Global identification from the equilibrium manifold under incomplete markets. (Draft for comments)

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Abstract

We show that even under incomplete markets, the equilibrium manifold identifies individual demands everywhere in their domains. For this, we assume conditions of smoothness, interiority and regularity, but avoid implausible observational requirements. It is crucial that there be date-zero consumption. As a by-product, we develop some duality theory under incomplete markets.

1 Introduction

The transfer paradox, first pointed out by Leontief (1936), and generalized by Donsimoni and Polemarchakis (1994), illustrates the importance of identifying the fundamentals of an economy from observable data. Under the hypothesis of general equilibrium, the aggregate demand function cannot be assumed to be observed: at equilibrium prices aggregate demand is, by definition, equal to aggregate endowment. Demand, either individual or aggregate, cannot be observed for out-of-equilibrium prices. One can observe, however, equilibrium prices and individual incomes. In this paper we address the problem of identifying individual preferences from the equilibrium manifold when asset markets are incomplete.

For the case of complete markets, positive results have been obtained by Balasko [1999], Chiappori et al [2000] and Matzkin [2003]. Balasko's result has been criticized for making very strong observational assumptions: that one can observe equilibrium prices in situations in which endowment is zero for all individuals but one. Under additional assumptions, Chiappori et al obtain local identification of individual demands using a constructive argument. Matzkin determines the largest class of fundamentals for which identification is possible. Her argument, however, is not constructive.

The case of incomplete markets is more cumbersome. Kubler et al [2000] and [2002] use the implicit function theorem to identify the aggregate demand function from the equilibrium manifold (hence they obtain a local identification of the aggregate demand function). They proceed to identify individual demands (locally) from the aggregate demand and finally, they use Geanakoplos and Polemarchakis [1990] to identify preferences from individual demand functions. Therefore, they are able to obtain local identification of individual preferences when asset markets are incomplete.

When we have real numeraire assets, we identify individual demands globally. For general real assets structures, we conjecture that our results hold (generically on prices and endowments). We extent Balasko's idea on how to recover the aggregate demand function from the equilibrium manifold to the case of incomplete asset markets, hence we avoid using the implicit function theorem. We then use a slightly different argument than Kubler et al. to identify individual demands from the aggregate demand function and we also avoid using Balasko's strong observational assumption pointed out before.

As a by product, we develop some basic duality theory for incomplete markets.

2 The Incomplete Markets Model

We consider the canonical, two period, multigood, incomplete markets model with financial assets. There are $S+1$ states of nature, $s = 0, ..., S$,[1] I individuals, $i = 1, ..., I$, and $L \geqslant 2$ commodities available in each state, $l = 1, ..., L$. We denote $L(S+1)$ by n and define the commodity space as $\mathbb{R}^n_+$.

A financial asset is a contract $v \in \mathbb{R}^S$ that promises to deliver at each state of nature $s = 1, ..., S$ an amount $v_s \in \mathbb{R}$ of the numeraire. Let good 1 be the numeraire of our economy and let $p \in \mathbb{R}^n_{++}$ denote the vector of spot prices where $p_s = (p_{s,1}, .., p_{s,L}) \in R^L_{++}$ and $p_{s,l}$ denotes the (current value of) price payable in state s for one unit of good l. Since assets are real, without loss of generality, we normalize price so that $p_{0,1} = 1$. Define, $S^{n-1}_{++} = \left\{p \in \mathbb{R}^n_{++} : p_{0,1} = 1\right\}$. We write any $p \in S^{n-1}_{++}$ as $p = (p_0, p_{\mathbf{1}})$, where $p_{\mathbf{1}} = (p_1, ..., p_S)$.

If $v^1, ..., v^J$ are $J \geq 1$ financial assets, we define $V(p_{\mathbf{1}})$ as the matrix of income transfers:

$$V(p_{\mathbf{1}}) = \begin{bmatrix} p_{1,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p_{S,1} \end{bmatrix} V$$

where $V = \begin{bmatrix} v^1_1 & \cdots & v^J_1 \\ \vdots & \ddots & \vdots \\ v^1_S & \cdots & v^J_S \end{bmatrix}$, and the space of income transfers $\langle V(p_{\mathbf{1}})\rangle$, as the column span of $V(p_{\mathbf{1}})$:

$$\langle V(p_{\mathbf{1}})\rangle = \left\{t \in \mathbb{R}^S : \left(\exists z \in \mathbb{R}^J\right) : t = V(p_{\mathbf{1}})\, z\right\}$$

[1] $s = 0$ is used to denote date zero.

Remark 1 *In general, as* $p_{\mathbf{1}}$ *changes,* $\langle V(p_{\mathbf{1}})\rangle$ *changes. If* $p_{\mathbf{1}} >> 0$*, then the dimension of* $\langle V(p_{\mathbf{1}})\rangle$ *remains unchanged.*

Let $q \in R^J$ be the price vector at which each one of these assets can be bought at $s = 0$.

