality. Proposition 3.3 establishes that there is an equilibrium in which these two effects exactly cancel each other, so that the total match value is maximized.

We do not intend Proposition 3.3 as a statement regarding constrained efficiency of the dynamic sorting outcome. To define constrained efficiency, one would need to be more rigorous about the restrictions on the search technology, and on the participation and acceptance decisions faced by a hypothetical social planner. It would seem reasonable to maintain the assumption of random pairwise meeting for the planner, but even with this restriction on the search technology, the planner can improve sorting efficiency by limiting participation of low types or by adopting a type-dependent acceptance rule in the first round. Jackson and Palfrey (1998) characterize constrained efficiency in a non-stationary, two-sided random matching environment with heterogeneous agents, but with a match value function arising from the buyer-seller bargaining problem.[8] While their techniques can be used to address the issue of constrained efficiency in our model, we will not pursue this line because our focus is on comparing sorting efficiency under costless and costly search. Before introducing costly search and endogenous participation, in the remainder of this section we provide separate extensions of our dynamic sorting result to asymmetric type distributions and multiple matching rounds. These extensions further illustrate the intuition of how dynamic sorting improves upon the random matching, but they are not critical for understanding our unraveling arguments in Sections 4 and 5.

Asymmetric type distributions. Now we relax our strong symmetry assumptions that the two sides of the market have the same type distribution and that the match value function takes the symmetric product form. Suppose the match value function is xy but the two sides, X and Y, have different type distributions, F_X and F_Y, on $[a_X, b_X]$ and $[a_Y, b_Y]$ respectively. Note that there is no loss of generality in assuming the match value function xy, as any multiplicatively separable match value function (with constant-sign cross derivatives) can be converted into xy if we redefine the types. In this asymmetric search model, an equilibrium is given by two acceptance thresholds k_X and k_Y, such that in the first round market k_X is the marginal type of X-agents that Y-agents are willing to accept, and k_Y is the marginal type of Y-agents that X-agents are willing to accept. In equilibrium matches are formed in the first round market when types $x \geq k_X$ and $y \geq k_Y$ meet with each other, with k_X equal to the expected type m_X of X-agents in the second round and k_Y equal to the expected type m_Y of Y-agents. As m_X and m_Y are functions of both k_X and k_Y, an equilibrium corresponds to a fixed point (k_X, k_Y) in the

[8] Jackson and Palfrey (1998) focus on a two-period model. Palfrey (1997) extends their characterization of constrained efficiency to an arbitrary sequence of periods.

mapping of m_X and m_Y. With the values of m_X and m_Y at $k_X = a_X$ or $k_Y = a_Y$ defined in the same way as in (3.1), the mapping of m_X and m_Y is continuous, implying the existence of an equilibrium by Brouwer's fixed point theorem. The following proposition shows that if a pair of acceptance thresholds maximizes the total match value, then it forms an equilibrium. The proof is similar to that of Proposition 3.3 and is relegated to the Appendix.

PROPOSITION 3.4. *If (k_X^*, k_Y^*) maximizes the total match value in the asymmetric model, then (k_X^*, k_Y^*) is a full participation equilibrium.*

Multiple matching rounds. To provide a general definition of equilibrium with T matching rounds, let G_t be the symmetric type distribution in round t and k_t be the acceptance threshold type for each $t = 1, \ldots, T$. In the first round of the market, G_1 is just F, the initial type distribution.

DEFINITION 3.5. *A sequence of threshold types $k_1, k_2, \ldots, k_T = a$ and a sequence of type distributions $G_1 = F, G_2, \ldots, G_T$, are an equilibrium if (i) for any $t = 1, 2, \ldots, T-1$,*

$$G_{t+1}(x)R_{t+1}(k_t) = \begin{cases} G_t(x), & \text{if } x \leq k_t; \\ G_t(k_t) + (G_t(x) - G_t(k_t))G_t(k_t), & \text{if } x > k_t; \end{cases} \tag{3.4}$$

where $R_{t+1}(k_t) = 1 - (1 - G_t(k_t))^2$ is the relative size of the round $t+1$ market with respect to the round t market; and (ii) for any $t = 1, \ldots, T-1$,

$$k_t = G_{t+1}(k_{t+1})k_{t+1} + \int_{k_{t+1}}^{b} x \, \mathrm{d}G_{t+1}(x). \tag{3.5}$$

According to the above definition, in each round t, only types higher than k_t have a positive probability of being matched.[9] Further, the second equilibrium condition implies

[9] An induction argument can be used to establish that equilibrium matching is determined as if acceptance thresholds are uniform in each round. To see this, consider the case of $T = 3$. We already know that in round 2 there is a uniform acceptance threshold, say k_2, which equals the expected type in round 3. In round 1, the acceptance threshold for types below k_2 is k_2, as they will not match in round 2. For types above k_2, the acceptance threshold, say k_1, is higher than k_2 because they have a positive probability of matching with types above k_2 in round 2. Thus, even though there are two different acceptance thresholds in round 1, only types above k_1 have a positive probability of being matched.

that $k_t > k_{t+1}$ for each t. Any equilibrium involves a decreasing sequence of acceptance thresholds $k_1, \ldots, k_T$, so that agents in equilibrium become increasingly less picky as matching proceeds over time, and correspondingly, the sequence of type distributions is ordered by stochastic dominance: G_t first order stochastically dominates G_{t+1} for each $t = 1, \ldots, T-1$. Finally, Definition 3.5 assumes that the market does not end before the final round T. The justification for this follows the same logic as Proposition 3.2. If the market were to end in round $t < T-1$, with $k_t = a$, then for any type distribution G_t at the beginning of round t, the expected match type from waiting for another round would be given by $(1/2)a + (1/2)\int_a^b xG_t(x)\ \mathrm{d}x$, which is greater than a. Thus, agents who were accepting types marginally higher than a in round t market were not making the optimal decision, implying that the market cannot end in round t.

Proposition 3.6. *(i) An equilibrium in the T-round model exists with $k_1 < b$; and (ii) equilibrium is unique if the initial distribution of types F is log-concave.*

The proof of Proposition 3.6 is rather involved and is relegated to the Appendix. The main technical difficulty lies in the fact that the matching decisions are determined by a backward induction through equation (3.5), while the evolution of matching opportunities is determined by a forward induction through equation (3.4). We overcome this difficulty by introducing an algorithm that iterates back and forth between equations (3.4) and (3.5) and reducing the equilibrium relations to a two-round problem.

For any sequence of acceptance thresholds $k_1, \ldots, k_{T-1}$, the expected total match value is given by:

$$\sum_{t=1}^{T} \Pi_{s=1}^{t} R_s(k_{s-1}) \left(\int_{k_t}^{b} x\ \mathrm{d}G_t(x) \right)^2,$$

where $R_1(k_0) = 1$, and where the sequence of type distributions $G_1 = F, G_2, \ldots, G_T$ satisfy equation (3.4). Imagine that a sequence of acceptance thresholds $k_1, \ldots, k_T$ is chosen to maximize the expected total match value. If the optimal sequence is a decreasing sequence, then it is an equilibrium in the T-round matching model. The proof is in the Appendix.

Proposition 3.7. *There is an equilibrium sequence of thresholds, $k_1, \ldots, k_T$, that maximizes the expected total match value among all decreasing sequences.*

With more rounds of matching, dynamic sorting becomes significantly more efficient. In our previous example of uniform type distribution on $[1, 2]$, with three rounds of matching, the unique equilibrium acceptance thresholds are $k_1 = 1.48$ and $k_2 = 1.32$. The resulting total match value is $V^{**} = 2.284$. According to the efficiency measure introduced earlier, in this example a matching market with three rounds achieves the efficiency level of $(V^{**} - V^0)/(V^\infty - V^0)$, which represents 40.7% of the available efficiency gain, compared to the efficiency gain of 26.5% with two rounds of matching.

4. Endogenous Participation and Unraveling

In the model of the previous section, agents do not choose to search. They appear in the first round market even if they have no chance of forming a match. This is innocuous if there is no cost of participating in the market. But by appearing in the market without any prospect of getting matched, agents of lower types impose a negative search externality on others who intend to match. Ironically, such negative externality turns out to be necessary for the market to perform the sorting function. High type agents who happen to meet a low type agent in the first round have to try their luck again in the second round market, so the externality imposed by low type agents helps preserve the quality of the pool in the second round market. In this section, we show that the externality is destroyed by a participation cost, and as a result, matching opportunities in the second round market deteriorate, leading to a collapse of the second round market.

The intuition of the unraveling argument in this section can be readily grasped when the match value function is additively separable. For example, suppose that the match value is $x + y$ to both a type x agent and a type y agent who decide to match, and imagine that each round of search costs c to an agent. Then, if the expected type in the second round market is m, a type y agent is acceptable to any type x agent in the first round if and only if $y \geq m - c$. Types lower than $m - c$ will not participate in the first round for any positive participation cost, since they would never be accepted. As no unacceptable types participate in the first round, there is no search externality, and all participating types will find an agreeable partner and exit the market after the first round. It follows that if all types above some threshold l participate in the first round market, the average participating type in the first round market is $Q(l)$ while the expected type m in the second round market is $q(l)$. But this kind of sorting cannot work for any $l > a$: we already know that l cannot be lower than $m - c$; l cannot be equal to $m - c$ either, because otherwise it would not be true that m equals $q(l)$; if instead $l > m - c$, then types just below l would be acceptable to all types in the first round market and would strictly prefer to enter.

