Measuring Segregation*

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Abstract

We define a segregation ordering as a ranking of cities from most segregated to least segregated. We propose a set of basic properties that any reasonable segregation ordering should have. We then fully characterize the class of segregation orderings that satisfy these basic properties. We prove that every such ordering is representable by a segregation index that has a particular simple form. Finally, we show that with one rarely used exception, each index defined in the literature violates one or more of our basic properties.

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1 Introduction

Segregation is a pervasive social issue. The segregation of men and women into different occupations helps explain the gender gap in earnings.[1] The continued racial

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[1] See Cotter et al [3], Lewis [8], and Macpherson and Hirsh [9].

segregation of schools appears to contribute to low educational achievement among minorities.[2] Residential segregation between blacks and whites has been blamed for black poverty, high black mortality, and increases in prejudice among whites.[3] In other contexts, segregation is viewed more positively. Many countries have ethnic minorities that seek separation and autonomy from other ethnic or religious groups. The formation of homogeneous living areas has been discussed as a solution to highly polarized conflicts in the Middle East and elsewhere.

Given the salience of segregation, its measurement is a critical issue. The approach of the existing literature has been to propose a segregation index, to show that it has a few desirable properties, and to proceed quickly to empirical work. (See Massey and Denton [10] for a survey.) This has the drawback that it is incomplete: other equally valid indices are not considered, and the full set of properties that characterize the index is not studied.

Our approach is different. We first define a segregation ordering as a ranking of cities from most segregated to least segregated. We propose a set of basic properties that any reasonable segregation ordering should have. We then fully characterize the class of segregation orderings that satisfy these basic properties. More precisely, we prove that every such ordering is representable by a segregation index that has a particular simple form. Finally, we show that with one exception, each index used in the literature violates one or more of our basic properties.

Our axioms rule out many segregation indices. However, they still leave the researcher with a choice of indices. Her choice depends on how she answers a central question. Suppose a segregated area in a city gains population and simultaneously becomes less segregated. How to weight these opposing developments? For any given increase in the area's size, how integrated must the area become for the city's level of segregation not to rise?

[2]See Meldrum and Eaton [11], Orfield [12], and Schiller [13].

[3]See Cutler and Glaeser [4], Collins and Williams [2], and Kinder and Mendelberg [7], respectively.

	Period 0			Period 1		
	Segregrated Area		Integrated Area	Segregrated Area		Integrated Area
	A	B	C	A	B	C
Blacks	100	0	100	120	30	50
Whites	0	100	100	30	120	50

Table 1: An example.

This basic tradeoff is illustrated in Table 1. The residents of a single city with three neighborhoods are surveyed in each of two periods. In period 0, there is one area of perfect segregation (neighborhoods A and B) and one area of perfect integration (neighborhood C). In period 1, the segregated area has gained population, but it has also become less segregated. If the researcher puts more weight on the segregated area's growth than on its change in composition, she would say that segregation has risen. If she weights the change in composition more, she would say that segregation has fallen. This is the essential choice that any researcher must face. The results of our paper provide guidance by showing how the choice between reasonable segregation indices (those that satisfy our set of axioms) depends solely on the researcher's decision of how much weight to put on these two factors.

The rest of the paper is organized as follows. After setting up some basic notation in Section 2, we introduce the notion of segregation orders in Section 3 and provide some known examples of segregation indices that represent various orders. Section 4 proposes some properties that a satisfactory segregation order should satisfy, and Section 5 characterizes the family of segregation orders that satisfy them all. Section 6 shows that the axioms used in the characterization are logically independent. In Section 7 we present rank correlations between various segregation indices using data from the 1990 U.S. Census.

2 Notation

Throughout the paper we use the language of urban racial segregation because it is the best known example. Our results apply in other contexts though: religious segregation, gender segregation, etc.

A neighborhood i is characterized by a pair (B_i, W_i) of non-negative real numbers. The first and second components are the numbers of blacks and whites, respectively, in i. A *city* is a finite set of neighborhoods. For example, $\{(1,2),(0,1)\}$ denotes a city with two neighborhoods, the first containing one black and two whites, and the second having just a single white. The set of neighborhoods of the city X is denoted $N(X)$.

Although we use set notation, a city can contain two distinct neighborhoods with identical numbers of blacks and whites. For example, $\{(1,2),(1,2)\}$ contains two distinct neighborhoods; it is different city from the city $\{(1,2)\}$, which contains only one. On the other hand, the order of neighborhoods does not matter; e.g., the city $\{(1,2),(3,4)\}$ can also be described just as well by $\{(3,4),(1,2)\}$.

Given a city X, we denote by $B(X)$ and $W(X)$ the total numbers of blacks and whites, respectively: $B(X) = \sum_{i \in N(X)} B_i$ and $W(X) = \sum_{i \in N(X)} W_i$. When it is clear to which city we are referring, we will write simply B and W. We restrict attention to cities in which $B > 0$ and $W > 0$. Also, the following notation will be useful.

$$
\begin{aligned}
P &= \frac{B}{B+W}\text{: the proportion of blacks in the city} \\
p_i &= \frac{B_i}{B_i+W_i}\text{: the proportion of blacks in neighborhood } i \\
T &= B+W\text{: the total population of the city} \\
t_i &= B_i+W_i\text{: the total population of neighborhood } i \\
b_i &= \frac{B_i}{B}\text{: the proportion of the city's blacks that live in neighborhood } i \\
w_i &= \frac{W_i}{W}\text{: the proportion of the city's whites that live in neighborhood } i.
\end{aligned}
$$

For any city X and any nonnegative constant c, cX denotes the city that results from multiplying the number of blacks and whites in each neighborhood of X by c: $cX = (cB_i, cW_i)_{i \in N(X)}$. For any two cities X and Y, let $X \uplus Y$ denote the union of X and Y. As in the case of individual cities, we keep identical neighborhoods separate. For instance, if $X = \{(1,2), (3,4)\}$ and $Y = \{(1,2)\}$ then $X \uplus Y = \{(1,2), (1,2), (3,4)\}$.

Neighborhood i is *representative* of the city if the proportion of the city's blacks in the neighborhood equals the proportion of the city's whites: if $p_i = P$. A neighborhood that is not representative of the city is said to be *unrepresentative*. If $p_i > P$, blacks are *overrepresented* in neighborhood i; if $p_i < P$, blacks are *underrepresented*.

3 Segregation orderings, and their measures

A segregation order, $\succcurlyeq$, is a complete and transitive binary relation on the set of cities. We interpret $X \succcurlyeq Y$ to mean "city X is at least as segregated as city Y." The relations $\sim$ and $\succ$ are derived from $\succcurlyeq$ in the usual way.[4]

Segregation orders are usually represented by segregation indices. A segregation index assigns to each city a nonnegative number which is meant to capture its level of segregation. Given a segregation index S, the associated segregation order is defined by $X \succeq Y \Leftrightarrow S(X) \geq S(Y)$. Clearly, a segregation order may be represented by more than one index.

3.1 Examples of segregation indices

The following indices have been used to study segregation (Massey and Denton [10]).[5]

[4] That is $X \sim Y$ if both $X \succcurlyeq Y$ and $Y \succcurlyeq X$; $X \succ Y$ if $X \succcurlyeq Y$ but not $Y \succcurlyeq X$.

[5] Massey and Denton [10] also survey several other indices that require additional information about neighborhoods' locations to be computed.

The Index of Dissimilarity This index measures the proportion of either racial group that would need to be reallocated across neighborhoods in order to obtain perfect integration. For example, if $b_i > w_i$, one needs to remove a proportion $b_i - w_i$ of the city's blacks from neighborhood i for the neighborhood to be representative; if $b_i < w_i$, one needs to add a proportion $w_i - b_i$ of the city's blacks to neighborhood i for the neighborhood to be representative. Thus, the Index of Dissimilarity equals:

$$D(X) = \frac{1}{2} \sum_{i \in N(X)} |b_i - w_i| \tag{1}$$

where we divide by 2 to avoid double counting. This index was introduced to the literature by Jahn et al [6].

