ANC Analytical Payoff Functions for Networks with Endogenous Bilateral Long Cheap Talk

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Abstract

We derive an almost non cooperative (ANC) analytical payoff function for all three-agent Aumann-Myerson (1988) games, and tractable ones exist for all three-agent A-M-like network games with any fixed valuation, in contrast to restricted results in the literature, if at all. Unlike link proposal game A-M and Myerson (1986), ANC has dynamic bilateral cooperation as we assume bilateral long cheap talk among three agents (differing from A-Hart (2003)), i.e., (1) pairs "smooth" Nash bargain during link discussions over credible expected payoffs induced by equilibria of a Nash demand-like game−where a link forms if the two agents match (2) double proposals, i.e., payoffs that sum up to their Myerson values in the prospective graph, and future bilateral coordination schemes. Thus, payoffs in the final graph of ANC yield a "variable Myerson value pair" which accounts for future possibilities of link formation. Instead, A-M has (a) "fixed" Myerson values (1977). In ANC, key (b) multiple equilibria in A-M−with conflicting requests of two agents to an indifferent third one−are solved, as coordination on requests by earlier linked pairs prevails if credible. This follows from (3) almost assuming that schemes are reminded chronologically "behind closed doors" as there is almost a natural first-mover advantage. (c) Inefficiency is possible in A-M, but not in ANC. We state some of the complete analytics for A-M. Also, in strictly superadditive games, only two-link graphs form. If only a link−coalition of two−forms then they achieve the grand coalition's worth.

*Thanks to Leonid Hurwicz; Maria Montero, Roger Myerson, Roberto Serrano-for referring me kindly to his related papers-, Jun Wako and David Levine-that made me aware that I may had been dealing with cheap talk. Originally: "An Extension of the Aumann-Myerson..."

1 Preliminaries

1.1 Introduction

The set of "payoff" equilibrium outcomes when agents can choose strategically with whom to establish *communication possibilities,* and thus determine who cooperates with whom, was first studied by Aumann-Myerson (1988) (A-M). They let pairs of agents propose non cooperatively *bilateral communication links* over time. However, no paper has addressed the effects, first on link selection and thus equilibrium *expansion*, and second, on how the set of equilibrium outcomes is *restricted*,[1] if *full dynamic bilateral communication possibilities (FBC)*−to be defined only *vaguely* (but "indirectly and formally" below) as bilateral communication, and hence cooperation, that happens across time periods−are added in "A-M-like" models. We are interested in evaluating in such models, for the three-agent case, if after adding *understandable*−assumption aimed at restricting equilibria−,[2] FBC, payoff equilibrium outcomes exist and are restricted so that they are unique.

The answer is yes. More importantly, solutions are analytical and tractable. As a way of example, an analytical payoff function is derived for A-M.

For getting our results, this paper adds first FBC to A-M *indirectly and formally* in the sense that the vague definition is modelled−as for some left out "ad hoc features"−in an *almost non cooperative way (ANC)*: Cheap talk is added through (2) *double proposals* of both payoffs and "future bilateral coordination schemes", (3) there are extra "chronological" substages with bilateral communication and (1) "two-agent smooth Nash (1950) demand bargaining" games; actually, we argue that the A-M model "*almost naturally*" implies FBC because the former almost naturally implies the ANC. After assuming that agents understand each other, we contribute, more in detail, along three issues in the A-M model by deriving an ANC analytical payoff function for all A-M normalized three-agent cooperative games in "characteristic function" form: we have a "variable" instead of (a) a *fixed* or static Myerson (1977) value as a way of evaluating "prospective" link structures or *graphs.* Payoff outcome predictions in the ANC model are unique and efficient instead of (b) multiple or maybe (c) inefficient in A-M.

Second, following the same procedure with all *A-M-like* three-agent games, i.e., "network" games played like in A-M *but* with any "fixed valuation function" or "payoff allocation rule", yields analogous ANC analytical payoff functions. It is also pointed out that the ANC model improves along even a fourth issue in related more general network models−(d) particular results.

In practice and as a way of illustration, an agent in "conflictive situations" may look like honoring an earlier bilateral "friendly" cheap talk, even though she doesn't

[1] As in Aumann and Hart (2003).

[2] See Myerson (1989) or Farrel and Rabin (1996) for a precise defintition and how a common language can select among equilibria.

gain or lose by doing so. Loosely, as in ANC, we almost assume a "first-mover advantage", there may be almost a rationale for this behavior that looks like a "social binding norm". If a pair understands friendship literally, they may talk cheaply about being best friends, and if this is credible, mean it later on and maybe decide independently not to propose another bilateral relationship—as a pair may end up talking "behind doors". If such behavior is expected, future uncertainty over what the indifferent friend would do if requested going one way or the other is reduced; if in addition, a pair can even bargain among these requests—and so can other possible future pairs—future uncertainty over outcomes induced by credible bilateral earlier cheap talk is eliminated.

Next, we present informally our extension of the A-M game by explaining more in detail the three ANC's key modelling features. It is also shown by means of "auxiliary models" how these features address partially or totally, but naturally, issues a, b and c in the A-M game. They do so in a more or less non cooperative way. Results are sketched. Because the extension of results to A-M like games is"straightforward", we just review the contribution to the related network literature in 1.3. A simple majority game is solved in section 1.2. All contributions and results are situated within the related literature—communication (including cheap talk), cooperation and network literature—in 1.3. Indexing follows in 1.4.