For $(p, q) \in R^n_{++} \times \mathbb{R}^J$ and $w \in \mathbb{R}^n_+$, let

$$\mathbf{B}(p, q, w; V) = \left\{x \in \mathbb{R}^n_+ : \exists z \in \mathbb{R}^J,\ p_0 \cdot (x_0 - w_0) \leq -qz \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) = V(p_{\mathbf{1}})\, z\right\}$$

where for every $(\rho_{\mathbf{1}}, \Delta_{\mathbf{1}}) = ((\rho_1, ..., \rho_S), (\Delta_1, ..., \Delta_S)) \in \mathbb{R}^{LS} \times \mathbb{R}^{LS}$:

$$\rho_{\mathbf{1}} \boxdot \Delta_{\mathbf{1}} = \begin{bmatrix} \rho_1 \cdot \Delta_1 \\ \vdots \\ \rho_S \cdot \Delta_S \end{bmatrix}.$$

Individual $i \in \mathcal{I}$ has preferences over consumption that are represented by utility functions $u^i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ and endowment denoted by $w^i \in \mathbb{R}^n_+$. Assume the following:

Condition 1 *For each individual* i*,* u^i *is continuous, monotone and strongly quasi-concave.*

For each individual i, define the individual demand function (in financial markets) $\mathbf{f}^i : S^{n-1}_{++} \times \mathbb{R}^J \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, as:

$$\mathbf{f}^i(p, q, w) = \arg\max\left\{u^i(x) : x \in \mathbf{B}(p, q, w; V)\right\}$$

Define also the aggregate demand function, $\mathbf{F} : S^{n-1}_{++} \times R^J \times \mathbb{R}^{nI}_+ \to \mathbb{R}^n_+$, as:

$$\mathbf{F}(p, q, w) = \sum_{i=1}^{I} \mathbf{f}^i(p, q, w^i)$$

Functions $\mathbf{f}^i$ and $\mathbf{F}$ are well defined since for $(p, q, w) \in S^{n-1}_{++} \times \mathbb{R}^J \times \mathbb{R}^n_+$, $B(p, q, w; V)$ is nonempty and compact (by remark 7) and each u^i is continuous and strongly quasi-concave.

A financial markets economy is: $E = E\left(\{u_i\}_{i\in\mathcal{I}}, \{w_i\}_{i\in\mathcal{I}}, V\right)$

Definition 1 *A financial markets equilibrium for the economy* E *is* $(x, z, p, q) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^J \times S^{n-1}_{++} \times \mathbb{R}^J$ *such that:*

1. *For every* i*,* $x^i = f^i(p, q, w^i)$*,* $p_0 \cdot (x^i_0 - w^i_0) = -qz^i$ *and* $p_{\mathbf{1}} \boxdot \left(x^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) = V(p_{\mathbf{1}})z^i$
2. $F(p, q, w) = \sum_{i=1}^{I} w^i$ *and* $\sum_{i=1}^{I} z^i = 0$

Remark 2 *If* V *is of full column rank, then* $\sum_{i=1}^{I} z^i = 0$ *is redundant in the previous definition.*

Condition 2 *Assume V is of full column rank.*

Definition 2 *The financial markets equilibrium manifold M_{FM} is:*

$$M_{FM} = \left\{ (p, q, w) \in S^{n-1}_{++} \times \mathbb{R}^J \times \mathbb{R}^{nI}_+ : \mathbf{F}(p, q, w) = \sum_{i=1}^{I} w^i \right\}$$

Remark 3 *What we observe in the real world is M_{FM}*

Let $P \in \mathbb{R}^n_{++}$ denote date-zero present value prices (see Magill and Shafer, p. 1534), where $P = (P_0, ..., P_S)$ and for every s, $P_s = (P_{s,1} ..., P_{s,L})$. Let $S^{n-1}_{++} = \left\{ P \in \mathbb{R}^n_{++} : P_{0,1} = 1 \right\}$. Normalizing prices to lie in S^{n-1}_{++} establishes date-zero first commodity as the numeraire.

For $P \in S^{n-1}_{++}$ and $w \in \mathbb{R}^n_+$, let

$$B(P, w; V) = \left\{ x \in \mathbb{R}^n_+ : \sum_{s=0}^{S} P_s \cdot (x_s - w_s) \leq 0 \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

We say that future consumption $x_{\mathbf{1}} \in \mathbb{R}^{LS}_+$ is financially feasible, at prices and endowments $(P_{\mathbf{1}}, w_{\mathbf{1}}) \in \mathbb{R}^{LS}_{++} \times \mathbb{R}^{LS}_+$, if the second condition in the definition of $B(P, w; V)$ is satisfied: there is a portfolio of assets, $z \in \mathbb{R}^J$, that delivers the transfers necessary to finance $x_{\mathbf{1}}$.

Remark 4 *If $\dim \langle V(P_{\mathbf{1}}) \rangle = S$ or equivalently $\dim \langle V \rangle = S$, the second condition that defines $B(P, w; V)$ is nonbinding. This is the case of complete markets.*

For each individual i, define the individual demand function $f^i : S^{n-1}_{++} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, as:

$$f^i(P, w) = \arg\max \left\{ u^i(x) : x \in B(P, w; V) \right\}$$

Define also the aggregate demand function, $F : S^{n-1}_{++} \times \mathbb{R}^{nI}_+ \to \mathbb{R}^n_+$, as:

$$F(P, w) = \sum_{i=1}^{I} f^i(P, w^i)$$

Remark 5 *f^i and F are well defined since for $(P, w) \in S^{n-1}_{++} \times \mathbb{R}^n_+$, $B(P, w; V)$ is nonempty and compact, and each u^i is continuous and strongly quasi-concave.*

Definition 3 *The no-arbitrage equilibrium for the economy E is a pair $(x, P) \in \mathbb{R}^{nI}_+ \times S^{n-1}_{++}$ such that:*

1. *For every i, $x^i = f^i(P, w^i)$*
2. *$F(P, w) = \sum_{i=1}^{I} w^i$*

Remark 6 *Since V has full column rank, in the previous definition we do not explicitly consider portafolios.*

