The trade-off between incentives and endogenous risk

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Abstract

Standard models of moral hazard predict a negative relationship between risk and incentives, but the empirical work has not confirmed this prediction. In this paper, we propose a model with adverse selection followed by moral hazard, where effort and the degree of risk aversion are private information of an agent who can control the mean and the variance of profits. For a given contract, more risk-averse agents supply more effort in risk reduction. If the marginal utility of incentives decreases with risk aversion, more risk-averse agents prefer lower-incentive contracts; thus, in the optimal contract, incentives are positively correlated with endogenous risk. In contrast, if risk aversion is high enough, the possibility of reduction in risk makes the marginal utility of incentives increasing in risk aversion and, in this case, risk and incentives are negatively related.

1 Introduction

Moral hazard plays a central role in problems involving delegation of tasks. When the principal cannot perfectly observe the effort exerted by a risk-averse agent, the payment must be designed taking into account the trade-off between incentives and risk sharing. As the optimal level of incentives depends on the variance of output, the relationship between risk and incentives is an important testable implication of incentive models.

Standard models of moral hazard predict a negative relationship between risk and incentives. The central reference is the model presented in Holmstrom and Milgrom (1987). They analyze the conditions in which optimal contracts are linear, that is, the agent's payoff is a fixed part plus a proportion of profits. In their model, the negative relationship between risk and incentives results from the interaction between these two variables in the risk premium of the agent. As the agent is risk averse and incentives put risk in agent's payoff, incentives incur a cost in utility. At the optimal incentive, an increase in risk is balanced by a reduction in incentives.

The empirical work does not verify the negative relationship between risk and incentives, and sometimes finds opposite results. Prendergast (2002) presents a survey of empirical studies in three application fields, namely, executive compensation, sharecropping and franchising. Positive or insignificant relationships are found in the three fields and negative relationship is found only in studies about executive compensation. The conclusion is that the evidence is weak. Similarly, in the insurance literature, the monotone relationship between risk and coverage is not verified as reported in Chiappori and Salanié (2000).

The lack of empirical support has stimulated the search for alternative models, compatible with the observed facts. Prendergast (2002) suggests a theoretical model that assumes monitoring is harder in riskier environments. As incentives are a substitute for monitoring, incentives and risk are positively related. His model departs from Holmstrom-Milgrom structure and risk aversion does not play any role. Ghatak and Pandey (2000) analyze contract forms in agriculture developing a moral hazard model with risk neutral agents and limited liability. Their model is related to ours as the agent controls mean and variance of output; however, as limited liability induces riskier behavior, they assume it is costly to the agent to increase the risk of the project.

We propose a model with adverse selection, moral hazard and multitask. Principal is risk neutral and agent is risk averse. Multitask models were first developed in Holmstrom and Milgrom (1991), but in these models, effort controls exclusively the mean of the profits. In our work, we consider the possibility of manager to control the variance of the profits. Note that the resulting variance is endogenous, and we can define two types of risk: the exogenous risk is the intrinsic risk of the firm, and the endogenous risk is the one resulting from the effort of the agent in reducing variance. Another feature in our model is the presence of adverse selection before moral hazard. The principal does not know the risk aversion of the agent and designs a menu of contracts so that self-selection reveals the type of the agent. Sung (1995) extends the Holmstrom-Milgrom model showing that linear contracts are optimal in moral hazard problems in which the agent controls risk. Sung (2002) shows that linear contracts are optimal for mixed models of adverse selection before moral hazard. His model is close to ours as variance is controllable, however, as he models an observable project choice, variance is assumed to be a contractible variable, while we assume the principal cannot observe the choice of variance. Although the optimality of linear contract is not established for our model, we assume linearity and restrict the analysis to the space of linear contracts.

When the agent cannot control the risk of the project, the marginal cost of incentive is higher for an agent with more risk aversion. For this reason, more risk-averse agents select lower-powered incentive contracts. However, when agents can exert effort in risk reduction, the direction of selection may change. An agent with high risk-aversion may prefer a high incentive contract, as he can reduce risk and the cost associated with risk. Technically speaking, our model does not have the single-crossing property. Consequently, the relationship between the incentive given to the agent and his risk aversion is ambiguous. We computed the optimal contracts for representative situations and found that the relationship between endogenous risk and incentives is ambiguous. For a set of agent types with high risk-aversion, incentives and endogenous risk are negatively related. Conversely, for a set of agents with low risk-aversion, the relationship is positive. With respect to exogenous risk, the Holmstrom-Milgrom result is preserved: exogenous risk and incentives are negatively related. In Araujo and Moreira (2001b), a model akin to the one presented here is applied to the insurance market and an ambiguous relationship between coverage and risk is found.

In Section 2, we present the general model. In Section 3, we give two examples. First, the single-task model is examined and the traditional relationship between risk and incentives is found. In the second example, we present a multitask model where the agent can control the risk. In Section 4, we compute the optimal contracts for relevant cases of multitask model and we find positive and negative relationships. In Section 5 we state the concluding remarks. In Appendix A, we discuss, in general terms, implementability and optimality without the single-crossing property, and, in Appendix B, we examine the technical conditions for computing the optimal contract in the multitask example.

2 The Model

The principal delegates the management of the firm to the agent, whose effort can affect the probability distribution of the profits. Let e be the vector of efforts and z be the profits, with normal distribution $N(\mu(e), \sigma^2(e))$. Let $c(e)$ denote the cost of the effort for the agent. The agent has exponential utility with risk aversion $\theta > 0$, uniformly distributed on $\Theta = [\theta_a, \theta_b]$. At the time of contracting, the agent knows his risk aversion, but the principal does not. We will occasionally refer to θ as the type of the agent. We assume the wage is a linear function of the profits, that is, $w = \alpha z + \beta$, $0 \leq \alpha \leq 1$. The contract parameter α is the proportion of the profits received by the agent and is called the incentive, or the power, of the contract. The parameter β is the fixed part of the contract which is adjusted in order to induce the agent to participate.

The timing of the problem is as follows: (1) the agent learns his type, then (2) the principal offers a menu of contracts $\{\alpha(\theta), \beta(\theta)\}_{\theta\in\Theta}$, (3) the agent chooses a contract, and (4) exerts effort accordingly, (5) the firm produces profit z and (6) the agent receives $w = \alpha z + \beta$ and the principal earns the net profit, $z - w$. The certainty equivalence of the agent's utility is

$$V_{CE}(\alpha, \beta, \theta, e) = \beta + \alpha\mu(e) - c(e) - \frac{\alpha^2}{2}\theta\sigma^2(e),$$

that is, the expected wage, minus the cost of the effort and the risk premium. The last term is the origin of the negative relationship between risk and incentives in pure moral hazard models. The risk premium acts as a cost because the principal must compensate the agent to induce him to participate. Since the marginal risk premium with respect to α is increasing in both α and σ^2, the principal compensates an increase of σ^2 by a reduction of α, and equates the marginal cost and the marginal benefit of incentive. With adverse selection preceding moral hazard, a similar effect exists: the principal has to compensate the agent for the costs, in order to induce participation and truth-telling.