Myerson (1977) argues that all gains from cooperation, summarized in the *characteristic function*—which associates to any coalition a maximum transferable utility (TU) *payoff* that its members can assure by cooperating—, are achievable if agents are communicating through bilateral links. He proposes a payoff *allocation rule* for graphs, the Myerson value (1977). A-M focus on games in characteristic function form where pairs of non myopic agents propose *indestructible* bilateral links following a bridge like rule order. The *prospective* graph—the one induced by the added link—is evaluated with the Myerson value. By looking at *subgame perfect*, i.e., credible, Nash equilibria of this endogenous communication game, they predict graph structures that induce "coalition structures" and individual payoffs.

The ANC is an extension of the A-M model. It is a multistage game with observed actions. (2) There are instead sequential bilateral link *discussions* which entail two types of simultaneous proposals at each *relevant* stage: payoffs in the prospective graph and as links are indestructible, *future bilateral coordination schemes,* i.e., requests[3] on future actions. (3) These schemes are to be *reminded* "chronologically" later on in extra substages "behind closed doors". Consistent with the A-M game, *double proposals* are not binding in "some sense" (See conclusions). This is just cheap talk—simultaneous, bilateral, long and bounded as in Aumann Hart (2003), but endogenous and among three agents. A link forms if double proposals match, i.e., if they "coincide" and the sum of the individual proposed payoffs add up to the sum of the pair's Myerson values in the prospective graph. (1) An "overlapping" bargaining game is allowed and derived from the double proposal game but now with induced "credible

[3] See Myerson (1989), where this term was taken from.

expected payoffs". An implicit "appropriate" *smoothed Nash* (1950) *demand game* solves the bargaining problem and yields the Nash bargaining rule solution (NBR). The *smooth game* fits naturally in the A-M game as commitment to take it or leave it offers is credibly supported by the rule of order.

We have a first auxiliary model, the *non cooperative extension* (*NCE*), which is the A-M game with only simultaneous payoff proposals. The second one, the *brute force* model, has in addition to the latter *future bilateral binding schemes*. The NCE improves over the fixed nature of the Myerson value however it has three types of multiple equilibria. Non credible equilibria are eliminated by requiring subgame perfection. The ANC model−a more non cooperative version of the brute force model−and the brute force model, address the other two types of multiplicity by allowing bargaining. This two latter models end up predicting in some sense a variable Myerson value.

<u>Using feature (1)</u> The first multiplicity problem is like the one in a "divide the dollar" game as infinite proposal matches are credible Nash equilibria in the *single* payoff proposal games in the NCE. This problem is solved by using the "two-agent" NBR either axiomatically (in the brute force model) or almost explicitly (in the ANC model).

<u>Using feature (2)</u>: Extra explicit (implicit) actions like proposing future bilateral coordination (binding) schemes in the ANC (brute force) model help solve the second multiplicity problem with "seemingly" conflicting requests, which we refer instead as a problem with *dynamic bilateral conflicts of interests*; this already exists in the A-M game: A second earlier linked agent may end up being indifferent−2 subgame perfect equilibria in the NCE−towards linking or not with a third one. Linking by the former agent say hurts the agent she linked earlier on−the first agent−and benefits the third one. However, the earlier linked pair can propose coordinating on the second agent entering or not link discussions with the third one and hence distinguish between these two outcomes. It follows that bilateral bargaining is well defined provided we deal with the seemingly conflicting request by the third agent to the second one to enter.

<u>Using feature (3)</u>: To solve the latter problem, in a preceding extra substage of the ANC model, the other earlier linked agent will remind the second one "behind closed doors" not to enter, if that is consistent with the NBR as of the *earlier* bargaining game! If so, it is an equilibrium outcome for the latter to obey the reminder and thus not enter link discussions with the third agent in the next added substage; in the brute force model, this scheme looks like an "equivalent" optimal credible binding scheme. *Almost assuming* this chronology in substages is justified for there is an almost−as for "one" exception−natural first-mover advantage; equivalently, in the brute force model, it is like almost assuming that credible earlier bilateral binding schemes are "enforced" first. In the previous example, the chronology is, nevertheless, natural because two agents' consent is needed for their link discussions This stands almost in contrast to Myerson (1989 page 295)) who in all related cases *assumes* a last-mover

advantage.

Results for (a): In case the outcome of bargaining is such that a given link is expected to be last to form, the associated credible expected payoff pair is not necessarily equal to the corresponding Myerson value pair in the prospective graph. Thus, we predict *variable* Myerson value pairs that account for future possibilities of link formation−given by the bridge like rule of order!

Results for (b): As existence of the ANC analytical payoff function for three-agent normalized games is proved. We do so by constructing well defined bargaining games from later histories to earlier ones even though the strategy sets are continuous. Also, we show that our three models are *payoff equivalent.*

Some of the complete results are stated in section 7: Analytical formulas for key "histories of play"; for strictly superadditive games, only two link graphs (grand coalition) form; if a one link (two-agent coalition) forms then the colluded can achieve what the grand coalition can. We refer to a companion paper (Nieva (March 2004)) for proofs of the complete results derived *with* the brute force model. Among others: The rule of order matters in general and the emptiness of the core is neither sufficient nor necessary for the grand coalition to form or not; and as we claimed before:

Results for (c): Payoff predictions are always efficient.

Note that the analytics enables testable hypothesis!

Finally, in some sense (See conclusions) there is a limitation because the Myerson value of the third agent is in some sense still fixed.