Remark 7 *It is well known that if (x, P) is a no-arbitrage equilibrium, then there exist portfolios $\left(z^1, ..., z^I\right)$, $z^i \in \mathbb{R}^J$ and an asset price vector $q \in \mathbb{R}^J$ such that $((x, z), (p, q))$ is a financial market equilibrium with the same asset structure V and viceversa. In fact, if $q \in R^J$ is a no-arbitrage asset price vector (see Magill and Shafer (1991)) then there exists a $\pi \in R^{S+1}_{++}$ such that $\pi_0 = 1$ and $q = \pi_{\mathbf{1}} V(p_{\mathbf{1}})$. It is easy to prove that $B(p, q, w; V) = B(P, w; V)$, where $P = \pi \boxdot p$.*

Definition 4 *The no-arbitrage equilibrium manifold (for short, Equilibrium Manifold) M is:*

$$M = \left\{ (P, w) \in S^{n-1}_{++} \times \mathbb{R}^{nI}_{+} : F(P, w) = \sum_{i=1}^{I} w^i \right\}$$

Henceforth we assume that there is a society of individuals that satisfy our assumptions and an asset structure V. We study whether from the financial markets equilibrium manifold M_{FM} (and given V) their unobserved fundamentals (i.e preferences) can be uniquely determined. We do not test the existence of such society (under the equilibrium hypothesis). In the following section we show that the equilibrium manifold uniquely determines aggregate demand. Then, we show that aggregate demand uniquely determines individual demands.

In real life we do not observe equilibrium date zero present value prices but rather, we observe (financial) equilibrium spot prices for commodities and assets. We now show how to define M from M_{FM} consistent with our previous definition of M.

Proposition 1 *Let $E = \left(\{u_i\}_{i\in\mathcal{I}}, \{w_i\}_{i\in\mathcal{I}}, V\right)$ be an economy and M_{FM} the associated financial markets equilibrium manifold and M the associated equilibrium manifold. Define the set*

$$\overline{M} = \{(P, \{w_i\}_{i\in\mathcal{I}})\} \in S^{n-1}_{++} \times \mathbb{R}^{nI}_{+} : (p, q, \{w_i\}_{i\in\mathcal{I}}) \in M_{FM}, p = P \text{ and } q = \sum_{s=1}^{S} V_s\}.$$

Then $\overline{M} = M$.

Proof. This follows from remark 7 by setting $\pi = [1, ..., 1]$. ■

Proposition 2 *Let $E = \left(\{u_i\}_{i\in\mathcal{I}}, \{w_i\}_{i\in\mathcal{I}}, V\right)$ and $\widetilde{E} = \left(\{\widetilde{u}_i\}_{i\in\mathcal{I}}, \{w_i\}_{i\in\mathcal{I}}, V\right)$ be two financial market economies (with possible different agents characteristics but equal endowements and financial structure) and let M_{FM} and $\widetilde{M}_{FM}$ be the associated financial markets equilibrium manifolds, respectively. If $M_{FM} = \widetilde{M}_{FM}$ then $M = \widetilde{M}$ (where M and $\widetilde{M}$ are the associated no-arbitrage equilibrium manifolds of the two economies respectively).*

Proof. Let $\mathbf{f}^i, \mathbf{F}$ be the individual and aggregate demands associated with economy E and let $\widetilde{\mathbf{f}}^i, \widetilde{\mathbf{F}}$ be the individual and aggreagte demands associated with economy $\widetilde{E}$. We first proof $M \subseteq \widetilde{M}$, the other inclusion is similar. Let $(P, w) \in M$ and define $p = P$ and $q = \sum_{s=0}^{S} V_s(P_{\mathbf{1}})$. It is easy to prove that $(p, q, w) \in M_{FM}$ (recall remark 7 with $\pi = [1, ..., 1]$), therefore $(p, q, w) \in \widetilde{M_{FM}}$ and by definition, for every i, $\widetilde{\mathbf{f}}^i(p, q, w^i) = \arg\max \left\{\widetilde{u}^i(x) : x \in \mathbf{B}(p, q, w^i; V)\right\}$ and $\widetilde{\mathbf{F}}(p, q, w) = \sum_{i=1}^{I} w^i$. Now, by remark 7 (take $\pi = [1, ..., 1]$) notice that $\widetilde{f}^i(p, q, w^i) = \arg\max \left\{\widetilde{u}^i(x) : x \in B(P, w^i; V)\right\}$, so$\widetilde{f}^i(P, w^i) = \widetilde{\mathbf{f}}^i(p, q, w^i)$. Finally,

$$\begin{aligned}
\widetilde{F}(P, w) &= \sum_{i=1}^{I} \widetilde{f}^i(P, w^i) \\
&= \sum_{i=1}^{I} \widetilde{\mathbf{f}}^i(p, q, w^i) \\
&= \widetilde{\mathbf{f}}(p, q, w) \\
&= \sum_{i=1}^{I} w^i,
\end{aligned}$$

hence, $(P, w) \in \widetilde{M}$. ■

It follows that if one cannot uniquely identify $\{u_i\}_{i \in \mathcal{I}}$ from M_{FM} and V, then one cannot uniquely identify $\{u_i\}_{i \in \mathcal{I}}$ from M and V.

We now show how to identify globally individual demands in the no-arbitrage economy.

3 From the Equilibrium Manifold to the Aggregate Demand Function

Let M be the equilibrium manifold. The next theorem shows that one can uniquely recover the aggregate demand function.

