A Theoretical Foundation for Bilateral Matching Mechanisms*

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ABSTRACT: This work introduces a rigorous set-theoretic foundation of bilateral matching mechanisms and studies their properties in a systematic manner. By providing a unified framework to study bilateral matching mechanisms, we formalize how different spatial/informational constraints can be implemented via a careful selection of matching mechanisms. In particular, this paper explains why and how various matching mechanisms generate different degrees of information isolation in the economy.

1 Introduction

There is a well-established research program in economics dealing with the efficiency of allocations achieved via decentralized and uncoordinated private decisions. Especially, the focus of this program has been on environments where trade is fragmented and subject to frictions. This literature has mainly relied on pairwise matching and trading, as a natural way to make spatial and informational constraints explicit.

This paper develops a rigorous set-theoretic foundation of bilateral matching mechanisms. It introduces a comprehensive definition of such mechanisms and formalizes systematically some of their basic properties, for economies populated by any arbitrary set of agents. It also provides an explicit specification of the mechanisms of operation of matching processes. In short, it presents a fully integrated theoretical approach to bilateral matching mechanisms.

Our study makes a contribution to a very large literature. For instance, pairwise

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trading frameworks have formed the basis of a literature studying how market frictions affect equilibrium output and unemployment, as in Diamond [5], the cyclical behavior of job creation and destruction, as in Pissarides [16] or Mortensen and Pissarides [15], and business cycles, as in Diamond and Fudenberg [6]. Bilateral matching has also been used to motivate the existence of obstacles—or complementary barriers—to the flow of information. This includes studies of the sustainability of cooperation in social games, as in Ellison [8], economic governance, as in Dixit [7], or the 'foundations of money' literature, as in Kiyotaki and Wright [11], Shi [17], Green and Zhou [9] or Lagos and Wright [14].

The need for a fully integrated theoretical approach to bilateral matching lies in the literature's fragmented treatment of such mechanisms. This lack of a unifying framework prevents a clear understanding of the exact connection between the environmental constraints imposed by the meeting technology, the frictions assumed in the environment and the possible allocations. One is often confronted with hazy explanations as to how, and to what extent, the desired geographical and informational constraints are a reflection, or an implication, of the mechanism by which agents meet each other. [1] In fact, understanding these aspects—in particular where informational constraints originate in a model—is critical. A clear example is provided by the work of Kocherlakota [12], who spells out why information frictions are central to monetary theory.

Using our machinery, we define an exact map between properties of pairwise matching mechanisms and degrees of informational isolation. We show how different types of informational constraints can be induced in a manner that is consistent with the physical description of the environment. For instance, we find that neither random matching nor unobservability of the partner's characteristics—both common assumptions in the literature (e.g., see [9] or [14])—are necessary to generate complete informational isolation. Strong anonymity, as we call it, can be achieved even when matched agents cannot hide their respective identities and actions and the matching rule is deterministic.

The paper is organized as follows. In Section 2 we introduce our notation. In Section 3 we define the notion of a bilateral matching rule for a single period and present a characterization that allows us to construct bilateral matching rules on any set of agents. Section 4 defines the notion of bilateral matching mechanisms, and formalizes the way in which different mechanisms can impose different levels of informational isolation in the economy. In Section 5 we discuss the method of operation of several bilateral matching mechanisms by means of various examples. Finally, in Section 6 we offer some concluding remarks.

[1]An exception is Corbae, Temzelides and Wright [4] who carefully describe a pairwise trading environment where matching rules are made explicit. Pairings are determined in equilibrium as a function of matching histories.

2 Notation

If A is any set, then the symbol $|A|$ will denote the cardinality of the set A. As usual, $|A| = \aleph_0$ means that A is a countable set and $|A| = \mathfrak{c}$ indicates that the cardinality of A is the continuum.

If a set A is a union of a pairwise disjoint family of sets $\{A_i\}_{i\in I}$, i.e., $A = \bigcup_{i\in I} A_i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$, then we shall denote this by the symbol $A = \bigsqcup_{i\in I} A_i$. That is, $A = \bigsqcup_{i\in I} A_i$ means that $A = \bigcup_{i\in I} A_i$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Throughout the paper the letter X will denote a non-empty set. We shall think of the set X as a collection of agents in the economy. We shall consider a discrete-time economy.

3 Bilateral Matching Rules

In this section, we shall discuss the properties of bilateral matching rules as they apply to agents in any period. We start with their definition.

Definition 1. *A* ***bilateral matching rule*** *for the set of agents X is simply a function $\phi\colon X \to X$ satisfying $\phi^2(x) = x$ for all $x \in X$, i.e., $\phi^2 = I$, the identity mapping on X.*

Here are two simple examples of bilateral matching rules.

a. Let $X = \mathbb{N} = \{1, 2, \ldots\}$, the set of natural numbers, and define $\phi\colon X \to X$ by

$$\phi(2a) = 2a - 1 \quad \text{and} \quad \phi(2a-1) = 2a\,.$$

b. Let $X = (0, \infty)$ and define $\phi\colon X \to X$ by $\phi(x) = \frac{1}{x}$.

It should be noted immediately that if $\phi\colon X \to X$ is a bilateral matching rule, then the function ϕ is invertible. That is, ϕ is a surjective function that is also one-to-one. Moreover, the inverse function of ϕ coincides with ϕ itself, i.e., $\phi^{-1} = \phi$.

A trivial bilateral matching rule is the identity mapping of X, that is, $\phi(x) = x$ for all $x \in X$. If $\phi\colon X \to X$ is a bilateral matching rule and $x \in X$ is an arbitrary agent, then we shall think any agent x as being matched with the agent $\phi(x)$. For this reason, we shall call $\phi(x)$ the **partner** of x; and, of course, by the symmetry of the situation $x = \phi\big(\phi(x)\big)$ is the partner of $\phi(x)$.

Definition 2. *A bilateral matching rule $\phi\colon X \to X$ is said to be* ***exhaustive*** *if $\phi(x) \neq x$ holds for all agents $x \in X$, i.e., whenever no agent is matched under ϕ to herself.*

Notice that the bilateral matching rule in example (a) above is exhaustive while the bilateral matching rule of example (b) is not. The next result reveals the structure of the bilateral matching rules. In fact, it characterizes the bilateral matching rules.

Theorem 3. *If $\phi\colon X \to X$ is a bilateral matching rule, then there exist three pairwise disjoint subsets A, B, and C of X such that:*

1. *$X = A \sqcup B \sqcup C$.*
2. *$\phi(a) = a$ for each $a \in A$.*
3. *$\phi(B) = C$ (or, equivalently, $\phi(C) = B$).*

Proof. Let $\phi\colon X \to X$ be a bilateral matching rule. Assume first that ϕ is exhaustive. So, in this case $A = \emptyset$. We shall establish the existence of the sets B and C using Zorn's lemma.[2]

To this end, let $\mathcal{C}$ denote the collection of all non-empty subsets B of X such that $B \cap \phi(B) = \emptyset$. Notice that for each $x \in X$ the set $B = \{x\}$ belongs to $\mathcal{C}$. Indeed, if $B \cap \phi(B) = \{x\} \cap \{\phi(x)\} \neq \emptyset$, then $x = \phi(x)$, which contradicts the fact that ϕ is exhaustive. It should be clear that the set $\mathcal{C}$ is partially ordered by the inclusion relation $\supseteq$.