1.2 An Example

Consider this mixed superadditive game as defined in section 7 (a simple majority rule game) with characteristic function v:

$v(\{1\}) = 0,$	$v(\{2\}) = 0,$	$v(\{3\}) = 0,$
$v(\{1,3\}) = 1 = a$	$v(\{2,3\}) = 1 = b$	$v(\{1,2\}) = 1 = c,$
$v(\{1,2,3\}) = 1 = d.$		

Graphs and Myerson values:

One-link	Values	Two-Link	Values	Complete	Values
g^{13}	$(\frac{3}{6}, 0, \frac{3}{6})$	g^{132}	$(\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$	g^N	$(\frac{2}{6}, \frac{2}{6}, \frac{2}{6})$
g^{23}	$(0, \frac{3}{6}, \frac{3}{6})$	g^{123}	$(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}))$		
g^{12}	$(\frac{3}{6}, \frac{3}{6}, 0)$	g^{213}	$(\frac{4}{6}, \frac{1}{6}, \frac{1}{6}))$		

Equivalently, theorem M.2 in section 7 could be used (Set $a = b = c = d = 1$) to prove the following claim :

Claim: The first link is the first and last to form with half each payoffs.

Alternative Proof:

Suppose that the first two links in the rule of order, say 12 and 23, were rejected.

1a. First note that a "half each" proposal match is best for both agents than payoffs in any credible scenario once link 13 forms. For a two-link graph to be the last to form, say agent 3, would have to give agent 2 at least $\frac{2}{6}$, so that agent 2 and 1 do not form the third link. Thus, the most agent 3 would get out of this in a credible way would be $\frac{3}{6}$. As payoffs are $\frac{2}{6}$ in the other credible scenario where the complete graph forms, it is always best for agent 1 or 2 not to link further.

1b. Something different than half each once link 13 forms would lead to a two link graph in which one would loose and the other would gain at the most $\frac{3}{6}$. The latter happens because agent 2 gets zero in g^{13}, there is one agent (1 or 3) who is getting less than $\frac{3}{6}$ and thus both would may gain more if a link 2 forms after bargaining. This will be the last one to form whenever agent 2 gets more than $\frac{2}{6}$. In the worst case scenario the agent linked with 2 gains $\frac{3}{6}$.

It follows, that if bargaining is possible among 1 and 3 to begin with, it will be strongly Pareto efficient—as the NBR assumes—for both of them to coordinate on the half each proposal match. In the worst case scenario in 1.b, say if agent 1 goes from having more than half to less than half and agent 3 goes from having less than half to gain a payoff of $\frac{3}{6}$, agent 1 would remind agent 3 not to enter link discussions with agent 2 and thus link 23 would not form.

3. Forming link 13 is better as other wise final payoffs would be zero.

4. As one of the agents would get zero because the next pair in the rule of order would form the first link, it is strongly Pareto efficient for the first link 12 to accept right away and not to link further.

1.3 Contributions to Related Literature

The literature review on communication is distinguished, depending on allowing or not for *coalitional communication*, that is, the possibility that agents within a coalition may communicate or not so as to "coordinate" actions and cooperate. The second division includes the study of endogenous communication, and thus cooperation, through "communication networks" and more general network literature.

1.3.1 No Coalitional Communication

From a dynamic perspective, the ANC has a multistage game with bilateral communication structures that induces unique payoff outcomes in contrast to Myerson's (1986) paper on multistage games with a *full* communication structure—every one is linked with each other. We don't have something analogous to a central mediator, like link specific mediators. As there is complete information in the ANC model, we think that the assumption of FBC being modelled as cheap talk is wlg.

In contrast to Aumann and Hart (2003)[4] that focus on *long* cheap talk as expanding the set of outcomes, our paper emphasizes instead the analysis of *long bounded* cheap talk—constrained, in particular, by the duration of link discussion or, more

[4] This paper also contains a concise updated review of related literature on strategic information transmission.

generally, by the A-M finite bridge like rule of order—as *restricting* the set of outcomes. The reason is that we assume that agents understand each other and solve the equilibrium selection problem in the A-M model.

Also, long bounded bilateral cheap talk is modelled explicitly in the ANC, but for some of all the possible bounded cheap talk within link discussions. There is no associated explicit bounded *talk phase* for simplicity. It is assumed—maybe not wlg.—that the *compromise* to play overlapping games would be reached after such phase.

Finally, in contrast to Aumann and Hart (2003), we focus instead on three-agent games and deal in an almost—as for one[5] exception—non axiomatic way with the possibility of conflicting requests.[6] In that respect, Myerson (1989 page 295) deals with the problem only axiomatically—he assumes a last-mover advantage.

1.3.2 Coalitional Communication, and Existence Results for Networks

The main objective of the paper was originally to add understandable FBC in the A-M game—i.e., in situations where communication possibilities, cheap talk, as a network of links, and thus cooperation is endogenous—and check for existence and uniqueness of payoffs equilibrium outcomes. Additionally, our payoff predictions turned out to be efficient. Thus, we review the associated literature in the context of equilibrium restriction and efficiency analysis. For our results can be extended to all A-M like three-agent network games, we do so with the related network literature.

In two agent *communication* games, the equilibrium selection problem and thus possible inefficiency persists even if words are understood—and equilibria in which communication is ignored are eliminated—and allowing for *explicit*, i.e., non cooperative, bargaining (Myerson, 1991, page 371, 456). In three-agent games, payoff prediction and efficiency analysis will depend in addition on agents cooperating in coalitions in a *coalition structure*—which is a set of disjoint coalitions that form a *partition* of all the agents in a game. Predictions and efficiency analysis are harder as the process of formation of coalition structures (See Aumann and Dreze (1974)) is not well understood. The authors, citing Mashler, argue that agents may act strategically and endogenize such process.

The communication network approach, that is used beginning with the A-M linking game, follows this endogenous reason and predicts coalition structures and thus payoffs—maybe inefficient. In the A-M model coalition structures are induced by networks of bilateral communication links or, as in the literature, *bilateral communication link structures* or more generally *cooperation structures*[7] that are evaluated

[5] Where we do assume assume a first-mover advantage (See 5.2).