Gini The Gini Index is defined as:[6]

$$G(X) = \frac{1}{2} \sum_{i \in N(X)} \sum_{j \in N(X)} |b_i w_j - b_j w_i| \tag{2}$$

This index is adapted from the income inequality index of the same name. It is related to the Lorenz curve, which plots the cumulative proportion of whites against the cumulative proportion of blacks, having sorted neighborhoods in increasing order of the percentage p_i of blacks. The Gini Index equals the area between this curve and the 45 degree line.

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[6]The Gini Index is usually defined as:

$$G = \frac{1}{2} \sum_{i \in N} \sum_{j=1}^{N} \frac{t_i t_j |p_i - p_j|}{T^2 P(1-P)}$$

but

$$\begin{aligned} \frac{t_i t_j |p_i - p_j|}{T^2 P(1-P)} &= \frac{(B_i + W_i)(B_j + W_j)\left|\frac{B_i}{B_i+W_i} - \frac{B_j}{B_j+W_j}\right|}{(B+W)^2 \frac{B}{B+W} \frac{W}{B+W}} \\ &= \frac{|B_i (B_j + W_j) - B_j (B_i + W_i)|}{BW} \\ &= \frac{|B_i W_j - B_j W_i|}{BW} = |b_i w_j - b_j w_i| \end{aligned}$$

Entropy The Entropy Index is defined as

$$H(X) = \frac{1}{TE} \sum_{i \in N(X)} t_i \left(E - E_i\right) \tag{3}$$

where

$$\begin{aligned} E_i &= p_i \ln\left(\frac{1}{p_i}\right) + (1-p_i)\ln\left(\frac{1}{1-p_i}\right) \\ E &= P \ln\left(\frac{1}{P}\right) + (1-P)\ln\left(\frac{1}{1-P}\right) \end{aligned}$$

This index, adapted from the information theory literature, was proposed by Theil and Finizza [16].

Atkinson The Atkinson Index, for constant $\beta \in [0,1]$, is defined as:[7]

$$A_\beta(X) = 1 - \left[\sum_{i \in N(X)} w_i^{1-\beta} b_i^{\beta}\right]^{\frac{1}{1-\beta}} \tag{4}$$

The Atkinson index was originally defined as a measure of income inequality (Atkinson [1]).

[7] The Atkinson Index with parameter β is usually defined as

$$A_\beta = 1 - \frac{P}{1-P}\left[\frac{1}{PT}\sum_{i \in N(X)} t_i (1-p_i)^{1-\beta} p_i^{\beta}\right]^{\frac{1}{1-\beta}}$$

This equals

$$\begin{aligned} & 1 - \frac{B}{W}\left[\frac{1}{B}\sum_{i \in N(X)} (B_i + W_i)\left(\frac{W_i}{B_i + W_i}\right)^{1-\beta}\left(\frac{B_i}{B_i + W_i}\right)^{\beta}\right]^{\frac{1}{1-\beta}} \\ = \; & 1 - \left[\left(\frac{1}{W}\right)^{1-\beta}\left(\frac{1}{B}\right)^{\beta}\sum_{i \in N(X)} (W_i)^{1-\beta}(B_i)^{\beta}\right]^{\frac{1}{1-\beta}} \\ = \; & 1 - \left[\sum_{i \in N(X)} w_i^{1-\beta} b_i^{\beta}\right]^{\frac{1}{1-\beta}} \end{aligned}$$

Isolation The Index of Isolation is written:

$$J(X) = \frac{\left(\sum_{i \in N(X)} \frac{B_i}{B} p_i\right) - P}{1 - P}. \tag{5}$$

A variant of this index was used by Cutler, Glaeser, and Vigdor [5] to measure the evolution of segregation in American cities.

4 Axioms

A variety of segregation indices are available for researchers. Are any of them more desirable than others? In this section we propose a number of properties that, in our view, a satisfactory segregation order must satisfy.[8] The next section characterizes the family of indices that satisfy all these properties.

The first axiom states that if whites become blacks and vice-versa, the segregation of a city does not change.

Race Symmetry (RS) The segregation in a city is unaffected by relabeling the races: $(B_i, W_i)_{i \in N(X)} \sim (W_i, B_i)_{i \in N(X)}$.

The next axiom states that the overall size of a group does not affect how segregated it is from another group. This is one of the five requirements that Jahn *et al* [6] say a satisfactory measure of segregation should satisfy.[9] To illustrate, suppose that

[8] With some abuse of language, we will say that a segregation index satisfies a property if its induced segregation order does.

[9] Jahn *et al* [6] write:

> A satisfactory measure of ecological segregation should (1) be expressed a single quantitative value so as to facilitate such statistical procedures as comparison, classification, and correlation; (2) be relatively easy to compute; (3) not be distorted by the size of the total population, the proportion of Negroes, or the area of a city; (4) be generally applicable to all cities; and (5) differentiate degrees of segregation in such a

the distribution of black albinos is the same as the distribution of black non-albinos across neighborhoods: for instance both have a 25% chance of living in neighborhood 1, a 13% chance of living in neighborhood 2, and so on. Then black albinos are as segregated from whites as black non-albinos are, even though the absolute number of black albinos is very small.

Scale Invariance (SI) The segregation in a city is unchanged if the number of agents of a given race is multiplied by the same nonzero factor in all neighborhoods: for any $\alpha, \beta > 0$, $(B_i, W_i)_{i \in N(X)} \sim (\alpha B_i, \beta W_i)_{i \in N(X)}$.

All the indices discussed in the previous section satisfy SI, except for the entropy and isolation indices (see section 6).

The following axiom gives conditions under which segregation rises if blacks are moved to neighborhoods where there are more of them. (Together with Race Symmetry, it implies an analogous property for whites.) It requires that moving $\varepsilon > 0$ blacks from one neighborhood to another should increase the level of segregation if the percentage of blacks is higher in the destination neighborhood than in the origin neighborhood.

Monotonicity (M) For any city X, let $i, j \in N(X)$ be two neighborhoods such that

$$1 > p_i \geq p_j > 0$$

and for some $\varepsilon \in (0, B_j]$, let X' be the city that is obtained from X by moving ε blacks from neighborhood j to neighborhood i. That is X' is that city $(B'_k, W'_k)_{k \in N(X)}$ in which $(B'_i, W'_i) = (B_i + \varepsilon, W_i)$, $(B'_j, W'_j) = (B_j - \varepsilon, W_j)$, and $(B'_k, W'_k) = (B_k, W_k)$ for all $k \neq i, j$, Then X', is more segregated than X: $X' \succ X$.

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way that the distribution of intmediate scores cover most of the possible range between the extremes of 0 and 100.

Property (3) is Scale Invariance.

All indices described in the previous section, except for the index of dissimilarity, satisfy monotonicity.

The next axiom states that under limited conditions, adjoining the same set of neighborhoods to each of two different cities does not affect which of the two cities is more segregated.

Independence (IND) Let X, Y, and Z be three cities. Suppose they all have the same proportion of blacks and that X and Y have the same total populations. Then $X \succcurlyeq Y$ if and only if $X \uplus Z \succcurlyeq Y \uplus Z$.

All indices described in the previous section satisfy IND except the Gini index.

The next property gives conditions under which two neighborhoods can be combined without affecting the level of segregation in a city: if two neighborhoods have the same proportion of blacks or at least one of them is empty, then combining them does not change the city's level of segregation.[10] One implication is that the presence of empty neighborhoods can have no effect on a city's level of segregation.

Composition Invariance (CI) Let X be a city in which, for some $i, j \in N(X)$, either $p_i = p_j$ or at least one of the neighborhoods i and j is empty. Let X' be result of combining neighborhoods i and j. Then $X \sim X'$.

All the indices described in the previous section satisfy CI.

Continuity is a technical condition that guarantees the existence of a segregation index.

Continuity (C) For any cities X, Y, and Z, where X and Y have the same proportion of blacks and the same total population, the sets

$$\{c \in [0,1] : cX \uplus (1-c)Y \succcurlyeq Z\} \text{ and } \{c \in [0,1] : Z \succcurlyeq cX \uplus (1-c)Y\}$$

are closed.