Theorem 1 *For each $(P, w) \in S_{++}^{n-1} \times \mathbb{R}_{+}^{nI}$, there exists $\left(\widehat{w}^i\right)_{i=1,\dots I} \in \mathbb{R}_{+}^{nI}$ such that*

1. $\left(P, \left(\widehat{w}^i\right)_{i=1,\dots I}\right) \in M$
2. *For all i, $\widehat{w}^i_{\mathbf{1}}$ is financially feasible at $\left(P_{\mathbf{1}}, w^i_{\mathbf{1}}\right)$ and $P \cdot \widehat{w}^i = P \cdot w^i$*

Moreover,

$$F(P, w) = \sum_{i=1}^{I} \widehat{w}^i$$

where $\left(\widehat{w}^i\right)_{i=1,\dots I}$ *is any one of the elements of* $\mathbb{R}_+^{nI}$ *that satisfy the previous two conditions.*

Proof. There is at least one $\left(\widehat{w}^i\right)_{i=1,\dots I} \in \mathbb{R}_+^{nI}$ satisfying the two conditions: define $\widehat{w}^i = f^i(P, w^i)$. Then $\left(P, \left(\widehat{w}^i\right)_{i=1,\dots I}\right)$ is a no-arbitrage equilibrium of the economy $E(u, \left(\widehat{w}^i\right)_{i=1,\dots I}; V)$ since, for all i, $\widehat{w}^i_{\mathbf{1}}$ is financially feasible at $\left(P_{\mathbf{1}}, w^i_{\mathbf{1}}\right)$ and, by strict monotonicity, $P \cdot \widehat{w}^i = P \cdot w^i$.

Now, if $\left(P, \left(\widehat{w}^i\right)_{i=1,\dots I}\right)$ is any element of M, such that $\widehat{w}^i_{\mathbf{1}}$ is financially feasible at $\left(P_{\mathbf{1}}, w^i_{\mathbf{1}}\right)$ and $P \cdot \widehat{w}^i = P \cdot w^i$ for all i, then, by the definition of equilibrium,

$$\sum_{i=1}^{I} \widehat{w}^i = \sum_{i=1}^{I} f^i(P, \widehat{w}^i)$$

and since $B\left(P, \widehat{w}^i; V\right) = B\left(P, w^i; V\right)$ then, $f^i(P, \widehat{w}^i) = f^i(P, w^i)$, which implies that $\sum_{i=1}^{I} \widehat{w}^i = \sum_{i=1}^{I} f^i(P, w^i)$. ■

Remark 8 *This is Balasko [1999] in incomplete markets. As in the complete markets case, one makes no use of any topological or differential property of the manifold* M *(strictly speaking, set* M*).*

4 From the Aggregate Demand to Individual Demand

If one is willing to assume that equilibrium prices are observable for situations in which the incomes of all individuals but one are zero, then it is straightforward that aggregate demand identifies individual demands: for all i, $f^i(P, w^i) = F(P, (\mathbf{0}, \mathbf{0}, \dots, w^i, \dots, \mathbf{0}))$. That is, when all agents different from i, have no income, the fact that prices are strictly positive implies no demand for agents different from i, and, therefore, that aggregate demand is agent i's individual demand.

Remark 9 *By the definition of aggregate demand and individual demand, for each set of prices and endowments* $\left(P, w^i\right) \in S_{++}^{n-1} \times \mathbb{R}_+^n$ *there is at least one portfolio of assets* z^i *such that* $f^i(P, w^i)$ *is financially feasible at* $\left(P, w^i\right)$. *When the asset structure* V *has nonredundant assets only, the portfolio of assets is unique (identified).*

We now show that under some additional assumptions one can identify an individual's demand without pegging everybody else's income at zero.

Condition 3 *For each individual* i*, in the interior of the commodity space* $\mathbb{R}_{++}^n$*,* u^i *is differentiably strictly monotone and differentiably strongly quasiconcave, and for all* $x \in \mathbb{R}_{++}^n$*,*

$$\left\{x' \in \mathbb{R}_+^n : u^i(x') \geq u^i(x)\right\} \subseteq \mathbb{R}_{++}^n.$$

Lemma 1 *For every* $(P,w) \in S^{n-1}_{++} \times \mathbb{R}^n_{++}$, $f^i(P,w) \in \mathbb{R}^n_{++}$.

Proof. It suffices to notice that $w \in B(P,w;V)$ and that $\left\{x \in \mathbb{R}^n_+ : u^i(x) \geq u^i(w)\right\} \subseteq \mathbb{R}^n_{++}$ ■

Lemma 2 f^i *is continuously differentiable.*

Proof. This follows from Duffie and Shafer (1985, p. 293). ■

As an auxiliary result, we first show that aggregate demand identifies individual demands up to a function of prices only.

Theorem 2 *For some* $\varphi^i : S^{n-1}_{++} \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$, *which is identified, and* $\phi^i : S^{n-1}_{++} \longrightarrow \mathbb{R}^n_+$,

$$f^i(P,w) = \varphi^i(P,w) + \phi^i(P)$$

for every $(P,w) \in S^{n-1}_{++} \times \mathbb{R}^n_+$.