Let $e^*(\alpha, \theta)$ denote the agent θ's optimal choice of effort, given α. Note that e^* is independent of β. The resulting indirect utility is $V(\alpha, \beta, \theta) = \beta + v(\alpha, \theta)$, where

$$v(\alpha, \theta) = \alpha\mu(e^*(\alpha, \theta)) - c(e^*(\alpha, \theta)) - \frac{1}{2}\alpha^2\theta\sigma^2(e^*(\alpha, \theta)) \tag{1}$$

is the non-linear term. Thus, the problem is reduced to an adverse selection problem where the agent has quasi-linear utility $V(\alpha, \beta, \theta)$.

We assume the principal is risk-neutral. Her utility, given θ, is the expectation of the net profit, that is, the profit after the wage is paid to the agent,

$$U(\alpha, \beta, \theta) = E[z - w] = (1 - \alpha)\mu(e^*(\alpha, \theta)) - \beta,$$

where the expectation is taken with respect to the conditional distribution of z, given the effort choice of the agent θ under the contract (α, β).

The adverse selection problem is to find the functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$(\alpha(\cdot), \beta(\cdot)) \in \arg\max E[U(\alpha(\theta), \beta(\theta), \theta)] \tag{2}$$

subject to

$$V(\alpha(\theta), \beta(\theta), \theta) \geq V(\alpha(\hat{\theta}), \beta(\hat{\theta}), \theta), \text{for all } \theta, \hat{\theta} \in \Theta, \tag{3}$$

$$V(\alpha(\theta), \beta(\theta), \theta) \geq 0, \text{for all } \theta \in \Theta. \tag{4}$$

The expectation in (2) is taken with respect to θ. The constraint (3) is the incentive compatibility condition (IC). A function $\alpha(\cdot)$ is called implementable, if there is a function $\beta(\cdot)$ that satisfies IC.

The constraint (4) is the participation constraint where the reservation utility is normalized to be zero.

Guesnerie and Laffont (1984) fully characterize the optimal contract under the assumption of single-crossing property, that is, the cross derivative $v_{\alpha\theta}$ has constant sign. The solution of the model involves the definition of the virtual surplus

$$f(\alpha,\theta) = \mu(e^*(\alpha,\theta)) - c(e^*(\alpha,\theta)) - \frac{1}{2}\alpha^2\theta\sigma^2(e^*(\alpha,\theta)) + (\theta - \theta_a)v_\theta(\alpha,\theta). \tag{5}$$

The four terms represent the costs and the benefits considered in the optimization: the average of profits, the cost of effort, the risk premium, and the informational rent. The pointwise maximization of this function, that is, $\alpha_1(\theta) = \arg\max f(\alpha,\theta)$, is the relaxed solution. The incentive assignment of the optimal contract is the best monotone combination of the relaxed solution and intervals of bunching.

In our model, we may use the envelope theorem to derive the marginal utility of incentive,

$$v_\alpha(\alpha,\theta) = \mu(e^*(\alpha,\theta)) - \alpha\theta\sigma^2(e^*(\alpha,\theta)).$$

It is the mean of the profits minus the marginal risk premium. As agents with higher risk aversion exert more effort in risk reduction, the marginal risk premium term may increase or decrease with the agent's risk aversion. Consequently, the cross derivative $v_{\alpha\theta}$ may have any sign. The characterization of the optimal contracts in adverse selection problems without the single-crossing property is analyzed in Araujo and Moreira (2001a), and Appendix A presents some results that are relevant for the solution of our model. When the single-crossing property does not hold, discrete pooling may occur: a discrete set of agent types may choose the same contract.

3 Two Examples: Single-Task and Multitask

We now examine two cases. In the single-task case, the agent effort affects only the mean of the profit. We show that the degree of incentives in the optimal contract decreases with risk. In the multitask case, the variance and the mean are under control of the agent. Since the marginal cost of incentives depends on the endogenous variance, the optimal contract may have a complex shape that must be found numerically. Optimal contracts were computed for the multitask case and are presented in Section 4.

3.1 Single-Task

We first analyze the single-task specification where agent's effort controls only the mean of the profits. Let e_μ denote the effort and assume the mean of the profits is linear in e_μ, $\mu(e_\mu) = \mu e_\mu$, and the cost of effort is quadratic, $c(e_\mu) = e_\mu^2/2$.

The first-order condition of the agent's problem provides the optimal effort, $e_\mu^* = \alpha\mu$. As expected, effort increases with the power of incentives. The non-linear term of indirect utility is

$$v(\alpha, \theta) = \frac{\alpha^2}{2}\left(\mu^2 - \theta\sigma^2\right),$$

and the marginal utility of incentive is $v_\alpha = \alpha\mu^2 - \alpha\theta\sigma^2$. An increase in incentives has positive and negative effects on the utility of the agent. The positive effect is the increase of the share of profits. The negative effect comes from the increase of risk in the wage. The single-crossing property holds for this case, since $v_{\alpha\theta} = -\alpha\sigma^2 < 0$. An agent with low risk aversion has high marginal utility of incentive and may choose a high-powered incentive contract.

The virtual surplus, as defined in (5), is a concave function and the solution of the relaxed problem is given by the first-order condition $f_\alpha(\alpha_1(\theta), \theta) = 0$. Thus,

$$\alpha_1(\theta) = \frac{\mu^2}{\mu^2 + (2\theta - \theta_a)\sigma^2}.$$

The function α_1 is decreasing in θ and $v_{\alpha\theta}$ is negative. In this case, the optimal contract of the problem coincides with the relaxed solution. The variance σ^2 has also a negative effect on α, since it increases the marginal cost of incentives present in the risk premium and in the informational rent.

The relationship between α and σ^2 is still negative, given θ. Therefore, adverse selection before moral hazard is not sufficient to change the traditional risk-incentive trade-off. If agent controls only the mean of the profits, risk does not affect the benefit of principal, because she is risk neutral, but increases the marginal cost, because she has to compensate for the risk premium and has to pay the informational rent. Consequently, the incentives are lower in riskier projects.