A more rigorous argument can be used to establish an unraveling result: with any positive participation cost, in equilibrium all types participate and are randomly matched in the first round market. However, this unraveling has no implications to matching efficiency, because any matching outcome yields the same total match value when the match value function is additively separable. In contrast, unraveling can have important effects when the match value function exhibits complementarity between types. This section establishes a similar unraveling result with the match value function xy: when c is arbitrarily close to zero, the equilibrium outcome becomes arbitrarily close to the random matching. A small participation cost thus dramatically reduces the sorting efficiency achieved by dynamic sorting. The argument is more complicated than in the case of $x + y$, because we need to prove that participation decisions are characterized by a threshold, and more importantly, acceptance decisions are no longer type-independent. Non-uniform acceptance decisions imply that some sorting is possible with a significant participation cost c. However, when c becomes arbitrarily small, acceptance decisions become almost uniform and the model behaves similarly as in the case of an additive separable match value function. An argument similar to the heuristic argument above then leads to the unraveling result.

For ease of exposition, we make two simplifying assumptions: the participation cost c is type-independent, and $c < a^2$. The second assumption ensures that even the lowest type agent will participate in the matching market at least once. Later in this section we show how both assumptions can be relaxed. Now we first consider participation decisions in the first round. The following lemma shows that the payoff gain from participating in the first round market satisfies a single-crossing property, and therefore participation decisions are characterized by a threshold.

Lemma 4.1. *There exists a threshold $l \in [a, b]$ such that types higher than l participate in the first round market, and types lower than l wait for the second round market.*

Proof. In the first round market, conditional on participation, a match between type x and type y is mutually agreeable if and only if $xy \geq xm - c$ and $xy \geq ym - c$. Consider the participation decision in the first round by an agent of type x. It is optimal for type x agent to participate in the first round market if

$$\mathrm{E}[p(x,y)xy + (1 - p(x,y))(xm - c)] - c \geq xm - c,$$

where the expectation is taken with respect to the distribution of y types that participate in the first round market, and $p(x, y)$ is the probability that agents of types x and y form a match. The above inequality can be written as:

$$\mathrm{E}[p(x,y)y + (1 - p(x,y))(m - c/x)] \geq m. \tag{4.1}$$

Any type $x' > x$ agent can follow the same acceptance strategy of type x, and can guarantee that $p(x', y) = p(x, y)$ for any y by rejecting any type y that is willing to accept type x' but not type x. Since $m - c/x$ is increasing in x, the above strategy implies that it is optimal for type x' to participate.[10] *Q.E.D.*

An agent who rejects a match in the first search round will incur the participation cost again in the second round. Since agents of higher types have relatively more to gain from finding a good match, they are more willing to incur the cost c. Unlike the model of Section 3, therefore, acceptance thresholds differ across participating types in the first round market. Fix an expected type $m \in [a, b]$ of the second round. For each type x, let

$$u(x) = m - \frac{c}{x}.$$

[10] We adopt the convention that an agent chooses participation when he is indifferent between participation and waiting. Otherwise, it is possible to construct equilibria with non-threshold participation decisions. In any such equilibrium, all first round participants are accepted with probability 1, and the expected participating type in the two rounds is the same, and hence equal to m^u. The expected total match value in any of these equilibria is the same as under the random matching.

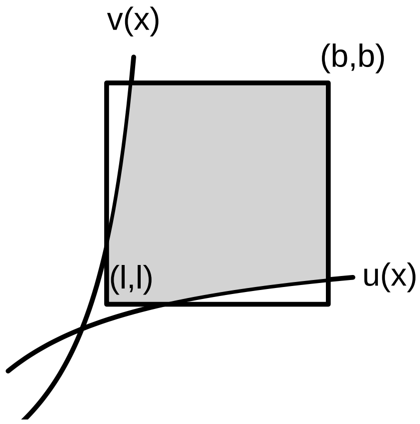

Figure 1

When $u(x)$ lies between a and b, it represents the lowest type that type x is willing to accept. Similarly, define

$$v(x) = \frac{c}{m - x}.$$

When $v(x)$ lies between a and b, it is the highest type that is willing to accept type x. Note that (i) $u(x)$ is increasing and concave, and $v(x)$ is increasing and convex; (ii) there are at most two intersections of $u(x)$ and $v(x)$; and (iii) $u(x) = x = v(x)$ at any intersection x. If the threshold for participation in the first round market is l, a match between participating types x and y is mutually acceptable if and only if

$$\min\{v(x), b\} \geq y \geq \max\{u(x), l\}. \tag{4.2}$$

Figure 1 shows the functions $u(x)$ and $v(x)$ when there is an intersection of u and v in $[a, b] \times [a, b]$. Also shown is a square box $[l, b] \times [l, b]$ which represents the pool of agents participating in the first round market, with l above the intersection. Random encounters that fall in the shaded region result in matches in the first round. The pool of agents in the second round market consists of all types below l, as well as types above l whose random encounter in the first round does not satisfy the matching rule (4.2). For fixed m and l, the second round type distribution, $G(x; m, l)$, is given by

$$G(x; m, l)R(m, l) = \begin{cases} F(x), & \text{if } x \leq l; \\ F(l) + \int_l^x (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(l)) \frac{\mathrm{d}F(x)}{1-F(l)}, & \text{if } x > l \end{cases} \tag{4.3}$$

where $R(m, l)$ is the size of the market in the second round, given by

$$R(m, l) = F(l) + \frac{1}{1 - F(l)} \int_l^b (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(l)) \, \mathrm{d}F(x).$$

The second round expected type resulting from m and l is then given by $\int_a^b x \, \mathrm{d}G(x; m, l)$. Note that the definition of $G(x; m, l)$ (equation 4.3) remains valid when u and v do not intersect in $[a, b] \times [a, b]$.

For other values of m and l, we can use (4.2) to define the resulting second round mean similarly. The only exception occurs when $l = a$ and $m \leq a + c/b$. In this case, the monotonicity of u implies that $u(x) < a$ for any $x \in [a, b]$. All types accept each other and exit in the first round, and hence $R(m, a) = 0$. As in equation (3.1), we use continuity to define the resulting second round expected type in this case as $\lim_{l \to a} \int_a^b x \, \mathrm{d}G(x; m, l)$. Since $m < l + c/b$ for any $l > a$, all participating types accept each other and exit in the first round. We have $R(m, l) = F(l)$, and

$$G(x; m, l) F(l) = \begin{cases} F(x), & \text{if } x \leq l; \\ F(l), & \text{if } x > l, \end{cases} \tag{4.4}$$

implying that for any $m \leq a + c/b$,

$$\int_a^b x \, \mathrm{d}G(x; m, a) = \lim_{l \to a} q(l) = a. \tag{4.5}$$

Definition 4.2. *An endogenous participation equilibrium is a participation threshold $l^e \in [a, b]$ and an expected type $m^e \in [a, b]$ for the second round market, such that (i) given m^e, any type $x \geq l^e$ prefers participating in the first round market and any type $x < l^e$ prefers waiting for the second round market; and (ii) $m^e = \int_a^b x \, \mathrm{d}G(x; m^e, l^e)$.*

We first construct an equilibrium that will play a prominent role in the discussions below. In such an equilibrium, $l^e = a$ and $m^e = a$, hence the second round market ceases to operate as all agents rush to form matches in the first round with anyone they happen to meet. This unraveling outcome is the same as the random matching.

Proposition 4.3. *For any participation cost c such that $0 < c < a^2$, $l^e = a$ and $m^e = a$ is an endogenous participation equilibrium.*

Proof. Condition (i) of Definition 4.2 is satisfied by $l = a$ and $m = a$. If $m = a$, then $u(b) < a$. By the monotonicity of u, we have $u(x) < a$ and $u(a) < x$ for any $x \in [a, b]$. Then, if $l = a$, type a is accepted with probability 1 and gets a payoff of m^u, which is strictly greater than $m = a$. By Lemma 4.1, all types strictly prefer participation in the first round. Condition (ii) of Definition 4.2 is satisfied, because $l^e = a$ and $m^e = a \le a + c/b$ imply that $\int_a^b x \, \mathrm{d}G(x; a, a) = a$ by (4.5). *Q.E.D.*

The construction of the unraveling outcome as an endogenous participation equilibrium relies on our definition that the second round mean is a for m and l such that $R(m, l) = 0$ (equation 4.5). Note that at $l = a$ and any $m < a + c/b$, any other definition would make the map from given m and l to the second round expected type discontinuous. This implies that as long as $c > 0$, the construction of the unraveling equilibrium in Proposition 4.3 is justified by continuity and is therefore robust to small perturbations to participation or acceptance decisions. On the other hand, at $l = a$ and $m = a + c/b$, the map from m and l to the second round expected type cannot be made continuous under any definition of the expected type, because $\lim_{l \to a} \int_a^b x \, \mathrm{d}G(x; a + c/b, l) = a$, while $\lim_{m \downarrow a + c/b} \int_a^b x \, \mathrm{d}G(x; m, a) > a$ (because for any $m > a + c/b$, but small enough so that $u(b) < v(a)$, the remaining types in the second round consist of an equal mass of types in $[a, u(b)]$ and in $[v(a), b]$.) As a result, when $c = 0$ we cannot resort to continuity to justify any definition of the second round mean when $l = a$ and $m = a$. Hence whether or not unraveling is an endogenous participation equilibrium outcome when the participation cost is 0 is entirely a matter of definition.[11] Further, as suggested in Section 3, when participation is costless, the notion of endogenous participation equilibria is not compelling.