Proof. Let $\varphi^i(p,w) = F(P,(\mathbf{1},\mathbf{1},...,w,...,\mathbf{1}))$, where w occupies the i^{th} position on its vector. Let $\phi^i(P) = -\sum_{j=1,j\neq i}^{I} f^j(P,\mathbf{1})$. Function φ^i is identified. ■

As in Chiappori et al (2002) and Kubler et al (2002), we impose the following:

Condition 4 (Regularity) *For every individual* i *and every* $P \in S^{n-1}_{++}$, *there exist* $w \in \mathbb{R}^n_+$, *and* $(s,l), (s',l') \in (\{0,...,S\} \times \{1,...,L\}) \setminus \{(0,1)\}$, *such that:*

$$\begin{vmatrix} \frac{\partial^2 f^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}(P,w) & \frac{\partial^2 f^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}(P,w) \\ \frac{\partial^3 f^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}(P,w) & \frac{\partial^3 f^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}(P,w) \end{vmatrix} \neq 0$$

Under regularity, global identification of individual demands is possible:

Theorem 3 *Aggregate demand identifies individual demands.*

Proof. It suffices to prove that the function ϕ^i of theorem 2 is also identified. From propositions 6 and 8 in the appendix, ignoring the arguments, it follows that for every $w \in \mathbb{R}^n_+$ and $(s,l),(s',l') \in (\{0,...,S\} \times \{1,...,L\}) \setminus \{(0,1)\}$:

$$\frac{\partial f^i_{s,l}}{\partial P_{s',l'}} + \left(f^i_{s',l'} - w^i_{s',l'}\right)\frac{\partial f^i_{s,l}}{\partial w^i_{0,1}} = \frac{\partial f^i_{s',l'}}{\partial P_{s,l}} + \left(f^i_{s,l} - w^i_{s,l}\right)\frac{\partial f^i_{s',l'}}{\partial w^i_{0,1}}$$

Substituting,

$$\begin{aligned} & \frac{\partial \varphi^i_{s,l}}{\partial P_{s',l'}} + \frac{\partial \phi^i_{s,l}}{\partial P_{s',l'}} + \left(\varphi^i_{s',l'} + \phi^i_{s',l'} - w^i_{s',l'}\right)\frac{\partial \varphi^i_{s,l}}{\partial w^i_{0,1}} \\ = \quad & \frac{\partial \varphi^i_{s',l'}}{\partial P_{s,l}} + \frac{\partial \phi^i_{s',l'}}{\partial P_{s,l}} + \left(\varphi^i_{s,l} + \phi^i_{s,l} - w^i_{s,l}\right)\frac{\partial \varphi^i_{s',l'}}{\partial w^i_{0,1}} \end{aligned}$$

Taking that $(s,l)\neq(0,1)$ and $(s',l')\neq(0,1)$ and deriving once and twice with respect to income gives us

$$\begin{aligned}&\frac{\partial^2\varphi^i_{s,l}}{\partial w^i_{0,1}\partial P_{s',l'}}+\left(\varphi^i_{s',l'}+\phi^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^2\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}\\ =\quad&\frac{\partial^2\varphi^i_{s',l'}}{\partial w^i_{0,1}\partial P_{s,l}}+\left(\varphi^i_{s,l}+\phi^i_{s,l}-w^i_{s,l}\right)\frac{\partial^2\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^3\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2\partial P_{s',l'}}+\frac{\partial\varphi^i_{s',l'}}{\partial w^i_{0,1}}\frac{\partial^2\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}+\left(\varphi^i_{s',l'}+\phi^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^3\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}\\ =\quad&\frac{\partial^3\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2\partial P_{s,l}}+\frac{\partial\varphi^i_{s,l}}{\partial w^i_{0,1}}\frac{\partial^2\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}+\left(\varphi^i_{s,l}+\phi^i_{s,l}-w^i_{s,l}\right)\frac{\partial^3\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}\end{aligned}$$

We can rewrite this system as

$$\Delta\begin{bmatrix}\phi^i_{s',l'}\\ \phi^i_{s,l}\end{bmatrix}=\Gamma$$

where

$$\Delta=\begin{bmatrix}\frac{\partial^2\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}(P,w) & -\frac{\partial^2\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}(P,w)\\ \frac{\partial^3\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}(P,w) & -\frac{\partial^3\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}(P,w)\end{bmatrix}$$

and Γ is a 2×1 matrix with first component

$$\frac{\partial^2\varphi^i_{s',l'}}{\partial w^i_{0,1}\partial P_{s,l}}-\frac{\partial^2\varphi^i_{s,l}}{\partial w^i_{0,1}\partial P_{s',l'}}+\left(\varphi^i_{s,l}-w^i_{s,l}\right)\frac{\partial^2\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}-\left(\varphi^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^2\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}$$

and second component

$$\begin{aligned}&\frac{\partial^3\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2\partial P_{s,l}}-\frac{\partial^3\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2\partial P_{s',l'}}+\frac{\partial\varphi^i_{s,l}}{\partial w^i_{0,1}}\frac{\partial^2\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}\\ &-\frac{\partial\varphi^i_{s',l'}}{\partial w^i_{0,1}}\frac{\partial^2\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}+\left(\varphi^i_{s,l}-w^i_{s,l}\right)\frac{\partial^3\varphi^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}-\left(\varphi^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^3\varphi^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}\end{aligned}$$