[6] As for our proof, we could have just simply *assumed* 1, 2 and 3. Actually, even assuming a last-mover advantage seems not to matter (See companion paper) at least for the three-agent case.

[7] In Myerson (1980), the autthor extends his paper (1977) to allow for conference structures in which bilateral communication links are just a conference of two agents.

with the Myerson (1977) value, an extension of the Shapley (1953) value.

The general *network* approach that first followed A-M is thought to be richer as, say, two link graphs among three agents may be valued differently than one with 3 links−in contrast to the Myerson (1977) value, where all gains of cooperation are achievable if agents are "directly or indirectly" linked. For these purposes, a *valuation function* is used that depends on the graph structure in a more general way. A *payoff allocation rule* assigns values based on the valuation function. Clearly, the A-M game is a particular case with an *implicit* valuation function associated to its characteristic function, and the Myerson value as a particular payoff allocation rule−it is in this sense that the A-M model is a *communication network* model.

As we cite below, first and later models that the communication network literature and, more generally, the network literature have proposed exhibit from one to four issues that we address. From the introduction, three of them (a,b,c) can be already found in the A-M solution. Now, we will focus on the following four types of games, look at their issues or at how they contribute to earlier problems and how the ANC has something new to add:

- The *Fixed Valuation Games (FV)*: These are games composed by network games, with fixed payoff allocation rules and valuation functions.

- *Network and Bargaining Models* (NB): These network models that include multilateral, sequential and simultaneous, sub sub types, (See Jackson (2003 page 23)) with only fixed valuation functions.

- *M-type:* [8] These network models consists of simultaneous and multilateral link formation games with fixed payoff allocation rules and valuation functions.

- A-M-like games: These are FV games that use the A-M linking game. This fourth type was created artificially to compare among equivalents and thus, emphasize the contribution of the ANC with respect to the FV models.

We distinguish the NB from the ANC for analytical purposes. In a different context, the ANC may belong to the NB sequential and bilateral sub sub type.(See Jackson (2003 page 23)).

(b) The Equilibrium Selection Problem In sharp contrast to the ANC model, no paper that we are aware of has solved the equilibrium selection problem at all or provide analytical formulas. Of course, there are M models that yield the Shapley value, "however" with equilibrium refinements[9].

[8] As originally the Myerson (1977) value was used (See Myerson 1991 page 448).

[9] For a short review see Jackson (2003 page 20-21, also Dutta, B. , a. van den Nouweland and S. Tijs, (1998).

(a) The Static Nature of the Payoff Allocation Rules The static nature of the payoff allocation rules in three-agent A-M-like games is also addressed by the sequential bilateral bargaining protocol of the ANC model. In general, in the former ones, its fixed allocation rules don't account for possibilities of extra link formation given by the bridge like rule of order. The first A-M-like game is the one associated to the first FV model in Jackson and Wolinsky (1996).[10]

In contrast to the ANC that addresses problem (a) by having sequential bilateral link discussions and bargaining constrained on the payoff allocation rule−recall that for a link to form payoff proposals have to add to both agents' payoff allocations, in particular!!, their Myerson values−, the NB literature, has approached the problem by allowing multilateral negotiation of links and payoffs *without* using payoff allocation rules. Bargaining, or better yet, proposals occur simultaneously (Slikker and Van de Noweland 2001) and sequentially (Currarini and Morelli 2000).[11] Navarro and Perea (2001) uses a bilateral sequential model. However, their goal is completely different from our's as their objective is to "implement" the Myerson value.

(c) Inefficiency In ANC, bargaining, sequentiality and transfer payments are important for efficiency. More importantly, a precise efficiency analysis for all ANC associated to A-M-like games, including games with externalities, should not be that difficult.

The FV literature has studied the conditions for efficient equilibrium outcomes. In some cases, results depend on types of valuation functions, payoff allocation rules for valuation functions, notions of equilibrium and efficiency.[12]. Within the NB literature there is an implementation perspective taken by Mutuswami and Winter (2002). In a positive analysis, Currarini and Morelli (2000) find efficient equilibrium outcomes for valuation functions that satisfy some monotonicity conditions. Both papers point out to bargaining and sequentiality as important for efficient equilibria as the ANC does. In a complementary way to our findings, Bloch and Jackson found recently (2004) types of side payments to be important for achieving efficiency in equilibrium in a simultaneous bargaining model that allows for different types of externalities.

(d) Particular Results The ANC is a result for all 3-agent games including games with heterogeneity. The n-agent case and extensions to ones with different sequential bargaining procedures look promising. In contrast, a general feature of all the reviewed literature that has tried to solve a,b and c, is that results are only for types of games, not including ones with heterogeneity because of intractability.[13]

Finally, in the ANC there is no restriction to stationary strategies as in the extensive form games literature (See Ray, D. and R. Vohra, (1999)).

[10]See Jackson (2003 page 36) for a review of the FV papers that follow.
[11]Navarro and Perea (2001) uses a bilateral sequential model to implement the Myerson value.
[12]For relevant research questions see Jackson (2003 page 36).
[13]See Bellaflamme (2000).

1.4 Section Indexing

As we study how adding communication restrict outcomes, a more detailed discussion on equilibrium restriction or selection for communication games with two and n-agents, and appropriate extended definitions, are added in section 2. We also explain and illustrate with an example the Myerson value as a payoff allocation rule for cooperation structures. As we use the communication network approach to predict payoffs in cooperative games, section 3, explains and illustrates the A-M solution. Problems and improvements of our three models over the A-M model are also illustrated informally with the same example. In section 4, we formalize the NCE and define more precisely the three types of multiple equilibria. In section 5, we have the cooperative extension that formalizes the brute force model and the ANC model in that order. Both address two types of multiple equilibria. In 5.7, conclusions include comments mostly on the key features that yield uniqueness (extra actions, substages and bargaining), an ad hoc limitation and future work. In section 6, existence of ANC analytical payoff function and payoff equivalence of our models for A-M are proved. In the straightforward corollary s1, the same is proved for all three-agent A-M like games. In section 7, we present a small sample of the main theorems and analytics derived from a companion paper (Nieva March 2004).