The unraveling equilibrium of $l^e = a$ and $m^e = a$ is the only equilibrium with the property that all participants in the first round market are accepted with probability 1. In fact, condition (ii) of Definition 4.2 is satisfied for any l when $m = q(l)$, because this

[11] According to our definition (4.5), $l^e = a$ and $m^e = a$ is an endogenous participation equilibrium when $c = 0$. But the conclusion is reversed if, as in equation (3.1), we compute the second round expected type by taking the limit as m converges to a while fixing l at a. Whether or not unraveling is an endogenous participation equilibrium when $c = 0$ does not affect our conclusion in Proposition 4.4 that it is the only limit equilibrium as c converges to 0.

implies $l \geq q(l) > u(b)$ and so all first round participants match and exit with probability 1, making the second round expected type equal to $q(l)$. However, as we will show formally in the proof of the next proposition, if $l > a$ then any type just below l would have strict incentives to participate in the first round market. Thus, $m = q(l)$ cannot be part of any equilibrium because it does not satisfy condition (i) of Definition 4.2.

Proposition 4.3 establishes unraveling as an equilibrium outcome, but it does not rule out the possibility that some sorting occurs in other equilibria. Of particular interest is whether there are equilibria that approach the level of sorting efficiency achieved by dynamic sorting when the participation cost becomes arbitrarily small. The following result states that the answer is no. Unraveling is the only limit equilibrium when the participation cost is arbitrarily small. That is, any equilibrium with small c must be close to the unraveling outcome with no sorting.

PROPOSITION 4.4. *As participation cost c converges to 0, $l^e = a$ and $m^e = a$ is the only limit endogenous participation equilibrium.*

PROOF. Suppose that there is an equilibrium other than $l^e = a$ and $m^e = a$ regardless of how small c is. Then, there is a sequence of c converging to 0 such that an equilibrium l^c and m^c different from the unraveling equilibrium exists for each c. First, we argue that each equilibrium l^c and m^c satisfies $l^c < u(b)$, or $m^c > l^c + c/b$. This property means that the threshold type l^c is accepted with probability strictly less than 1. Suppose instead $m^c \leq l^c + c/b$. From the monotonicity of u, we have $u(x) < y$ and $u(y) < x$ for all $x, y \in [l, b]$, so that all participating types accept each other and exit with probability 1. The type distribution in the second round is then given by (4.4), implying $m^c = q(l^c)$. Now, if $l^c > a$, then any type x between $q(l^c)$ and l^c would be accepted with probability 1 in the first round as $u(b) = m^c - c/b = q(l^c) - c/b < x$. Such type x would strictly prefer to join the first round of search since they expect a partner of average type $Q(l^c)$, compared to an average type $m^c = q(l^c)$ if they wait for the second round. Therefore, $l^c = a$, and hence $m^c = q(a) = a$, contradicting the assumption that the equilibrium l^c and m^c is different from the unraveling equilibrium.

Next, type l^c must also be accepted with a strictly positive probability, so we have $c/b < m^c - l^c < c/l^c$. Thus, $m^c - l^c$ converges to 0 as c converges to 0. It then follows that

$u(b) - l^c \to 0$. Further, in any equilibrium we have $v(l^c) > m^c$; otherwise type l^c would strictly prefer not to participate since the highest type $v(l^c)$ that would accept l^c is lower than the expected type in the second round. This implies that l^c is greater than the larger intersection of u and v if they intersect (because $v(x) = x = u(x) < m^c$ at any intersection x). Hence the second round type distribution $G(x; m^c, l^c)$ is given by (4.3). Since $u(b) - l^c \to 0$, which is equivalent to $v(l^c) - b \to 0$, we have $1 - F(\min\{v(l^c), b\}) \to 0$ and $F(\max\{u(b), l^c\}) - F(l^c) \to 0$. As u and v are increasing functions, $1 - F(\min\{v(x), b\}) \to 0$ and $F(\max\{u(x), l^c\}) - F(l^c) \to 0$ for every $x > l^c$. Thus $G(x; m^c, l^c)$ converges pointwise to the distribution function given by (4.4). Thus, $m^c - q(l^c) \to 0$, which is consistent with $m^c - l^c \to 0$ only if $l^c \to a$ and $m^c \to a$. *Q.E.D.*

We illustrate Proposition 4.4 with the earlier example of uniform type distribution on $[1, 2]$. In this example, besides the unraveling equilibrium with $l^e = 1$ and $m^e = 1$, which exists for any cost $c < 1$, there is a sequence of equilibria converging to the unraveling equilibrium. In these endogenous participation equilibria, some sorting takes place because relative to the random matching, high types have a higher probability of matching with each other as some of the low types do not participate in the first round. For example, when $c = 0.04$, we have $l^c = 1.05$ and $m^c = 1.08$, with a corresponding total match value of $V = 2.262$, compared to $V^0 = 2.25$ with no sorting. According to the measure introduced earlier, the sorting efficiency gain is $(V - V^0)/(V^\infty - V^0) = 14.35\%$. Note that in this example, we have $m^c - c/l^c < l^c < m^c - c/b$ so that type l^c is accepted by some but not all participants in the first round. See Figure 1 for an illustration. In fact, the presence of sufficiently many first round participants that are accepted with probability less than 1 is critical for sorting to occur in an endogenous participation equilibrium. These agents, who are unacceptable to the highest types, create the search externality needed to maintain the average quality of the second round pool. Without this search externality, the second round mean m would be close to the average quality of the non-participants, which would motivate more types to participate in the first round and reduce the level of sorting. Moreover, unlike in dynamic sorting of Section 3 where search externality is guaranteed by the assumption of full participation, it is more delicate to ensure the search externality with costly and endogenous participation, because the first round participants must be at the same time accepted with a strictly positive probability. Indeed, the 14.35% efficiency gain is the maximum that can be achieved in an endogenous participation equilibrium, compared to 26.5% achieved in dynamic sorting. When c becomes small, it becomes increasingly difficult to create the search externality in equilibrium, as the $u(x)$ function becomes almost horizontal while the $v(x)$ function becomes almost vertical. This means that for any expected type m in the second round, first round participating types have almost identical acceptance thresholds, and the set of types that are acceptable to some but not all agents shrinks to the empty set in the limit when c converges to 0.

In the remainder of this section, we show that our unraveling result is robust to alternative assumptions of type-dependent participation costs, binding outside options, asymmetric type distributions, and multiple matching rounds. These robustness checks further illustrate how endogenous participation destroys the negative search externality and causes the market to unravel. Readers who are more interested in how our unraveling result depends on the richness of the type space can continue directly to Section 5.

Type-dependent participation costs. We have so far assumed that the participation cost c is uniform. Although there is no presumption as to whether and how the cost should change with type, it is important to check if the unraveling result is robust. To do so, suppose that the participation cost of type x on either side of the market is given by a continuous function $c\theta(x)$, where c is a positive parameter. As c converges to 0, the whole cost function becomes arbitrarily small. We claim that if $\theta(x)/x$ is a decreasing function for $x \in [a, b]$, then our earlier analysis of unraveling goes through in the same way, and all results remain valid. To see this, note that because $\theta(x)/x$ decreases with x, from equation (4.1) we have that participation in the first round market by a lower type implies participation by a higher type. Lemma 4.1 still holds, and any equilibrium is associated with a threshold l of participation.[12] The mutual acceptance region is now defined by the

[12] If the match value function is not xy but some other separable function, the condition that is sufficient to imply Lemma 4.1 will change. For example, if the match value function is $w_X(x)w_Y(y)$, then the corresponding condition is that $\theta(x)/w_X(x)$ decreases with x and $\theta(y)/w_Y(y)$ decreases with y. Lemma 4.1 cannot be obtained in the same way if the match value function is not separable.

following two functions:

$$u(x) = m - \frac{c\theta(x)}{x},$$
$$v(x) = \frac{c\theta(v(x))}{m - x}.$$

Since $\theta(x)/x$ decreases with x, both $u(x)$ and $v(x)$ remain increasing. Equilibrium can then be defined as in Definition 4.2. As c converges to 0, $u(x)$ becomes horizontal and $v(x)$ becomes vertical in Figure 1, as what happens when $\theta(x)$ is constant. This is sufficient to imply that the only limit equilibrium is $l = a$ and $m = a$ as c converges to 0.

Binding outside options. In making our main unraveling argument, we have assumed that $c < a^2$, so that in the second round all remaining agents participate. This assumption is innocuous when agents' outside option of remaining unmatched is low relative to potential match values, as we are primarily interested in what happens when the participation cost is small. To reconsider our unraveling result when the outside option value is relatively high, assume that $a = 0$. Then, for any positive participation cost, however small, there will be agents that never participate. This implies that for any second round expected participating type m, in the first round the waiting payoff to a type x is $\max\{xm - c, 0\}$, instead of xm. This does not affect Lemma 4.1, and so the first round participation decisions are still governed by a threshold rule. But now given any participation threshold l, expected type m, and the resulting second round type distribution $G(x; m, l)$, the second round participation decisions are also determined by a threshold. An equilibrium pair of first round participation threshold l^e and second round mean m^e can be defined in the same way as in Definition 4.2, with the second condition replaced by: (ii′) given l^e, m^e and the resulting $G(x; m^e, l^e)$, either the second round expected type m^e uniquely satisfies

$$m^e(1 - G(c/m^e; m^e, l^e)) = \int_{c/m^e}^{b} x \ \mathrm{d}G(x; m^e, l^e),$$

or if no such value exists, implying that no remaining agent participates in the second round, we have $m^e = 0$. Let ϕ be the unique solution to the equation $\phi Q(\phi) = c$. In words, type ϕ is the threshold type if the market operates for only one round. Note that ϕ is an increasing function of c. Corresponding to Proposition 4.3, the unraveling outcome here is $l = \phi$ and $m = 0$. This follows from arguments analogous to Proposition 4.3; in particular, for any $m < \phi + c/b$, the participation threshold that satisfies condition (i) in Definition 4.2 is ϕ. Finally, we can show that the counterpart of Proposition 4.4 holds: $l = \phi$ and $m = 0$ is the only limit equilibrium as c converges to 0. Since ϕ converges to 0 as c converges 0, we have the same unraveling outcome for the case of $a = 0$.