3.2 Multitask

We introduce the possibility for the agent to control the variance of the profits. Let e_μ and e_σ be the effort exerted in mean increase and in variance reduction, respectively. We assume cost is quadratic and separable, $c(e) = \frac{1}{2}(e_\mu^2 + e_\sigma^2)$. Let $\mu(e) = \mu e_\mu$ and $\sigma^2(e) = (\sigma_0 - e_\sigma)^2$, where the exogenous variance, σ_0, is the variance when no effort is provided to reduce it. Given these functional forms, the optimal choices of effort are

$$e_\mu^* = \alpha\mu, \quad \text{and} \quad e_\sigma^* = \frac{\alpha^2\theta}{1+\alpha^2\theta}\sigma_0 < \sigma_0.$$

The effort in mean e_μ^* is higher, the higher is the incentive. The effort in variance reduction e_σ^* is higher, the higher is the incentive, the risk aversion and the exogenous variance of the profits. This is the expected result, since higher α provides incentive to the agent increase average profits, but, simultaneously, increases the risk of his payoff. The risk-averse agent is induced to reduce risk increasing e_σ^*, and this effect is stronger, the higher is the risk aversion. So, the endogenous variance, $\sigma^2(e^*) = \left(\frac{1}{1+\alpha^2\theta}\right)^2 \sigma_0^2 < \sigma_0^2$, is decreasing in α, for a given σ_0^2 and θ.

The non-linear term of indirect utility is

$$v(\alpha,\theta) = \frac{1}{2}\alpha^2\mu^2 - \frac{\alpha^2\theta\sigma_0^2}{2(1+\alpha^2\theta)}.$$

More intuitive expressions are obtained by the use of the envelope theorem:

$$v_\alpha = \mu e_\mu^* - \alpha\theta(\sigma_0 - e_\sigma^*)^2,$$
$$v_\theta = -\frac{\alpha^2}{2}(\sigma_0 - e_\sigma^*)^2 < 0.$$

The former states that the utility increases with α due to the mean of the profits, but decreases due to the risk premium. The latter states that informational rent decreases with risk aversion. From the former, the cross derivative is

$$v_{\alpha\theta} = \underbrace{\mu\frac{\partial e_\mu^*}{\partial\theta}}_{=0} \underbrace{- \alpha(\sigma_0 - e_\sigma^*)^2}_{<0} + \underbrace{2\alpha\theta(\sigma_0 - e_\sigma^*)\frac{\partial e_\sigma^*}{\partial\theta}}_{>0}.$$

The first term is zero, that is, the marginal utility is not affected by the effort in the mean of the profits. The other two terms stem from risk premium. The direct effect, $-\alpha(\sigma_0 - e_\sigma^*)^2$, has an interpretation similar to the one in the single-task case: the higher is the risk aversion, the higher is the effect of incentive on risk premium. The effect via effort, $2\alpha\theta(\sigma_0 - e_\sigma^*)\frac{\partial e_\sigma^*}{\partial\theta}$, acts in opposite direction; marginal utility increases with θ because more risk-averse agents exert more effort in risk reduction. In our example,

$$v_{\alpha\theta} = -\frac{\alpha(1-\theta\alpha^2)\sigma_0^2}{(1+\theta\alpha^2)^3} \tag{6}$$

and the function $\alpha_0(\theta) = 1/\sqrt{\theta}$ defines a decreasing border between $v_{\alpha\theta} > 0$ and $v_{\alpha\theta} < 0$ regions, with $v_{\alpha\theta} > 0$, for $\alpha > \alpha_0$. For less risk-averse agents, the direct effect dominates and the marginal utility of incentive decreases with risk aversion. For more risk-averse agents, the effort produces a stronger effect, such that the second term dominates and $v_{\alpha\theta} > 0$. This changes the self-selection direction, that is, an agent with a higher degree of risk aversion has a higher marginal utility of incentive, and chooses contracts with more power in incentives.

The next step is to define the virtual surplus and find the solution of the relaxed problem, $\alpha_1(\theta)$. The incentive schedule of the optimal contract is $\alpha_1(\theta)$, whenever the incentive compatibility constraint is satisfied. As the single-crossing property does not hold, two points have to be observed: first, the incentive compatibility cannot be trivially checked; and, second, if $\alpha_1(\theta)$ is not implementable, the computation of optimal contract must follow the procedure presented in Appendix A. The optimal incentive schedule may have a complex form, resulting from a combination of $\alpha_1(\theta)$, discrete pooling and continuous bunching.

We restrict the analysis to parameters values that satisfy the conditions in Araujo and Moreira (2001a), as explained in Appendix B. For given σ_0, μ and $[\theta_a, \theta_b]$, we compute the optimal contract $\alpha^*(\theta)$ and the endogenous risk $\sigma^2(e^*(\alpha^*(\theta), \theta))$, then we plot the function $\alpha^*(\theta)$, and the risk-incentive curve. In Section 4, the results for three representative cases are reported.

The relationship between incentives and endogenous risk is connected to the relationship between incentives and risk aversion. Note that

$$\sigma^2(e^*(\alpha(\theta), \theta)) = \left(\frac{1}{1 + \theta\alpha^2(\theta)} \right)^2 \sigma_0^2.$$

When $v_{\alpha\theta} > 0$, $\alpha(\theta)$ is increasing and, consequently, risk is decreasing in θ. Therefore the relationship between endogenous risk and incentives is negative. On the other hand, when $v_{\alpha\theta} < 0$, $\alpha(\theta)$ is decreasing and risk and incentives may be positively related if $\theta\alpha^2(\theta)$ is increasing in θ. That is, the endogenous risk decreases with risk aversion, provided that $\alpha(\theta)$ does not decrease too fast.

We show in Appendix B that the incentive in the relaxed solution is decreasing in σ_0, therefore the relationship between incentives and exogenous risk is negative when optimal contract coincides with relaxed solution. For more complex contract schedules, the relationship is obtained numerically.

4 Results

The equations above for the multitask example were numerically implemented for three cases that generate increasing, decreasing and mixed relationship between incentives and risk aversion. The parameter values, $\sigma_0 = 0.91$ and $\mu = 1$, are the same for the three cases, and the values of θ_a and θ_b change for each case. These values were chosen in order to generate functions that are tractable by the procedure detailed in Araujo and Moreira (2001a).