2 Graphs and The Myerson Value

After formalizing sets of links as graphs, it is explained that whenever two agents try to predict payoffs out of cooperation induced by communication, they face multiple equilibria even when words have a literal meaning and thus, an axiomatic approach is ultimately advised. Extra definitions are given and it is pointed out that the ANC, which adds understandable dynamic communication to three-agent A-M games, is a first step towards a counter example—as the latter already deals, within the communication network approach, with key extra prediction problems induced by agents negotiating in coalition structures. Before we show that the ANC is so, we continue by describing cooperation possibilities in an n-agent set up by a characteristic function. As a second step, the Myerson (1977) value is explained as a payoff allocation rule for communication graphs that induce cooperation structures and thus coalition structures. Our claim is illustrated in section 3 where we begin by explaining the A-M game that endogenizes communication graphs. Formal modelling follows.

2.1 Notation for Graphs

Denote by $N = \{1, 2, 3\}$ the set of agents. We assume the first two agents are females and the third one a male agent.

A graph g is a set of unordered pairs of distinct agents belonging to N. Each pair is represented by a link (non-directed segment) between the two agents (nodes).

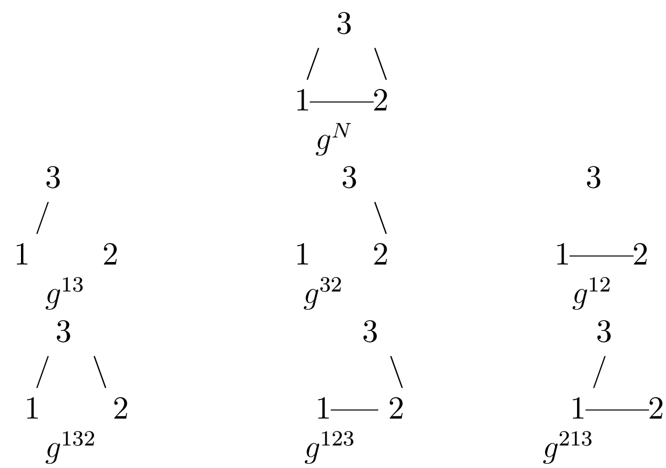
Thus, g stands for the set of links for graph g.

We denote by ij ,or equivalently ji, the link that joins agents i and j, where $i \neq j \neq l,\ i, j, l \in N$. If $ij \in g$, we say that i and j are *directly linked* in graph g. Iff $ij,\ jl \in g$, we say that i and l are *indirectly* linked by j.

We use often ij as a superscript for referring to the graph g that contains only link ij, say g^{ij}. In turn, the superscript ijl would refer to the graph where only player j is directly linked to two agents. Later on, we will distinguish among different ordering of ijl representing the order in which links have been formed. To save on notation, we sometimes use g as superscript when its components or the ordering of its components are not important or can be inferred from the context.

The graph where every pair is directly linked, or linked from now on, is called the complete graph, and is denoted by g^N. The empty graph where no pair is linked is represented by $g^{\emptyset}$.The set G of all possible graphs on N is $\{g : g \subseteq g^N\}$. We use, $g + ij$ when referring to the graph that results to adding link ij to g; $g - ij$ is to be understood as g without link ij. If we care about the resulting graph when adding ij to graph g^{il}we write g^{il+ij}or equivalently $g^{il} + ij$. Diagram 1 illustrates the possible graphs when $N = 3$.

Diagram 1



2.2 Payoffs in Cooperation Structures as Graphs

2.2.1 Payoffs Predictions in Two-Agent Cooperative Games

A pair of agents may coordinate actions in a game by means of communication possibilities or contracts so that to expand the set of possible payoff outcomes. For example, the couple in the battle of the sexes game adds *free communication possibilities* (as in Myerson (1986) following Aumann (1974)) if in addition to the standard strategies they talk, cyber chat, have the same language and so on. In particular, they may be able to alternate by going to the football game or the ballet concert. Instead, agents have *free contracting possibilities* if there is an entity that would enforce contracts over action profiles. In either case, agents can *coordinate effectively.*

However, coordinating effectively doesn't imply a certain outcome out of cooperation. In the couple above, if words are not taken by their literal meanings (See Myerson 1989) any of the original Nash equilibria is an equilibrium. In a more general set up, even if words have an absolute meaning, the set of left out correlated equilibria−that refers to all possible outcomes achievable if there is full communication and moral hazard−is in general not a singleton.

A pair of agents may be inclined to allow for more communication or contracting so that to have *negotiation* over multiple equilibrium outcomes. In the literature that tries to predict rational behavior in these situations, such negotiation, whenever modelled explicitly, is denoted as *explicit bargaining* and requires the use of a non cooperative solution. Otherwise, we have *axiomatic bargaining* that requires only a "reasonable" axiomatic bargaining solution. If there is implicit negotiation or implicit coordination, the game is played *cooperatively* and the associated axiomatic solution concepts are then *cooperative solutions.* Games that make explicit the means leading to cooperation are called *cooperative transformations* of the cooperative game as in Myerson (1991, page 371). If in a cooperative transformation, coordination or negotiation is implicit and explicit, it is a *hybrid* cooperative transformation with an associated hybrid solution concept. Finally, if the game is played in its original form, it is played *non cooperatively.* As for the many cooperative solutions that have been proposed, the implicit and the explicit approach are now complementary following the *Nash's program* for predicting outcomes in games with cooperation possibilities. This program is a guideline for selecting over cooperative solutions proposed as it consists on defining cooperative solutions for any game such that they are a Nash equilibrium of a cooperative transformation of the game.

