Equilibrium Directed Search with Multiple Applications*

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1 Introduction

In this paper, we construct an equilibrium model of directed search in a large labor market. Unemployed workers, observing the wages posted at all vacancies, direct their applications towards the vacancies they find most attractive. At the same time, (the owners of) vacancies post wages taking into account that their posted wages influence the number of applicants they attract. In our model, each unemployed worker makes a fixed number of applications, a. Each vacancy (among those receiving applications) then chooses one applicant to whom it offers its job. When $a > 1$, there is a possibility that more than one vacancy will want to hire the same worker. In this case, we assume that the vacancies in question can compete for this worker's services.

When $a = 1$, our model is the same as the limiting version of Burdett, Shi, and Wright (2001) (hereafter BSW), albeit translated to a labor market setting. BSW derive a unique symmetric equilibrium in which all vacancies post a wage between zero (the monopsony wage) and one (the competitive wage). The value of this common posted wage depends on the number of unemployed, u, and the number of vacancies, v, in the market. Letting $u, v \to \infty$ with $v/u = \theta$, the equilibrium posted wage is an increasing function of θ, i.e., of labor market tightness. This is a model of competitive search equilibrium, as in Moen (1997). As is well known, this means that equilibrium is constrained efficient. Assume there is a cost per vacancy created. A social planner would choose a level of vacancy creation – or, equivalently in a large labor market, a level of labor market tightness – to trade off the cost of vacancy creation against the benefit of making it easier for workers to match. Moen (1997) shows that the θ the social planner would choose is the same as the one that arises in free-entry equilibrium.

Our model is also related to Julien, Kennes, and King (2000) (hereafter JKK). JKK assume that each unemployed worker posts a minimum wage at which he or she is willing to work, i.e., a "reserve wage," and that each vacancy, observing all posted reserve wages, then makes an offer to one worker. If more than one vacancy wants to hire the same worker, then, as in our model, there is *ex post* competition for that worker's services. This is equivalent to a model in which each worker applies to every vacancy, i.e., $a = v$, sending the same reserve wage in each application. Each vacancy then chooses one worker at random to whom it offers a job. If a worker has more than one offer, then there is competition for his or her services. In a finite labor market, JKK show that the unique, symmetric equilibrium reserve wage lies between the monopsony and competitive levels. There is thus equilibrium wage dispersion in their model. Those workers who receive only one offer are employed at the reserve wage, while those who receive multiple offers are employed at the competitive wage. In the limiting labor market version of JKK, the symmetric equilibrium reserve wage converges to zero, and free-entry equilibrium is again constrained efficient.

When $a \in \{2, ..., A\}$, where A is fixed, i.e., when each worker makes a finite number of multiple applications, our results differ radically from those of BSW and Moen (1997). We show that all vacancies post the monopsony wage in the unique symmetric equilibrium. As in JKK, this leads to equilibrium wage dispersion. Some workers (those who receive exactly one offer) are employed at the monopsony wage, and some workers (those who receive multiple offers) have their wages bid up to the competitive level. The key difference, however, is that free-entry equilibrium is inefficient; there is excessive vacancy creation. Equivalently, the expected wage paid is too low in equilibrium.

In the next section we derive our basic positive results in a single-period framework, and in Section 3, we give our results on constrained efficiency. In Section 4, we present a steady-state version of our model for the case of $a \in \{2, .., A\}$. The key to the steady-state analysis is that a worker who receives only one offer in the current period has the option to reject that offer in favor of waiting for a future period in which more than one vacancy bids for his or her services. Allowing for free entry of vacancies, this leads to a tractable model in which labor market tightness and the equilibrium wage distribution are simultaneously determined. The normative results that we derived in the single-period model continue to hold in the steady-state setting. Finally, in Section 5, we conclude.

2 The Basic Model

We consider a game played by u homogeneous unemployed workers and (the owners of) v homogeneous vacancies. This game has several stages:

1. Each vacancy posts a wage.

2. Each unemployed worker observes all posted wages and then submits a applications with no more than one application going to any one vacancy.

3. Each vacancy that receives at least 1 application randomly selects one to process. Any excess applications are returned as rejections.

4. A vacancy with a processed application offers the applicant the posted wage. If more than one vacancy makes an offer to a particular worker, then those vacancies can bid against one another for that worker's services.

5. A worker with one offer can accept or reject that offer. A worker with more than one offer can accept one of the offers or reject all of them.

Workers who fail to match with a vacancy and vacancies that fail to match with a worker receive payoffs of zero. The payoff for a worker who matches with a vacancy is w, where w is the wage that he or she is paid. A vacancy that hires a worker at a wage of w receives a payoff of $1 - w$.

Before we analyze this game, some comments on the underlying assumptions are in order. First, this is a model of directed search in the sense that workers observe all wage postings and send their applications to vacancies with attractive wages and/or where relatively little competition is expected. We assume that vacancies cannot pay less than their posted wages. If they could, directed search would not make sense. Second, we are treating a as a parameter of the search technology; that is, the number of applications is taken as given. In general, $a \in \{1, 2, ..., A\}$. Third, we assume that it takes a period for a vacancy to process an application. This is why vacancies return excess applications as rejections. This processing time assumption is important for our results. It captures the idea that when workers apply for several jobs at the same time, firms can waste time and effort pursuing applicants who ultimately go elsewhere. Finally, we assume that 2 or more vacancies that are competing for the same worker can engage in *ex post* Bertrand competition for that worker. This means that workers who receive more than one offer will have their wages bid up to $w = 1$, the competitive wage. There are, of course, other possible "tie-breaking" assumptions. For example, one might assume that vacancies hold to their posted wages, that is, refuse to engage in *ex post* bidding.