In Figure 1, for $\theta \in [2.5, 3.5]$, the dotted line $\alpha_0(\theta)$ is the border between the $v_{\alpha\theta} < 0$ region to the left, and the $v_{\alpha\theta} > 0$ region to the right. The relaxed solution $\alpha_1(\theta)$ is increasing in Θ, and coincides with the optimal contract. Figure 2 is the corresponding plot for risk and incentives. An agent with higher risk aversion exerts more effort in risk reduction and this behavior reduces the marginal cost from risk premium. This effect more than compensates the increase in marginal cost due to higher risk aversion. The net effect is that more risk-averse agents choose higher-powered incentive contracts and the relationship between risk and incentives is negative as in Holmstrom and Milgrom (1987).

The contract for a set of types with lower risk aversion, $\theta \in [0.5, 1.4]$, is shown in Figure 3. The relaxed solution is implementable as $v_{\alpha\theta}(\alpha_1(\theta_a), \theta_b) < 0$. The optimal contract coincides with the relaxed solution, but this time the relationship is reversed. More risk-averse agents have higher marginal cost of incentives, thus they prefer lower-powered incentive contracts. At the same time, more risk-averse agents exert more effort in risk reduction and the variance is lower. As is seen in Figure 4, the risk and incentives are positively related.

For a broader interval of types, that encompasses $v_{\alpha\theta}$ of both signs, the discrete pooling is possible and the optimal contract presents a U-shaped form. In Figure 5, the optimal contract for $\theta \in [0.7, 3.0]$ is plotted.[1] Computational procedures found the optimal contract that combines relaxed solution, discrete pooling and continuous bunching. Incentives and risk aversion are positively related for more risk-averse agents and negatively related for less risk-averse agents. The U-shape of the optimal contract is also present in risk-incentive graph, as we can see in Figure 6.

The results above are concerned with the endogenous risk. The relationship between exogenous risk and incentives is negative for the first two cases, since the optimal contracts coincide with the relaxed solutions. For the third case, the sensitivity $d\alpha/d\sigma_0$ was numerically calculated and plotted in Figure 7. Note that the sensitivity is negative, which suggests that the incentives decrease with exogenous risk.

[1] As prescribed in Appendix B, the validity of assumptions A2 and A3 were checked numerically.

5 Conclusion

The negative relationship between risk and incentives, found in standard models of moral hazard, is not preserved in the presence of adverse selection, if the agent can control the variance. A more risk-averse agent exerts more effort in reduction of risk. The relationship between risk and incentives is positive if more risk-averse agents select lower-powered incentive contracts. This is true when the marginal utility of incentive is decreasing with respect to the agent's risk aversion. However, if risk aversion is high enough, the possibility of risk reduction may reverse this effect and the traditional negative relationship between risk and incentives may be found. The optimal contract may also be U-shaped, such that agents with intermediate degrees of risk aversion choose contracts with low incentives, and agents with extremely high or extremely low degree of risk aversion choose high-powered incentive contracts. These conclusion holds for endogenous risk. With respect to the exogenous risk, the numerical calculations suggest that the relationship between incentives and risk remains negative.

Apendix A

A Adverse Selection without the Single-Crossing Property

The general model presented in Section 2 reduces to the maximization problem (2) subject to incentive compatibility and participation constraints. It differs from the traditional adverse selection model because the objective function does not have the single-crossing property. We present below the main steps toward the solution, stressing the peculiarities that arise when single-crossing property is absent. Most of the results are developed in Araujo and Moreira (2001a).

A.1 Incentive Compatibility and Participation Constraint

When $\alpha(\cdot)$ and $\beta(\cdot)$ are differentiable, the incentive compatibility may be locally checked by the first and second order conditions. These conditions are necessary but not sufficient for incentive compatibility. The first order condition gives

$$v_\alpha(\alpha(\theta), \theta)\alpha'(\theta) + \beta'(\theta) = 0, \tag{7}$$

which states that indifference curves of type θ agent must be tangent to an implementable contract on $\alpha \times \beta$ plane, at point $(\alpha(\theta), \beta(\theta))$.

The second order condition gives

$$v_{\alpha\alpha}(\alpha(\theta), \theta)[\alpha'(\theta)]^2 + v_\alpha(\alpha(\theta), \theta)\alpha''(\theta) + \beta''(\theta) \leq 0, \tag{8}$$

and, after differentiating (7) with respect to θ, the expression (8) simplifies to the condition

$$v_{\alpha\theta}(\alpha(\theta), \theta)\alpha'(\theta) \geq 0, \tag{9}$$

which implies the monotonicity of $\alpha(\theta)$, in the single-crossing context.

Given the menu of implementable contracts $\{\alpha(\theta), \beta(\theta)\}_{\theta \in \Theta}$, the level of utility achieved by the agent with risk aversion θ is his informational rent and denoted $r(\theta)$, that is, $r(\theta) = v(\alpha(\theta), \theta) + \beta(\theta)$. Using (7), we get

$$r'(\theta) = v_\theta(\alpha(\theta), \theta), \tag{10}$$

and applying the envelope theorem on equation (1), we have $v_\theta(\alpha, \theta) = -\frac{1}{2}\alpha^2\sigma^2(e^*) < 0$. Consequently, the agent with the highest the risk aversion has the lowest informational rent and the participation constraint is active for him, that is, $r(\theta_b) = 0$.

Thus, the fixed component of the wage can be isolated by integration of $r'(\theta)$,

$$\beta(\theta) = -\int_\theta^{\theta_b} v_\theta(\alpha(\tilde{\theta}), \tilde{\theta})d\tilde{\theta} - v(\alpha(\theta), \theta), \tag{11}$$

which allows us to eliminate $\beta(\cdot)$ from the problem and focus on the characterization of $\alpha(\cdot)$.

A.2 Implementability without the Single-Crossing Property

Since the single-crossing property is not ensured, the first and the second order conditions are necessary but they are not sufficient. The following points must be observed:

1. The function $\alpha(\theta)$ may be non-monotone. The same contract may be chosen by a discrete set of agents. We call this situation as discrete pooling. In this case, the pooled types follow the conjugation rule

$$v_\alpha(\alpha(\theta), \theta) = v_\alpha(\alpha(\theta'), \theta'), \tag{12}$$

whenever $\alpha(\theta) = \alpha(\theta')$, which states that the indifference curves of θ and θ' are both tangent at the same point to the menu of contracts on $\alpha \times \beta$ plane.

2. The incentive compatibility must be globally checked. When the single-crossing property holds, local incentive compatibility implies global incentive compatibility, that is, if types in the neighborhood of θ is not better with the contract assigned to θ, no other type will be better. This means that the first and second order conditions are sufficient for incentive compatibility. On the other hand, when the single-crossing property is violated, types out of the neighborhood of θ may prefer the contract assigned to θ. In this case, the first and second order conditions are not sufficient and further conditions must be imposed to obtain implementability.