However, even cooperative transformations that allow for explicit bargaining may not be enough to solve the equilibrium selection problem an axioms may be indispensable. According to Myerson (1991, page 371), the Nash program may then fail by itself to determine a unique cooperative solution as any explicit bargaining game may have as prediction not only the cooperative solution prediction that is being tested but others more. For predicting what a pair of rational agents would do in cooperative situations, the author suggests (Myerson 1991, page 456) assuming as part of bargaining theory notions of equity and efficiency. In other words, agents would bargain focusing (following Schelling (1960)) according to such notions[14] on one equilibrium payoff outcome of some implicit bargaining game[15]. If that assumption holds, Myerson says agents *cooperate effectively.* The associated cooperative solutions for two-agent games like the Nash bargaining rule (NBR) solution is under this interpretation a normative and efficient prediction for two-agent games with free contracting or communication possibilities.

Our hybrid ANC model is a first step as a counterexample to the multiplicity

[14]See Myerson (1991, page 372-75).

[15]In terms of Hurwicz (1993),this cooperation will be consistent with a *social goal correspondence.* In terms of Binmore (1998), we would have *a social decision function.*

problem as long as it is assumed that words have an absolute meaning (Myerson 1989) and provided we deal with the possibility of coalition formation and the extra problems in prediction that this possibility induces. If that is the case, then it is a first step towards an almost "pure" non cooperative foundation[16] of some cooperative solution for even 3-agent more general network games as for corollary s1 in section 7.

For illustrating why our claim holds, we move on to the three-agent set up, and from now on, we assume that words have an absolute meaning.

2.2.2 Payoffs in Three-Agent Cooperative Games

We are interested in existence of a payoff outcome equilibrium and uniqueness after adding full dynamic bilateral communication possibilities (FBC) in three-agent games, where cooperation possibilities depends on agents choosing strategically whom to communicate with . As a first step in that direction, we describe cooperation possibilities in three-agent coalitional games by a characteristic function. As a second step, the Myerson value is explained as a way of dealing "more or less"[17] with a key problem in cooperative theory, as it is a reasonable cooperative solution function defined for all coalitional games with fixed communication structures and thus cooperation structures.

Formally, let a coalitional game v be given with N as agent set. A characteristic function $v : CL \to \mathbb{R}$ associates the maximum wealth or transferable utility (TU) achievable if the coalition $B \in CL$ forms and coordinates effectively, so as to achieve that maximum wealth. We reserve the term of effective cooperation as defined in 2.2.1 when referring to cooperation possibilities of the grand coalition, i.e. when $B = N$ forms.[18]

There are intermediate cases between N-agent games that are played cooperatively and non-cooperatively. For predicting payoff outcomes in these cases, Myerson (1977) assumes that effective coordination[19] can occur if pairs of agents establish at least bilateral agreements or friendship relationships that are represented by links of communication. For example a link between two agents lets them get all the benefits of effective coordination. It is also assumed, in contrast to the network literature (See Jackson 2003), that a two link graph yields effective coordination among the 3 agents where one is the intermediary. In this context a set of links is denoted equivalently

[16] In costrast to Myerson, Binmore (1998, ch. 1) argues that Rubinstein's alternating offers model is indeed consistent with the Nash program as it yields "in the limit" and "realistically" a unique equilibrium payoff prediction: the NBR. In our opinion, Binmore and Myerson disagree as for their definition of realism. However, they coincide as for the need of some common sense realistic non cooperative foundation. We think that the ANC model has such common sense realism.

[17] Recall from the previous subsection that ideally, we would like a non cooperative solution See Navarro and Perea (2001) where the objective is to implement the Myerson value.

[18] In the cooperative literature, effective cooperation in the grand coalition and thus predicted payoffs in it will depend on the threats of effective coordination by other coalitions summarized in the characteristic function.

[19] Myerson also uses the term *effective negotiation*.

as a cooperation, communication or *effective coordination structure*. Extending the definitions in 2.2.1 , we could say that before *free bilateral communication* or *free bilateral contracting* is possible, pairs of agents need first to agree to communicate bilaterally. Of course this extra definition is meaningful when $N > 2$.

The reader familiar with the Myerson value (1977) should skip the paragraphs that follows until the illustration of that concept with an example used often in the paper.

Myerson derives axiomatically a cooperative solution (as in 2.2.1) for given cooperation structures, i.e., a graph g for $g \subseteq g^N$. For example, the complete graph g^N has a *full* cooperation structure, as every pair of agents can communicate bilaterally. Informally, the complete graph may be thought as everyone sitting in the bargaining table. As an extension of the Shapley value (1953), the Myerson value coincides with the latter for the case of the complete graph g^N. In the other extreme, an agent, say i, who is totally isolated (no links with anyone) will get nothing beyond his own worth. In general, the more links an agent has with others, the better the predicted payoffs.

Formally, given N, let CL be the set of all coalitions (nonempty subsets) of N, $CL = \{B \subseteq N, B \neq \varnothing\}$. Let $B \subseteq N$, $g \subseteq G$, $i \in B$, $j \in B$ be given. Agents i and j are *connected* in B by g iff there is a *path* in g from i to j and stays within B. That is, iff i and j are directly or indirectly linked under some g', where g' is such that $g' \subseteq g$ and $g' \subseteq G'$, and G' is the set of all graphs of B.