We consider symmetric equilibria in which all vacancies post the same wage and all workers use the same strategy to direct their applications. We do this in a large labor market in which we let $u, v \to \infty$ with $v/u = \theta$ keeping $a \in \{1, 2, ..., A\}$ fixed. We show that for each (θ, a) combination there is a unique symmetric equilibrium and we derive the corresponding equilibrium matching probability and posted wage. Assuming (for the moment) the existence of a symmetric equilibrium, we begin with the matching probability.

Let $M(u, v; a)$ be the expected number of matches in a labor market with u unemployed workers and v vacancies when each unemployed workers submits a applications. Then $m(\theta;a) = \lim_{u,v\to\infty, v/u=\theta} \frac{M(u,v;a)}{u}$ is the matching probability for an unemployed worker in a large labor market.

Proposition 1 *Let $u, v \to \infty$ with $v/u = \theta$ and $a \in \{1, ..., A\}$ fixed. The probability that a worker finds a job converges to*

$$m(\theta;a) = 1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a. \tag{1}$$

The proof is given in Albrecht, et. al (2003); see also Philip (2003). Here we sketch the idea of the proof to clarify the relationship between our matching probability and the finite-market matching functions presented in BSW (the standard urn-ball matching function) and JKK (the urn-ball matching function with the roles of u and v reversed).

We compute $m(\theta;a)$ as follows. The probability that a worker finds a job is one minus the probability that he or she gets no job offers. Consider a worker who applies to a vacancies, and let the random variables $X_1, X_2, ..., X_a$ be the number of competitors that he or she has at vacancy 1, vacancy 2, ..., vacancy a. The probability that the worker gets no job offers can be expressed as

$$\sum ... \sum \frac{x_1}{x_1+1}\frac{x_2}{x_2+1}...\frac{x_a}{x_a+1}P[X_1 = x_1, X_2 = x_2, ...X_a = x_a].$$

In general, the random variables $X_1, X_2, ..., X_a$ are not independent, making the computation of the joint probability a difficult one. (Albrecht, et. al. 2003 and Philip 2003 give an expression for the joint probability.) The intuition for dependence is simple. Consider, for example, a labor market in which u and v are small and in which each worker makes $a = 2$ applications. Then, if a worker has relatively many competitors at the first vacancy to which he or she applies, it is more likely that his or her second application has relatively few competitors. The key to Proposition 1 is that this dependence vanishes in the limit. In that case, the fact that a worker has an unexpectedly large number of competitors at one vacancy says next to nothing about the number of competitors he or she faces elsewhere. The joint probability then equals the product of the marginals, and the probability a worker gets at least one offer can be computed as $1 - \left(\sum \frac{x}{x+1}P[X = x]\right)^a$, which, letting $u, v \to \infty$ with $v/u = \theta$, leads to equation (1).

If $a = 1$, there is no problem of dependence. The number of competitors that a worker has at the vacancy to which he or she applies is a $bin(u-1, \frac{1}{v})$

random variable. The probability that a worker gets an offer is then

$$1-\sum_{x=0}^{u-1}\frac{x}{x+1}\binom{u-1}{x}\left(\frac{1}{v}\right)^x\left(1-\frac{1}{v}\right)^{u-1-x}=\frac{v}{u}\left[1-(1-\frac{1}{v})^u\right].$$

With the notational change of $m=v$ and $n=u$, this result is the same as the one given in Proposition 2 of BSW. Taking the limit of this matching probability as $u,v\to\infty$ with $v/u=\theta$ gives $m(\theta;1)=\theta(1-e^{-1/\theta})$, as equation (1) implies. The case of $a=v$ is the polar opposite. In this case, $X_1=X_2=...=X_a=u-1$ with probability one, so the probability a worker gets an offer is $1-(\frac{u-1}{u})^v$, as in JKK. Taking the limit as $u,v\to\infty$ with $v/u=\theta$ gives

$$m(\theta)=1-e^{-\theta}. \tag{2}$$

The same expression can be derived by taking the limit of $m(\theta;a)$ as $a\to\infty$ in equation (1).