Asymmetric type distributions. When the two sides of the market X and Y have different type distributions, to define an equilibrium, we need two pairs of first round participation thresholds and second round expected types, one for each side of the market. Let the four variables be l_X, m_X, l_Y and m_Y. Given m_X and m_Y, the lowest type that x is willing to accept in the first round is $u(x) = m_Y - c/x$, and the highest type that is willing to accept x is $v(x) = c/(m_X - x)$. The roles of u and v are reversed for Y-agents. The mutual acceptance region is completely described by the two functions u and v. The two functions are no longer symmetric around the main diagonal in the $[a_X, b_X] \times [a_Y, b_Y]$ diagram, but the crucial property is retained that u and v become almost horizontal and vertical respectively when c becomes arbitrarily small. The assumption of different type distributions also calls for an extension of our search technology, because in general the size of participants can differ for the two sides of the first round market. In particular, the probability of finding a match cannot be 1 for all participants on the long side (the side with more participants). In any natural extension, agents on the short side of the market find a match with probability 1. With this restriction, the unraveling result of Proposition 4.4 can be derived in a similar way. We sketch the argument as follows. The critical step is to show that $l_X^c - m_X^c$ and $l_Y^c - m_Y^c$ converge to 0 in any sequence of equilibria indexed by the participation cost c. Take any subsequence of equilibria $(l_X^{c_i}, m_X^{c_i}, l_Y^{c_i}, m_Y^{c_i})$ such that $l_X^{c_i} - m_X^{c_i}$ and $l_Y^{c_i} - m_Y^{c_i}$ converge. Clearly, $\lim_{c_i \to 0} l_X^{c_i} - m_X^{c_i} \geq 0$ and $\lim_{c_i \to 0} l_Y^{c_i} - m_Y^{c_i} \geq 0$; otherwise, for sufficiently small participation cost, threshold types $l_X^{c_i}$ and $l_Y^{c_i}$ would be incurring the cost in the first round without having any chance of being accepted. Next, at least for the short side of the first round market, say type x agents, $\lim_{c_i \to 0} l_X^{c_i} - m_X^{c_i} \leq 0$; otherwise, for sufficiently small cost, types just below $l_X^{c_i}$ would strictly prefer participating because they would find a match with probability 1 and would be acceptable to all agents. Thus, $\lim_{c_i \to 0} l_X^{c_i} - m_X^{c_i} = 0$. Together with the fact that the second round type distribution converges to the conditional distribution below l_X, we have $\lim_{c_i \to 0} l_X^{c_i} =$

a_X and $\lim_{c_i \to 0} m_X^{c_i} = a_X$. Since all type x agents participate in the first round market and since they are on the short side of the market, we also have $\lim_{c_i \to 0} l_Y^{c_i} = a_Y$ and $\lim_{c_i \to 0} m_Y^{c_i} = a_Y$.[13] Since the above holds for all convergent subsequences of equilibria, the only possible limit equilibrium is the unraveling outcome, with l_X and m_X both converging to a_X, and l_Y and m_Y converging to a_Y.

Multiple matching rounds. Consider how our unraveling result is affected when there are more than two matching rounds. The improvement in sorting efficiency afforded by multiple matching rounds described in Section 3 does not extend to the case with endogenous participation. A simple induction argument makes this point clear. In round $T-1$, our two-round unraveling result in Proposition 4.4 applies: for c converging to 0, in the only limit equilibrium all remaining agents participate in round $T-1$ and accept anyone they meet. But then in round $T-2$, agents should anticipate that the market will effectively close in the next round if c is arbitrarily small. So round $T-2$ is just like the next-to-last round. Our two-round unraveling result again applies, and so on. Thus, for any finite T, the only limit equilibrium with T matching rounds when the participation cost per round converges to 0 is that the market operates only for the first round in which all agents participate and accept whomever they meet. Thus, when agents choose when to search, adding more matching rounds only serves to hasten the date of search and contracting for all market participants, with no increase in matching efficiency.

5. Sorting and Unraveling with a Discrete Type Space

So far we have assumed that there is a continuum of types in a finite-horizon matching model. This modeling choice allows us to produce clean insights about how dynamic sorting improves matching efficiency and how it depends critically on the search externality. Implicit in our choice of a continuum of types is the assumption that the type space is infinitely richer than the potential matching opportunities afforded by a finite number

13 Recall that we have assumed that the two sides of the market have the same size at the outset. If the two sides have different sizes, then the unraveling result holds for the short side of the market. The long side will unravel to the point that all acceptable agents participate in the first round market.

of rounds. Do our conclusions about dynamic sorting and unraveling apply to matching markets with a finite number of types?

First consider a symmetric, costless participation model with two rounds of matching and N types, where $N \geq 2$ is a positive integer. Let the types be $x^1 > x^2 > \ldots > x^N$. Each type x^i, $i = 1, \ldots, N$, consists of a continuum of agents, and has a fraction $f^i > 0$ in the population. The match value to a type x^i agent, if matched with a type x^j agent from the other side, is $x^i x^j$. As in Section 3, there is a common acceptance threshold in the first round market: if m is the expected match type in the second round market, each type accepts a potential match x^i if $x^i \geq m$. Given the first round threshold type x^k (the highest type accepted), the second round type distribution is given by

$$g^i R = \begin{cases} f^i & \text{if } i > k; \\ f^i \sum_{j>k} f^j & \text{if } i \leq k \end{cases}$$

where $R = 1 - \left(\sum_{j \leq k} f^j \right)^2$ is the relative size of the second round market, and g^i is the fraction of x^i types agents in the second round market population. The expected type in the second round is then $m = \sum_{i=1}^{N} g^i x^i$.[14] A full participation equilibrium can be characterized by a threshold type x^k such that $x^{k+1} < m \leq x^k$. Existence of a full participation equilibrium can be easily established, and we can extend the analysis to the case of more than two rounds as in Section 3. In general, multiple equilibria exist, and some equilibria may involve a probability between 0 and 1 of each type rejecting x^k. In any of these equilibria, the negative search externality that low types impose on high types allows the market to perform a dynamic sorting function.

Before considering how costly search affects the sorting function of the market, it is important to note that when the type space is discrete, there are equilibria with sorting that do not rely on the negative search externality. In these equilibria, types choose to enter the market sequentially even though participation is costless.[15] For example, when there are N

[14] As in equation (3.1), we can define the expected type in an "empty" second round market, to be the limit of the second round expected type as the probability that each type rejects x^N converges to 0.

[15] In the continuous type case of Section 3, there may exist equilibria where not all types participate in the first round market. For example, with two matching rounds, one such equilibrium is defined by a participation threshold l and an acceptance threshold $k > l$, such that the expected type m in the

types and $T \geq N$ matching rounds, it is an equilibrium that for each $i = 1, \ldots, N$, type x^i skips the first $i-1$ rounds, and in each round j, $j \geq i$, type x^i participates and accepts type x^j and above. The equilibrium outcome is the perfect sorting for N types. As suggested in Section 3, these perfect sorting equilibria are difficult to justify when participation is costless. However, because these equilibria do not rely on the search externality, our unraveling argument in Section 4 does not apply. In this section, we establish that when there are at least as many matching rounds as there are types, the perfect sorting can be approximated arbitrarily closely by sequential participation as the participation cost converges to zero. This finding contrasts our unraveling result in Section 4: the sorting function provided by sequential participation does not disappear as the participation cost becomes small. However, the size of the efficiency gain afforded by sequential participation crucially depends on the richness of the types space. We will show that, consistent with our analysis in the continuous type case of Section 4, for any fixed number of rounds, as the number of types becomes large, no equilibrium can achieve a level of sorting efficiency that is significantly higher than the unraveling outcome.

We now consider a general symmetric model of $N \geq 2$ types and $T \geq 2$ rounds by backward induction. The cost of participation is c; we assume that $\sqrt{c} < x^N$. In round $T-1$, let g^i_{T-1}, $i = 1, \ldots, N$, be the type distribution of remaining agents. Unlike in the continuous type case, participation and acceptance decisions can be probabilistic; indeed, we will construct mixed-strategy equilibria so that the type distribution g_{T-1} is non-degenerate. Let $x^{l_{T-1}}$ be the lowest type that participates in round $T-1$ with positive probability, and let $\pi_{T-1} > 0$ be the participation probability. From an argument identical to that in Lemma 4.1, we know that types above $x^{l_{T-1}}$ participate with probability 1. Next, let $x^{v_{T-1}}$ be the lowest type that rejects type $x^{l_{T-1}}$ with positive probability, and let $\gamma_{T-1} > 0$ be the rejection probability. Types above $x^{v_{T-1}}$ reject type $x^{l_{T-1}}$ with probability 1. Given l_{T-1} and π_{T-1}, the probability that any participating type meets type $x^{l_{T-1}}$ in

second round is exactly k. Unlike the endogenous participation equilibria with sequential sorting that we construct below, these equilibria rely on the negative search externality and disappear when participation is costly.

round $T-1$ is

$$\mu_{T-1}^{l_{T-1}} = \frac{g_{T-1}^{l_{T-1}} \pi_{T-1}}{g_{T-1}^{l_{T-1}} \pi_{T-1} + \sum_{j<l_{T-1}} g_{T-1}^{j}},$$

and the probability of meeting any type x^i, $i = 1, 2, \ldots, l_{T-1} - 1$, is

$$\mu_{T-1}^{i} = \frac{g_{T-1}^{i}}{g_{T-1}^{l_{T-1}} \pi_{T-1} + \sum_{j<l_{T-1}} g_{T-1}^{j}}.$$