3. The function $\alpha(\theta)$ may be discontinuous. The possibility of discrete pooling creates jumps in the optimal assignment of contracts, so we allow the contract to be piecewise continuous. Where jump occurs, the agent must be indifferent between the start and the end point of the jump. If, for example, the agent θ were strictly better with the end point than the start point, then, for a small $\varepsilon > 0$, the agents with type in $[\theta - \varepsilon, \theta]$ would strictly prefer the end point, and no jump could exist in θ.

The following definition will be useful for global analysis of incentive compatibility. For a given contract $\alpha(\theta)$ define the integral $\Phi(\theta, \hat{\theta})$ as

$$\Phi(\theta, \hat{\theta}) = \int_{\theta}^{\hat{\theta}} \left[\int_{\alpha(\tilde{\theta})}^{\alpha(\hat{\theta})} v_{\alpha\theta}(\tilde{\alpha}, \tilde{\theta}) d\tilde{\alpha} \right] d\tilde{\theta}. \tag{13}$$

It can be shown, using (10), that $\Phi(\theta, \hat{\theta}) = V(\alpha(\theta), \beta(\theta), \theta) - V(\alpha(\hat{\theta}), \beta(\hat{\theta}), \theta)$, thus $\Phi(\theta, \hat{\theta})$ is the difference for agent θ between the utility of the contract assigned to himself and the one assigned to $\hat{\theta}$. The incentive compatibility constraint can be stated as

$$\Phi(\theta, \hat{\theta}) \geq 0, \quad \text{for all } \theta, \hat{\theta} \in \Theta,$$

that is, the agent with risk aversion θ is not better pretending to be an agent with risk aversion $\hat{\theta}$. The function $\Phi(\theta, \hat{\theta})$ is appropriate for a graphical analysis, since the signal of $v_{\alpha\theta}$ is known and the integration is performed in the region between the constant $\alpha(\hat{\theta})$ and the curve $\alpha(\tilde{\theta})$.

A.3 Virtual Surplus and the Principal's Problem

We follow the standard procedure and define the social surplus,

$$S(\alpha, \theta) = \mu(e^*(\alpha, \theta)) - c(e^*(\alpha, \theta)) - \frac{1}{2}\alpha^2\theta\sigma^2(e^*(\alpha, \theta)), \tag{14}$$

and virtual surplus,

$$f(\alpha,\theta) = S(\alpha,\theta) + (\theta - \theta_a)v_\theta(\alpha,\theta). \tag{15}$$

The maximization of social surplus for each θ gives the first best of the model. The virtual surplus is the social surplus plus the informational rent term. This term is negative and represents a cost that takes into account the rent that is paid to the agents with risk aversion in $[\theta_a, \theta]$, in order to preserve implementability when agent θ receives $\alpha(\theta)$.

As types are uniformly distributed, the expectation of integral term in (11) may be simplified by Fubini's theorem to, $E\left[\int_\theta^{\theta_b} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta}\right] = E\left[v_\theta(\alpha(\theta),\theta)(\theta-\theta_a)\right]$. Thus, $\beta(\theta)$ can be eliminated from the principal's objective function, which can be rewritten as $E[f(\alpha(\theta),\theta)]$. After the optimal incentive, $\alpha^*(\theta)$, is found, the fixed part of optimal contract, $\beta^*(\theta)$, can be calculated using (11).

The maximization problem of principal without the constraints is called relaxed problem. Its solution, denoted $\alpha_1(\theta)$, satisfies

$$f_\alpha(\alpha_1(\theta),\theta) = 0 \quad \text{and} \quad f_{\alpha\alpha}(\alpha_1(\theta),\theta) < 0.$$

Since $f_\alpha(\alpha_1(\theta),\theta) = S_\alpha(\alpha_1(\theta),\theta) + (\theta-\theta_a)v_{\alpha\theta}(\alpha_1(\theta),\theta)$, the relaxed solution provides less incentive than the first best when $v_{\alpha\theta} < 0$, and more incentive when $v_{\alpha\theta} > 0$. This distortion occurs because the cross derivative is associated with the marginal cost of informational rent. For example, when $v_{\alpha\theta} < 0$, the cost of informational rent is increasing with respect to α, therefore the principal pays less incentive.

A.4 Optimality without the Single-Crossing Property

In the standard adverse selection model, the single-crossing property ensures that $\alpha_1(\theta)$ is the optimal contract if (9) is satisfied, that is, $\alpha_1(\theta)$ is non-increasing when $v_{\alpha\theta} < 0$, or non-decreasing when $v_{\alpha\theta} > 0$. When $\alpha_1(\theta)$ is non-monotone, the optimal contract is the best combination of $\alpha_1(\theta)$ and intervals of bunching so that (9) is satisfied. Such procedure is not suitable in the absence of the single-crossing property. As before, $\alpha_1(\theta)$ is the optimal contract if it is implementable. However, monotonicity condition (9) is no more sufficient for implementability and global incentive condition must be checked.

When $v_{\alpha\theta}$ changes its sign, the discrete pooling is possible and $\alpha_1(\theta)$ is not the optimal contract for the pooled types. The assignment of contracts to the discretely pooled types must take into account the conjugation of types according to the constraint (12). Let $\alpha_u(\theta)$ denote the optimum assignment of contracts with discrete pooling. Then the joint maximization of pooled types results in the condition

$$\frac{f_\alpha(\alpha_u(\theta),\theta)}{v_{\alpha\theta}(\alpha_u(\theta),\theta)} = \frac{f_\alpha(\alpha_u(\theta'),\theta')}{v_{\alpha\theta}(\alpha_u(\theta'),\theta')}. \tag{16}$$

where θ' is given by $v_\alpha(\alpha_u(\theta),\theta) = v_\alpha(\alpha_u(\theta'),\theta')$ and $\alpha_u(\theta) = \alpha_u(\theta')$. The optimal contract will be a combination of $\alpha_1(\theta)$, bunching and $\alpha_u(\theta)$.

We follow Araujo and Moreira (2001a) and restrict the solution $\alpha^*(\theta)$ to the closure of the continuous functions. It means that when there is a jump in $\alpha(\theta)$ all the intermediate contracts in the jump is offered to the agent. The optimal contract with discrete pooling can be characterized under the following assumptions:

A1. $v_{\alpha\theta}(\alpha,\theta) = 0$ defines a decreasing function $\alpha_0(\theta)$, $v_{\alpha\theta}$ is positive above and negative below $\alpha_0(\theta)$, for all $\theta \in \Theta$.