For future reference, we note the following properties of $m(\theta;a)$:
(i) $m(\theta;a)$ is increasing and concave in θ, $\lim_{\theta\to 0}m(\theta;a)=0$, and $\lim_{\theta\to\infty}m(\theta;a)=1$;
(ii) $\dfrac{m(\theta;a)}{\theta}$ is decreasing in θ, $\lim_{\theta\to 0}\dfrac{m(\theta;a)}{\theta}=1$, and $\lim_{\theta\to\infty}\dfrac{m(\theta;a)}{\theta}=0$.[1]

The effect of a on $m(\theta;a)$ is less clearcut. Treating a as a continuous variable, we find that $m_a(\theta;a)\gtrless 0$ as $\dfrac{a}{1-q}\dfrac{\partial q}{\partial a}-\ln(1-q)\gtrless 0$ where $q=\dfrac{\theta}{a}(1-e^{-\frac{a}{\theta}})$. For moderately large values of θ ($\theta>\frac{1}{2}$, approximately), $m(\theta;a)$ first increases and then decreases with a. This nonmonotonicity reflects the double coordination problem that arises when workers apply to more than one but not all vacancies. The first coordination problem is the standard one associated with urn-ball matching, namely, that some vacancies can receive applications from more than one worker, while others receive none. With multiple applications, there is a second coordination problem, this time among vacancies. When workers apply for more than one job at a time, some workers can receive offers from more than one vacancy, while others receive none. Ultimately, a worker can only take one job, and the vacancies that "lose the race" for a worker will have wasted time and effort

[1] Interestingly, $\dfrac{m(\theta;a)}{\theta}$ is not convex in θ, as can be seen immediately by considering the case of $a=1$. The properties of $m(\theta;a)$ and $\dfrac{m(\theta;a)}{\theta}$ given in (i) and (ii) are the minimal ones required for our normative results in Sections 3 and 4 below.

while considering his or her application. The matching function derived in BSW captures only the urn-ball friction, while the one derived in JKK captures only the multiple application friction. Our matching probability incorporates both the urn-ball and the multiple application frictions, and the interaction between these two frictions provides new insights.

Proposition 1 and its implications are only interesting if a symmetric equilibrium exists. We now turn to the existence question.

Proposition 2 *Consider a large labor market in which $u, v \to \infty$ with $v/u = \theta$. There is a unique symmetric equilibrium to the wage posting game. When $a = 1$, all vacancies post a wage of*

$$w(\theta; 1) = \frac{e^{-1/\theta}}{\theta(1 - e^{-1/\theta})}. \tag{3}$$

When $a \in \{2, ..., A\}$, $w(\theta; a) = 0$, and the fraction of wages paid equal to one is

$$\gamma(\theta; a) = \frac{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a - \theta(1 - e^{-a/\theta})(1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}}{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a}. \tag{4}$$

The proof is given in the Appendix. The basic idea is as follows. To prove the existence of a symmetric equilibrium, we show that when $a = 1$, the wage given in equation (3) has the property that if all vacancies, with the possible exception of a "potential deviant," post that wage, then it is also in the interest of the deviant to post that wage. When $a \in \{2, ..., A\}$, then no matter what the common wage posted by other vacancies, it is always in the interest of the deviant to undercut that common wage. This forces the wage down to the monopsony level, which in our single-period model is $w = 0$.

The equilibrium wage for the case of $a = 1$ is equal to one minus the limit of the price given in Proposition 2 in BSW – again with the appropriate notational change. The tradeoff that leads to a well-behaved equilibrium wage, $w \in (0, 1)$, when $a = 1$ is the standard one in equilibrium search theory. As any particular vacancy increases its posted wage, holding the wages posted at other vacancies constant, the profit that this vacancy generates conditional on attracting an applicant decreases. At the same time, however, the probability that it will attract at least one applicant also increases. This tradeoff varies smoothly with θ; so the equilibrium wage varies smoothly between zero and one. Thus, as emphasized in BSW (p. 1069), there is a sense in which frictions "smooth" the operation of the labor market.

When $a \in \{2, ..., A\}$, no matter what the value of θ, the posted wage collapses to the Diamond (1971) monopsony level. The intuition for this result is based on the change in the tradeoff underlying equilibrium wage determination. It is still the case that as any particular vacancy increases its posted wage, holding the wages posted at other vacancies constant, the profit that this vacancy generates conditional on hiring the applicant at its posted wage decreases. Likewise, the probability that the vacancy will attract at least one applicant also increases. It is no longer certain, however, that attracting applicants will lead to a filled job – the applicant chosen by the vacancy may ultimately take another job. Further, even if the vacancy is able to hire its chosen applicant, it may be able to do so only by engaging in Bertrand competition with one or more rivals. Essentially, the cost of increasing the posted wage is the same as in the case of $a = 1$; the expected benefit is lower.

Despite the fact that the posted equilibrium wage is zero when $a \in \{2, ..., A\}$, there is still a sense in which "the wage" varies smoothly with θ. The expected fraction of wages paid equal to one, $\gamma(\theta; a)$, has the following properties:
(i) $\gamma(\theta; a)$ is increasing in θ and in a;
(ii) $\lim_{\theta \to 0} \gamma(\theta; a) = 0$ and $\lim_{\theta \to \infty} \gamma(\theta; a) = 1$.
The fact that γ is increasing in θ is exactly as one would expect – as the labor market gets tighter, the chance that an individual worker gets multiple offers increases. To understand why γ is also increasing in a, it is important to remember that $\gamma(\theta; a)$ is the expected wage for those workers who match with a vacancy; in particular, those workers who fail to match are not treated as receiving a wage of zero. Finally, defining $\gamma(\theta) = \lim_{a \to \infty} \gamma(\theta; a)$, we can show

$$\gamma(\theta) = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - e^{-\theta}}. \tag{5}$$

This is the expected wage in a large labor market when each worker sends out an arbitrarily large number of applications.