For notational brevity, let $\alpha_{T-1}^{l_{T-1}} = \sum_{j=v_{T-1}+1}^{l_{T-1}} \mu_{T-1}^{j} + \mu_{T-1}^{v_{T-1}}(1-\gamma_{T-1})$ be the total acceptance probability for the threshold type $x^{l_{T-1}}$ conditional on participation. Given l_{T-1}, v_{T-1}, π_{T-1} and γ_{T-1}, round T type distribution is:

$$g_T^i R_T = \begin{cases} g_{T-1}^{i} & \text{if } i > l_{T-1}; \\ g_{T-1}^{l_{T-1}}(1-\pi_{T-1}\alpha_{T-1}^{l_{T-1}}) & \text{if } i = l_{T-1}; \\ 0 & \text{if } l_{T-1} > i > v_{T-1}; \\ g_{T-1}^{v_{T-1}} \mu_{T-1}^{l_{T-1}} \gamma_{T-1} & \text{if } i = v_{T-1}; \\ g_{T-1}^{i} \mu_{T-1}^{l_{T-1}} & \text{if } i < v_{T-1} \end{cases}$$

where g_T^i is the fraction of type x^i in round T market, and

$$R_T = \sum_{j>l_{T-1}} g_{T-1}^{j} + g_{T-1}^{l_{T-1}}(1-\pi_{T-1}\alpha_{T-1}^{l_{T-1}}) + g_{T-1}^{v_{T-1}} \mu_{T-1}^{l_{T-1}} \gamma_{T-1} + \sum_{j<v_{T-1}} g_{T-1}^{j} \mu_{T-1}^{l_{T-1}}$$

is the relative size of the round T market. In writing the above expressions, we have implicitly assumed that type $x^{l_{T-1}-1}$ and above are accepted with probability 1 by all participating types. In other words, the threshold type $x^{l_{T-1}}$ is the only participating type that faces a positive rejection probability. This must hold when c is sufficiently small. To see this, note that for type $x^{l_{T-1}}$ to be acceptable to some types but not to all, we need $x^{l_{T-1}}$ to be close to m^T, the round T expected type, when c becomes sufficiently small. Then, we have $x^{l_{T-1}-1} > m_T - c/x^1$ when c small enough, and so all types above $x^{l_{T-1}}$ are accepted with probability 1. We modify Definition 4.2 to have the following:

Definition 5.1. *Given a round $T-1$ type distribution g_{T-1}, a continuation equilibrium in round $T-1$ is l_{T-1}, π_{T-1}, v_{T-1}, γ_{T-1} and m_T such that (i) type $x^{l_{T-1}}$ is the lowest type that weakly prefers to participate in round $T-1$ market; (ii) type $x^{v_{T-1}}$ is the lowest type that weakly prefers to reject type $x^{l_{T-1}}$; and (iii) the round T mean m_T is given by $m_T = \sum_{j=1}^{N} g_T^j x^j$.*

The above definition can be applied recursively to define an endogenous participation equilibrium. When $T = 2$, we have an endogenous participation equilibrium by setting $g^i_{T-1} = f^i$.[16] As in the continuous type case, there is an equilibrium corresponding to the unraveling outcome: $l_{T-1} = N$ with $\pi_{T-1} = 1$, and $\upsilon_{T-1} = 1$ with $\gamma_{T-1} = 0$. In this equilibrium, all types participate and accept each other with probability 1. Unlike the continuous type case, however, even when c is arbitrarily small, there exist other equilibria. In particular, consider $l_{T-1} = N$ and $\upsilon_{T-1} = N - 1$. If type x^N agents are accepted by type x^{N-1} with positive probability, then they can be indifferent between participating and not participating if the cost c is small enough and $x^{N-1} > m^T$. Type x^{N-1} can be indifferent between accepting and rejecting type x^N, because a round T mean m_T greater than x^N due to the rejection of x^N compensates the cost of participating again in the matching. The intuition of this construction is verified in the lemma below. Moreover, in this equilibrium, as the participation cost c converges to 0, the participation probability π_{T-1} for type x^N converges to 0 and the rejection probability γ_{T-1} for type x^{N-1} converges to 1. If there are only two types ($N = 2$), then such equilibrium outcome would be the perfect sorting in the limit.

Lemma 5.2. *Given a round $T-1$ type distribution g_{T-1}, for c sufficiently small, a continuation equilibrium in round $T-1$ exists with $l_{T-1} = N$, $\upsilon_{T-1} = N-1$ and $\pi_{T-1}, \gamma_{T-1} \in (0, 1)$. Further, as c converges to 0, $\lim_{c\to 0} \pi_{T-1} = 0$, and $\lim_{c\to 0} \gamma_{T-1} = 1$.*

The proof of the above lemma is in the Appendix. As long as $T \geq N$, we can apply the construction of the continuation equilibria in Lemma 5.2 recursively to obtain an equilibrium through sequential participation, which converges to the perfect sorting as c converges to 0.

Proposition 5.3. *Suppose that $T \geq N$. There exists a sequence of endogenous participation equilibria such that the equilibrium matching converges to the perfect sorting as c converges to 0.*

[16] For $T > 2$, recursive application of Definition 5.1 requires an implicit assumption that the participation decisions in each round are governed by a threshold rule. This assumption is satisfied by construction in equilibria with sequential sorting and bottom sorting that we develop below, and is not required in the proof of Proposition 5.5.

PROOF. We prove the proposition by construction. Fix any sufficiently small c, and consider the following form of strategies. The market closes in round N, when all remaining agents participate and accept each other. In each active round before the market closes, types participate in sequence: type x^1 agents start full participation from round 1 and are always accepted; for each $t = 2, \ldots, N$, type x^t agents skip the first $t-2$ rounds, where they would be rejected with probability 1, participate with a small probability in round $t-1$, where they are rejected with a high probability, and fully participate from round t onwards and are thereafter always accepted.

When c becomes arbitrarily close to 0, each active round t, $t = 1, \ldots, N$, becomes an exclusive matching place for x^t, and the matching converges to the perfect sorting. It remains to show that there exist strategies of the form described above that constitute an endogenous participation equilibrium for any sufficiently small c. We do this by induction. Without loss of generality, assume $T = N$, and set $l_T = N$ with $\pi_T = 1$, and $v_T = 1$ with $\gamma_T = 0$. By Lemma 5.2, we only need to show that (i) for sufficiently small c, there exists a round t continuation equilibrium with $l_t = t+1$ and $v_t = t$ and $\pi_t, \gamma_t \in (0,1)$, where the expected payoffs in round $t+1$ are given by the round $t+1$ continuation equilibrium with $l_{t+1} = t+2$ and $v_{t+1} = t+1$ and $\pi_{t+1}, \gamma_{t+1} \in (0,1)$; and (ii) as c converges to 0, $\lim_{c\to 0} \pi_t = 0$ and $\lim_{c\to 0} \gamma_t = 1$. To establish this step, we note that when c becomes sufficiently small, by induction types below x^{l_t} play no role in determining the expected payoffs of type x^{l_t} and above. The round t continuation equilibrium can be identified in the same way as the round $T-1$ continuation equilibrium in Lemma 5.2. *Q.E.D.*

If the number of matching rounds T is smaller than the number of types N, then the endogenous participation equilibria constructed in the above proof, which we will refer to as "sequential sorting," cannot approximate the perfect sorting. Matching inefficiency then arises. The important question is: how great is the inefficiency when $T < N$? An upper bound of the inefficiency can be obtained in the following way. For sufficiently small c, an equilibrium with "bottom sorting" exists which looks just like sequential sorting, except that the types that fully participate from round 1 onwards are type x^{N-T+1} and above, instead of the single highest type x^1 in sequential sorting. When c converges to 0, the bottom types (x^{N-T+2} through x^N) are almost perfectly sorted through sequential participation, while all higher types (x^{N-T+1} through x^1) are randomly matched to each other and exit in the first round. Unlike the full participation equilibria with dynamic sorting that exist when there is no participation cost, the endogenous participation equilibria with bottom sorting do not unravel when c becomes arbitrarily small. However, as N becomes larger for fixed T, bottom sorting becomes more inefficient, because the fraction of types that are randomly matched in round 1 becomes larger. Although there may exist other equilibria more efficient than bottom sorting, we show that as N becomes arbitrarily large and c arbitrarily small for fixed T, there is almost no sorting in any of these equilibria.

As in the proof of Proposition 4.4, a necessary condition for any continuation equilibrium in round $T-1$ with $l_{T-1} < N$ is that $m_T > x^{l_{T-1}}$. Otherwise, type $x^{l_{T-1}}$ would be accepted with probability 1 by all participating types and would strictly prefer to participate. This implies that $m_T \leq x^{l_{T-1}+1}$ since all higher types would participate in round $T-1$ and exit. But then type $x^{l_{T-1}+1}$ would be accepted by higher types with probability 1 in round $T-1$ and would therefore strictly prefer participation, contradicting the assumption that type $x^{l_{T-1}}$ is the participation threshold. The lemma below provides a necessary condition for $m_T > x^{l_{T-1}}$. The proof is in the Appendix.