Both Δ and Γ are identified, as they depend only on φ^i. Moreover, by Regularity, for some $w\in\mathbb{R}^n_+$, $s,s'\in\{1,...,S\}$ and $l,l'\in\{1,...,L\}$, matrix Δ is invertible, which identifies $\phi^i_{s,l}$ and $\phi^i_{s',l'}$. For every other

$$(l'',s'')\in(\{1,...,L\}\times\{0,...,S\})\setminus\{(0,1)\}$$

$\phi^i_{l'',s''}$ is

$$\frac{\frac{\partial^2 \varphi^i_{l'',s''}}{\partial w^i_{0,1} \partial P_{s,l}} - \frac{\partial^2 \varphi^i_{s,l}}{\partial w^i_{0,1} \partial P_{l'',s''}} + \left(\varphi^i_{s,l} + \phi^i_{s,l} - w^i_{s,l}\right) \frac{\partial^2 \varphi^i_{l'',s''}}{\partial \left(w^i_{0,1}\right)^2} - \left(\varphi^i_{l'',s''} - w^i_{l'',s''}\right) \frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}}{\frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2}}$$

where we have assumed, without loss of generality, that $\frac{\partial^2 \varphi^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2} \neq 0$. Finally, $\phi^i_{0,1}$ can be identified by Walras' law. ■

Remark 10 *Since V is of full rank by condition 2, the identification of individual assets demand is straightforward.*

So far we have shown that individual financial markets demand functions are identified but moreover, we can define them in terms of the the individual demands that we have just identified from the no-arbitrage equilibrium manifold. Let $(p, q, w) \in S^{n-1}_{++} \times \mathbb{R}^J \times \mathbb{R}^{nI}_{+}$ then $f^i\,(p, q, w) = f^i(P, w)$ where $P = \pi \boxdot p$ and $\pi \in \mathbb{R}^{S+1}_{++}$is any vector such that $q = \pi_{\mathbf{1}} V(p_{\mathbf{1}})$.

Remark 11 *Since V is of full rank by condition 2 and we have identified individual financial markets demand the identification of individual assets demand (in financial markets) follows.*

5 Appendix: duality in incomplete markets

Fix an individual i.

Define $U \subseteq \mathbb{R}$ as the image of $\mathbb{R}^n_{++}$ under u^i:

$$U = \left\{\mu \in \mathbb{R} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\,(x) = \mu\right\}$$

For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, let $D\,(w_{\mathbf{1}}, \mu) \subseteq S^{n-1}_{++}$ be defined as follows:

$$D\,(w_{\mathbf{1}}, \mu) = \left\{P \in S^{n-1}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\,(x) = \mu \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

Proposition 3 *For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, $D\,(w_{\mathbf{1}}, \mu)$ is diffeomorphic to*

$$\left\{\left((P_{0,2}, ..., P_{0,L}), P_{\mathbf{1}}\right) \in \mathbb{R}^{n-1}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : u\,(x) = \mu \textit{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

which is open.

Proof. Let D denote the latter set. That $D\,(w_{\mathbf{1}}, \mu)$ and D are diffeomorphic is straightforward. We now show that D is open. Let $P \in D$. By definition, for some $x \in \mathbb{R}^n_{++}$, $u\,(x) = \mu$ and $P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle$, whereas using the implicit function theorem, for some $\varepsilon > 0$, $B_\varepsilon\,(x_{\mathbf{1}}) \subseteq \mathbb{R}^{LS}_{++}$ and

$$\left(\forall \widetilde{x}_{\mathbf{1}} \in B_\varepsilon\,(x_{\mathbf{1}})\right)\left(\exists \widetilde{x}_0 \in \mathbb{R}^L_{++}\right) : u^i\,(\widetilde{x}_0, \widetilde{x}_{\mathbf{1}}) = u^i(x)$$

Given that $\forall (s,l) \in \{1,...,S\} \times \{1,...,L\}$,

$$\lim_{\delta > 0} \frac{\delta (w_{s,l} - x_{s,l})}{\frac{P_{s,l}}{P_{s,1}} + \delta} = 0$$

there exists $\overline{\delta}_{s,l} > 0$ such that

$$|\delta| < \overline{\delta}_{s,l} \Longrightarrow \frac{|\delta| |w_{s,l} - x_{s,l}|}{\left| \frac{P_{s,l}}{P_{s,1}} + \delta \right|} < \frac{\varepsilon}{\sqrt{LS}}$$

Define

$$\overline{\delta} = \min_{(s,l) \in \{1,...,S\} \times \{1,...,L\}} \{\overline{\delta}_{s,l}\}$$

and consider the function $\phi : \mathbb{R}^{n-1}_{++} \to \mathbb{R}^{n-1}_{++}$, $\phi(P) = \left((P_{0,2}, ..., P_{0,L}), \frac{P_1}{P_{1,1}}, ... \frac{P_S}{P_{S,1}} \right)$. The function h is continious, therefore there is a $\delta > 0$ such that for all $P' \in B_\delta(P)$, $\|h(P') - h(P)\| < \overline{\delta}$, in particular $\left| \frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right| < \overline{\delta}$.

Define $x'_{\mathbf{1}} \in \mathbb{R}^{LS}$ as follows: $\forall (s,l) \in \{1,...,S\} \times \{1,...,L\}$,

$$x'_{s,l} = \frac{\frac{P_{s,l}}{P_{s,1}} x_{s,l} + \left(\frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right) w_{s,l}}{\frac{P'_{s,l}}{P'_{s,1}}}$$

Then,

$$\left| x'_{s,l} - x_{s,l} \right| = \frac{\left| \frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right| |w_{s,l} - x_{s,l}|}{\left| \frac{P'_{s,l}}{P'_{s,1}} \right|}$$

and, since $P' \in B_\delta(P)$, it follows that $\left| \frac{P'_{s,l}}{P'_{s,1}} - \frac{P_{s,l}}{P_{s,1}} \right| < \overline{\delta} \leqslant \overline{\delta}_{s,l}$, from where

$$\left| x'_{s,l} - x_{s,l} \right| < \frac{\varepsilon}{\sqrt{LS}}$$

and, hence, $\|x'_{\mathbf{1}} - x_{\mathbf{1}}\| < \varepsilon$. This implies that $x'_{\mathbf{1}} \in B_\varepsilon(x_{\mathbf{1}})$ and, therefore, that there exists $x'_0 \in \mathbb{R}^L_{++}$ such that $u^i(x'_0, x'_{\mathbf{1}}) = u^i(x)$.