A2. α_1 is U-shaped, crosses α_0 in an increasing way, $\alpha_1(\theta_a) \leq \alpha_1(\theta_b)$, $f_\alpha(\alpha,\theta)$ is negative above and positive below $\alpha_1(\theta)$, for all $\theta \in \Theta$.

A3. For each $\hat{\theta}$, the equations $v_\alpha(\alpha_1(\cdot),\cdot) = v_\alpha(\alpha_1(\cdot),\hat{\theta})$ have at most one solution in the decreasing part of α_1, on $v_{\alpha\theta} < 0$ region.

Under these assumptions, the optimal contract, $\alpha^*(\theta)$, will have one of the following forms:

$$\alpha^*(\theta) = \begin{cases} \alpha_u(\theta), & \text{if } \theta < \theta_1, \\ \alpha_1(\theta), & \text{if } \theta \geq \theta_1, \end{cases} \tag{17}$$

where θ_1 is defined by $\alpha_u(\theta_1) = \alpha_u(\theta_a)$, [2] or

$$\alpha^*(\theta) = \begin{cases} \alpha_1(\theta), & \text{if } \theta < \theta_2, \\ \min\{\bar{\alpha}, \alpha_u(\theta)\}, & \text{if } \theta \geq \theta_2, \end{cases} \tag{18}$$

[2] To be rigorous, we should consider the case in which the jump transition from α_u-segment to α_1-segment takes place in $\theta_j < \theta_1$. In this case, the contracts for $[\theta_a, \hat{\theta}_j]$, where $\hat{\theta}_j$ is the conjugate of θ_j, are the conjugates of the contracts in the vertical line, at the jump. For the examples worked in this paper, the characterization above suffices. For further details see Araujo and Moreira (2001a)

where $\bar{\alpha}$ is the incentive of the continuous bunching and θ_2 is defined by $\alpha_1(\theta_2) = \bar{\alpha}$. The set of bunched types, $J = \{\theta \in \Theta : \alpha(\theta) = \bar{\alpha}\}$, satisfies

$$\int_J f_\alpha(\bar{\alpha}, \theta) p(\theta) d\theta = 0.$$

Apendix B

B Optimal Contract in the Multitask Specification

The following expression is the virtual surplus of the problem,

$$f(\alpha, \theta) = \frac{\alpha(2-\alpha)}{2}\mu^2 - \frac{\alpha^2(\alpha^2\theta^2 + 2\theta - \theta_a)}{2(1+\alpha^2\theta)^2}\sigma_0^2.$$

The derivative with respect to α is

$$f_\alpha(\alpha, \theta) = (1-\alpha)\mu^2 - \frac{\alpha[\theta(1+\alpha^2\theta_a) + (\theta - \theta_a)]}{(1+\alpha^2\theta)^3}\sigma_0^2$$

and the relaxed solution $\alpha_1(\theta)$ is given by $f_\alpha(\alpha_1(\theta), \theta) = 0$ and $f_{\alpha\alpha}(\alpha_1(\theta), \theta) < 0$. Note that $f_\alpha(0, \theta) > 0$ and $f_\alpha(1, \theta) < 0$, so relaxed problem has an interior solution and $f_\alpha(\cdot, \theta)$ has at least one root in the interval $[0, 1]$. If $f(\cdot, \theta)$ is not concave in α, the incentive that maximizes the virtual surplus must be correctly chosen among solutions of the first order condition.

Writing f_α as a function of σ_0, it is ease to see that $\partial f_\alpha / \partial \sigma_0 < 0$, and, as $f_{\alpha\alpha}(\alpha_1(\theta), \theta) < 0$, the application of the theorem of implicit function on $f_\alpha(\alpha_1(\theta), \theta) = 0$ gives $d\alpha_1/d\sigma_0 < 0$. That is, for a given θ, an increase of exogenous risk reduces incentives on relaxed solution.

When $v_{\alpha\theta}(\alpha_1(\theta), \theta)$ has ambiguous sign, the optimal contract must consider the possibility of discrete pooling. When θ and $\hat{\theta}$ are discretely pooled at incentive α, the conjugation rule (12) relates the pooled types by $\hat{\theta}(\alpha, \theta) = 1/\theta\alpha^4$. Then, working on condition (16), we obtain the discrete pooling segment $\alpha_u(\theta)$ as the solution of the equation

$$(1-\alpha)(1+\theta\alpha^2)^2(1+\theta^2\alpha^4) = 2\theta^2\alpha^3\frac{\sigma_0^2}{\mu^2}.$$

The numerical examples presented in Section 4 correspond to three cases for which we can characterize the optimal contract.

(a) $\alpha_1(\theta)$ is increasing and $v_{\alpha\theta}(\alpha_1(\theta), \theta) > 0$.

Since $\alpha_0(\theta)$ is decreasing, the integral in $\Phi(\theta, \hat{\theta})$ takes values in $v_{\alpha\theta} > 0$ region. Therefore $\Phi(\theta, \hat{\theta}) > 0$ and $\alpha_1(\theta)$ is the optimal contract.

(b) $\alpha_1(\theta)$ is decreasing and $v_{\alpha\theta}(\alpha_1(\theta),\theta) < 0$.

A sufficient condition for implementability is $v_{\alpha\theta}(\alpha_1(\theta_a),\theta_b) < 0$. As $\alpha_0(\theta)$ is a decreasing function, the integral in $\Phi(\theta,\hat{\theta})$ takes values in $v_{\alpha\theta} < 0$ region. Then $\Phi(\theta,\hat{\theta}) > 0$ and $\alpha_1(\theta)$ is the optimal contract.

(c) $v_{\alpha\theta}(\alpha_1(\theta),\theta)$ changes sign only once.

In this case, the optimal contract can be computed by the procedure in Appendix A, if assumptions A1, A2 and A3 hold. Assumption A1 holds since, from equation (6), the function $\alpha_0(\theta) = 1/\sqrt{\theta}$ defines a decreasing border between $v_{\alpha\theta} > 0$ and $v_{\alpha\theta} < 0$ regions, with $v_{\alpha\theta} > 0$, for $\alpha > \alpha_0$. The following lemma shows that the first part of assumption A2 holds.

Lemma 1 Let θ_x be defined by $\alpha_1(\theta_x) = \alpha_0(\theta_x)$. If θ_x exists, $\alpha_1'(\theta_x) > 0$.