LEMMA 5.4. *Given a round $T-1$ type distribution g_{T-1}, for c sufficiently small, for any $l_{T-1} < N$, a necessary condition for a continuation equilibrium with participation threshold type $x^{l_{T-1}}$ is*

$$\frac{g_{T-1}^{l_{T-1}}}{\sum_{j=1}^{l_{T-1}} g_{T-1}^j} > \frac{\sum_{j>l_{T-1}} g_{T-1}^j (x^{l_{T-1}} - x^j)}{\sum_{j<l_{T-1}} g_{T-1}^j (x^j - x^{l_{T-1}})}. \tag{5.1}$$

The left-hand-side of the inequality (5.1) is the largest probability of meeting the threshold type $x^{l_{T-1}}$ in round $T-1$, computed under the assumption that $\pi_{T-1} = 1$. This probability represents the greatest possible negative search externality imposed on the participants in round $T-1$, as only the threshold type is rejected with a positive probability. The inequality thus requires that the search externality imposed by the single threshold type be sufficiently large, so that enough higher types remain unmatched after round $T-1$ to keep m_T above $x^{l_{T-1}}$. Note that (5.1) is automatically satisfied if $l_{T-1} = N$

(as the right-hand-side is zero): if the threshold type is the lowest type x^N then m_T is greater than x^N for arbitrarily small search externality. This explains why the continuation equilibrium with $l_{T-1} = N$ (bottom sorting) always exists.

The intuition behind Lemma 5.4 is key to understanding the next proposition. For any fixed T, as N becomes arbitrarily large and c arbitrarily small, the search externality that can be imposed by any single threshold type becomes negligible. Thus, no equilibrium sorting can differ significantly from bottom sorting, which in turn becomes closer to the unraveling outcome in terms of matching inefficiency. For the following proposition, we consider sequences of type distributions as more types are added. Let f_N be the type distribution with N different types, and denote $F_N(x) = \sum_{i:x^i \leq x} f_N^i$. We assume that (i) the support of each f_N is contained in $[a, b] \subset (0, \infty)$; (ii) $\liminf_{N\to\infty} F_N(x) > 0$ for any $x > a$; and (iii) $\lim_{N\to\infty} \sup_i f_N^i = 0$. These assumption ensure that the type distribution becomes atomless and a is a limit point when N is arbitrarily large.[17]

Proposition 5.5. *For any fixed T, in any endogenous participation equilibrium, the first round participation threshold converges to a as N becomes arbitrarily large and c arbitrarily small.*

The proof of the proposition is in the Appendix. We use the following discretized version of our previous example in Sections 3 and 4 to illustrate Proposition 5.5. There are N types evenly spaced between 1 and 2, with $x^j = 1 + (N - j)/(N - 1)$ for each $j = 1, \ldots, N$, and the type distribution is uniform, with $f^j = 1/N$. Suppose that $T = 2$ and $c = 0$. For each N we compute both the full participation equilibrium with dynamic sorting that achieves the highest level of efficiency according to the measure introduced in Section 3, and the most efficient endogenous participation equilibrium. The efficiency measure corresponding to the dynamic sorting outcome varies little as N increases, centering around 27%. For the best-performing endogenous participation equilibria, the efficiency measure starts at 100% when $N = 2$ (the perfect sorting), but it drops to below 27% when $N = 60$, and approaches zero as N increases further. For example, if $N = 101$, the threshold type in

[17] If a is not a limit point then the following proposition applies to the smallest limit point as N becomes arbitrarily large.

dynamic sorting is $x^k = 1.39$, with a total match value of $V^d = 2.273$, implying an efficiency measure of $(V^d - V^0)/(V^{101} - V^0) = 27.05\%$, where $V^0 = 2.25$ for the random matching and $V^{101} = 2.335$ for the perfect sorting with 101 types. In contrast, for the endogenous participation equilibrium, the participation threshold type x^l is 1.09 ($l_1 = 91$, $v_1 = 90$, $\pi_1 = 0$ and $\gamma_1 = 1$), with a total match value of $V^b = 2.27$, and an efficiency measure of 24.12%. When the number of types is so numerous that an atomless continuous type model is a close approximate, the conclusion of Section 4 applies, and almost no sorting takes place in any endogenous participation equilibrium. With more matching rounds, efficiency improves notably in dynamic sorting, while the improvement is slower under endogenous participation. For example, if $T = 3$ and $N = 101$, the participation threshold types in the most efficient dynamic sorting equilibrium are 1.49 in round 1 and 1.33 in round 2, with an efficiency measure of 41.18%. The gain over two rounds of matching is obtained through substantially more selective acceptance decisions in the first round. In contrast, the participation threshold types in the endogenous participation equilibrium are 1.12 and 1.03, with an efficiency measure of 31.36%. Participation thresholds change only marginally compared to two rounds of matching. There is limited efficiency gain over two rounds of matching, because sorting occurs only at the bottom of the type distribution.

6. Conclusion

Economists have long recognized that in a matching market both matching decisions and search decisions involve externalities and can cause market inefficiency. The existing literature (Diamond 1982; Mortensen 1982; Hosios 1990) has focused on the search externalities by assuming homogeneity on the two sides of the market. The research on the search externalities culminates in the so-called Hosios (1990) condition for search efficiency, which requires an agent's bargaining power to equal the elasticity of the matching function. A recent paper by Shimer and Smith (2001) examines the implications of search and matching externalities in a model with heterogeneous agents. The Hosios condition does not hold in the model of Shimer and Smith: in the decentralized market attractive types search too little and match too readily, while unattractive types search too much and match too infrequently. In a different setup with posted prices and directed search, Shi (2002) finds efficiency with heterogeneous agents.

The papers on search and matching inefficiencies mentioned above focus on steady-state stationary analysis, which greatly reduces the distributional complexity of search and matching dynamics. Our model is motivated by the concern that the steady state need not be the relevant model in many entry level markets for professional workers. We posit that the dynamics in this kind of markets are better captured by a finite-horizon model with no replacement of the types that have formed matches and left the market. Two different sorting mechanisms emerge from our analysis of a non-stationary dynamic matching environment, dynamic sorting and sequential sorting. In the former, agents can afford to be selective in early rounds of matching, because the negative search externality imposed by the presence of low types maintains sufficiently high quality in later rounds. This search externality makes it impossible for types to sort perfectly, but is necessary for dynamic sorting to function. When a small participation cost is introduced, lower types are forced to skip initial matching rounds, making them exclusive markets for higher types. But these exclusive markets tend to unravel, as the highest types in the later markets have incentives to switch to early markets. In contrast, sequential sorting operates by creating exclusive markets through sequential participation. Lower types skip initial rounds because they would be rejected by higher types in early rounds. Each exclusive market must be homogeneous, for otherwise they would be unraveled. Since sequential sorting does not rely on search externality, sorting is perfect when there are enough many rounds to create one exclusive market for each type, and it is robust to the introduction of a small participation cost. However, when there are not enough many rounds, only the types at the bottom of the distribution can be sorted. Sorting becomes increasingly inefficient, and eventually indistinguishable from the unraveling outcome.

The endogenous evolution of trading opportunities poses difficult problems for equilibrium analysis. Characterizing trading opportunities in matching markets is complicated because agents are heterogeneous and sorting is important. We are able to make some progress by ruling out side payments and restricting the search technology and the match value function. Relaxing these assumptions in a tractable way remains a challenge.

Appendix

Proof of Proposition 3.4. For any pair of thresholds k_X and k_Y, let

$$R(k_X, k_Y) = 1 - (1 - F_X(k_X))(1 - F_Y(k_Y))$$

be the size of second round market. Define $Q_X(k_X) = \mathrm{E}[x \mid x \geq k_X]$ and $Q_Y(k_Y) = \mathrm{E}[y \mid y \geq k_Y]$. Let $m_X(k_X, k_Y)$ be the mean of X-agents in the second round market, and define $m_Y(k_X, k_Y)$ similarly. An equilibrium is characterized by k_X^e and k_Y^e such that $k_X = m_X(k_X, k_Y)$ and $k_Y = m_Y(k_X, k_Y)$. Consider the problem of choosing k_X and k_Y to maximize the total match value $V(k_X, k_Y)$, given by

$$(1 - R(k_X, k_Y))Q_X(k_X)Q_Y(k_Y) + R(k_X, k_Y)m_X(k_X, k_Y)m_Y(k_X, k_Y).$$

For notational convenience, we drop the variable k in the functions. Taking derivatives,

$$\frac{\partial V}{\partial k_X} = (1 - R)Q'_X Q_Y + Rm_Y \frac{\partial m_X}{\partial k_X} + Rm_X \frac{\partial m_Y}{\partial k_X} + (m_X m_Y - Q_X Q_Y)\frac{\partial R}{\partial k_X}.$$

Since $(1 - R)Q_X + Rm_X = m_X^u$ for any k_X, where m_X^u is the unconditional mean of x,

$$(1 - R)Q'_X + R\frac{\partial m_X}{\partial k_X} + (m_X - Q_X)\frac{\partial R}{\partial k_X} = 0.$$

Similarly, if m_Y^u is the unconditional mean of y, we have $(1 - R)Q_Y + Rm_Y = m_Y^u$, and so

$$R\frac{\partial m_Y}{\partial k_X} + (m_Y - Q_Y)\frac{\partial R}{\partial k_X} = 0.$$

We follow similar steps as those in the proof of Proposition 3.3 to show that $\partial V/\partial k_X = 0$ if $m_X = k_X$. Similarly, $\partial V/\partial k_Y = 0$ at any equilibrium threshold k_Y^e. *Q.E.D.*

Proof of Proposition 3.6. For the following proof of Propositions 3.6 and 3.7, it is convenient to write $\tilde{G}_t(x) = \int_a^x G_t(z)\ \mathrm{d}z$, $S_t(x) = 1 - G_t(x)$, and $\tilde{S}_t(x) = \int_x^b S_t(z)\ \mathrm{d}z$ for any t. Also, for each round t, threshold k_t and type distribution G_t, let

$$m(k_t; G_t) = G_t(k_t)k_t + \int_{k_t}^b x\ \mathrm{d}G_t(x),$$

so that the second equilibrium condition can be written as $k_{t-1} = m(k_t; G_t)$. Whenever confusion does not arise, we write m_t instead of $m(k_t; G_t)$.
(i) The existence of equilibrium can be shown with an induction argument. We know from Proposition 3.2 that for any initial type distribution G an equilibrium exists when $T = 2$. Suppose that an equilibrium exists in a model with $T \geq 2$ rounds, and let $k_1(T; G)$ be the largest equilibrium threshold in the first round market with the initial type distribution G. Then, consider the following algorithm for finding an equilibrium with $T+1$ rounds of the market in total and the initial type distribution F: start with a first round threshold type $k_1 \in (k_1(2; F), b)$; set the type distribution G_1 in the first round market to F; use G_1 and k_1 to compute the type distribution G_2 in round 2 according to the first equilibrium condition (3.4); use k_1 and G_2 to determine a round 2 threshold k_2 from the second equilibrium condition (3.5). If $k_2 = k_1(T; G_2)$, then an equilibrium has been found by combining this particular k_1 with the sequence of T thresholds that starts with the resulting k_2, with the resulting G_2 as the initial type distribution.