Next, we claim that the partially ordered set $\mathcal{C}$ satisfies the condition of Zorn's lemma. That is, we claim that every chain of $\mathcal{C}$ has an upper bound in $\mathcal{C}$. To see this, let $\{B_j\}_{j\in J}$ be a chain of $\mathcal{C}$, that is, for any pair of indices $i, j \in J$ we either have $B_i \supseteq B_j$ or $B_j \supseteq B_i$. Let $B = \bigcup_{j\in J} B_j$, and we claim that $B \in \mathcal{C}$. To establish this claim, assume by way of contradiction that $B \cap \phi(B) \neq \emptyset$. Fix some $b \in B \cap \phi(B)$ and let $a \in B$ be such that $b = \phi(a)$. Choose $i, j \in J$ such that $a \in B_i$ and $b \in B_j$. Since either $B_i \supseteq B_j$ or $B_j \supseteq B_i$ is true, we can assume without loss of generality that $a, b \in B_j$. In particular, we have $b = \phi(a) \in B_j \cap \phi(B_j) = \emptyset$, which is impossible. This contradiction shows that $B \cap \phi(B) = \emptyset$, and so $B \in \mathcal{C}$.

According to Zorn's lemma there exists a maximal element in $\mathcal{C}$, say B^*. We claim that $B^* \sqcup \phi(B^*) = X$. To see this, assume by way of contradiction that $B^* \sqcup \phi(B^*) \neq X$. So, there exists some $x \in X$ such that $x \notin B^* \sqcup \phi(B^*)$. Now consider the set $B' = B^* \cup \{x\}$. Clearly, the set B' properly contains B^* and we claim that $B' \cap \phi(B') = \emptyset$. Indeed, if

$$y \in B' \cap \phi(B') = \big[B^* \cup \{x\}\big] \cap \big[\phi(B^*) \cup \{\phi(x)\}\big] = B^* \cap \{\phi(x)\},$$

then we have $y = \phi(x) \in B^*$. This implies, $x = \phi\big(\phi(x)\big) = \phi(y) \in \phi(B^*)$, contrary to $x \notin B^* \sqcup \phi(B^*)$. Thus, $B' \cap \phi(B') = \emptyset$ must be the case, which contradicts the maximality property of the set B^*. Therefore, $B^* \sqcup \phi(B^*) = X$. This shows that in this case the desired conclusion is true with $A = \emptyset$, $B = B^*$, and $C = \phi(B^*)$.

Now consider the general case. That is, assume that $\phi\colon X \to X$ is an arbitrary bilateral matching rule. Let $A = \{x \in X\colon\ \phi(x) = x\}$ and put $X_1 = X \setminus A$. If $x \in X_1$, then notice that $\phi(x) \in X_1$. Otherwise, $\phi(x) \in A$ implies $x = \phi^2(x) = \phi\big(\phi(x)\big) = \phi(x)$ or $x \in A$, which is impossible. It follows that $\phi\colon X_1 \to X_1$ is an exhaustive bilateral matching rule,

[2]For a rigorous discussion regarding Zorn's Lemma see [10].

and so by the previous part there exist two disjoint sets B and C with $B \sqcup C = X_1$ and $\phi(B) = C$. Now notice that the sets A, B, and C satisfy the desired properties. ■

The interpretation of Theorem 3 is the following: If $\phi\colon X \to X$ is a bilateral matching rule, then by deleting the set of fixed points of ϕ (i.e., the set of agents that are matched to themselves by ϕ) we can split the remaining set of agents into two sets (the sets B and C) having the same cardinality such that ϕ maps B onto C (that represents a complete matching of the agents in the set $X \setminus A$).

The partition $X = A \sqcup B \sqcup C$ is not unique. Of course, the set A (as being the set of fixed points of ϕ) is uniquely determined. The sets B and C need not be uniquely determined. For instance, if $X = \{1, 2, 3, 4\}$ and the exhaustive bilateral matching rule $\phi\colon X \to X$ is defined by $\phi(1) = 2$, $\phi(2) = 1$, $\phi(3) = 4$, and $\phi(4) = 3$, then we have the following decompositions $X = \{2, 4\} \sqcup \{1, 3\} = \{2, 3\} \sqcup \{1, 4\}$.

Any partition $X = A \sqcup B \sqcup C$ as described in Theorem 3 will be referred to as an (A, B, C)-*decomposition of X with respect to the bilateral matching rule ϕ*. For instance, if $X = (0, \infty)$ and the bilateral matching rule $\phi\colon X \to X$ is defined by $\phi(x) = \frac{1}{x}$, then an (A, B, C)-decomposition is given by $A = \{1\}$, $B = (0, 1)$ and $C = (1, \infty)$. Notice that a bilateral matching rule ϕ is exhaustive if and only if $A = \emptyset$.

Theorem 3 demonstrates how one can construct bilateral matching rules on any set X. For an example, let $X = [0, 1]$ and consider the partition of X determined by the sets $A = \left[\frac{1}{2}, \frac{3}{4}\right]$, $B = \left[0, \frac{1}{2}\right)$, and $C = \left(\frac{3}{4}, 1\right]$. If we take any surjective and one-to-one function $f\colon B \to C$, then the function $\phi\colon X \to X$, defined by

$$\phi(x) = \begin{cases} x & \text{if } \ x \in A \\ f(x) & \text{if } \ x \in B \\ f^{-1}(x) & \text{if } \ x \in C\,, \end{cases}$$

is clearly a bilateral matching rule for the set X.

Finite sets with an odd number of agents and compact convex subsets of Hausdorff locally convex spaces do not admit continuous exhaustive bilateral matching rules.

Lemma 4. *For a set of agents X we have the following.*

1. *If X is a finite set with an odd number of agents, then X does not admit any exhaustive bilateral matching rule.*

2. *If X is a non-empty compact convex subset of a Hausdorff locally convex space, then X does not admit any continuous exhaustive bilateral matching rule.*

Proof. (1) Assume that X is a finite set with an odd number of agents and let $\phi\colon X \to X$ be a bilateral matching rule. If $X = A \sqcup B \sqcup C$ is an (A, B, C)-decomposition of X with respect to ϕ, then we have $|X| = |A| + |B| + |C| = |A| + 2|B|$. Since $|X|$ is an odd number, we get $|A| \neq 0$, and this shows that ϕ cannot be an exhaustive bilateral matching rule.

(2) If X is a non-empty compact convex subset of some Hausdorff locally convex space, then according to the classical Brouwer–Schauder–Tychonoff fixed point theorem every continuous function $\phi\colon X \to X$ must have a fixed point; see, for instance [1, Corollary 16.52, p. 550]. ■

4 Bilateral Matching Mechanisms

We start with the definition of a bilateral matching mechanism.