[10]For example, $X = \{(1,2),(2,4)\}$ is just as segregated as the city that contains the single neighborhood $(3,6)$.

5 Main Results

We believe the above axioms are intuitive and desirable properties. Each of them, in isolation, is satisfied by most of the segregation measures used in prior literature. However, with one exception, *none* of the indices used by researchers satisfies *all* of these properties. More precisely, each of the five indices discussed in section 3.1 violates exactly one of the axioms listed in section 4, except the Atkinson index with a particular parameter. We state this fact as the following observation, whose proof is a corollary of our main characterization theorem and of the results of section 6.

Observation 1 *Of the indices defined in section 3.1, only the Atkinson index with parameter β equal to 1/2 satisfies all the axioms RS, SI, M, I, CI, and C.*

The above observation holds not because the axioms are collectively difficult to satisfy. Indeed, as our main theorem shows, there is a continuum of segregation measures that satisfy them all:

Theorem 1 *The segregation ordering $\succcurlyeq$ satisfies axioms RS, SI, M, I, CI, and C, if and only if there is a function $f : [0,1] \times [0,1] \to \Re$ with the following properties:*

1. *For all cities X and Y,*

$$X \succcurlyeq Y \text{ if and only if } \sum_{i \in N(X)} f(b_i, w_i) \geq \sum_{j \in N(Y)} f(b_j, w_j).$$

2. *f is symmetric, homogeneous of degree 1, and strictly convex on the simplex $\Delta = \{(b, w) \in [0, 1] : b + w = 1\}$.*

In addition, the function $f(c, 1-c)$ is unique up to a positive affine transformation. That is f and g both satisfy properties 1 and 2 if and only if there is are constants $\alpha \in (0, \infty)$ and $\beta \in \Re$, such that

$$f(c, 1-c) = \alpha g(c, 1-c) + \beta \qquad \forall c \in [0, 1].$$

5.1 Discussion

Some remarks are in order:

1. The fact that f is symmetric, homogeneous of degree 1, and strictly convex on the simplex implies that $f(b,w)$ has a strict global minimum at $b = w$: when the neighborhood is representative of the city.

2. The uniqueness of f up to positive affine transformations allows us to choose f so that

   - For any completely integrated city X (in which $b_i = w_i$ for all i),

   $$\sum_{i \in N(X)} f(b_i, w_i) = 0$$

   - For any completely segregated city X (in which, for all i, either $b_i = 0$ or $w_i = 0$), $\sum_{i \in N(X)} f(b_i, w_i) = 1$.[11]

The value $f(b,w)$ represents the contribution of a neighborhood that contains a proportion b of the city's blacks and a proportion w of the city's whites to the overall segregation of the city. Since f is homogeneous of degree one, this contribution can be decomposed into two factors. The first one is its size relative to other neighborhoods, as measured by $b + w$. The other is related to the degree of dissimiliarity of the neighborhood, and it is captured by the normalized difference in the proportions of the city's blacks and whites who live in the neighborhood, $d = |b - w|/(b + w)$. More formally, let $g : [0,1] \to \Re$ be defined by

$$g(d) = f((1 + d)/2, (1 - d)/2) \tag{6}$$

[11] One can easily verify that these two properties hold if and only if $f(1/2, 1/2) = 0$ and $f(1,0) = 1/2$.

Then, the neighborhood's contribution to the city's segregation is

$$\begin{aligned} f(b,w) &= (b+w)f\big(\frac{b}{b+w}, \frac{w}{b+w}\big) \\ &= (b+w)g(d). \end{aligned}$$

The size component $b+w$ enters linearly: the contribution of a neighborhood to the city's segregation is proportional to the neighborhood's size. In contrast, the neighborhood's degree of dissimilarity, d, may enter nonlinearly, but g is increasing and strictly convex by (6) and property 2 of Theorem 1. A simple example is $g(d) = d^n$ for any real $n > 1$. The index of dissimilarity corresponds to $n = 1$, which is ruled out by Monotonicity: if blacks are moved from a neighborhood in which they are overrepresented to one in which they are even more overrepresented, the index of dissimilarity does not rise. The axiom of Monotonicity states that segregation must rise, but not by how much. Accordingly, there are indices that are arbitrarily close to the index of dissimilarity that do satisfy all of our axioms; take $n = 1 + \varepsilon$ for instance.

This example relates to our discussion in the introduction. The presence of an unrepresentative neighborhood (in which $b \neq w$) contributes to a city's degree of segregation in the amount $f(b,w) = (b+w)g(d)$. The extent of this contribution depends on the tradeoff between two aspects of the neighborhood. The first is its size relative to other neighborhoods. This is represented by the sum of the proportions of the city's blacks and whites in the neighborhood, $b+w$. The second factor is the degree to which the neighborhood is unrepresentative. This is captured by the degree of dissimilarity, d. The elasticity of the neighborhood's contribution with respect to size is 1 while its elasticity with respect to unrepresentativeness depends on g; when $g(d) = d^n$, it is n. An increase in the second elasticity makes the segregation index more sensitive to a given percentage increase in a neighborhood's unrepresentativeness, without changing its sensitivity to the neighborhood's size. Thus, it captures the tradeoff between a neighborhood's unrepresentativeness and its size.

An alternative interpretation of the function f is as follows. Noting that $b_i = (p_i/P)(t_i/T)$ and $w_i = \big((1-p_i)/(1-p)\big)(t_i/T)$, by the homogeneity of f we have

$$f(b_i, w_i) = \frac{t_i}{T} f\big(\frac{p_i}{P}, \frac{1-p_i}{1-P}\big).$$

That is, the contribution of neighborhood i to the city's segregation can be decomposed into two factors. The first one is the size of the neighborhood relative to the whole city, t_i/T. This relative size enters linearly in the segregation index. The second one depends on the ratio p_i/P of the proportion of blacks in i, and the proportion of blacks in the whole city. The farther away this proportion is from one, the higher the contribution of neighborhood i to the city's segregation. And since f is convex, the marginal segregation caused by a further departure of p_i/P from 1 is increasing.

5.2 Proof of Theorem 1.

We first prove the "only if" part. Assume the segregation ordering $\succcurlyeq$ satisfies the axioms. We now build a segregation index that represents $\succcurlyeq$. First, Lemmas 1 and 2 show that the order $\succcurlyeq$ has maximal elements (the set of cities with no mixed neighborhoods) and minimal elements (the set of cities in which every neighborhood is representative).

Lemma 1 *All cities in which every neighborhood is representative have the same degree of segregation. Any such city is strictly less segregated than any city in which some neighborhood is unrepresentative.*

Proof. Consider any city in which at least one neighborhood is unrepresentative. Suppose one progressively moves the agents who are overrepresented in each neighborhood to neighborhoods in which they are underrepresented, until all neighborhoods are representative. By M, this procedure makes the city strictly less segregated. By CI, one can then merge all of the city's neighborhoods into a single neighborhood without changing the city's degree of segregation. Finally, by SI, any city with a single neighborhood is as segregated as any other city with a single neighborhood. Q.E.D.

Lemma 2 *All cities that have no mixed neighborhoods*[12] *have the same degree of segregation, and are strictly more segregated than any city in which some neighborhood is mixed.*

Proof. Start with any city that has at least one mixed neighborhood. Now progressively move agents from neighborhoods in which they are underrepresented to neighborhoods in which they are overrepresented. Continue until no neighborhood is racially mixed. By M, the resulting city must be strictly more segregated. By CI, one can then combine all the black (white) neighborhoods into a single black (white) neighborhood without changing the degree of segregation in the city. Finally, by SI, every city with only two neighborhoods, one of which contains only whites and the other only blacks, is as segregated as any other such city. Q.E.D.

By Lemmas 1 and 2, no city is more segregated than the city $\overline{X} = \{(1,0),(0,1)\}$ while none is less segregated than the city $\underline{X} = \{(1,1)\}$. Lemma 3 shows that every city X is as segregated as the union of the scaled cities $\alpha\overline{X}$ and $(1-\alpha)\underline{X}$ for a unique weight α that lies between zero and one.