The above process is well-defined, because for each k_1 and F, the type distribution G_2 in the second round is uniquely defined according to (3.4). Further, (3.5) uniquely defines a second round threshold k_2 for any $k_1 \geq k_1(2; F)$. To see this latter point, rewrite the condition as follows:

$$k_1 = \int_a^b x \, \mathrm{d}G_2(x) + \int_a^{k_2} (k_2 - x) \, \mathrm{d}G_2(x).$$

Using integration by parts and equation (3.4) for G_2, that noting that $k_2 \leq k_1$, we can further rewrite the above as $\tilde{F}(k_2) = \tilde{F}(k_1) - F(k_1)\tilde{S}(k_1)$. Since $k_1(2; F)$ is the largest equilibrium threshold in the first round market with $T = 2$ and the initial type distribution F, we have $\tilde{F}(k_1) \geq F(k_1)\tilde{S}(k_1)$ for any $k_1 \geq k_1(2; F)$, with equality if and only if $k_1 = k_1(2; F)$. Thus, the first equilibrium condition uniquely defines k_2 for any $k_1 \geq k_1(2; F)$.

Now, from Definition 3.1 (or equivalently Definition 3.5) we know that $k_2 = a$ when $k_1 = k_1(2; F)$, so $k_2 < k_1(T; G_2)$ if we start the process with $k_1 = k_1(2; F)$. On the other hand, from (3.5) we have $k_2 = b$ when $k_1 = b$, so $k_2 > k_1(T; G_2)$ if we start with $k_1 = b$. Continuity of k_2 and $k_1(T; G_2)$ in k_1 then implies that the algorithm yields at least one $k_1 \in (k_1(T; G_2), b)$ such that $k_2 = k_1(T; G_2)$, which identifies an equilibrium with $T + 1$ rounds from the induction assumption.

(ii) To prove the uniqueness of the equilibrium, consider the following algorithm for finding an equilibrium with T rounds of the market in total and the initial type distribution F: start with a first round threshold type k_1; use the initial distribution $G_1 = F$ to compute the type distribution G_2 in round 2 from the first equilibrium condition (3.4); use k_1 and G_2 to determine a round 2 threshold k_2 from the second equilibrium condition (3.5); use k_2 and G_2 to find G_3; repeat this process for all $t = 3, \ldots, T-1$, until we find k_{T-1} and G_T. (It can be shown by an induction argument that this algorithm is well-defined.) Since the algorithm defines a sequence of decreasing thresholds, we have $g_t(x)/G_t(x) = f(x)/F(x)$ for all $x \leq k_{t-1}$ and for each $t = 2, \ldots, T-1$, and is therefore a decreasing function due to log-concavity of F. If $k_{T-1} = \int_a^b x \, \mathrm{d}G_T(x)$, we have found an equilibrium.

To show that there is a unique equilibrium, we need to compute the derivatives with respect to k_1. Recognizing that k_1 determines both the sequence of thresholds k_t and the sequence of distributions G_t, we use the following iterative method. For each $t = 1, \ldots, T-2$, using integration by parts, we can rewrite (3.5) as follows:

$$\tilde{G}_t(k_{t+1}) = \tilde{G}_t(k_t) - G_t(k_t)\tilde{S}_t(k_t).$$

Since the algorithm determines a decreasing sequence of thresholds, the above becomes:

$$\tilde{F}(k_{t+1}) = \tilde{F}(k_t) - F(k_t)\tilde{S}_t(k_t). \tag{A.1}$$

For $t = 1$, taking derivatives implies that

$$\frac{dk_2}{dk_1} = \frac{F(k_1)(1 + S(k_1)) - f(k_1)\tilde{S}(k_1)}{F(k_2)}.$$

Note that $dk_2/dk_1 > 1$ if $F(k_2) < F(k_1) - f(k_1)\tilde{S}(k_1)$. For $t = 2, \ldots, T-1$, we use another way of rewriting the second equilibrium condition for round $t-1$, again with integration by parts, to get

$$k_{t-1} = k_t + \tilde{S}_t(k_t). \tag{A.2}$$

Combining (A.1) and (A.2), we have for $t = 2, \ldots, T-1$

$$\tilde{F}(k_{t+1}) = \tilde{F}(k_t) - F(k_t)(k_{t-1} - k_t).$$

The above equation can be used to compute each dk_t/dk_1 recursively, starting from dk_2/dk_1. Taking derivatives, we have for $t = 2, \ldots, T-1$

$$F(k_{t+1})\frac{dk_{t+1}}{dk_1} = (2F(k_t) - f(k_t)\tilde{S}_t(k_t))\frac{dk_t}{dk_1} - F(k_t)\frac{dk_{t-1}}{dk_1}.$$

An equilibrium is defined by $k_{T-1} = \int_a^b x \, \mathrm{d}G_T(x)$, or equivalently,

$$k_{T-2} = k_{T-1} + \frac{\tilde{F}(k_{T-1})}{F(k_{T-1})}.$$

The above can be viewed as an equation in k_1. Since F is log-concave, $\tilde{F}/F$ is increasing, and it follows that a unique fixed-point in k_1 exists if $dk_{T-1}/dk_1 > dk_{T-2}/dk_1$. Thus, $G_t(k_{t+1}) < G_t(k_t) - g_t(k_t)\tilde{S}_t(k_t)$ if for any each $t = 1, \ldots, T-2$, then we obtain $dk_{t+1}/dk_1 > dk_t/dk_1$ recursively, starting from $dk_2/dk_1 > 1$, and therefore the equilibrium is unique.

It remains to argue that for any distribution G, any thresholds $k > k'$, such that $k > k_1(2; G)$, k' is determined by $\tilde{G}(k') = \tilde{G}(k) - G(k)\tilde{S}(k)$ and $g(x)/G(x)$ is decreasing for any $x < k$, we have

$$G(k') < G(k) - g(k)\tilde{S}(k). \tag{A.3}$$

This condition is sufficient, because even though changes in k_1 affect all distributions G_t, the stated condition applies to all G, k and k' that are linked through the equilibrium conditions and is therefore stronger than $G_t(k_{t+1}) < G_t(k_t) - g_t(k_t)\tilde{S}_t(k_t)$ for any each $t = 1, \ldots, T-1$. To see why (A.3) is true, note that since $g(x)/G(x)$ is decreasing in x, we have $g(x) > G(x)\frac{g(k)}{G(k)}$ for any $x < k$. Integrating from k' to k (note that $k' < k$ by assumption) gives

$$G(k) - G(k') > (\tilde{G}(k) - \tilde{G}(k'))\frac{g(k)}{G(k)}.$$

Since $\tilde{G}(k') = \tilde{G}(k) - G(k)\tilde{S}(k)$, we have (A.3), as desired. *Q.E.D.*

Proof of Proposition 3.7. The efficiency of an equilibrium can be established with an induction argument. Fix a market with a total of T rounds and the initial distribution F. For any sequence of decreasing thresholds $k_1, \ldots, k_{T-1}$, let G_t, $t = 2, \ldots, T$, be defined according to equation (3.4), starting from $G_1 = F$. Let V_T be the expected total match value for round T, with type distribution G_T:

$$V_T = \left(\int_a^b x \, \mathrm{d}G_T(x) \right)^2 .$$

For each $t = 1, \ldots, T-1$, recursively define the expected match value from t onward:

$$V_t = \left(\int_{k_t}^b x \, \mathrm{d}G_t(x) \right)^2 + R_{t+1}(k_t) V_{t+1},$$

which, by integration by parts, can be more conveniently written as

$$V_t = \left(k_t S_t(k_t) + \tilde{S}_t(k_t) \right)^2 + R_{t+1}(k_t) V_{t+1}.$$

The objective is then to maximize V_1.

We restrict attention to decreasing sequence of thresholds. Our induction argument starts with the observation from Proposition 3.3 that for the two-round case, any optimal threshold satisfies equation (3.5). Now, assume that this holds for any $T-1$ rounds, so that for each $t = 2, \ldots, T-1$, any sequence of decreasing thresholds that maximizes V_t satisfies the equilibrium condition that $k_{t-1} = m_t$. Then, for a sequence of thresholds $k_1, \ldots, k_{T-1}$ to maximize V_1, it is necessary that $k_{t-1} = m_t$ for all $t \geq 3$, and that $\partial V_1 / \partial k_1$, evaluated at $k_1, \ldots, k_{T-1}$, is equal to 0. We will show that these necessary conditions imply that $k_1 = m_2$, which establishes the proposition by induction.