Definition 5. *A* ***bilateral matching mechanism*** *on a set of agents X is a sequence $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ such that:*

a. For each $t \geq 1$ the function $\phi_t\colon X \to X$ is a bilateral matching rule of X.

b. The function $\phi_0\colon X \to X$ is the identity mapping, i.e., $\phi_0(a) = a$ for each $a \in X$.

As before, the agent $\phi_t(a)$ is called the ***partner*** *of agent a at period t; and, of course, by the symmetry of the situation a is the partner of $\phi_t(a)$ at period t.*

The period 0 can be viewed as the "idle" period before the process of trading starts in period 1, i.e., at period 0 we consider that each agent is matched to himself or herself. For concreteness, we assume that an agent a in a match observes the identity of his partner, $\phi(a)$, with whom he can voluntarily exchange information on past matches or objects available to the agent in the match. That is, although agents cannot observe directly the outcome, or identities of individuals, of matches in which they were not directly involved, they can acquire or provide this information through their partners.

Note that Definition 5 *does not* imply that a bilateral matching mechanism pairs every agent to someone else at every date. However, this is often assumed for practical purposes; see, for instance, Kiyotaki and Wright [11]. We now formalize this special matching scheme in the definition below.

Definition 6. *A bilateral matching mechanism $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ on a set of agents X is said to be* ***exhaustive*** *if for each $t \geq 1$ the bilateral matching rule ϕ_t is exhaustive.*

Now let $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ be a bilateral matching mechanism on a set of agents X. For each $t \geq 0$ we shall denote by $P_t(a)$ the set of all partners of an agent $a \in X$ in periods up to and including period t. That is,

$$P_t(a) = \{\phi_0(a), \phi_1(a), \phi_2(a), ..., \phi_t(a)\} .$$

The life-long collection of all partners of a is the set:

$$P(a) = \{\phi_0(a), \phi_1(a), \phi_2(a), ...\} .$$

Since $\phi_0(a) = a$ note that $a \in P_t(a)$ holds for all $t \geq 0$ and all agents a. Also observe that $P_0(a) = \{a\}$ for each $a \in X$. Two agents a and b are said to have a **common partner** if there exists some agent c different than a and b such that

$$c \in P(a) \cap P(b) .$$

Before proceeding further, we formalize some terminology in the context of our model.

Definition 7. *We shall say that two agents a and b:*

1. *Share a **direct partner**, if there exist periods $t_1 < t_2 < t_3$ and an agent c different than a and b such that:*

$$\begin{aligned} \phi_{t_1}(a) &= b\,, \\ \phi_{t_2}(b) &= c\,, \\ \phi_{t_3}(c) &= a\,. \end{aligned}$$

2. *Share an **indirect partner**, if there exist periods $t_1 < t_2 < t_3 < \cdots < t_k$ and agents $a_1, a_2, \ldots, a_{k-2}$ different than a and b, where $k \geq 4$ such that:*

$$\begin{aligned} \phi_{t_1}(a) &= b\,, \\ \phi_{t_2}(b) &= a_1\,, \\ \phi_{t_3}(a_1) &= a_2\,, \\ &\vdots \\ \phi_{t_{k-1}}(a_{k-3}) &= a_{k-2}\,, \\ \phi_{t_k}(a_{k-2}) &= a\,. \end{aligned}$$

This helps us define the way in which information or objects may flow across agents, over time. Specifically, suppose a and b meet, at some date, and b wants to transfer information to a after the match breaks, via a third agent. The "direct partner case" considers an occurrence in which a meets agent c, after b has met both of them. Here we say that c is a direct partner of b. Thus, agent c can transfer information to a from b after their match has ended.

The "indirect partner case" deals with exchange of information by means of a sequence of matches among agents. That is, agent a meets a_{k-2} who has never met b. However, a_{k-2} has met someone who was in direct or indirect contact with b, in the past. Here we say that in period t_k agent a_{k-2} is an indirect partner of b. This, too, can allow information transfers from b to a, across time.

We need to introduce one more notation. We shall denote by $\Pi_t(a)$ the set of all of a's past and current partners (including a herself), the past partners of a's current partner, the partners that a's partner in period $t-1$ met until period $t-2$, and so on. This set of agents is given by the recursive formula

$$\Pi_0(a) = P_0(a) = \{a\}\,, \quad \text{and} \quad \Pi_t(a) = \Pi_{t-1}(a) \bigcup \Pi_{t-1}\big(\phi_t(a)\big) \ \text{ for } \ t = 1, 2, \ldots \ .$$

From the above recursive formula and an easy inductive argument it follows that $\Pi_t(a)$ is a finite set since it includes a finite set of matching dates and partners. In particular, we note that $\Pi_t(a)$ does *not* include the agents that a's partners have met *after* matching with a and until the current period t. Also, it should be clear that $P_t(a) \subseteq \Pi_t(a)$ holds for all agents $a \in X$ and all periods $t \geq 0$.

We now have all the necessary machinery to introduce several properties of bilateral matching mechanisms. More precisely, we can formalize the different restrictions on the exchange of information that are commonly assumed throughout the matching literature.

Definition 8. *A bilateral matching mechanism on a set of agents X is said to be:*

1. ***Eventually weakly anonymous**, if for each agent a there exists a period t (depending on a) such that:*

   (a) the partners of a after period t are all distinct, and

   (b) $P_t(a) \cap \{\phi_{t+1}(a), \phi_{t+2}(a), \ldots\} \subseteq \{a\}$.

2. ***Weakly anonymous**, if the lifetime partners of any agent a are distinct. That is, for each agent a and each $t \neq \tau$ with $\phi_t(a) \neq a$ we have $\phi_t(a) \neq \phi_\tau(a)$.*

3. ***Anonymous**, if for each agent a that satisfies $\phi_{t+1}(a) \neq a$ in some period $t \geq 1$ the agents a and $\phi_{t+1}(a)$ do not have a common partner up to and including period t, that is,*
$$P_t(a) \cap P_t\big(\phi_{t+1}(a)\big) = \emptyset\ .$$

4. ***Strongly anonymous**, if for each agent a that satisfies $\phi_{t+1}(a) \neq a$ in some period $t \geq 1$ we have*
$$\Pi_t(a) \cap \Pi_t\big(\phi_{t+1}(a)\big) = \emptyset\ .$$

A key implication of our notions of anonymity, is that different degrees of anonymity provide different levels of informational isolation between any two partners. Before proceeding with a formalization of this claim, it may be helpful to provide some intuition.

In a model with eventually weak anonymity, two agents may be paired repeatedly to each other over time, but their match will eventually break down and it cannot be reconstituted. This matching mechanism is commonly adopted in the labor-search literature where ongoing bilateral worker-firm matches are affected by a process of job-destruction; see, for instance, Pissarides [16] or Mortensen and Pissarides [15]. It has also been used in the monetary literature, to study the interaction between money and credit (as in Corbae and Ritter [3]), or the evolution of the equilibrium price in markets with decentralized price formation mechanisms; see for example Binmore and Herrero [2].