Lemma 3 *For any city X, there is a unique $\alpha_X \in [0,1]$ such that*

$$X \sim \alpha_X \overline{X} \uplus (1-\alpha_X)\underline{X} \tag{7}$$

Proof: See Appendix.

We define the segregation index S as this weight: $S(X) = \alpha_X$ for any city X, where α_X is the number identified in Lemma 3. By Lemma 3, S is indeed a function. (It has a single value for each argument.) Lemma 4 states that an increase

[12] This is the set of cities X such that for all neighborhoods $i \in N(X)$, either $b_i = 0$ or $w_i = 0$.

in segregation corresponds to an increase in S: for any cities X and Y, $X \succcurlyeq Y$ if and only if $S(X) \geq S(Y)$. In other words, the index S represents the segregation ordering $\succcurlyeq$.

Lemma 4 *Let $1 \geq \alpha > \beta \geq 0$. Then*

$$\alpha\overline{X} \uplus (1-\alpha)\underline{X} \succ \beta\overline{X} \uplus (1-\beta)\underline{X}$$

Proof: See Appendix.

It remains to show that this segregation index that represents $\succcurlyeq$ has the requisite form. For any city X, let $T(X)$ denote the population of X. Lemma 5 shows that the index is linear:

Lemma 5 *For any cities X and Y with equal proportions of blacks,*

$$S\left(X \uplus Y\right) = \frac{T(X)}{T(X)+T(Y)}S(X) + \frac{T(Y)}{T(X)+T(Y)}S(Y) \tag{8}$$

Proof: See Appendix.

We now build a function $f(b,w)$, with all the required properties of this function, such that the index S is just the sum of the function f evaluated at each neighborhood of X:

$$\text{for all cities } X,\ S(X) = \sum_{i \in N(X)} f(b_i, w_i) \tag{9}$$

There are two cases:

1. If $b+w=1$, let X be the symmetric city $\{(b,w),(w,b)\}$ and set $f(b,w)$ equal to $S(X)/2$. By Race Symmetry, $f(b,w) = f(w,b)$.

2. If $b+w \neq 1$, let

$$f(b,w) = \begin{cases} (b+w)f\left(\frac{b}{b+w}, \frac{w}{b+w}\right) & \text{if } b+w>0 \\ 0 & \text{if } b+w=0. \end{cases}$$

The function f is clearly symmetric. By construction it is homogeneous of degree 1. Also by construction, $f(\alpha, \alpha) = 0$ and $f(\alpha, 0) = f(0, \alpha) = \alpha/2$.

Lemma 6 *Equation (9) holds for the function f defined above.*

Proof: See Appendix.

It remains to show that the function f is strictly convex on the simplex Δ. Let (b, w) be in the interior of Δ; assume, without loss of generality, that $b \geq w$. Let X be the symmetric city $\{(b, w), (w, b)\}$. Let $\varepsilon < w = 1 - b$ and consider the city $X' = ((b + \varepsilon, w - \varepsilon), (w - \varepsilon, b + \varepsilon))$, in which a proportion ε of the city's blacks have moved to the first neighborhood, in which they are weakly overrepresented, and the same proportion of the city's whites have moved to the second neighborhood, in which *they* are weakly overrepresented. By Monotonicity, we know that X' is strictly more segregated than X. In other words, $S(X') > S(X)$, which implies

$$f(b + \varepsilon, w - \varepsilon) + f(w + \varepsilon, b - \varepsilon) > f(b, w) + f(w, b)$$

or, using the symmetry of f,

$$\frac{f(b + \varepsilon, w - \varepsilon) + f(b - \varepsilon, w + \varepsilon)}{2} > f(b, w)$$

Strict convexity of f on the simplex now follows from the following Lemma, letting $x = (b + \varepsilon, w - \varepsilon)$ and $y = (b - \varepsilon, w + \varepsilon)$.

Lemma 7 *Let $g : [0, 1]^2 \to \Re_+$ be homogeneous of degree 1 and satisfy the following property: for any $x = (x_1, x_2) \in \Delta$ and any $y = (y_1, y_2) \in \Delta$ such that $x \neq y$, $\frac{g(x)+g(y)}{2} > g\left(\frac{x+y}{2}\right)$. Then g is convex on $[0, 1]^2$. Moreover, for any $x, y \in [0, 1]^2$ that do not lie on the same ray through the origin (i.e., such that there is no $c \in \Re$ such that $x = cy$ or $y = cx$), g is strictly convex along the line segment that connects x and y.*

Proof: See Appendix.

The above results establish the "only if" part. The proof of the "if" direction is as follows. Let $f : [0,1]^2 \to [0,\infty)$ be a symmetric function that is strictly convex on the simplex Δ and homogenous of degree one. Define the function S on the set of cities by

$$S(X) = \sum_{i \in N(X)} f\left(\frac{B_i}{B}, \frac{W_i}{W}\right).$$

Now define $\succeq$ from S as follows: for any cities X and Y, $X \succeq Y$ if and only if $S(X) \geq S(Y)$. We now show that $\succeq$ satisfies all the axioms:

1. Race Symmetry, because f is symmetric;

2. Scale Invariance because for all $\alpha, \beta > 0$, $\left(\frac{\alpha B_i}{\alpha B}, \frac{\beta W_i}{\beta W}\right) = \left(\frac{B_i}{B}, \frac{W_i}{W}\right)$;

3. Monotonicity: Assume that city X is such that $1 > B_i/(B_i + W_i) \geq B_j/(B_i + W_j) > 0$ for some $i, j \in N$. Suppose we move $\delta \in (0, B_j]$ blacks from j to i. The segregation of the city goes up if

$$f(b_i + \varepsilon, w_i) + f(b_j - \varepsilon, w_j) > f(b_i, w_i) + f(b_j, w_j),$$

where $\varepsilon = \delta/B$. Equivalently, the segregation goes up if

$$\frac{f(b_i + \varepsilon, w_i) - f(b_i, w_i)}{\varepsilon} > \frac{f(b_j, w_j) - f(b_j - \varepsilon, w_j)}{\varepsilon}.$$

Multiplying the numerator and denominator of the right-hand side of the inequality by w_i/w_j and using the homogeneity of f, we conclude that segregation goes up if

$$\frac{f(b_i + \varepsilon, w_i) - f(b_i, w_i)}{\varepsilon} > \frac{f(b'_i, w_i) - f(b'_i - \varepsilon', w_i)}{\varepsilon'} \qquad (10)$$

where $b'_i = b_j \frac{w_i}{w_j}$ and $\varepsilon' = \varepsilon \frac{w_i}{w_j}$. But $B_i/(B_i + W_i) > B_j/(B_i + W_j)$ implies $b_i/b_j = B_i/B_j \geq W_i/W_j = w_i/w_j$, so $b_i \geq b'_i$. This implies $b_i + \varepsilon > b'_i$ and

$b_i > b_i' - \varepsilon'$. Consequently, inequality (10) can be interpreted as saying that the "partial derivative" of f with respect to its first argument is increasing. By Lemma 7, f is strictly convex on $[0,1]^2$, except along rays through the origin. Equation (10) follows from this property.