3 Efficiency

We now turn to the question of constrained efficiency. The result suggested by the efficiency of competitive search equilibrium holds in our setting when $a = 1$; however, when workers make a fixed number of multiple applications, this result breaks down.

Suppose vacancies are set up at the beginning of the period and that each vacancy is created at cost c. The efficient level of labor market tightness[2] is determined as the solution to

$$\max_{\theta>0} -c\theta + m(\theta;a).$$

The first-order condition for this maximization is

$$c = m_\theta(\theta^*;a). \tag{6}$$

The equilibrium level of labor market tightness is determined by free entry. When $a = 1$, this means

$$c = \frac{m(\theta^{**};1)}{\theta^{**}}(1 - w(\theta^{**};1)), \tag{7}$$

whereas for $a \in \{2,...,A\}$, the condition is

$$c = \frac{m(\theta^{**};a)}{\theta^{**}}(1 - \gamma(\theta^{**};a)). \tag{8}$$

Equations (7) and (8) reflect the condition that entry (vacancy creation) occurs up to the point that the cost of vacancy creation is just offset by the value of owning a vacancy. This value equals the probability of hiring a worker times the expected surplus generated by a hire – equal to 1 minus the posted wage when $a = 1$ and to 1 minus the expected wage when $a \in \{2,...,A\}$.

Note that θ^* denotes the constrained Pareto efficient level of labor market tightness and θ^{**} denotes the equilibrium level of labor market tightness. At issue is the relationship between θ^* and θ^{**}.

Proposition 3 *Let $u, v \to \infty$ with $v/u = \theta$ and $a \in \{1,...,A\}$ fixed. For $a = 1$, $\theta^* = \theta^{**}$. For $a \in \{2,...,A\}$, $\theta^{**} > \theta^*$.*

Proof. Differentiating equation (1) with respect to θ gives

$$m_\theta(\theta;a) = (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}). \tag{9}$$

[2] In a finite labor market with u given, the social planner chooses v to maximize $-cv + M(u,v;a)$; i.e., expected output (equal to the expected number of matches since each match produces an output of 1) minus the vacancy creation costs. Dividing the maximand by u and letting $u, v \to \infty$ gives the maximand in the text.

For the case of $a = 1$, equation (6) becomes

$$c = 1 - e^{-1/\theta} - \frac{1}{\theta} e^{-1/\theta}.$$

From equations (1) and (3), in equation (7) we have

$$\frac{m(\theta; 1)}{\theta}(1 - w(\theta; 1)) = 1 - e^{-1/\theta} - \frac{1}{\theta} e^{-1/\theta}.$$

Thus, equations (6) and (7) imply $\theta^* = \theta^{**}$.

When $a \in \{2, ..., A\}$, equation (9) implies that θ^* solves

$$c = (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta} - \frac{a}{\theta} e^{-a/\theta}), \tag{10}$$

whereas, using equations (1) and (4), θ^{**} (equation 8) solves

$$c = (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta}). \tag{11}$$

The right-hand sides of both (10) and (11) are decreasing in θ. Since the right-hand side of (11) is greater than that of (10) for all $\theta > 0$, it follows that $\theta^{**} > \theta^*$. ■

Posting a vacancy has the standard congestion and thick-market effects in our model – adding one more vacancy makes it more difficult for the incumbent vacancies to find workers but makes it easier for the unemployed to generate offers. A striking result of the competitive search equilibrium literature is that adding one more vacancy causes the wage to adjust in such a way as to balance these external effects correctly. One way to interpret this is to say that competition leads to a wage equal to the one that would be dictated by the Hosios (1990) condition in a Nash bargaining model. Equivalently, one can say (Moen, 1997, p. 387) that the competitive search equilibrium wage has the property that the marginal rate of substitution between labor market tightness and the wage is the same for vacancies as for workers. The first part of Proposition 3 shows that this result holds when one uses an explicit urn-ball ($a = 1$) microfoundation for the matching function. However, when workers make multiple applications, the result that $\theta^{**} > \theta^*$ indicates that the equilibrium level of vacancy creation is too high. Equivalently, the equilibrium expected wage is below the level that would be indicated by the Hosios condition. The effects of the marginal vacancy are more complicated with multiple applications than in the urn-ball model. Adding one more vacancy makes it less likely that each incumbent vacancy will attract any applicants but, conditional on attracting an applicant, makes it more likely that the incumbent vacancy "wins the race" for that applicant. Adding another vacancy to the market puts upward pressure on the (expected) wage but not to the extent required to achieve the efficient level of entry.

It is interesting to note that the equilibrium outcome is again Pareto efficient when we let $a \to \infty$. To see this, simply substitute the expressions for $m(\theta)$ and $\gamma(\theta)$ from equations (2) and (5) into the efficiency and equilibrium conditions. This is Proposition 2.5 in JKK. In a companion paper, Julien, Kennes, and King (2002) show that equilibrium in a finite labor market with $a = v$ is also constrained efficient if one *assumes* a particular wage determination mechanism; namely, vacancies offering jobs to workers who have no other offers receive all of the surplus ($w = 0$) but vacancies offering jobs to workers who do have other offers receive none of the surplus ($w = 1$). Julien, Kennes, and King (2002) interpret this result in terms of what they call the Mortensen rule (Mortensen 1982) – that efficiency in matching is attained if the "initiator" of the match gets the total surplus.[3] By mimicking our proof of Proposition 2, we can show that this assumed wage determination mechanism is in fact the symmetric equilibrium outcome in a directed search model with wage posting when $a = v$.