Finally, by construction, $(\frac{P'_1}{P'_{1,1}}, ... \frac{P'_S}{P'_{S,1}}) \boxdot (x'_{\mathbf{1}} - w_{\mathbf{1}}) = (\frac{P_1}{P_{1,1}}, ... \frac{P_S}{P_{S,1}}) \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V \rangle$, and, hence, $P' \in D$. ■

For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$ such that $D(w_{\mathbf{1}}, \mu) \neq \varnothing$, define the Hicksian demand function $h(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}^n_{++}$, as:

$$h(P; w_{\mathbf{1}}, \mu) = \arg\min \left\{ \sum_{s=0}^{S} P_s \cdot x_s : u^i(x) \geq \mu \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}}) \rangle \right\}$$

and the expenditure function $e(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}$ as:

$$e(P; w_{\mathbf{1}}, \mu) = P \cdot h^i(P; w_{\mathbf{1}}, \mu)$$

Remark 12 *By the first part of condition 3, any solution lies in* $\mathbb{R}^n_{++}$ *and is unique.*

Now, define for each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, $M(w_{\mathbf{1}}) \subseteq S^{n-1}_{++} \times \mathbb{R}_{++}$ as follows:

$$M(w_{\mathbf{1}}) = \left\{(P,m) \in S^{n-1}_{++} \times \mathbb{R}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : \sum_{s=0}^{S} P_s \cdot x_s \leqslant m \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

Proposition 4 *For each* $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, $M(w_{\mathbf{1}})$ *is diffeomorphic to*

$$\left\{(((P_{0,2},...,P_{0,L}), P_{\mathbf{1}}), m) \in \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} : \left(\exists x \in \mathbb{R}^n_{++}\right) : \sum_{s=0}^{S} P_s \cdot x_s \leqslant m \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

which is nonempty and open.

Proof. This is straightforward. ■

For each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, define the conditional individual demand function $\widetilde{f}(\cdot,\cdot;w_{\mathbf{1}}) : M(w_{\mathbf{1}}) \longrightarrow \mathbb{R}^n_{++}$ as

$$\widetilde{f}(P,m;w_{\mathbf{1}}) = \arg\max\left\{u^i(x) : \sum_{s=0}^{S} P_s \cdot x_s \leq m \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle\right\}$$

Remark 13 *By the first part of condition 3, any solution lies in* $\mathbb{R}^n_{++}$ *and is unique. Obviously,*

$$\widetilde{f}(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}) = f^i(P,w)$$

Following is the standard duality result, extended to the case of incomplete markets. It contains three parts:

1. Given endowments w, If x^* solves the utility maximization problem at prices and $P \in S^{n-1}_{++}$, then x^* solves the expenditure minimization problem at prices P and minimum utility $u^i(x^*)$.

2. Given endowments $w_{\mathbf{1}}$ and utility μ, if x^* solves the expenditure minimization problem at prices $P \in D(w_{\mathbf{1}}, \mu)$, then x^* solves the utility maximization problem at prices P and endowments x^*.

3. Given endowments $w_{\mathbf{1}}$ and utility μ, if x^* solves the expenditure minimization problem at prices $P \in D(w_{\mathbf{1}}, \mu)$, then x^* solves the conditional utility maximization problem at prices P and income $e^i(P,w,\mu)$.That is

Proposition 5 1. *For every* $w = (w_0, w_{\mathbf{1}}) \in \mathbb{R}^n_{++}$ *and every* $P \in S^{n-1}_{++}$,

$$u^i(f^i(P,w)) \in U$$

$$P \in D\left(w_{\mathbf{1}}, u^i(f^i(P,w))\right)$$

and

$$h^i\left(P; w_{\mathbf{1}}, u^i(f^i(P,w))\right) = f^i(P,w)$$

2. *Given* $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$*, for every* $P \in D(w_{\mathbf{1}}, \mu)$*,*

$$f^i\left(P, h^i(P; w_{\mathbf{1}}, \mu)\right) = h^i(P; w_{\mathbf{1}}, \mu)$$

3. *Given* $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$*, for every* $P \in D(w_{\mathbf{1}}, \mu)$*,*

$$\left(P, e^i(P, w, \mu)\right) \in M(w_{\mathbf{1}})$$

and

$$\widetilde{f}^i\left(P, e^i(P, w, \mu); w_{\mathbf{1}}\right) = h^i(P; w_{\mathbf{1}}, \mu)$$

Proof. Part (1) is straightforward given lemma 1 and condition 3: argue by contradiction and use strict monotonicity of the utility function.