4. Independence: Let X, Y, and Z be three cities as in the statement of the axiom. Letting

$$K = \frac{B(X)}{B(X)+B(Z)} = \frac{W(X)}{W(X)+W(Z)} = \frac{B(Y)}{B(Y)+B(Z)} = \frac{W(Y)}{W(Y)+W(Z)}$$

and $\quad K' = \dfrac{B(Z)}{B(X)+B(Z)} = \dfrac{W(Z)}{W(X)+W(Z)} \quad$ we have

$$\begin{aligned}
\sum_{i \in N(X)} f(b_i, w_i) &\geq \sum_{j \in N(Y)} f(b_j, w_j) \Leftrightarrow \\
\sum_{i \in N(X)} f(Kb_i, Kw_i) &\geq \sum_{j \in N(Y)} f(Kb_j, Kw_j) \Leftrightarrow \\
\begin{array}{c} \sum_{i \in N(X)} f(Kb_i, Kw_i) \\ + \sum_{k \in N(Z)} f(K'b_k, K'w_k) \end{array} &\geq \begin{array}{c} \sum_{j \in N(Y)} f(Kb_j, Kw_j) \\ + \sum_{k \in N(Z)} f(K'b_k, K'w_k) \end{array} \Leftrightarrow \\
\sum_{m \in N(X \uplus Z)} f(b_m, w_m) &\geq \sum_{n \in N(Y \uplus Z)} f(b_n, w_n)
\end{aligned}$$

5. Composition Invariance because f is homogeneous of degree one;

6. Continuity: If $B(X) = B(Y)$ and $W(X) = W(Y)$, then $S(cX \uplus (1-c)Y)$ is a linear function of c:

$$S(cX \uplus (1-c)Y) = c \sum_{i \in N(X)} f(b_i, w_i) + (1-c) \sum_{j \in N(Y)} f(b_j, w_j)$$

Thus, $\{c \in [0,1] : cX \uplus (1-c)Y \succcurlyeq Z\}$ and $\{c \in [0,1] : Z \succcurlyeq cX \uplus (1-c)Y\}$ are each closed intervals.

As for uniqueness, let us say that the function $f : [0,1]^2 \rightarrow \Re_+$ represents the segregation order $\succcurlyeq$ if it satisfies properties 1 and 2 of Theorem 1. Suppose $g : [0,1]^2 \rightarrow \Re_+$ represents the segregation order $\succcurlyeq$. Define the function $f : \Delta \rightarrow \Re$ by $f(c, 1-c) = \alpha g(c, 1-c) + \beta$ for $c \in [0,1]$; extend f in a homogeneous-of-degree-1 way to the rest of of $[0,1]^2$. Then f also represents $\succcurlyeq$: for all cities X and Y,

$$\begin{aligned} \sum_{i\in N(X)} f\left(b_i, w_i\right) &\geq \sum_{j\in N(Y)} f\left(b_j, w_j\right) \Longleftrightarrow \\ \sum_{i\in N(X)} \left(b_i + w_i\right)\left[\alpha g\left(\frac{b_i}{b_i+w_i}, \frac{w_i}{b_i+w_i}\right) + \beta\right] &\geq \sum_{j\in N(Y)} \left(b_j + w_j\right)\left[\alpha g\left(\frac{b_j}{b_j+w_j}, \frac{w_j}{b_j+w_j}\right) + \beta\right] \Longleftrightarrow \\ \sum_{i\in N(X)} g\left(b_i, w_i\right) &\geq \sum_{j\in N(Y)} g\left(b_j, w_j\right) \Longleftrightarrow X \succcurlyeq Y \end{aligned}$$

Conversely, assume that both f and g represent $\succcurlyeq$: for all cities X and Y,

$$\begin{aligned} X \succcurlyeq Y \quad &\Leftrightarrow \quad \sum_{i\in N(X)} f(b_i, w_i) \geq \sum_{j\in N(Y)} f(b_j, w_j) \\ &\Leftrightarrow \quad \sum_{i\in N(X)} g(b_i, w_i) \geq \sum_{j\in N(Y)} g(b_j, w_j). \end{aligned}$$

Let $X = \{(c, 1-c), (1-c, c)\}$ for some $c \in [0,1]$. By Lemma 3, there is a unique $\alpha_X \in [0,1]$ such that

$$X \sim \alpha_X \overline{X} + (1-\alpha_X)\underline{X}.$$

This implies, using symmetry and homogeneity of f and g, that

$$f(c, 1-c) = \alpha_X f(1,0) + (1-\alpha_X) f\left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$g(c, 1-c) = \alpha_X g(1,0) + (1-\alpha_X) g\left(\frac{1}{2}, \frac{1}{2}\right).$$

Hence, $f(c, 1-c) = \alpha g(c, 1-c) + \beta$, where

$$\alpha = \frac{f(1,0) - f\left(\frac{1}{2}, \frac{1}{2}\right)}{g(1,0) - g\left(\frac{1}{2}, \frac{1}{2}\right)} \text{ and } \beta = f\left(\frac{1}{2}, \frac{1}{2}\right) - \alpha g\left(\frac{1}{2}, \frac{1}{2}\right),$$

and $\alpha \in (0, \infty)$ by the strict convexity of f and g. Q.E.D.

6 Analysis of various indices and the independence of the axioms

In this section we will show that except for the Atkinson index with parameter $\beta = 1/2$, each of the indices described in Subsection 3.1 fails to satisfy one of the axioms used in Theorem 1. Two additional examples will complete the proof that all the axioms are logically independent. We state our results as a series of claims, which are proved in the appendix.

Claim 1 *The Dissimilarity index D satisfies RS, SI, IND, CI, and C, but fails M.*

Claim 2 *The Gini index G satisfies RS, SI, M, CI, and C, but fails IND.*

Claim 3 *The Entropy index H satisfies RS, M, I, CI, and C, but fails SI.*

Claim 4 *The Atkinson index A_β satisfies SI, M, I, CI, and C. However, it violates RS unless $\beta = 1/2$.*[13]

Claim 5 *The Isolation index J satisfies RS, M, I, CI, and C, but violates SI.*

The previous claims show that M, SI, RS, and IND are each logically independent of the other axioms. We now show that CI and C are also independent.

[13]The Atkinson Index with parameter $\beta = 1/2$ satisfies all of our axioms. Indeed, it induces the same segregation ordering as $1 - (1 - A_{1/2})^{1/2}$, a monotonic transformation of $A_{1/2}$ that can be written in the form $\sum f(b_i, w_i)$:

$$\begin{aligned} 1 - (1 - A_{1/2})^{1/2} &= 1 - \sum_{i=1}^{N} w_i^{1/2} b_i^{1/2} \\ &= \sum_{i=1}^{N} \left(\frac{b_i + w_i}{2} - w_i^{1/2} b_i^{1/2} \right) \end{aligned}$$

It is easily verified that $f(b_i, w_i)$ satisfies all of the properties of Theorem 1.

To show that CI is logically independent, consider the following segregation ordering:

$$X \succeq Y \Leftrightarrow \begin{cases} |N(X)| > |N(Y)| \\ \text{or} \\ |N(X)| = |N(Y)| \text{ and } A_{1/2}(X) \geq A_{1/2}(Y) \end{cases},$$

where for any city Z, $|N(Z)|$ is the number of neighborhoods of Z, and $A_{1/2}$ is the Atkinson index with parameter 1/2. Clearly, $\succeq$ does not satisfy CI. On the other hand, since both $A_{1/2}$ and $|N|$ satisfy RS, SI, and I, so does $\succeq$. Since the migration considered in the axiom of Monotonicity does not change the number of neighborhoods, and since $A_{1/2}$ satisfies Monotonicity, so does $\succeq$. Finally, $\succeq$ satisfies continuity: for any cities X, Y, and Z, where X and Y have the same proportion of blacks and the same total population, let $S = \{c \in [0,1] : cX \uplus (1-c)Y \succcurlyeq Z\}$. There are two cases:

1. if $|N(X)| + |N(Y)| \neq |N(Z)|$, then S is either the empty set or the whole interval $[0,1]$, both of which are closed;
2. if $|N(X)|+|N(Y)| = |N(Z)|$, then S equals $\{c \in [0,1] : A_{1/2}(cX \uplus (1-c)Y) \geq A_{1/2}(Z)\}$, which is closed since $A_{1/2}$ satisfies continuity.