An intuition for why we find constrained efficiency with $a = 1$ and as $a \to \infty$ but not with a fixed, finite number of multiple applications is that with $a = 1$ and as $a \to \infty$, only one coordination problem affects the operation of the labor market, whereas with a fixed $a \in \{2, ..., A\}$, the urn-ball and the multiple applications coordination problems operate simultaneously. Adjusting the wage can only solve one coordination problem at a time.

4 Steady State

We now turn to steady-state analysis for a labor market with directed search and multiple applications. We work with the limiting case in which $u, v \to \infty$ with $v/u = \theta$ and $a \in \{2, ..., A\}$ fixed. Since only the ratio of v to u matters in the limiting case, we normalize the labor force to 1; thus, u is interpreted

[3] The intuitions for constrained efficiency (i) in a large labor market when $a = 1$ and (ii) when $a = v$ are thus quite different. When $a = 1$, constrained efficiency is a result of competition, and competition requires a labor market sufficiently large that individual vacancies have negligible market power. When $a = v$, constrained efficiency is a result of perfect monopoly power – the entire surplus goes to the vacancy if there is no competition for the applicant it selects and to the worker if he or she winds up having the monopoly power. The monopoly intuition does not require that the labor market be large.

as the unemployment rate.

In steady-state, workers flow into employment with probabililty $m(\theta; a)$ per period. We assume that matches break up exogenously with probability δ, giving the countervailing flow back into unemployment. Similarly, jobs move from vacant to filled with probability $\dfrac{m(\theta; a)}{\theta}$ and back again with probability δ. Steady-state analysis thus allows us to endogenize vacancies and unemployment. More importantly, moving to the steady state means that those unemployed who fail to find an acceptable job in the current period can wait and apply again in the future. In the case of $a = 1$, this isn't particularly interesting since, in equilibrium, there is no gain to waiting. However, with multiple applications, the ability of the unemployed to hold out for a situation in which vacancies engage in Bertrand competition for their services, albeit at the cost of delay, implies a positive reservation wage. This leads to a simple and appealing model in which labor market tightness and the reservation wage are simultaneously determined. On the one hand, the lower is the reservation wage of the unemployed, the more vacancies firms want to create. On the other, as the labor market becomes tighter, i.e., as θ increases, the unemployed respond by increasing their reservation wage.

The analysis proceeds as follows. Suppose the unemployed set a reservation wage R. With multiple applications, the wage-posting problem for a vacancy is qualitatively the same as in the one-period game. Whatever common wage might be posted at other vacancies, each individual vacancy has the incentive to undercut. In the one-period game, this implies a monopsony wage posting of $w = 0$; in the steady state, this same mechanism implies a dynamic monopsony wage posting of $w = R$. In addition, the probability that an unemployed worker finds a job in any period and the probability that he or she is hired at the competitive wage, conditional on finding a job, are the same as in the single-period model; i.e., equations (1) and (4) for $m(\theta; a)$ and $\gamma(\theta; a)$ continue to apply.

We begin by examining the value functions for jobs and for workers. A job can be in one of three states – vacant, filled paying the competitive wage, and filled paying R. Let V, $J(1)$, and $J(R)$ be the corresponding values. The value of a vacancy is

$$V = -c + \frac{1}{1+r}\{\frac{m(\theta; a)}{\theta}[\gamma(\theta; a)J(1) + (1 - \gamma(\theta; a))J(R)] + (1 - \frac{m(\theta; a)}{\theta})V\}.$$

Maintaining a vacancy entails a cost c, which is incurred at the start of each period. Moving to the end of the period, and thus discounting at rate r, the vacancy has hired a worker with probability $\frac{m(\theta;a)}{\theta}$. With probability $\gamma(\theta;a)$, the worker who was hired had his or her wage bid up to the competitive level, thus implying a value of $J(1)$. With probability $1-\gamma(\theta;a)$ the worker was hired at $w=R$, thus implying a value of $J(R)$. Finally, with probability $1-\frac{m(\theta;a)}{\theta}$, the vacancy failed to hire, in which case the value V is retained.

Free entry implies $V=0$. Given $V=0$, there is no incentive for vacancies competing for a worker to drop out of the Bertrand competition before the wage is bid up to $w=1$ (thus justifying the notation $J(1)$). This in turn implies that we also have $J(1)=0$. Inserting these equilibrium conditions into the expression for V gives

$$\frac{m(\theta;a)}{\theta}(1-\gamma(\theta;a))J(R)=c(1+r).$$

At the same time, the value of employing a worker at $w=R$ is

$$J(R)=(1-R)+\frac{1}{1+r}[(1-\delta)J(R)+\delta V].$$

Again using $V=0$, we have

$$J(R)=\frac{1+r}{r+\delta}(1-R).$$

Combining these equations gives the first steady-state equilibrium condition,

$$c=\frac{m(\theta;a)}{\theta}(1-\gamma(\theta;a))\frac{1-R}{r+\delta}. \tag{12}$$