To show $k_1 = m_2$, we recursively derive the expressions of V_1 and $\partial V_1 / \partial k_1$, both evaluated at the optimal sequence of thresholds $k_1, \ldots, k_{T-1}$. To start, from the induction assumption that $k_{T-1} = m_T$, we have $V_T = k_{T-1}^2$. To compute $\partial V_T / \partial k_1$, we rewrite

$$V_T = \left(k_{T-1} + \frac{G_{T-1}(k_{T-1}) \tilde{S}_{T-1}(k_{T-1}) - \tilde{G}_{T-1}(k_{T-1})}{R_T(k_{T-1})} \right)^2 .$$

Taking derivatives with respect to k_1, and evaluating at $k_{T-1} = m_T$, which, by integration by parts is equivalent to $G_{T-1}(k_{T-1}) \tilde{S}_{T-1}(k_{T-1}) = \tilde{G}_{T-1}(k_{T-1})$, we find that $\partial V_T / \partial k_1$ is given by

$$\frac{2k_{T-1}}{R_T(k_{T-1})} \left(G_{T-1}(k_{T-1}) \frac{\partial \tilde{S}_{T-1}(k_{T-1})}{\partial k_1} + \tilde{S}_{T-1}(k_{T-1}) \frac{\partial G_{T-1}(k_{T-1})}{\partial k_1} - \frac{\partial \tilde{G}_{T-1}(k_{T-1})}{\partial k_1} \right) .$$

Since $k_{T-1} < k_1$, we have

$$\tilde{S}_{T-1}(k_{T-1})\frac{\partial G_{T-1}(k_{T-1})}{\partial k_1} = \frac{\partial \tilde{G}_{T-1}(k_{T-1})}{\partial k_1},$$

and therefore

$$\frac{\partial V_T}{\partial k_1} = 2k_{T-1}\frac{G_{T-1}(k_{T-1})}{R_T(k_{T-1})}\frac{\partial \tilde{S}_{T-1}(k_{T-1})}{\partial k_1}.$$

Now, we can proceed to round $T-1$, and so on. In recursively computing $\partial V_t/\partial k_1$, we treat the thresholds $k_2, \ldots, k_{T-1}$ as independent variables, and recognize that the choice of k_1 affects only the sequence of distributions $G_2, \ldots, G_T$. We have

$$V_2 = m_2^2 - \sum_{t=2}^{T-1} 2k_t\tilde{S}_t(k_t)G_2(k_t),$$

and its derivative

$$\frac{\partial V_2}{\partial k_1} = 2m_2\frac{\partial \tilde{S}_2(k_2)}{\partial k_1} + \sum_{t=2}^{T-1} 2k_t\tilde{S}_t(k_t)\frac{\partial S_2(k_t)}{\partial k_1}.$$

Since for each $t = 2, \ldots, T-1$,

$$\frac{\partial S_2(k_t)}{\partial k_1} = f(k_1)S(k_1)\frac{G_2(k_t)}{R_2(k_1)},$$

and since $m_2 = k_2 + \tilde{S}_2(k_2)$, we have

$$\frac{\partial V_1}{\partial k_1} = -2f(k_1)(k_1S(k_1) + \tilde{S}(k_1))k_1 + 2f(k_1)S(k_1)m_2^2 + 2R_2(k_1)m_2\frac{\partial m_2}{\partial k_1}.$$

Using the definition of m_2 and integration by parts, we can rewrite m_2 as

$$m_2 = k_1 + \frac{1}{R_2(k_1)}(F(k_1)\tilde{S}(k_1) - \tilde{F}(k_1) + \tilde{F}(k_2)).$$

The derivative of m_2 with respect to k_1 is

$$\frac{\partial m_2}{\partial k_1} = \frac{f(k_1)}{R_2(k_1)}(\tilde{S}(k_1)F^2(k_1) + 2S(k_1)(\tilde{F}(k_1) - \tilde{F}(k_2))).$$

Substituting m_2 and $\partial m_2/\partial k_1$, we have

$$\frac{\partial V_1}{\partial k_1} = \frac{2f(k_1)}{R_2(k_1)}(F(k_1)\tilde{S}(k_1) + S(k_1)(\tilde{F}(k_1) - \tilde{F}(k_2)))(m_2 - k_1).$$

Thus, the optimal first round threshold k_1 satisfies $k_1 = m_2$, completing the induction argument. *Q.E.D.*

Proof of Lemma 5.2. For notational convenience, we drop the subscript $T-1$ from l_{T-1}, v_{T-1}, π_{T-1}, γ_{T-1} and μ_{T-1}. With $l = N$, $v = N-1$, and $\pi, \gamma \in (0,1)$, the condition for type x^{N-1} to be indifferent between accepting and rejecting type x^N is:

$$x^{N-1}x^N = x^{N-1}m_T - c. \tag{A.4}$$

For any $\gamma \in (0,1)$, equation (A.4) determines at least one $\pi \in (0,1)$. To see this, note that $m_T = x^N$ if $\pi = 0$ (because only types x^{N-1} and above participate in round $T-1$ and they accept each other and exit) so that the left-hand-side of equation (A.4) is strictly larger than the right-hand-side. On the other hand, $m_T > x^N$ if $\pi = 1$ (because types x^{N-1} and above have a positive probability of meeting and rejecting type x^N in round $T-1$), so that the left-hand-side is strictly smaller than the right-hand-side when c is sufficiently small. Furthermore, equation (A.4) implies m_T converges to x^N as c converges to 0. Hence any π that satisfies the equation becomes arbitrarily small.

Next, consider the condition for type x^N to be indifferent between participating in round $T-1$ and waiting for round T:

$$-c + \mu^N(x^N)^2 + \mu^{N-1}(1-\gamma)x^N x^{N-1} = (\mu^N + \mu^{N-1}(1-\gamma))(x^N m_T - c). \tag{A.5}$$

For any π, if $\gamma = 1$, then type x^N strictly prefers waiting for round T, as the left-hand-side of the above equation is strictly less than the right-hand-side. For c and π sufficiently small, if $\gamma = 0$, the left-hand-side of equation (A.5) is strictly greater than the right-hand-side. To see this, note that μ^N converges to 0 as π becomes close to 0, and so m_T converges to x^N for any γ. It follows that for sufficiently small c, at $\gamma = 0$, type x^N strictly prefers participating in round $T-1$.

By continuity of the solutions to equations (A.4) and (A.5), there is a pair $\pi, \gamma \in (0,1)$ that satisfies the two equations for sufficiently small c. Further, we know from equation (A.4) that m_T converges to x^N and π converges to 0 as c becomes close to 0, so from equation (A.5) we obtain that $\lim_{c\to 0}\gamma = 1$. *Q.E.D.*

Proof of Lemma 5.4. For notational convenience, we drop the subscript $T-1$ from l_{T-1}, v_{T-1}, π_{T-1}, γ_{T-1}, μ_{T-1} and g_{T-1}. For any $j = 1, \ldots, N$, let $q^j = \mathrm{E}_{T-1}[x \mid x \leq x_j]$ and $Q^j = \mathrm{E}_{T-1}[x \mid x \geq x_j]$. Given l, π, v and γ, we can write m_T as:

$$m_T = \underline{\omega} q^{l-1} + \omega_l x^l + \omega_v x^v + \overline{\omega} Q^{v+1},$$

where the weights are given by $R_T \underline{\omega} = \sum_{j>l} g^j$, $R_T \omega_v = g^v \mu^l \gamma$, $R_T \overline{\omega} = \mu^l \sum_{j<v} g^j$, and $R_T \omega_l = (1-\pi) g^l + \pi g^l \left(\gamma \mu^v + \sum_{j<v} \mu^j \right)$. We can rewrite $m_T > l$ as

$$R_T \omega_v (x^v - x^l) + R_T \overline{\omega} (Q^{v-1} - x^l) > R_T \underline{\omega} (x^l - q^{l+1}).$$

The left-hand-side of the above inequality is increasing in γ. It is also increasing in π because μ^l increases with π. Finally, we can verify that when v increases by 1, the left-hand-side changes by $\mu^l g^v (x^v - x^l)(1-\gamma) + \mu^l g^{v+1} (x^{v+1} - x^l)\gamma$, which is positive so long as $v < l-1$. Thus, the left-hand-side increases with v. Since the right-hand-side is constant, a necessary condition for $m_T > x^l$ for some π, γ and v is that m_T evaluated at $\pi = 1$, $\gamma = 1$ and $v = l-1$ is strictly greater than x^l. Substitution and manipulation of the terms in m_T give the inequality stated in the lemma. *Q.E.D.*

Proof of Proposition 5.5. We first prove the proposition for $T = 2$. For each N, set g_{T-1} to f_N in Lemma 5.4. We claim that for any threshold type $x > a$, the inequality (5.1) cannot be satisfied for sufficiently large N, implying that x cannot be an equilibrium participation threshold for any c sufficiently small. To see this, note that when $x > a$ the left-hand-side of the inequality in Lemma 5.4 converges to 0 as N becomes arbitrarily large. The numerator of the right-hand-side becomes arbitrarily close to the difference between x and the conditional mean below x, $q_N(x)$. We can bound $q_N(x)$ by

$$q_N(x) \leq \frac{F_N(x')}{F_N(x)} x' + \left(1 - \frac{F_N(x')}{F_N(x)}\right) x$$

for any $x' \in (a, x)$. By assumption, $F_N(x')/F_N(x)$ is bounded away from 0 as N becomes large, hence $q_N(x)$ is strictly smaller than x.

Next, suppose that the proposition is true for some $T = T'$. Then, for $T = T' + 1$, when N is sufficiently large and c is sufficiently small, in round 1 each type x faces a continuation payoff that is arbitrarily close to $xm_2 - c$, where m_2 the mean type among all agents remaining in round 2. Hence, Lemma 5.4 applies and the inequality (5.1) is necessary for any type x^l to be an equilibrium participation threshold. We have already shown that (5.1) cannot be satisfied for any $x^l > a$ when N is arbitrarily large. The proposition then follows from induction. *Q.E.D.*

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