Finally, we build an index that satisfies all the axioms except for C. Consider the following segregation ordering:

$$X \succeq Y \Leftrightarrow \begin{cases} A_{1/2}(X) > A_{1/2}(Y) \\ \text{or} \\ A_{1/2}(X) = A_{1/2}(Y) \text{ and } D(X) \geq D(Y) \end{cases}$$

where D is the Index of Dissimilarity and $A_{1/2}$ is the Atkinson index with parameter 1/2. Since both the Atkinson and the Dissimilarity indices satisfy RS, SI, IND, and CI, so does $\succeq$. The order $\succeq$ satisfies M because A does. However, $\succeq$ does nor satisfy continuity. To see this, consider the cities $\overline{X} = \{(1,0),(0,1)\}$, $\underline{X} =$

$\{(1/2, 1/2), (1/2, 1/2)\}$, and $X = \{(1/5, 4/5), (4/5, 1/5)\}$. Note that $A_{1/2}(c\overline{X} \uplus (1 - c)\underline{X}) = c$ while $A_{1/2}(X) = 1/5$, and that $D(c\overline{X} \uplus (1-c)\underline{X}) = c$ while $D(X) = 3/5$. Consequently,

$$\{c \in [0, 1] : c\overline{X} \uplus (1 - c)\underline{X} \succeq X\} = (1/5, 1].$$

7 Empirical Correlations

In this section we present empirical correlations between the various segregation indices studied in this paper. In addition to the five indices discussed in section 3.1, we consider the indices $I^n(N) = \sum_{i \in N(X)} (b_i + w_i) \left| \frac{b_i - w_i}{b_i + w_i} \right|^n$ for $n = 2, 4, 8, 16$. These indices, which satisfy our axioms for any $n > 1$, are discussed after Theorem 1. The universe is the 313 Metropolitan Statistical Areas (MSA's) present in the U.S. Census in 1990; the dataset is from Cutler, Glaeser, and Vigdor [5].

Table 2 presents rank correlations between these indices for the 313 MSA's in this dataset. Most of the indices are highly correlated with most of the other indices, even those that violate some of our axioms. A striking exception is the index of isolation (J), which violates the principle of scale invariance.

Appendix

Lemma 3 relies on Lemma 4, so we prove Lemma 4 first.

Proof of Lemma 4: By CI,

$$\alpha\overline{X} \uplus (1-\alpha)\underline{X} \sim \beta\overline{X} \uplus (\alpha - \beta)\overline{X} \uplus (1-\alpha)\underline{X}$$

and

$$\beta\overline{X} \uplus (1-\beta)\underline{X} \sim \beta\overline{X} \uplus (\alpha - \beta)\underline{X} \uplus (1-\alpha)\underline{X}$$

	I^2	I^4	I^8	I^{16}	D	H	A	J	G
I^2	1	0.99	0.96	0.90	0.99	0.95	0.99	0.81	0.98
I^4	0.99	1	0.99	0.94	0.97	0.95	0.998	0.83	0.96
I^8	0.96	0.99	1	0.98	0.92	0.93	0.98	0.81	0.91
I^{16}	0.90	0.94	0.98	1	0.85	0.97	0.95	0.75	0.84
D	0.99	0.97	0.92	0.85	1	0.93	0.97	0.79	0.99
H	0.95	0.95	0.93	0.97	0.93	1	0.94	0.96	0.92
A	0.99	0.998	0.98	0.95	0.97	0.94	1	0.80	0.96
J	0.81	0.83	0.81	0.75	0.79	0.96	0.80	1	0.79
G	0.98	0.96	0.91	0.84	0.99	0.92	0.96	0.79	1

Table 2: Rank correlations of segregation indices for 313 MSA's in 1990 census.

By M, $(\alpha-\beta)\overline{X} \succ (\alpha-\beta)\underline{X}$. Since the numbers of blacks and whites are equal in city $\overline{X}$ ($\underline{X}$), they are also equal in city $c\overline{X}$ ($c\underline{X}$) for any $c > 0$. So by IND,

$$\beta\overline{X} \uplus (\alpha-\beta)\overline{X} \uplus (1-\alpha)\underline{X} \succ \beta\overline{X} \uplus (\alpha-\beta)\underline{X} \uplus (1-\alpha)\underline{X}$$

The result follows by transitivity. Q.E.D.

Proof of Lemma 3: By C, $\left\{\alpha \in [0,1] : \alpha\overline{X} \uplus (1-\alpha)\underline{X} \succcurlyeq X\right\}$ and $\left\{\alpha \in [0,1] : X \succcurlyeq \alpha\overline{X} \uplus (1-\alpha)\underline{X}\right\}$ are closed sets. Any α_X satisfies (7) if and only if it is in the intersection of these two sets. The sets are each nonempty by Lemmas 1 and 2. Their union is the whole unit interval since $\succcurlyeq$ is complete. Since the interval $[0,1]$ is connected, the intersection of the two sets must be nonempty. By Lemma 4, their intersection cannot contain more than one element. Thus, their intersection contains a single element α_X. Q.E.D.

Proof of Lemma 5: Let $\omega = W(X)/B(X) = W(Y)/B(X)$ be the common ratio of whites to black is each city, and let $X' = \omega X$ and $Y' = \omega Y$. Note that the proportion of blacks is 1/2 both in X' and in Y'. Since $X' \uplus Y' = \omega(X \uplus Y)$, by SI,

$X \uplus Y \sim X' \uplus Y'$. Therefore,

$$S\left(X \uplus Y\right) = S\left(X' \uplus Y'\right)$$

We need to show that $S\left(X' \uplus Y'\right)$ equals the right-hand side of (8). By SI, $X' \sim X$ and $Y' \sim Y$. Thus, by SI,

$$X' \sim \alpha_X \overline{X} \uplus (1-\alpha_X)\underline{X} \sim \alpha_X \frac{T(X')}{2}\overline{X} \uplus (1-\alpha_X)\frac{T(X')}{2}\underline{X}$$

and

$$Y' \sim \alpha_Y \overline{X} \uplus (1-\alpha_Y)\underline{X} \sim \alpha_Y \frac{T(Y')}{2}\overline{X} \uplus (1-\alpha_Y)\frac{T(Y')}{2}\underline{X}$$

where the first and third city in each equation have equal proportions of blacks and whites and equal total populations. Hence,

$$\begin{aligned}
X' \uplus Y' &\sim \left(\alpha_X \frac{T(X')}{2}\overline{X} \uplus (1-\alpha_X)\frac{T(X')}{2}\underline{X}\right) \uplus Y' \\
&\sim \left(\alpha_X \frac{T(X')}{2}\overline{X} \uplus (1-\alpha_X)\frac{T(X')}{2}\underline{X}\right) \uplus \left(\alpha_Y \frac{T(Y')}{2}\overline{X} \uplus (1-\alpha_Y)\frac{N^{Y'}}{2}\underline{X}\right) \\
&\sim \left(\alpha_X \frac{T(X')}{2} + \alpha_Y \frac{T(Y')}{2}\right)\overline{X} \uplus \left((1-\alpha_X)\frac{T(X')}{2} + (1-\alpha_Y)\frac{T(Y')}{2}\right)\underline{X} \\
&\sim \left(\frac{\alpha_X T(X') + \alpha_Y T(Y')}{T(X')+T(Y')}\right)\overline{X} \uplus \left(\frac{(1-\alpha_X)T(X') + (1-\alpha_Y)T(Y')}{T(X')+T(Y')}\right)\underline{X}
\end{aligned}$$

where the first and second equivalences follow from I, the third from CI, and the fourth from SI. Since the weights on the two cities in the last line add to one,

$$\begin{aligned}
S\left(X' \uplus Y'\right) &= \frac{\alpha_X T(X') + \alpha_Y T(Y')}{T(X')+T(Y')} \\
&= \frac{T(X')}{T(X')+T(Y')}S(X) + \frac{T(Y')}{T(X')+T(Y')}S(Y) \\
&= \frac{T(X)}{T(X)+T(Y)}S(X) + \frac{T(Y)}{T(X)+T(Y)}S(Y).
\end{aligned}$$

Q.E.D.