Weak anonymity implies that agents are not paired longer than one period. Furthermore, once an agent a meets a partner b at some period t, a will never meet b again. Therefore, a and b cannot **directly** exchange information or objects over time. In every $t \geq 1$ and for each a we have $\phi_{t+1}(a) \notin P_t(a)$. An implication of this property, for example, is that direct credit arrangements, such as the direct redemption of IOU's, cannot take place. However, the door is open to the possibility that a and b, although never meeting again, may share a **direct partner** c.

The next result shows how matching mechanisms with stronger degrees of anonymity remove all direct and indirect links between agents.

Lemma 9. *If the bilateral matching mechanism is:*

a. anonymous, then no pair of agents will share any direct partner over their lifetimes.

b. strongly anonymous, then no pair of agents will share any direct or indirect partner over their lifetimes.

Proof. (a) Let $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ be an anonymous bilateral matching mechanism and assume by way of contradiction that two agents a and b share a direct partner. This means that there exist three periods $t_1 < t_2 < t_3$ and an agent c such that:

$$\text{(i)}\ \ \phi_{t_1}(a) = b, \quad \text{(ii)}\ \ \phi_{t_2}(b) = c, \quad \text{and} \quad \text{(iii)}\ \ \phi_{t_3}(c) = a\,.$$

Clearly, we have

$$t_1 < t_2 \leq t_3 - 1\,. \tag{$\star$}$$

Now note that (iii) yields $a = \phi_{t_3}(c) = \phi_{(t_3-1)+1}(c) \neq c$ and so by the anonymity of Φ, we get $P_{t_3-1}(c) \cap P_{t_3-1}(\phi_{(t_3-1)+1}(c)) = \emptyset$ or

$$P_{t_3-1}(c) \cap P_{t_3-1}(a) = \emptyset\ . \tag{$\star\star$}$$

Using (ii) we obtain $b = \phi_{t_2}(c)$ and a glance at ($\star$) guarantees that $b \in P_{t_3-1}(c)$. Next, observe that (i) in conjunction with ($\star$) implies $b \in P_{t_3-1}(a)$. So $b \in P_{t_3-1}(c) \cap P_{t_3-1}(a)$ contrary to ($\star\star$). This contradiction establishes the validity of (a).

(b) Assume that $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ is a strongly anonymous bilateral matching mechanism and suppose first by way of contradiction that two agents a and b share an indirect partner. This means that there exist periods $t_1 < t_2 < t_3 < \cdots < t_k$ and agents

$a_1, a_2, \ldots, a_{k-2}$ different than a and b, where $k \geq 4$ such that:

$$\begin{aligned}
\phi_{t_1}(a) &= b\,, \\
\phi_{t_2}(b) &= a_1\,, \\
\phi_{t_3}(a_1) &= a_2\,, \\
&\vdots \\
\phi_{t_{k-1}}(a_{k-3}) &= a_{k-2}\,, \\
\phi_{t_k}(a_{k-2}) &= a\,.
\end{aligned}$$

Clearly, we have

$$t_1 < t_2 < t_3 < \cdots < t_{k-1} \leq t_k - 1\,. \tag{†}$$

From $a = \phi_{t_k}(a_{k-2}) = \phi_{(t_k-1)+1}(a_{k-2}) \neq a_{k-2}$ and the strong anonymity of Φ, it follows that $\Pi_{t_k-1}(a_{k-2}) \cap \Pi_{t_k-1}(\phi_{(t_k-1)+1}(a_{k-2})) = \emptyset$ or

$$\Pi_{t_k-1}(a_{k-2}) \cap \Pi_{t_k-1}(a) = \emptyset\,. \tag{††}$$

Now note that $a_{k-2} \in \Pi_{t_k-1}(a_{k-2})$ is trivially true. On the other hand, it is not difficult to see that $a_{k-2} \in \Pi_{t_k-1}(a)$. But then we have $a_{k-2} \in \Pi_{t_k-1}(a_{k-2}) \cap \Pi_{t_k-1}(a)$, contrary to (††).

Finally, to establish that no pair of agents share a direct partner in their life times, use part (a) in conjunction with the fact that strong anonymity implies anonymity. (See also the proof of Lemma 10.) ■

Lemma 9 shows that in an anonymous matching mechanism any two agents a and b cannot exchange information (or objects) over time indirectly, by means of a common partner c. An example is the Townsend Turnpike model [18]; see also Example 14 in Section 5. However, a possibility still exists that agents a and b may share an **indirect partner** d.[3] This possibility is ruled out by strong anonymity.

A strongly anonymous matching mechanism is characterized by the most severe restriction on information flows among agents. It rules out the possibility that an agent meets former partners or any agents that his former partners might have been in contact with (directly or indirectly) before matching with him. Perhaps, this is obvious from the definition of strong anonymity, and from prior research; our definition replicates assumption (A2) in Kocherlakota [12] for $v \neq \omega$ under bilateral matching.

Lemma 9 main contribution, however, is it brings out a more subtle, but very important, implication. Strong anonymity rules out any chances that an arbitrary agent a may

[3]For example, suppose that agent 3 met 4 in t. Suppose 3 is matched to 6 in $t+j$. Then it is possible that in the periods between t and $t+j$, agent 4 has met 2, then agent 2 has met agent 1, and agent 1 has met agent 6. In this case, agent 2 is an indirect partner of both agents 6 and 3. In practical terms, this means that agent 2 could have communicated to agent 1 something he heard from 4 about agent 3. Agent 1 can then pass this on to 6 before he meets 3.

meet *in the future* someone who has been in direct or indirect contact with any of a's former partners. In short, strong anonymity insures total information isolation between any two partners at any point in time, past *and* future. This is unlike weaker forms of anonymity, as they only provide a weaker restriction on the feasible pattern of future matches.

The assumption of strong anonymity features prominently in the foundations of money literature; see the original model of Kiyotaki and Wright [11], or the more recent matching models of Shi [17], Lagos and Wright [14], or Green and Zhou [9]. In this class of models, as observed by Kocherlakota [12], strong restrictions on information flows make money essential in expanding the allocation set. Similar severe informational frictions have also been exploited in the social games literature concerned with the study of cooperative equilibria (as in Ellison [8]), or the interaction between long-term exchange relationships and anonymous market exchange (as in Kranton [13]).

Interestingly, in the above models (as well as virtually in all other models in the literature) strong anonymity is implemented by assuming that agents are unable to recognize their partners' identities, preferences, and the like or by assuming random matching. Our formalization of anonymous matching emphasizes that strong anonymity need not rely on these stringent assumptions. In fact, we later prove how complete informational isolation can be achieved by a careful specification of the matching mechanism (see Subsection 5.1), when partners cannot hide their respective identities or actions from each other and when the matching mechanism is deterministic.

As expected, the more restrictive anonymity properties subsume the less restrictive.

Lemma 10. *We have the following implications:*

$$\begin{aligned} \textit{Strong Anonymity} &\implies \textit{Anonymity} \\ &\implies \textit{Weak Anonymity} \\ &\implies \textit{Eventual Weak Anonymity} \end{aligned}$$

In general, no reverse implication is true.