Proof: By definition, $\alpha_1(\theta)$ satisfies $f_\alpha(\alpha_1(\theta),\theta) = 0$. Using the implicit function theorem,

$$\alpha_1'(\theta) = -\frac{f_{\alpha\theta}(\alpha_1(\theta),\theta)}{f_{\alpha\alpha}(\alpha_1(\theta),\theta)},$$

and, as second order condition states that $f_{\alpha\alpha}(\alpha_1(\theta),\theta) < 0$, $\alpha_1'(\theta)$ has the same sign as $f_{\alpha\theta}(\alpha_1(\theta),\theta)$. Differentiating f_α with respect to θ,

$$f_{\alpha\theta}(\alpha,\theta) = \frac{-2\alpha[1 - 2\alpha^2(\theta-\theta_a) - \alpha^4\theta\theta_a]}{(1+\alpha^2\theta)^4}$$

and manipulating this expression, we conclude that $\alpha_1'(\theta)$ has the same sign as

$$h(\alpha,\theta) = \theta - \frac{1+2\alpha^2\theta_a}{\alpha^2(2+\alpha^2\theta_a)}.$$

On $\alpha_0(\theta)$, $\alpha = 1/\sqrt{\theta}$. Then, $h(\alpha_0(\theta_x),\theta_x) = \theta_x(1-\theta_a/\theta_x)(2+\theta_a/\theta_x)$, which is positive for $\theta_x > \theta_a$. Therefore $\alpha_1'(\theta_x) > 0$. ¥

However, the second part of A2, and A3 is not valid for every value of parameters and must be checked before the application of the procedure in Appendix A.

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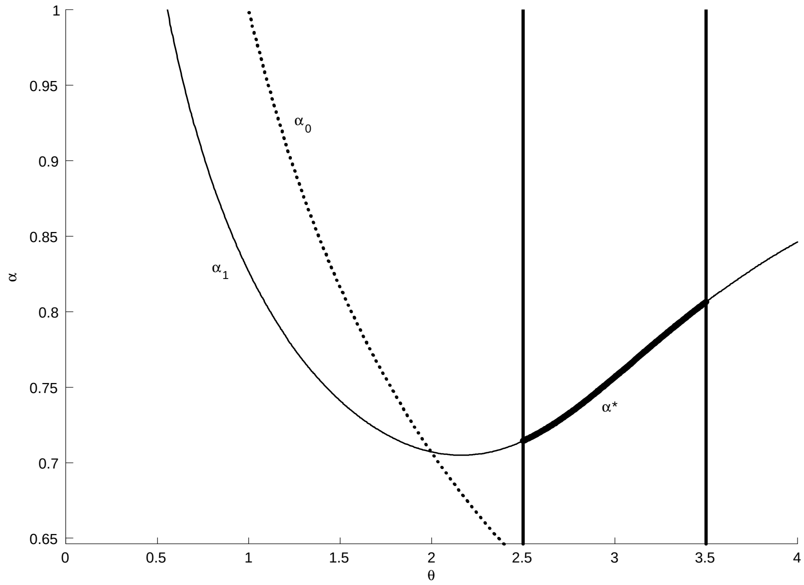

Figure 1: Optimal contract. $\Theta = [2.5, 3.5]$.

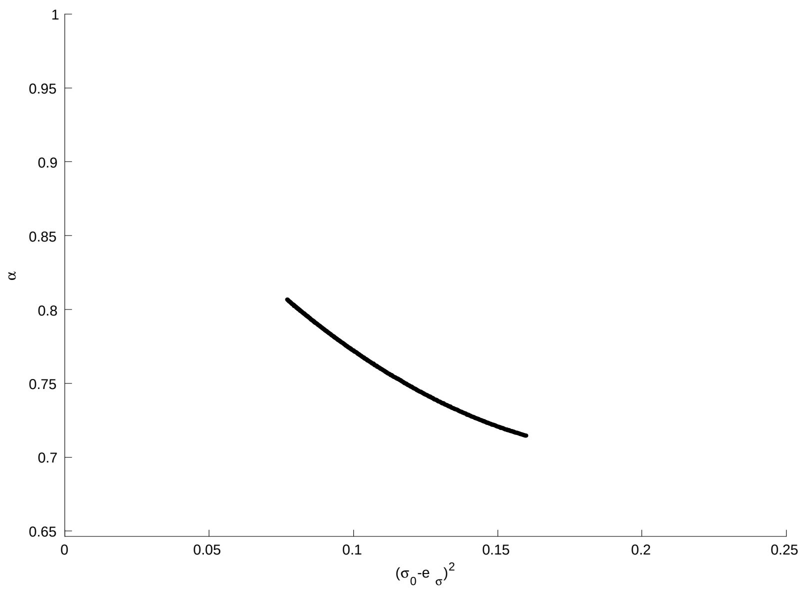

Figure 2: Risk × incentives. $\Theta = [2.5, 3.5]$.

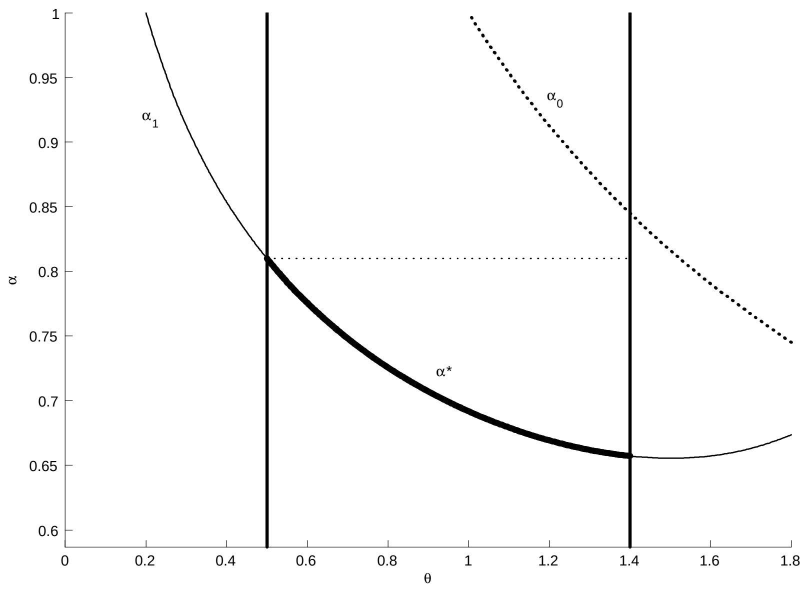

Figure 3: Optimal contract. $\Theta = [0.5, 1.4]$.