Proof of Lemma 6: For any $c \in [0,1]$, let X^c be the symmetric, 2-neighborhood city $\{(c, 1-c), (1-c, c)\}$. By definition of f and Race Symmetry, $f(c, 1-c) =$

$f(1-c,c) = S(X^c)/2$. Consequently,

$$S(X^c) = f(c, 1-c) + f(1-c, c). \tag{11}$$

Now let $X\,(B_i, W_i)_{i \in N(X)}$ be any city and assume for the moment that it has equal numbers of blacks and whites ($B = W$). Let X' be the city that results from swapping blacks and whites: $X' = (W_i, B_i)_{i \in N(X)}$. By RS, $S(X) = S(X')$. By IND, $S(X \uplus X) = S(X \uplus X')$. By CI and SI, $S(X \uplus X) = S(X)$. Thus, $S(X \uplus X') = S(X)$. We can partition $X \uplus X'$ into $|N(X)|$ symmetric subcities, where subcity i is $X_i = \{(B_i, W_i), (W_i, B_i)\}$. That is, $X \uplus X' = \biguplus_{i \in N(X)} X_i$. Note that each subcity has the same proportion of blacks. Therefore, by Lemma 5,

$$S(X) = S(X \uplus X') = \sum_{i \in N} \frac{B_i + W_i}{B + W} S(X_i) \tag{12}$$

By SI, $S(X_i) = S(X^{c_i})$ where $c_i = B_i/(B_i + W_i)$. Hence,

$$\begin{aligned}
S(X) &= \sum_{i \in N} \frac{B_i + W_i}{B + W} S\left(X^{c_i}\right) \\
&= \sum_{i \in N} \frac{B_i + W_i}{B + W} \left[f\left(c_i, 1 - c_i\right) + f\left(1 - c_i, c_i\right)\right] \\
&= \sum_{i \in N} 2 \frac{B_i + W_i}{B + W} f\left(c_i, 1 - c_i\right) \\
&= \sum_{i \in N} f\left(2 \frac{B_i + W_i}{B + W} c_i, 2 \frac{B_i + W_i}{B + W} (1 - c_i)\right) \\
&= \sum_{i \in N} f\left(b_i, w_i\right)
\end{aligned}$$

where the second equality follows from equation (11), and the last line follows since $B = W$.

For general cities X, we can make the citywide proportions of blacks and whites equal by multiplying the number of blacks in each neighborhood by W/B. By SI, the index of segregation remains unchanged. Moreover, the proportions of the city's blacks and whites who reside in each neighborhood, b_i and w_i, are unchanged as well. Thus, the preceding formula holds for any city. Q.E.D.

Proof of Claim 1. The Index of dissimilarity can be written as $D(X) = \sum_{i \in N(X)} f(b_i, w_i)$ where for all $(b, w) \in [0,1]^2$, $f(b, w) = |b - w|/2$. Note that f is an homogeneous of degree one and symmetric function. Therefore, since the proof of the "if" part of Theorem 1 uses the assumption of strict convexity on the simplex only to show monotonicity, that same proof shows that D satisfies RS, SI, CI, I, and C. It is well-known that D fails Monotonicity: if $1 > p_i \geq p_j > 0$ and a small number ε of blacks move from neighborhood j to neighborhood i, then D rises only if $p_i \geq P \geq p_j$. Otherwise, D is unchanged. Q.E.D.

Proof of Claim 2. Clearly, the Gini index G satisfies RS and SI. It satisfies C because G is a continuous function of the proportions of blacks and whites that live in each neighborhood. As a result, for any two cities X and Y, $G(cX \uplus (1-c)Y)$ is a continuous function of $c \in [0,1]$, and consequently, the sets $\{c \in [0,1] : G(cX \uplus (1-c)Y) \geq k\}$ and $\{c \in [0,1] : G(cX \uplus (1-c)Y) \leq k\}$ are closed. It also satisfies Monotonicity: assume that $1 > p_i \geq p_j > 0$ and that $\varepsilon \in (0, B_j]$ blacks move from neighborhood j to neighborhood i. We need to show that G must rise. The terms of G that can change are:

$$\sum_{k \in N} (|b_i w_k - b_k w_i| + |b_j w_k - b_k w_j|)$$

For k such that $b_k/w_k > b_i/w_i$ or $b_k/w_k < b_j/w_j$, the term $|b_i w_k - b_k w_i| + |b_j w_k - b_k w_j|$ does not change. For k such that $b_i/w_i \geq b_k/w_k \geq b_j/w_j$ (e.g., $k = i, j$), the term rises. Thus, G satisfies M. G also satisfies Composition Invariance: if neighborhoods i and j have the same proportion of blacks, this implies that there is a constant c such that $b_i = cb_j$ and $w_i = cw_j$. The combined neighborhood $i \wedge j$ contains a proportion

$b_{i\wedge j} = (c+1)b_j$ of the city's blacks and $w_{i\wedge j} = (c+1)w_j$ of the city's whites. But

$$\begin{aligned} & \sum_{k\in N} \left(|b_i w_k - b_k w_i| + |b_j w_k - b_k w_j|\right) \\ = & \sum_{k\in N} \left(c\,|b_j w_k - b_k w_j| + |b_j w_k - b_k w_j|\right) \\ = & \sum_{k\in N} |b_{i\wedge j} w_k - b_k w_{i\wedge j}| \end{aligned}$$

so the sum of the terms in G that relate to neighborhood i and j remains the same if the neighborhoods are combined.

However, G does not satisfy Independence. Let $X = Z = \{(1,2), (3,2)\}$, $Y = \{(2,1), (2,3)\}$, and $Z = X$. The three cities have equal numbers of blacks and whites, and the same total populations. But while $G(X) = G(Y) = 1/4$, $G(X \uplus Z) = 1/4 \neq G(Y \uplus Z) = 9/32$. Q.E.D.

Proof of Claim 3. The Entropy index, H, clearly satisfies RS and C. For two neighborhoods i and j in which $p_i = p_j$, we have $E_i = E_j = E_{i\wedge j}$ and $t_{i\wedge j} = t_i + t_j$, where $i \wedge j$ denotes the result of merging i and j into a single neighborhood. So H satisfies CI. It is also straightforward to verify that H satisfies Independence. To see that H satisfies Monotonicity, assume that $1 > p_i \geq p_j > 0$ and that $\varepsilon \in (0, B_j]$ blacks move from neighborhood j to neighborhood i. We need to show that H must rise. This holds if $t_i E_i + t_j E_j$ falls. But

$$\begin{aligned} t_i E_i &= -\left(B_i \ln\left(\frac{B_i}{B_i + W_i}\right) + W_i \ln\left(\frac{W_i}{B_i + W_i}\right)\right) \\ \Longrightarrow & \quad \frac{\partial\left(t_i E_i\right)}{\partial B_i} = -\ln\left(\frac{B_i}{B_i + W_i}\right) \end{aligned}$$

Hence, $t_i E_i + t_j E_j$ must fall: M holds. To see that H violates Scale Invariance, note that $H\left((1,9), (9,1)\right) = 0.53$ while $H\left((1,90), (9,10)\right) = 0.44$. Q.E.D.

Proof of Claim 4. The Atkinson index with $\beta \neq 1/2$ satisfies all the axioms except for RS. To see this, note that the Atkinson index A_β is ordinally equivalent to $1 - (1 - A_\beta)^{1-\beta}$, which can be written, for any city X, as $\sum_{i\in N(X)} f(b_i, w_i)$ where,

$f(b,w) = \frac{b+w}{2} - b^\beta w^{1-\beta}$. Consequently, an argument analogous to the one of the "if" part of Theorem 1 shows that it satisfies SI, I, CI, and C. To verify M, assume that

$$1 > p_i \geq p_j > 0$$

and that $\varepsilon \in (0, B_j]$ blacks move from neighborhood j to neighborhood i. We need to show that A_β cannot fall. The new index is

$$c - w_i^{1-\beta}(b_i + \varepsilon)^\beta - w_j^{1-\beta}(b_j - \varepsilon)^\beta$$

where c is a constant that is unchanged by the migration. The derivative of this quantity with respect to ε is

$$\beta w_i^{1-\beta}(b_i + \varepsilon)^{\beta-1} - \beta w_j^{1-\beta}(b_j - \varepsilon)^\beta = \beta\left(\left(\frac{w_i}{b_i+\varepsilon}\right)^{1-\beta} - \left(\frac{w_j}{b_j-\varepsilon}\right)^{1-\beta}\right)$$

This is strictly negative since $\varepsilon > 0$ and $b_i/w_i \geq b_j/w_j$. So M holds. To see that A_β does not satisfy RS for $\beta \neq 1/2$, consider the symmetric cities $X = ((1,0),(1,2))$ and $Y = ((0,1),(2,1))$. It can be checked that $A_\beta(X) \neq A_\beta(Y)$ unless $\beta = 1/2$. Q.E.D.