Given that u^i is continuous, for parts (2) and (3) it suffices to prove that $u^i(h^i(P; w_{\mathbf{1}}, \mu)) = \mu$. For this, suppose not: $u^i(h^i(P; w_{\mathbf{1}}, \mu)) > \mu$. Define $x = h^i(P; w_{\mathbf{1}}, \mu) - (\varepsilon, 0, ..., 0)$, where $\varepsilon \in \mathbb{R}_{++}$. By construction, $x_{\mathbf{1}} = h^i_{\mathbf{1}}(P; w_{\mathbf{1}}, \mu)$, from where $P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle$, and $\sum_{s=0}^{S} P_s \cdot x_s < e(P; w_{\mathbf{1}}, \mu)$, whereas since $h^i(P; w_{\mathbf{1}}, \mu) \in \mathbb{R}^n_{++}$, for ε small enough $x \in \mathbb{R}^n_+$ and, by continuity, $u^i(x) \geqslant \mu$, which is a contradiction. ■

Proposition 6 (Shepard's Lemma) *For every* $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$*, the function* $e(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}_{++}$ *is differentiable and*

$$\partial_P(e(P; w_{\mathbf{1}}, \mu)) = h(P; w_{\mathbf{1}}, \mu)$$

Proof. This is an immediate consequence of the Duality Theorem (see Mas-Colell et al, Proposition 3.F.1): let

$$K = \{x \in \mathbb{R}^n_+ : u^i(x) \geq \mu \text{ and } P_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(P_{\mathbf{1}})\rangle$$

Then, K is closed and $e^i(P; w_{\mathbf{1}}, \mu)$ is the support function of K. ■

Proposition 7 *For every* $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$*, the function* $\widetilde{f}(\cdot, \cdot; w_{\mathbf{1}}) : M(w_{\mathbf{1}}) \longrightarrow \mathbb{R}^n_{++}$ *is differentiable.*

Proof. This can be argued in the same way as fact 5 in Duffie and Shafer (1985). ■

Proposition 8 (Slutsky Equation in incomplete markets) *. Let* $(P, w) \in S^{n-1}_{++} \times \mathbb{R}^n_+$ *and* $\mu = u^i(f^i(P, w))$*. Then,* $h(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}^n_{++}$ *is differentiable and for all* $(s, l), (s', l') \in (\{0, ..., S\} \times \{1, ..., L\}) \setminus \{(0, 1)\}$*, we have:*

$$\frac{\partial h_{s,l}(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}} = \frac{\partial f^i_{s,l}(P, w)}{\partial P_{s',l'}} + \frac{\partial f^i_{s,l}(P, w)}{\partial w_{0,1}}(f^i_{s',l'}(P, w) - w_{s',l'})$$

Proof. That $h(\cdot; w_{\mathbf{1}}, \mu)$ is differentiable follows from propositions 5 and 7.

Also from proposition 5, we have that $h(P; w_{\mathbf{1}}, \mu) = \widetilde{f}\left(P, e\left(P; w_{\mathbf{1}}, \mu\right); w_{\mathbf{1}}\right)$. Therefore,

$$\frac{\partial h_{s,l}(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}} = \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P; w_{\mathbf{1}}, \mu\right); w_{\mathbf{1}}\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P; w_{\mathbf{1}}, \mu\right); w_{\mathbf{1}}\right)}{\partial m} \frac{\partial e(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}}$$

By proposition 6, we have:

$$\frac{\partial h_{s,l}(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}} = \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P; w_{\mathbf{1}}, \mu\right); w_{\mathbf{1}}\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P; w_{\mathbf{1}}, \mu\right); w_{\mathbf{1}}\right)}{\partial m} h_{s',l'}(P; w_{\mathbf{1}}, \mu)$$

Now, since $f^i(P, w) = \widetilde{f}^i\left(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}\right)$, then

$$\frac{\partial f^i_{s,l}\left(P, w\right)}{\partial P_{s',l'}} = \frac{\partial \widetilde{f}_{s,l}\left(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}\right)}{\partial m} w_{s',l'}$$

Under monotonicity (condition 3), at $\mu = u^i\left(f^i\left(P, w\right)\right)$, $e\left(P; w_{\mathbf{1}}, \mu\right) = \sum_{s=0}^{S} P_s \cdot w_s$ and, hence,

$$\frac{\partial f^i_{s,l}\left(P, w\right)}{\partial P_{s',l'}} = \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P, w, \mu\right); w_{\mathbf{1}}\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P, w, \mu\right); w_{\mathbf{1}}\right)}{\partial m} w_{s',l'}$$

Solving for

$$\frac{\partial \widetilde{f}^i_{s,l}\left(P, e\left(P, w, \mu\right); w_{\mathbf{1}}\right)}{\partial P_{s',l'}}$$

and replacing gives us

$$\frac{\partial h_{s,l}(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}} = \frac{\partial f^i_{s,l}\left(P, w\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, e\left(P, w, \mu\right); w_{\mathbf{1}}\right)}{\partial m} (h_{s',l'}(P; w_{\mathbf{1}}, \mu) - w_{s',l'})$$

By proposition 5, since $\mu = u^i(f^i(P, w))$,

$$\frac{\partial h_{s,l}(P; w_{\mathbf{1}}, \mu)}{\partial P_{s',l'}} = \frac{\partial f^i_{s,l}\left(P, w\right)}{\partial P_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}\right)}{\partial m} (f^i_{s',l'}\left(P, w\right) - w_{s',l'})$$

Finally, notice that

$$\frac{\partial f^i_{s,l}\left(P, w\right)}{\partial w_{0,1}} = \frac{\partial \widetilde{f}_{s,l}\left(P, \sum_{s=0}^{S} P_s \cdot w_s; w_{\mathbf{1}}\right)}{\partial m}$$

Substitution gives us the desired result. ■

6 References

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