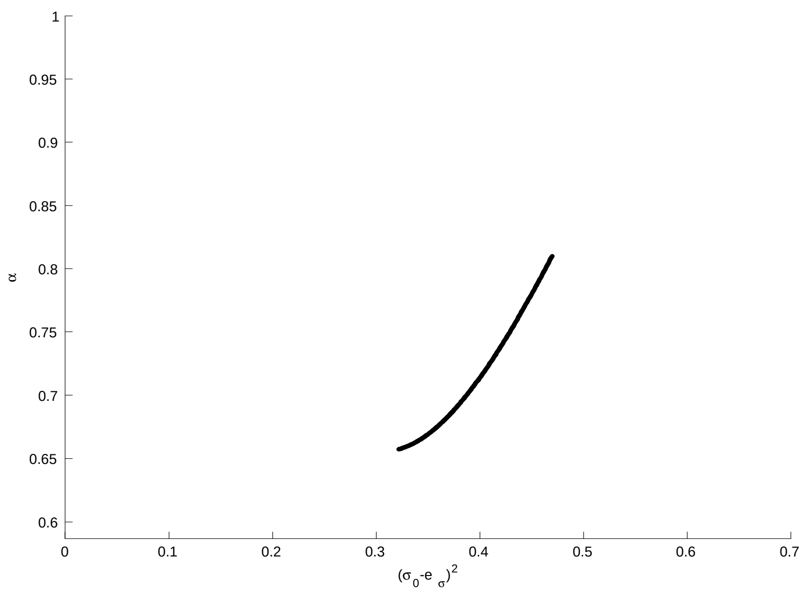

Figure 4: Risk $\times$ incentives. $\Theta = [0.5, 1.4]$.

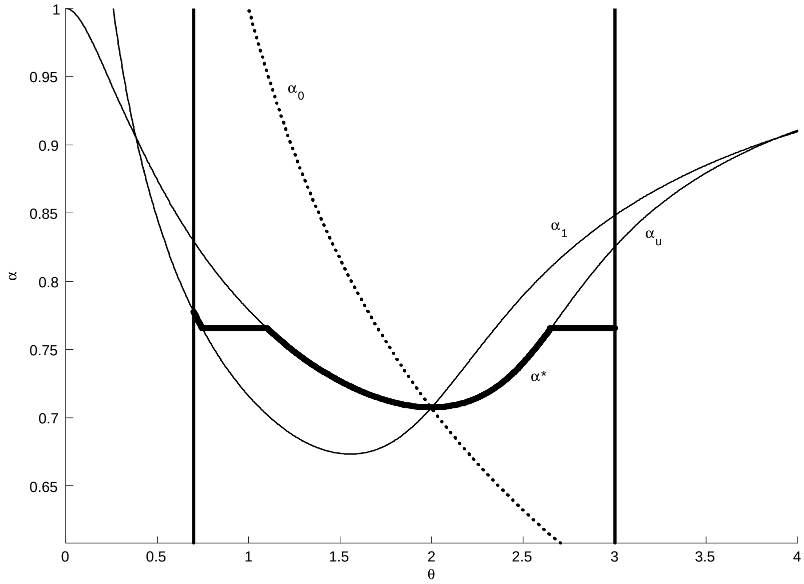

Figure 5: Optimal contract. $\Theta = [0.7, 3.0]$.

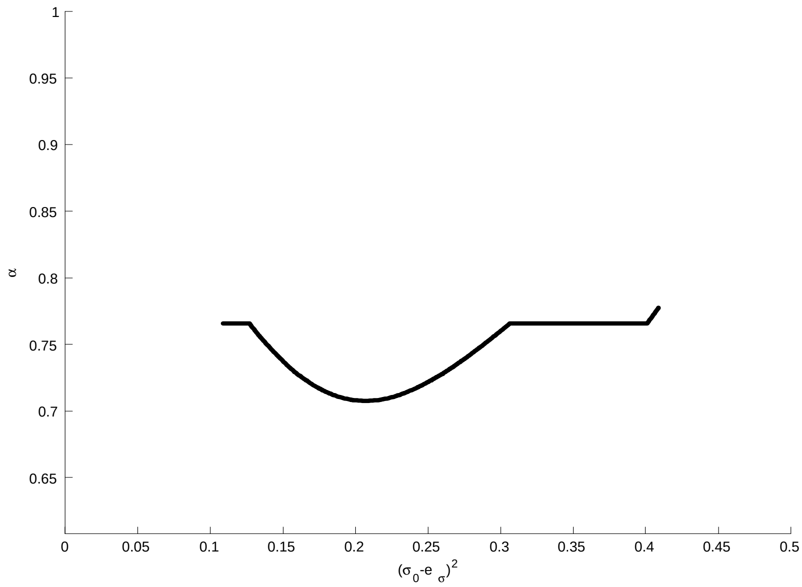

Figure 6: Risk × incentives. $\Theta = [0.7, 3.0]$.

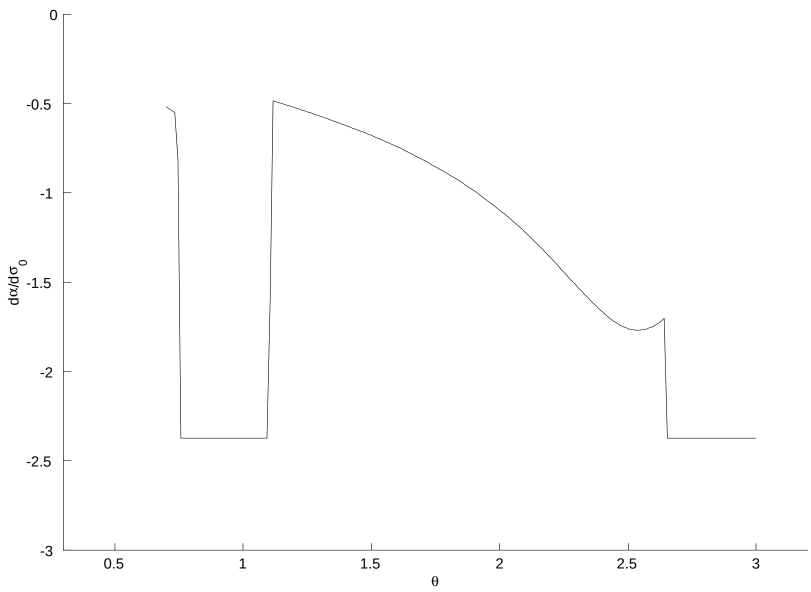

Figure 7: Exogenous risk × incentives. $\Theta = [0.7, 3.0]$.