Proof. Let $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ be a bilateral matching mechanism on a set of agents X and fix some agent $a \in X$.

Assume first that Φ is strongly anonymous. If $\phi_{t+1}(a) \neq a$, then from

$$P_t(a) \cap P_t\big(\phi_{t+1}(a)\big) \subseteq \Pi_t(a) \cap \Pi_t\big(\phi_{t+1}(a)\big) = \varnothing\,,$$

it follows that $P_t(a) \cap P_t\big(\phi_{t+1}(a)\big) = \varnothing$. This shows that Φ is an anonymous bilateral matching mechanism.

Now suppose that Φ is anonymous. Assume by way of contradiction that for some $1 \leq t < \tau$ and some agent a we have $\phi_t(a) \neq a$ and $\phi_t(a) = \phi_\tau(a)$. Let $t^* = \tau - 1$ and $b = \phi_{t^*+1}(a) = \phi_\tau(a) \neq a$. Clearly, $t \leq t^*$. Now note that $b \in P_{t^*}(b)$ and that

$b = \phi_\tau(a) = \phi_t(a) \in P_{t^*}(a)$, contrary to $P_{t^*}(a) \cap P_{t^*}(b) = \emptyset$. This contradiction shows that anonymity implies weak anonymity.

The fact that weak anonymity implies eventual weak anonymity is obvious. To see that no reverse implication holds true, see Examples 12, 13, and 14 in Section 5. ■

In general, although the opposite implication is not true, there are cases in which less restrictive anonymity implies more stringent anonymity. For example, the next lemma presents a condition under which weak anonymity implies anonymity.

Lemma 11. *Let $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3, \ldots)$ be a weakly anonymous bilateral matching mechanism. Assume that there exists a partition $X = B \sqcup C$ of X such that $\phi_t(B) = C$ for each $t \geq 1$, i.e., for each $t \geq 1$ the bilateral matching rule ϕ_t maps B onto C. Then the bilateral matching mechanism Φ is anonymous.*

Proof. Notice that by the symmetry of the situation we also have that $\phi_t(C) = B$ for each $t \geq 1$. This implies ϕ_t is an exhaustive bilateral matching rule for each $t \geq 1$. Assume by way of contradiction that there exists some agent a such that $P_t(a) \cap P_t(b) \neq \emptyset$ holds true for some $t \geq 1$, where $b = \phi_{t+1}(a)$. Without loss of generality, we can assume that $a \in B$; and so $b = \phi_{t+1}(a) \in C$. Clearly, $b \neq a$.

Since $\phi_t(a) \in C$, $\phi_t(b) \in B$ and $B \cap C = \emptyset$, it follows from $P_t(a) = \{a, \phi_1(a), \ldots, \phi_t(a)\}$, $P_t(b) = \{b, \phi_1(b), \ldots, \phi_t(b)\}$, and $P_t(a) \cap P_t(b) \neq \emptyset$ that there exists some $1 \leq \tau \leq t$ such that either $a = \phi_\tau(b)$ or $b = \phi_\tau(a)$. In either case, we have $\phi_\tau(a) = b = \phi_{t+1}(a)$. However, the latter conclusion contradicts the weak anonymity of Φ. Consequently, the bilateral matching mechanism Φ is anonymous. ■

An example is as follows. Divide a set of agents into two sets with the same cardinality. Call these two sets, for example, "sellers" and "buyers," and in each period match every seller to a different buyer. Then the agents cannot share any common direct partner as no seller is ever matched to another seller, and no buyer is ever matched to another buyer. Thus these matches are anonymous. An example of this type of matching mechanism, as noted above, is the Townsend's Turnpike [18].

5 Examples

In this section, we shall present a variety of bilateral matching mechanisms. Start by observing that only infinite sets of agents can admit exhaustive eventually weakly anonymous (and hence weakly anonymous, anonymous, and strongly anonymous) bilateral matching mechanisms.

We start by presenting an exhaustive eventually weakly anonymous bilateral matching mechanism that is not weakly anonymous.

Example 12. Let $\mathbb{N} = \bigsqcup_{n=1}^{\infty} \mathbb{N}_n$ be a partition of the natural numbers $\mathbb{N}$ such that each $\mathbb{N}_n$ is countable.[4] For each $t \in \mathbb{N}$ the set $\bigsqcup_{i \neq t} \mathbb{N}_i$ is countable. So, there exists a one-to-one surjective function $f_t \colon \bigsqcup_{i \neq t} \mathbb{N}_i \to \mathbb{N}_t$. By Theorem 3, this function f_t defines an exhaustive bilateral matching rule $\phi_t \colon \mathbb{N} \to \mathbb{N}$. Letting $\phi_0 = I$, it follows that $(\phi_0, \phi_1, \phi_2, \ldots)$ is an exhaustive bilateral matching mechanism on the set $\mathbb{N}$. Now fix some $a \in \mathbb{N}$. Then there exists a unique $k \in \mathbb{N}$ such that $a \in \mathbb{N}_k$. In particular, since for each $t \neq k$ we have $a \in \bigsqcup_{i \neq t} \mathbb{N}_i$, it follows that $\phi_t(a) \in \mathbb{N}_t$ for each $t \neq k$. This implies that the set of agents in the set $\{\phi_i(a)\colon\ i \neq k\}$ are all distinct. However, there is a possibility (and it is easy to construct examples) that $\phi_k(a) = \phi_r(a)$ for some $r \neq k$. If we let $\tau = \max\{r, k\}$, then we have $P_\tau(a) \cap \{\phi_{\tau+1}(a), \phi_{\tau+2}(a), \ldots\} = \emptyset$, and this shows that $(\phi_0, \phi_1, \phi_2, \ldots)$ is an exhaustive eventually weakly anonymous bilateral matching mechanism that might fail to be weakly anonymous. ■

The next example is an example of a weakly anonymous bilateral mechanism that fails to be anonymous.

Example 13. We consider $X = \mathbb{N}$. As usual, we let ϕ_0 be the identity mapping on $\mathbb{N}$. The next three exhaustive matching rules are defined by the following matrices:

$$\phi_1 = \begin{bmatrix} 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & \ldots \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & \ldots \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & \ldots \\ 3 & 7 & 6 & 8 & 13 & 15 & 9 & 11 & 21 & 23 & 17 & 19 & \ldots \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & \ldots \\ 7 & 3 & 8 & 21 & 6 & 23 & 17 & 19 & 13 & 15 & 9 & 11 & \ldots \end{bmatrix}$$

The remaining exhaustive matching rules $\phi_4, \phi_5, \phi_6, \ldots$ will be constructed by induction. As a matter of fact if

$$\begin{aligned} B &= \{1, 2, 4, 5, 10, 12, 14, 16, 18, 20, 22, 24, \ldots\}, \quad \text{and} \\ C &= \{7, 3, 8, 21, 6, 23, 17, 19, 13, 15, 9, 11, \ldots\}, \end{aligned}$$

then for each $t \geq 4$ the bilateral matching rule ϕ_t will map B onto C. (Notice that the agents of the set C are ordered as shown in the second row of the matrix ϕ_3.)