A worker also passes through three states – unemployed, employed at the competitive wage, and employed at R. The value of unemployment is defined by

$$U=\frac{1}{1+r}\{m(\theta;a)[\gamma(\theta;a)N(1)+(1-\gamma(\theta;a))N(R)]+(1-m(\theta;a))U\},$$

where $N(1)$ and $N(R)$ are the values of employment at $w=1$ and $w=R$, respectively. These latter two values are in turn defined by

$$\begin{aligned} N(1) &= 1+\frac{1}{1+r}\{(1-\delta)N(1)+\delta U\} \\ N(R) &= R+\frac{1}{1+r}\{(1-\delta)N(R)+\delta U\}. \end{aligned}$$

The reservation wage property, i.e., $N(R) = U$, then implies

$$\begin{aligned} U &= \frac{1+r}{r} R \\ N(1) &= \frac{(1+r)}{r(r+\delta)}(r + \delta R). \end{aligned}$$

Inserting these expressions into the expression for U and rearranging gives the second steady-state equilibrium condition,

$$R = \frac{m(\theta; a)\gamma(\theta; a)}{r + \delta + m(\theta; a)\gamma(\theta; a)}. \tag{13}$$

The final equation for the steady-state equilibrium is the standard flow (Beveridge curve) condition for unemployment. Since the labor force is normalized to 1, this is

$$u = \frac{\delta}{\delta + m(\theta; a)}. \tag{14}$$

Equations (13) and (14) show that, as is common in this class of models, once labor market tightness (θ) is determined, the other endogenous variables – in this case, R and u – are easily determined. Using equation (13) to eliminate R from equation (12) gives the equation that determines the steady-state equilibrium value of θ, namely,

$$c = \frac{m(\theta; a)}{\theta} \frac{1 - \gamma(\theta; a)}{r + \delta + m(\theta; a)\gamma(\theta; a)}. \tag{15}$$

Using our results on the properties of $m(\theta; a)$ and $\gamma(\theta; a)$, we can show that the right-hand side of equation (15) equals $\frac{1}{r+\delta}$ as $\theta \to 0$, that it goes to zero as $\theta \to \infty$, and that its derivative with respect to θ is negative for all $\theta > 0$. Equation (15) thus has a unique solution for each $c \in (0, \frac{1}{r+\delta}]$.

The natural next step is to compare equilibrium steady-state labor market tightness with the constrained efficient value of θ. The planner's problem is to choose the level of labor market tightness that maximizes the discounted value of output net of vacancy costs for an infinitely lived economy. That is, the planner's problem is to maximize

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (1 - u_t - c\theta_t u_t)$$

subject to

$$u_{t+1} - u_t = m(\theta_t; a)u_t - \delta(1 - u_t)$$

with u_0 given.

The current-value Hamiltonian for this problem is

$$H(\theta, u) = 1 - u - c\theta u + \lambda[m(\theta; a)u - \delta(1 - u)]$$

with necessary conditions

$$\begin{aligned} \frac{\partial H}{\partial \theta} &= -cu + \lambda m_\theta(\theta; a)u = 0 \\ r\dot{\lambda} &= -\frac{\partial H}{\partial u} = 1 + c\theta - \lambda[m(\theta; a) + \delta]. \end{aligned}$$

Evaluating at the steady-state, and eliminating λ, gives

$$c = \frac{(1 + c\theta)m_\theta(\theta; a)}{r + \delta + m(\theta; a)}. \tag{16}$$

Now we can compare the levels of labor market tightness implied by equations (15) and (16). Using equations (1) and (4), equation (15) can be rewritten as

$$c(r + \delta + m(\theta; a)) = (1 + c\theta)(1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta}). \tag{17}$$

Using equation (9), equation (16) can be rewritten as

$$c\,(r + \delta + m(\theta; a)) = (1+c\theta)(1-\frac{\theta}{a}(1-e^{-a/\theta}))^{a-1}(1-e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}). \tag{18}$$

As in the single period analysis, let θ^* be the constrained efficient level of labor market tightness, i.e., the value of θ that solves equation (16), and let θ^{**} be the equilibrium level of labor market tightness, i.e., the value of θ that solves equation (15). Comparing equations (15) and (16) yields the following:

Proposition 4 *Let $u, v \to \infty$ with $v/u = \theta$ and $a \in \{2, ..., A\}$ fixed. Then in steady state, $\theta^{**} > \theta^*$.*

Proposition 4 indicates that, as in the single-period analysis, when the unemployed make a fixed number of multiple applications per period ($a \in \{2, ..., A\}$), equilibrium is constrained inefficient. Specifically, there is too much vacancy creation. This result holds even though the ability of the unemployed to reject offers in favor of waiting for a more favorable outcome in some future period implies a dynamic monopsony wage above the single-period monopsony wage of zero.