We will define $B|g$ as the unique partition of B in which groups of players are together iff they are connected in B by g. Loosely speaking, it is the collection of smaller coalitions, or connected *components* of $B|g$, into which B would break up, if players could only coordinate along the links in g.

Let a coalitional game v be given with N as agent set and g as the cooperation structure. For each player i and given the graph g and the characteristic function v, the Myerson value for player i is denoted by $\phi_i^g = \phi_i^g(v)$ and it is determined by the following axioms[20]:

Axiom 1 (fairness). If a graph g is obtained from another graph h by adding a single link, namely the one between players i and j, then i and j gain (or lose) equally by the change; that is,

$\phi_i^g - \phi_i^h = \phi_j^g - \phi_j^h$

Axiom 2 (efficiency). If $B \in N|g$, then the sum of the values of the agents in B is the worth of B; that is,

$\sum_{i \in B} \phi_i^g(v) = v(B)$.

Myerson (1977) proves that this value is unique and if v is superadditive, then two agents who form a new link never lose by it (*stability*). Note that in the case of superadditivity, the two sides of the equation in Axiom 1 are nonnegative.

Myerson (1977) also established the following practical method: Given v and g,

[20] Part of what follows are extracts form A-M (1988) or Myerson (1977).

define a coalitional game v^g by

$v^g(S) := \sum v^g(S_j)$,

where the sum ranges over the connected components S_j^g of $S|g$. Then

$\phi_i^g(v) = \phi_i(v^g)$

where ϕ_i denotes the ordinary Shapley (1953) value for player i.

We illustrate now, how the axioms determine the Myerson value with an example of a cooperative game with characteristic function v presented in detail in section 3.2. It is also shown how the Myerson value, as an extension of the Shapley value, has an *average marginal contribution* interpretation. Recall, an agent's Shapley value is an average of marginal contributions to an existing coalition of that given agent, where all such contributions are calculated for all orderings of agents. Let v be defined as:

$v(\{1\}) = 1, \qquad v(\{2\}) = 1, \qquad v(\{3\}) = 0, \qquad v(\{1,2\}) = 2,$

$v(\{1,3\}) = 3, \qquad v(\{2,3\}) = 3, \qquad v(\{1,2,3\}) = 4.$

Graphs and Myerson values are respectively: g^{13}, $(2,1,1)$, g^{23}, $(1,2,1)$, g^{12}, $(1,1,0)$, g^{132}, $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$ g^{123}, $(1,2,1)$, g^{213}, $(2,1,1)$, g^N, $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$.

Suppose a graph has only link 13. The Myerson value predicts the triplet (2,1,1), where the first two components correspond to the identical female agents 1 and 2 respectively. Half each for pair $(1,3)$ is a fair deal (fairness axiom). Finally, Myerson (1977) assumes that no transfers are possible between not (directly or indirectly) linked players (efficiency axiom). No wonder agent 2 gets only her outside option $v(\{2\}) = 1$.

Let us add link 12 to g^{13} so that to have $g^{312} = g^{13} + 12$. The Myerson value triplet is the same $(2,1,1)$ even though agents 1 and 2 communicate indirectly through 3 and thus, transfers out of the gains of the grand coalition (equal to 4) are possible now among the three of them. Consistent with the first axiom, the extra gain of agent 1 linking with agent 2 and vice versa is the same, actually, zero. The zero extra gain can be explained "easily" by using the marginal contribution interpretation of the Shapley value adapted in an appropriate way. Note that $v(\{1\}) + v(\{2\}) = v(\{1,2\}) = 2$. Thus, formation of link 12 would maintain any agent's "bargaining position" measured, informally, by a given agent's average marginal contribution[21]. No wonder, all agents get the same Myerson value as the one associated with g^{13}, $(2,1,1)$.

If we would add a third link, 23, agents 3 and 2 would get out of coordinating effectively $v(\{2,3\}) = 3$. In contrast, without that link they would get only $v(\{2\}) + v(\{3\}) = 1$. The formation of link 23 would increase agents 2 and 3's "bargaining positions". When everyone is linked, the identical agents have identical communication possibilities and thus identical "bargaining positions". The Myerson value, that coincides with the Shapley value when every one is linked, predicts equal payoffs for the identical agents, $\left(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3}\right)$.

[21] Say, if agent 1 is the existing singleton coalition and next in the rule of order is agent 2, the extra addition is a TU of 1 idependently if they are linked or not.

A graph with only links 13 and 23 would yield a Myerson value of $\left(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3}\right)$ for the same intuitive reasons above and in addition because the grand coalition forms whenever we have a two-link graph and thus, the total TU to share is 4.

3 Endogenizing Coalition Structures: The A-M, NCE, Brute Force and ANC Models

In three-agent games, there are extra problems when predicting payoffs as for the possibility of agents negotiating in coalition structures (Aumann and Dreze (1974)). As we want to illustrate−within the communication network approach−how the ANC model improves over the A-M solution when coalition structures are endogenous, preliminaries for standard cooperative games have been given first. Second, the Myerson (1977) value has been explained as a payoff allocation rule for coalition structures induced by fixed cooperation or communication structures represented by graphs. In 3.1, as a third step, the A-M game that predicts coalition structures and payoffs is explained. In 3.2, the latter is illustrated with an example. In 3.3, as promised, our extensions are presented informally as improving over the problems in the A-M game with the same example. The NCE "partially" (See 5.7.3) corrects the fixed nature of the Myerson value in the A-M game. However, it yields three different types of equilibrium selection problem. Two of them motivated the brute force and the ANC model which entail three key natural modelling features (extra stage actions, sub-stages and bargaining). A more precise definition of FBC follows and it is natural in the A-M game as the extensions are natural. The issue of stability and efficiency is more delicate. Details are given in the companion paper. As we backward solve in our proof, we do so partially with the same example in 3.4.