Proof of Claim 5. To see that the Isolation index, J, satisfies CI, note first that we can rewrite J as follows:

$$J = \sum_{i \in N(X)} \frac{B_i}{B}\left(\frac{p_i - P}{1-P}\right),$$

and consider two neighborhoods i and j such that $p_i = p_j = p$. Then for the merged neighborhood $i \wedge j$ we have $p_{i\wedge j} = p$ too. Then

$$\begin{aligned} \frac{B_i}{B}\left(\frac{p_i - P}{1-P}\right) + \frac{B_j}{B}\left(\frac{p_j - P}{1-P}\right) &= \frac{B_i + B_j}{B}\left(\frac{p-P}{1-P}\right) \\ &= \frac{B_{i\wedge j}}{B}\left(\frac{p_{i\wedge j} - P}{1-P}\right) \end{aligned}$$

which immplies that J satisfies CI. For M, note that

$$\begin{aligned} J &= \sum_{i \in N(X)} b_i \left(\frac{\frac{B_i}{B_i+W_i} - \frac{B}{B+W}}{1 - \frac{B}{B+W}}\right) = \sum_{i \in N(X)} b_i \left(\frac{B_i(B+W) - B(B_i+W_i)}{W(B_i+W_i)}\right) \\ &= \sum_{i \in N(X)} b_i \left(\frac{B_iW - BW_i}{W(B_i+W_i)}\right) = \sum_{i \in N(X)} b_i \left(\frac{b_i - w_i}{b_i + w_i\frac{W}{B}}\right) \end{aligned}$$

But

$$\frac{\partial}{\partial \frac{b_i}{w_i}} \frac{\partial}{\partial b_i} \left[b_i \left(\frac{b_i - w_i}{b_i + w_i \frac{W}{B}} \right) \right] = \frac{2\frac{W}{B}\left(\frac{W}{B}+1\right)}{\left(\frac{W}{B}+\frac{b_i}{w_i}\right)^3} > 0$$

Thus, if $b_i/w_i > b_j/w_j$ and ε blacks are moved from neighborhood j to neighborhood i, J must increase: J satisfies M. It is straightforward to verify that J satisfies IND and C as well. J also satisfies RS. To see this, let $p_i' = 1 - p_i$ and $P' = 1 - P$; let J' be the Index of Isolation computed after swapping black and white:

$$J' = \sum_{i \in N(X)} \frac{W_i}{W} \left(\frac{p_i' - P'}{1 - P'} \right)$$

We have

$$\begin{aligned}
J - J' &= \sum_{i \in N(X)} \frac{B_i}{B} \left(\frac{p_i - P}{1 - P} \right) - \sum_{i \in N(X)} \frac{W_i}{W} \left(\frac{p_i' - P'}{1 - P'} \right) \\
&= \sum_{i \in N(X)} \frac{B_i}{B} \left(\frac{p_i - P}{1 - P} \right) + \sum_{i \in N(X)} \frac{W_i}{B} \left(\frac{p_i - P}{1 - P} \right) \\
&= \frac{1}{B(1-P)} \sum_{i \in N(X)} (B_i + W_i) \left(\frac{B_i}{B_i + W_i} - \frac{B}{B + W} \right) \\
&= \frac{1}{B(1-P)} \left(\sum_{i \in N(X)} B_i - \frac{B}{B+W} \sum_{i \in N(X)} (B_i + W_i) \right) = 0
\end{aligned}$$

so J satisfies RS. Finally, since $J(\{(1,2),(3,2)\}) = 1/15$ while $J(\{(2,2),(6,2)\}) = 1/16$, J violates SI. Q.E.D.

Proof of Lemma 7. We first show that g is strictly convex along the line segment joining any $x = (x_1, x_2) \in \Delta$ and any $y = (y_1, y_2) \in \Delta$ such that $x \neq y$: that

$$\text{for any } c \in (0,1), \quad (1-c)g(x) + cg(y) > g((1-c)x + cy) \tag{13}$$

Define $h(c) = g((1-c)x + cy) - [(1-c)g(x) + cg(y)]$. Let

$$k = \sup_{c \in [0,1]} h(c)$$

This is the maximum vertical distance between g and the chord connecting $(x, g(x))$ to $(y, g(y))$. Obviously (setting $c = 0$ or $c = 1$), $k \geq 0$. We claim that $k = 0$. Otherwise for any $\varepsilon > 0$ there is a c^ε such that $h(c^\varepsilon) > k - \varepsilon > 0$. Let $x' = (1 - c^\varepsilon)x + c^\varepsilon y$. Without loss of generality assume $c^\varepsilon \leq 1/2$. Let $y' = (1 - 2c^\varepsilon)x + 2c^\varepsilon y$, which lies on the line segment connecting x and y and hence lies in Δ. By assumption, $\frac{g(x)+g(y')}{2} > g\left(\frac{x+y'}{2}\right) = g(x')$, so $g(y') > 2g(x') - g(x)$, so

$$\begin{aligned} h(2c^\varepsilon) &= g(y') - [(1 - 2c^\varepsilon)g(x) + 2c^\varepsilon g(y)] \\ &> 2g(x') - g(x) - [(1 - 2c^\varepsilon)g(x) + 2c^\varepsilon g(y)] \\ &= 2g(x') - 2\left[(1 - c^\varepsilon)g(x) + c^\varepsilon g(y)\right] \\ &= 2h(c^\varepsilon) > 2(k - \varepsilon) \end{aligned}$$

which exceeds k for small enough ε. This is a contradiction, so $k = 0$.

Now suppose that $h(c) = 0$ for some $c \in (0, 1)$; assume w.l.o.g. that $c \leq 1/2$. An argument analogous to the above implies that $h(2c) > 0$, which contradicts the prior result that $k = 0$. Hence, $h(c) < 0$ for all $c \in (0, 1)$, which establishes (13).

Since g is homogeneous, it is (weakly) convex along any ray through the origin. Thus, if we show the last claim of the lemma, we will be done. Consider any $x, y \in [0, 1]^2$ that do not lie on the same ray through the origin. This implies, in particular, that neither x nor y is the origin $(0, 0)$. We will show that g is strictly convex along the line segment joining x and y: that for any $c \in (0, 1)$, the point $z = (1 - c)x + cy = (z_1, z_2)$ satisfies $g(z) < (1 - c)g(x) + cg(y)$. Let $x' = \frac{1}{x_1+x_2}x$, $y' = \frac{1}{y_1+y_2}y$, and $z' = \frac{1}{z_1+z_2}z$. We have

$$\begin{aligned} (z_1 + z_2)\, z' \quad &= \quad (1 - c)\,(x_1 + x_2)\, x' + c\,(y_1 + y_2)\, y' \\ &\Longrightarrow \quad z' = (1 - c)\left(\frac{x_1 + x_2}{z_1 + z_2}\right) x' + c\left(\frac{y_1 + y_2}{z_1 + z_2}\right) y' \\ &\Longrightarrow \quad z' = (1 - c')x' + c'y' \end{aligned}$$

where $c' = c\left(\frac{y_1+y_2}{z_1+z_2}\right)$ which exceeds 0 since $y \neq (0, 0)$ and is less than 1 since

$$z_1 + z_2 = (1 - c)(x_1 + x_2) + c(y_1 + y_2)$$

and $x \neq (0,0)$ and $c < 1$. In addition, x', y', and z' all lie in the simplex Δ. Hence, by the preceding result,

$$g(z') < (1-c')g(x') + c'g(y')$$

By homogeneity of g,

$$\begin{aligned} g(z) &= (z_1+z_2)g(z') \\ &< (1-c')(z_1+z_2)g(x') + c'(z_1+z_2)g(y') \\ &= (1-c)g(x) + cg(y) \end{aligned}$$

Q.E.D.

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