5 Concluding Remarks

In this paper, we construct an equilibrium search model of a large labor market in which workers, after observing all posted wages, submit a fixed number of applications, $a \in \{1, ...A\}$, to the vacancies that they find most attractive. We derive the symmetric equilibrium matching probability and the common posted wage. When $a = 1$, our analysis is a large labor market version of BSW. However, when $a \in \{2, ...A\}$, i.e., when workers make multiple applications, the symmetric equilibrium of our model is radically different. With multiple applications, the match probability in our model reflects the interplay of two coordination failures – an urn-ball failure among workers and a multiple-application failure among vacancies. In addition, when workers make more than one application, all vacancies post the monopsony wage, but there is dispersion in wages paid. Workers who receive only one job offer are paid the monopsony wage, but those who receive multiple offers get the competitive wage. When workers make a single application or when they apply to an arbitrarily large number of vacancies, equilibrium is constrained efficient; but when workers make a finite number of multiple applications, too many vacancies are posted. These results, both positive and normative, carry over from the single-period model to a steady-state framework.

Directed search is an appealing way to model equilibrium unemployment and wage dispersion. In reality, workers do direct their applications to attractive vacancies, but unemployment nonetheless persists as a result of coordination failures on both sides of the labor market. In addition, those workers who are lucky enough to generate competition for their services do in fact have their wages bid up. The contribution of this paper is to show that these realistic features can be captured in a tractable equilibrium model.

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6 Appendix – Proof of Proposition 2

As discussed in the text, we need to show that when $a = 1$, $w(\theta; 1)$ has the property that if all vacancies, with the possible exception of a "potential deviant," post that wage, then it is also in the interest of the deviant to post that wage. When $a \in \{2, ..., A\}$, we need to show that no matter what the common wage posted by other vacancies, it is always in the interest of the deviant to undercut that common wage, thus driving $w(\theta; a)$ to zero.

Let D denote the potential deviant, posting a wage of w', and let N denote the nondeviant vacancies, each posting the common wage, w. D's expected profit is

$$\pi(w'; w) = (1-w')P[D \text{ gets applicant}]P[\text{selected applicant doesn't have another offer}]$$

Let k be the probability that any individual applies to D, where $k \in [0, 1]$. In symmetric equilibrium, k must be the same for all workers. As $u \to \infty$, k must go to zero; otherwise, (i) D gets an applicant with probability one and (ii) any applicant to D has an infinity of competitors and therefore gets the job with probability zero. Therefore, we set $k = \xi/u$. We characterize ξ below, but for now we take ξ as given.

Let X^D be the number of applications received by D. As $u \to \infty$, X^D is approximately Poisson (ξ), so

$$P[D \text{ gets applicant}] = P[X^D > 0] = 1 - e^{-\xi}.$$

Note that the probability that a worker who applies to D is offered that job is

$$\sum_{x=0}^{\infty} \frac{1}{x+1} \frac{e^{-\xi}\xi^x}{x!} = \frac{1}{\xi}(1 - e^{-\xi}).$$

Call this probability q^D.

Next, consider a worker who has applied to and been offered a job at D. We need to find the probability that none of this worker's other applications (all of which go to N vacancies) also lead to an offer. The probability that a worker applies to any particular N vacancy is $\frac{a - \frac{\xi}{u}}{v-1}$. (A worker sends $a-1$ applications to N vacancies with probability 1; the a^{th} application goes to an N vacancy with probability $1 - \frac{\xi}{u}$. There are $v - 1$ N vacancies.) Consider an N vacancy to which the worker offered a job by D also applied. Let X^N be the number of applications received by that vacancy in addition to that of the D applicant. X^N is approximately Poisson ($\frac{a}{\theta}$). (The number of other potential applicants goes to infinity; the probability that any one potential applicant actually applies to the N vacancy in question goes to zero, and the product of these 2 terms goes to a/θ.) The probability that the applicant offered the D job also gets the N offer is

$$\sum_{x=0}^{\infty} \frac{1}{x+1} \frac{e^{-a/\theta}(\frac{a}{\theta})^x}{x!} = \frac{\theta}{a}(1 - e^{-a/\theta})$$

Call this probability q^N. Then

$$P[\text{selected applicant doesn't have another offer}] = (1 - q^N)^{a-1}.$$

We thus have

$$\pi(w'; w) = (1 - w')(1 - e^{-\xi})(1 - q^N)^{a-1}.$$

Note that the intensity of applications to D, ξ, depends on w', through the indifference condition, but the final term, $(1 - q^N)^{a-1}$, does not.

Each worker must be indifferent between (i) sending all a applications to N and (ii) sending 1 application to D and the other $a - 1$ to N. The possible payoffs for a worker who sends all a applications to N are

(i) 1 if 2 or more of these applications are accepted; this occurs with probability $1 - (1 - q^N)^a - aq^N(1 - q^N)^{a-1}$

(ii) w if only 1 application is accepted; this occurs with probability $aq^N(1 - q^N)^{a-1}$

(iii) 0 if neither application is accepted; this occurs with probability $(1 - q^N)^a$.