To define ϕ_4 consider C as a pairwise disjoint union of sets each of which contains twelve agents. That is, we write $C = C^4 = \bigsqcup_{n=1}^{\infty} A_n^4$, where A_1^4 consists of the first twelve elements of C, A_2^4 consists of the next twelve elements of C (as ordered above), A_3^4 consists of the

[4] One way of constructing by induction such a partition is as follows. Start with $\mathbb{N}_1 = \{2, 4, 6, \ldots\}$ and assume that $\mathbb{N}_n$ has been selected so that $\mathbb{N} \setminus \mathbb{N}_n = \{n_1, n_2, n_3, \ldots\}$ is countable, where $n_1 < n_2 < n_3 < \cdots$. Now to complete the inductive argument let $\mathbb{N}_{n+1} = \{n_1, n_3, n_5, \ldots\}$.

next twelve elements of C, and so on. Notice that $A_1^4 = \{7,3,8,21,6,23,17,19,13,15,9,11\}$. Next, we reorder C as follows:

$$C = C^4 = A_2^4 \sqcup A_1^4 \sqcup A_4^4 \sqcup A_3^4 \sqcup \cdots = \bigsqcup_{n=1}^{\infty} \big[A_{2n}^4 \sqcup A_{2n-1}^4\big] = \bigsqcup_{n=1}^{\infty} A_n^5\,,$$

where $A_n^5 = A_{2n}^4 \sqcup A_{2n-1}^4$ and A_n^5 is considered as an ordered set by having first the ordered elements of A_{2n}^4 followed by the ordered elements of the set A_{2n-1}^4. Notice that each ordered set A_n^5 consists of $24 = 2 \times 12$ elements. The exhaustive matching rule ϕ_4 is now given by the matrix:

$$\phi_4 = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & \ldots \\ A_2^4 & A_1^4 & A_4^4 & A_3^4 & A_6^4 & A_5^4 & \cdots & A_{2n}^4 & A_{2n-1}^4 & \cdots \end{bmatrix}.$$

Notice that we can replace each A_n^t with anyone of its permutations.

The construction of the $\phi_5, \phi_6, \ldots$ can be completed by induction following the above process. More specifically, assume that for some $t \geq 4$ the set C has been ordered as follows: $C = C^t = \bigsqcup_{n=1}^{\infty} A_n^t$, where each A_n^t consists of $2^{t-4} \times 12$ elements. Now consider the ordered sets $A_n^{t+1} = A_{2n}^t \sqcup A_{2n-1}^t$ (each of which has $2^{(t+1)-4} \times 12$ elements) and obtain the new ordering of the set C given by $C = C^{t+1} = \bigsqcup_{n=1}^{\infty} A_n^{t+1}$. Now to complete the induction define the exhaustive matching rule ϕ_{t+1} via the matrix

$$\phi_{t+1} = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & \ldots \\ A_2^{t+1} & A_1^{t+1} & A_4^{t+1} & A_3^{t+1} & A_6^{t+1} & A_5^{t+1} & \cdots & A_{2n}^{t+1} & A_{2n-1}^{t+1} & \cdots \end{bmatrix}.$$

It is easy to check that $(\phi_0, \phi_1, \phi_2, \ldots)$ is an exhaustive weakly anonymous bilateral matching mechanism. If we let $a = 3$, then we have $b = \phi_{2+1}(3) = \phi_3(3) = 2$. Now notice that

$$P_2(2) = \{2,1,7\} \quad \text{and} \quad P_2(3) = \{3,4,1\}\,.$$

This shows that $(\phi_0, \phi_1, \phi_2, \ldots)$ is not anonymous. ■

The next example is an example of an exhaustive anonymous bilateral matching mechanism that is not strongly anonymous and is due to R. Townsend [18].

Example 14 (Townsend [18]). The matching mechanism in Townsend's turnpike model of exchange [18] is an example of matching mechanism that is anonymous but not strongly anonymous. It has countably many agents. Each agent is assumed to be located into one of the countably many of spatially separated islands. The bilateral matching mechanism is such that "any two agents are paired at most once during their lifetimes" (i.e., it satisfies weak anonymity), and "they share no common third agent as a trading partner" (i.e., it satisfies anonymity). At each time period each agent travels on a turnpike, either east or west, moving by one position. See Townsend [18] for a figure depicting such a bilateral matching mechanism. As in Kocherlakota [12], we can interpret this economy as having countably many islands (or "trading posts") located at the integer points along the real line. At any time period each island is populated by two agents, one "stayer" and one "mover."

The set of agents is $X = \mathbb{Z} \setminus \{0\} = \{\ldots, -3, -2, -1, 1, 2, 3, \ldots\}$, i.e., the set of integers deprived of the zero. Without loss of generality we can identify the stayers with even numbers and the movers with odd numbers. The exhaustive bilateral matching mechanism $(\phi_0, \phi_1, \phi_2, \ldots)$ is defined as follows. As usual, we let $\phi_0 = I$, the identity mapping on X. Now let $B = \{\ldots, -4, -2, 2, 4, \ldots\}$, the set of all non-zero even integers, and $C = \{\ldots, -3, -1, 1, 3, \ldots\}$, the set of all odd integers. For $t \geq 1$ the exhaustive matching rules $\phi_t \colon B \to C$ are defined via the formulas:

$$\phi_1(a) = \begin{cases} a - 1 & \text{if } \ 0 < a \in B \\ a + 1 & \text{if } \ 0 > a \in B\,, \end{cases}$$

and

$$\phi_t(a) = \phi_1(a) - 4(t-1) \ \text{ if } \ t > 1\,.$$

The following table describes the above exhaustive bilateral matching mechanism.

	$\cdots$	-16	-14	-12	-10	-8	-6	-4	-2	2	4	6	8	10	12	14	16	$\cdots$
1	$\cdots$	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	$\cdots$
2	$\cdots$	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	$\cdots$
3	$\cdots$	-23	-21	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	$\cdots$
4	$\cdots$	-27	-25	-23	-21	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	$\cdots$

It is a routine matter to verify that $(\phi_0, \phi_1, \phi_2, \ldots)$ is a weakly anonymous bilateral matching mechanism—which is also anonymous. (A quick way of seeing this is by using Lemma 11.) This bilateral matching mechanism is not strongly anonymous since some partners of some agents have common partners. For instance, using the above table, it is easy to see that

$$\begin{aligned} \Pi_2(2) &= \Pi_1(2) \cup \Pi_1\big(\phi_2(2)\big) = \Pi_0(2) \cup \Pi_0\big(\phi_1(2)\big) \cup \Pi_0\big(\phi_2(2)\big) \cup \Pi_0\big(\phi_1(\phi_2(2))\big) \\ &= P_0(2) \cup P_0(1) \cup P_0(-3) \cup P_0(-4) \\ &= \{2\} \cup \{1\} \cup \{-3\} \cup \{-4\} = \{-4, -3, 1, 2\}\ , \end{aligned}$$

and that

$$\begin{aligned} \Pi_2(\phi_3(2)) &= \Pi_2(-7) = \Pi_1(-7) \cup \Pi_1\big(\phi_2(-7)\big) \\ &= \Pi_0(-7) \cup \Pi_0\big(\phi_1(-7)\big) \cup \Pi_0\big(\phi_2(-7)\big) \cup \Pi_0\big(\phi_1(\phi_2(-7))\big) \\ &= P_0(-7) \cup P_0(-8) \cup P_0(-4) \cup P_0(-3) \\ &= \{-7\} \cup \{-8\} \cup \{-4\} \cup \{-3\} = \{-8, -7, -4, -3\}\ . \end{aligned}$$

Since $-3 \in \Pi_2(2) \cap \Pi_2(\phi_3(2))$, it follows that $\Pi_2(2) \cap \Pi_2(\phi_3(2)) \neq \emptyset$, so that the bilateral matching mechanism is not strongly anonymous.

Note that the Townsend bilateral matching mechanism allows for *indirect* links among agents. To see the reason for this, consider the first three periods in the above table. Agent 2 meets 1 in $t = 1$, and -3 in $t = 2$. Agent -7 meets -8 in $t = 1$, -4 in $t = 2$, and 2 in $t = 3$. Furthermore, -4 and -3 are partners in $t = 1$. Therefore, a partner of 2 and a partner of -7 met before period $t = 3$. Hence, this economy could possibly admit the following (non-monetary) transfer scheme: in period $t = 1$ agent -4 makes a transfer to -3, expecting that (i) in period $t = 2$ agent -7 will make a transfer to -4, while -3 will make a transfer to 2, and (ii) in period $t = 3$ agent 2 will make a transfer to -7. ■

5.1 Strongly anonymous bilateral matching mechanisms

We start by exhibiting examples of strongly anonymous bilateral matching mechanisms. Let X denote again an infinite set of agents. Assume that there exists a partition $\{A_1, A_2, \ldots\}$ of X, i.e., $X = \bigsqcup_{n=1}^{\infty} A_n$, such that all the A_n have the same cardinality. At each period $t \geq 1$ we can also partition X as follows (the brackets indicate the partition sets in each period):

Period	Partition of the set of agents X
0	$X = [A_1] \sqcup [A_2] \sqcup [A_3] \sqcup [A_4] \sqcup [A_5] \sqcup [A_6] \sqcup \cdots$
1	$X = \big[A_1 \sqcup A_2\big] \sqcup \big[A_3 \sqcup A_4\big] \sqcup \big[A_5 \sqcup A_6\big] \sqcup \cdots$
2	$X = \big[A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4\big] \sqcup \big[A_5 \sqcup A_6 \sqcup A_7 \sqcup A_8\big] \sqcup \cdots$
3	$X = \big[A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4 \sqcup A_5 \sqcup A_6 \sqcup A_7 \sqcup A_8\big] \sqcup$ $\big[A_9 \sqcup A_{10} \sqcup A_{11} \sqcup A_{12} \sqcup A_{13} \sqcup A_{14} \sqcup A_{15} \sqcup A_{16}\big] \sqcup \cdots$
$\vdots$	$\vdots$
t	$X = \bigsqcup_{n=1}^{\infty} \big[A_{(n-1)2^t+1} \sqcup A_{(n-1)2^t+2} \sqcup \cdots \sqcup A_{n2^t}\big] = \bigsqcup_{n=1}^{\infty} \bigsqcup_{k=1}^{2^t} A_{(n-1)2^t+k} = \bigsqcup_{n=1}^{\infty} B_n^t = \big[B_1^t \sqcup B_2^t\big] \sqcup \big[B_3^t \sqcup B_4^t\big] \sqcup \cdots = \bigsqcup_{n=1}^{\infty} \big[B_{2n-1}^t \sqcup B_{2n}^t\big] = \bigsqcup_{n=1}^{\infty} B_n^{t+1}$
	$\vdots$,

where we let $B_n^t = \bigsqcup_{k=1}^{2^t} A_{(n-1)2^t+k}$ for each $n = 1, 2, \ldots$ and each $t \geq 1$. It should be clear that for each $t \geq 1$ the sequence $\{B_1^t, B_2^t, B_3^t, \ldots\}$ is pairwise disjoint and the sets B_n^t, $n = 1, 2, \ldots$, all have the same cardinality. Moreover, note that $B_n^{t+1} = B_{2n-1}^t \sqcup B_{2n}^t$ holds for all $n = 1, 2, \ldots$ and all $t \geq 1$.

For each $n = 1, 2, \ldots$ and each $t \geq 1$, let $f_n^t \colon B_{2n-1}^t \to B_{2n}^t$ be a one-to-one and surjective function. Also, let $g_n^t \colon B_{2n}^t \to B_{2n-1}^t$ be the inverse of f_n^t. Next, for each $t \geq 1$ define $\phi_t \colon X \to X$ by

$$\phi_t(x) = \begin{cases} f_n^t(x) & \text{if } x \in B_{2n-1}^t \\ g_n^t(x) & \text{if } x \in B_{2n}^t\,. \end{cases} \tag{✠}$$

We let $\phi_0 = I$, the identity on X.

Clearly, for each $t \geq 1$ the function $\phi_t \colon X \to X$ is an exhaustive bilateral matching rule on X. Moreover, from $B_n^{t+1} = B_{2n-1}^t \sqcup B_{2n}^t$, it follows that for each n the set B_n^{t+1} is ϕ_t-invariant, i.e., $\phi_t(B_n^{t+1}) \subseteq B_n^{t+1}$. As a matter of fact, it is easy to see that ϕ_t restricted to each B_n^{t+1} is an exhaustive bilateral matching rule. More generally, we have the following.

Lemma 15. *For each $t \geq 0$ and each $\tau = 0, 1, \ldots, t$ the sets B_n^{t+1}, $n = 1, 2, \ldots$, are ϕ_τ-invariant. In fact, for each $\tau = 1, \ldots, t$ the restriction of ϕ_τ to each B_n^{t+1} is an exhaustive bilateral matching rule.*

Proof. We shall use induction on t. For $t = 0$ the conclusion is obvious. Therefore, for the induction step, assume that the conclusion is true for some $t \geq 0$. For each n we have $B_n^{t+2} = B_{2n-1}^{t+1} \sqcup B_{2n}^{t+1}$ and by our induction hypothesis for each $i = 1, \ldots, t$ the functions $\phi_i \colon B_{2n-1}^{t+1} \to B_{2n-1}^{t+1}$ and $\phi_i \colon B_{2n}^{t+1} \to B_{2n}^{t+1}$ are exhaustive bilateral matching rules. It easily follows that for each $i = 1, \ldots, t$ the function $\phi_i \colon B_n^{t+2} \to B_n^{t+2}$ is itself an exhaustive bilateral matching rule on the set B_n^{t+2}. By the preceding discussion, we also know that ϕ_{t+1} restricted to each B_n^{t+2} is an exhaustive bilateral matching rule. Hence, for each $\tau = 1, \ldots, t+1$ the function ϕ_τ restricted to B_n^{t+2} is an exhaustive bilateral matching rule. This completes the induction and the proof of the lemma. ■

We are now ready to show that the exhaustive bilateral matching mechanism defined above is strongly anonymous.