The expected payoff for a worker who sends both applications to N vacancies is thus

$$1 - (1 - q^N)^a - aq^N(1 - q^N)^{a-1} + waq^N(1 - q^N)^{a-1}.$$

The possible payoffs for a worker who sends 1 application to D and the other $a - 1$ to N are

(i) 1 if 2 or more applications are accepted; this occurs with probability

$$\begin{aligned} & q^D(1 - (1 - q^N)^{a-1}) + (1 - q^D)(1 - (1 - q^N)^{a-1} - (a - 1)q^N(1 - q^N)^{a-2}) \\ = \; & 1 - (1 - q^N)^{a-1} - (1 - q^D)(a - 1)q^N(1 - q^N)^{a-2}, \end{aligned}$$

(ii) w' if only the application to D is successful; this occurs with probability $q^D(1 - q^N)^{a-1}$

(iii) w if only one application to N is successful; this occurs with probability $(1-q^D)(a-1)q^N(1-q^N)^{a-2}$

(iv) 0 if no applications are successful; this occurs with probability $(1-q^D)(1-q^N)^{a-1}$

The expected payoff for a worker who sends 1 application to D and $a-1$ to N is thus

$$1-(1-q^N)^{a-1}-(1-q^D)(a-1)q^N(1-q^N)^{a-2}+w'q^D(1-q^N)^{a-1}+w(1-q^D)(a-1)q^N(1-q^N)^{a-2}.$$

Equating the two expected payoffs implicitly defines $\xi(w';w,\theta)$. Differentiating with respect to w', taking into account that $\frac{dq^D}{d\xi}=-\frac{1-e^{-\xi}-\xi e^{-\xi}}{\xi^2}$, and substituting for q^D and q^N gives

$$\frac{d\xi}{dw'}=\frac{\xi(1-e^{-\xi})(1-\frac{\theta}{a}(1-e^{-a/\theta}))}{(1-e^{-\xi}-\xi e^{-\xi})\left((a-1)\frac{\theta}{a}(1-e^{-a/\theta})(1-w)+w'(1-\frac{\theta}{a}(1-e^{-a/\theta}))\right)}$$

Since $1-e^{-x}-xe^{-x}>0$ for all $x>0$ and $1\geq w$, we have $\frac{d\xi}{dw'}>0$ (as expected) and $\frac{d^2\xi}{dw'^2}<0$.

Turning back to D's optimization problem, $\pi(w';w)$ is proportional to $(1-w')(1-e^{-\xi})$. Maximizing with respect to w', the first-order (Kuhn-Tucker) condition is

$$-(1-e^{-\xi})+(1-w')e^{-\xi}\frac{d\xi}{dw'}\leq 0 \text{ with equality if } w'>0.$$

Note that if there is an interior solution, the second-order condition holds.

We are interested in the possibility of an interior solution at $w'=w$. Consider first the case of $a=1$. If $w'=w$, then $\xi=1/\theta$. Substituting and solving gives

$$w(\theta;1)=\frac{e^{-1/\theta}}{\theta(1-e^{-1/\theta})}$$

Consider next the case of $a\in\{2,...,A\}$. In this case $w'=w$ implies $\xi=a/\theta$. Substituting the expression for $\frac{d\xi}{dw'}$ into the first-order condition gives.

$$\frac{(1-w)\xi e^{-\xi}(1-\frac{1}{\xi}(1-e^{-\xi}))}{(1-e^{-\xi}-\xi e^{-\xi})\left((a-1)\frac{1}{\xi}(1-e^{-\xi})(1-w)+w(1-\frac{1}{\xi}(1-e^{-\xi}))\right)}\leq 1$$

This can be rewritten as

$$(1-w)e^{-\xi}(\xi^2-\xi(1-e^{-\xi})) \leq \left(1-e^{-\xi}-\xi e^{-\xi}\right)\left((a-1)(1-e^{-\xi})(1-w)+w(\xi-(1-e^{-\xi}))\right)$$

Or

$$\frac{\xi^2 e^{-\xi}+(a-2)\xi e^{-\xi}(1-e^{-\xi})-(a-1)^2(1-e^{-\xi})^2}{(1-e^{-\xi})} \leq w(\xi-a(1-e^{-\xi})+(a-1)^2\xi(1-e^{-\xi})^2)$$

Only a corner solution exists with $w(\theta;a)=0$ if this is a strict inequality.

Note that as $\xi \to 0$, the $RHS \to 0$ and, using a L'Hôpital's Rule argument, so does the LHS. Note also that

$$\frac{dRHS}{d\xi} = w(1-ae^{-\xi}+(a-1)^2(1-e^{-\xi})^2+2(a-1)^2\xi(1-e^{-\xi})e^{-\xi}) > 0,$$

while

$$\frac{dLHS}{d\xi} = \frac{-e^{-\xi}((1-e^{-\xi})^2((a-1)(a-2)+\xi(a-2))+(1-e^{-\xi}-\xi)^2}{(1-e^{-\xi})^2},$$

which is negative for $a \in \{2, ..., A\}$. Thus, in this case, we have a corner solution with $w(\theta;a)=0$.

Finally to derive $\gamma(\theta;a)$, note that in symmetric equilibrium $q^N = q^d \equiv q = \frac{\theta}{a}(1-e^{-a/\theta})$. A fraction $1-(1-q)^a$ of all workers get a job. A fraction $1-(1-q)^a - a(1-q)^{a-1}$ of all workers receive multiple offers. Thus, a fraction

$$\frac{1-(1-q)^a - a(1-q)^{a-1}}{1-(1-q)^a}$$

of the workers who find a job receive the competitive wage. Substituting for q gives equation (4). QED