Theorem 16. *Any bilateral matching mechanism $(\phi_0, \phi_1, \phi_2, \ldots)$ as defined by (✠) is strongly anonymous.*

Proof. Let $a \in X$ be an arbitrary agent and fix $t \geq 1$. Let k be the unique natural number such that $a \in B_k^{t+1}$. According to Lemma 15, we have $\phi_i(a) \in B_k^{t+1}$ for each $i = 0, 1, \ldots, t$. This easily implies $\Pi_t(a) \subseteq B_k^{t+1}$. Now according to the definition of the bilateral matching rule ϕ_{t+1} either we have $b = \phi_{t+1}(a) \in B_{k-1}^{t+1}$ or $b = \phi_{t+1}(a) \in B_{k+1}^{t+1}$. In particular, as above, either $\Pi_t(b) \subseteq B_{k-1}^{t+1}$ or $\Pi_t(b) \subseteq B_{k+1}^{t+1}$. Since $B_{k-1}^{t+1} \cap B_k^{t+1} = \emptyset$, $B_{k+1}^{t+1} \cap B_k^{t+1} = \emptyset$ and $\Pi_t(a) \subseteq B_k^{t+1}$, we infer that $\Pi_t(a) \cap \Pi_t(b) = \emptyset$. Consequently, $(\phi_0, \phi_1, \phi_2, \ldots)$ is a strongly anonymous bilateral matching mechanism. ■

It is now easy to construct exhaustive bilateral matching mechanisms on sets of agents X. The only thing that is needed is a countable partition of the set X for which all sets of the partition have the same cardinality. Here are some examples of partitions on various sets:

$$\begin{aligned} X &= (0,1] = \bigsqcup_{n=1}^{\infty} \big(\tfrac{1}{n+1}, \tfrac{1}{n}\big] \\ \mathbb{N} &= \{1,2,\ldots\} = \bigsqcup_{n=1}^{\infty} \{n\} \\ \mathbb{N} &= \{1,2,\ldots\} = \bigsqcup_{n=1}^{\infty} \{2n-1, 2n\} \\ X &= [0,\infty) = \bigsqcup_{n=1}^{\infty} [n-1, n)\,. \end{aligned}$$

An example of an exhaustive strongly anonymous bilateral matching mechanism on $\mathbb{N}$ that corresponds to the partition $\mathbb{N} = \bigsqcup_{n=1}^{\infty}\{n\}$ is given by the matrix shown below. The matrix describes how the even agents (the first row) are paired to odd agents in periods $t = 1, 2, 3, 4$.

t	2	4	6	8	10	12	14	16	18	20	$\cdots$
1	1	3	5	7	9	11	13	15	17	19	$\cdots$
2	5	7	1	3	13	15	9	11	21	23	$\cdots$
3	13	15	9	11	5	7	1	3	29	31	$\cdots$
4	29	31	25	27	21	23	17	19	13	15	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

Finally, we mention that it is easy to modify the preceding examples of exhaustive bilateral matching mechanisms to yield non-exhaustive bilateral matching mechanisms. As an example, we shall modify the last example to produce examples of (not necessarily exhaustive) strongly anonymous bilateral matching mechanisms.

Example 17. Let X be an infinite set of agents such that $X = \bigsqcup_{n=1}^{\infty} A_n$, where all the A_n have the same infinite cardinality. In particular, note that for each $t \geq 1$ the sets B_n^t $(n = 1, 2, \ldots)$ all have the same infinite cardinality. For each $t \geq 1$ and each $n \in \mathbb{N}$ let F_n^t be a (possibly empty) subset of B_{2n-1}^t such that the sets $B_{2n-1}^t \setminus F_n^t$ and B_{2n}^t have the same (infinite) cardinality. For each n and $t \geq 1$ let $f_n^t \colon B_{2n-1}^t \setminus F_n^t \to B_{2n}^t$ be a one-to-one and surjective function. Also, let $g_n^t \colon B_{2n}^t \to B_{2n-1}^t \setminus F_n^t$ be the inverse of f_n^t.

Next, for each $t \geq 1$ define $\phi_t \colon X \to X$ by

$$\phi_t(x) = \begin{cases} x & \text{if } \ x \in F_n^t \\ f_n^t(x) & \text{if } \ x \in B_{2n-1}^t \setminus F_n^t \\ g_n^t(x) & \text{if } \ x \in B_{2n}^t \,. \end{cases}$$

We also let $\phi_0 = I$. Clearly, for each $t \geq 1$ the function ϕ_t is a bilateral matching rule whose set of idle agents is $\bigsqcup_{n=1}^{\infty} F_n^t$. Therefore, $\Phi = (\phi_0, \phi_1, \phi_2, \ldots)$ is a bilateral matching mechanism of the set X. In addition, it should be clear that ϕ_t restricted to the set $B_n^{t+1} = B_{2n-1}^t \sqcup B_{2n}^t$ is also a bilateral matching rule having as an (A, B, C)-decomposition the partition $B_n^{t+1} = F_n^t \sqcup \big(B_{2n-1}^t \setminus F_n^t\big) \sqcup B_{2n}^t$.

We claim that Φ is strongly anonymous. To see this, assume that an agent $a \in X$ satisfies $\phi_{t+1}(a) \neq a$. Let $n \in \mathbb{N}$ be the unique natural number such that $a \in B_n^{t+1}$. If n is even (say $n = 2k$), then it follows from the definition of ϕ_{t+1} that $\phi_{t+1}(a) \in B_{2k-1}^{t+1} \setminus F_k^{t+1}$ and so $\phi_{t+1}(a) \in B_{n-1}^{t+1}$. If n is odd (say $n = 2k-1$), then $\phi_{t+1}(a) \neq a$ implies $a \notin F_k^{t+1}$ and so $a \in B_{2k-1}^{t+1} \setminus F_k^{t+1}$, from which it follows that $\phi_{t+1}(a) \in B_{2k}^{t+1} = B_{n+1}^{t+1}$. Using the latter conclusion and arguing as in Lemma 15, it is easy to see that the sets B_n^{t+1} are all ϕ_τ-invariant for all $0 \leq \tau \leq t$. From this, and another easy argument (as in Theorem 16), we see that $\Pi_t(a) \cap \Pi_t(\phi_{t+1}(a)) = \emptyset$. Therefore, Φ is a strongly anonymous bilateral matching mechanism. ■

6 Concluding remarks

We have presented a unified framework to study bilateral matching mechanisms and demonstrated how different geographical and informational constraints can be implemented by an appropriate choice of the mechanism by which agents meet each other. It is our belief that the bilateral matching framework we have presented, can be used as the fundamental building block to study rigorously more general classes of matching mechanisms. For example, one could consider extending our basic formalization to include matching mechanisms that involve coalitions of more than two agents, continuous-time matching processes and matching rules that are functions of prior realizations of matches.

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