3.1 The A-M Linking Game

Aumann and Dreze (1974) outline possible exogenous and endogenous reasons for the formation of coalition structures, which are important if we want to predict payoffs in cooperative games. Using the Myerson (1977) value, A-M (1988) contribute on one of Aumann and Drezes' arguments and model a situation where agents may act strategically when thinking of forming coalition structures by adding communication possibilities so that to endogenize cooperation structures−the communication network approach. They focus on games in characteristic function form, say v, where pairs of non myopic agents have added a non cooperative proposal game:

This consists of pairs of agents proposing indestructible bilateral communication links following a bridge like rule order and evaluating induced cooperation structures using the Myerson value. Links are formed when both parties agree. As in bridge, after the last link has been formed, each of the pairs must be given a final opportunity to form an additional link. This game is of perfect information. Hence, it has subgame perfect equilibriums in pure strategies. Each equilibrium has a unique graph formed at the end of play.

How can agents propose if they cannot communicate to begin with? The way we interpret the A-M is by saying that in order for the *total technology of bilateral communication* to be "switched to on" (and thus for pairs of agents to be able to coordinate actions effectively), they first have to go over a *partial communication technology* that allows them to discuss everything but to coordinate actions.

The best analogy we can think of so far is the one of a "child" belonging to the B-type peer group acting by choice totally indifferent against some of her A-type peers. That means that she is not even listening to them even though she could. Maybe she communicated partially before so as to foresee the effects of *total communication.* She might have not liked the induced lower bargaining power when her total communicating induces her B-type peers to communicate also totally with these A-type peers that were originally ignored by all children belonging to the B-type peers to begin with.

Using our definitions and the previous analogy, we could say that A-M (1988) are assuming that when the link is being proposed or *discussed* with this partial communication technology, link discussions occur in a non cooperative way. We reserve the term coordination or negotiation for its usual meaning. [22]

The ANC model follows A-M and will assume instead that link discussions happen in a hybrid way but so that to allow for FBC. While constructing our cooperative transformations, we expect to generate multiple equilibria as for the arguments in section 2.2. Surprisingly, the final prize will be an analytical payoff function. The costs are clear too whenever there is no explicit non cooperative game as foundation (See footnote 3 and section 5.7.3.).[23]

3.2 The Enforcer Game and the A-M Solution

The following is a particular case of the Principal's (Enforcer) Double Extortion Empty Core Game (Nieva June 2003, October 2002). This example motivated us to extend the A-M game, as it is the only case where the induced cooperative game by the enforcer has a non empty core and a two-agent coalition never forms. In those papers, the enforcer colludes with any of the identical females iff he "sets or induces" $v(\{1,2\}) > \frac{3}{4}$, that is, iff the two identical females are in some sense "stronger" enough against the male enforcer: "Divide and rule". It can be shown that the ANC model predicts identical coalition structures in all cases. The associated characteristic function v is:

[22] That is, negotiations can occur iff agents can communicate fully with the total communication technology.

[23] Following 2.2, our implicit cooperative solution is not in one sense an effective cooperative solution by definition as coalitions of agents don't necessarily coordinate effectively. In a second sense, it is a an effective cooperative solution as we assume that the grand coalition forms whenever we use the Myerson value. It should be clear that there is no contradiction in this two statements.

$v(\{1\}) = 1, \quad v(\{2\}) = 1, \quad v(\{3\}) = 0, \quad v(\{1,2\}) = 2,$
$v(\{1,3\}) = 3, \quad v(\{2,3\}) = 3, \quad v(\{1,2,3\}) = 4.$

Graphs and Myerson values are respectively: g^{13}, $(2,1,1)$, g^{23}, $(1,2,1)$, g^{12}, $(1,1,0)$, g^{132}, $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$ g^{123}, $(1,2,1)$, g^{213}, $(2,1,1)$, g^N, $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$.

Claim: In any subgame perfect equilibrium outcome, only the two-link graph g^{132} or the complete graph g^N forms with payoffs $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$.

Proof:

Suppose we are at a stage of the game where the history of play is such that either the two-link graph g^{123} or g^{213} forms with Myerson values $(1,2,1)$ and $(2,1,1)$ respectively. The agents that are not linked would like to form their link and thus the complete graph g^N as they would get more than 1; indeed $1\frac{1}{3}$ each. In g^{132}, the agents that are not linked are indifferent to form another link. So, an inevitably consequence of building a second link is ending up with g^N or with g^{132} and, in any case, with payoffs, $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$.

Assume that either link 13, 32, or 12 is formed. There is always a pair that would agree to form the second link as the ultimate payoff is $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$. The latter will indeed happen because every pair has a last opportunity to propose.

Wlg, suppose that there are no links to begin with and that we have $(1,2)$ $(2,3)$ and $(1,3)$ as the rule of order. Let us say that the first two pairs refused. The last pair would get linked because, thinking ahead, this will lead to a payoff of $1\frac{1}{3}$ each. Otherwise, agents 1 and 3 would get $v(\{1\}) = 1$ and $v(\{3\}) = 0$. One stage backward, if agents 2 and 3 are proposing after link 12 has been rejected, both agents 2 and 3 are indifferent between accepting and rejecting because in either case final payoffs would be $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$. At the first stage of the game, we have the same situation. Both agent 1 and 2 are indifferent between forming or not their link. Thus, there are several subgame perfect equilibrium outcomes with different histories of acceptance and rejection. Our claim follows■

3.3 Problems with the Myerson Value and our Solution

Even though the A-M predicts both payoffs and coalition structures, there are three issues that have prompted research: (a) the Myerson value is fixed or static in nature; (b) multiple equilibria; (c) inefficiency is possible. Next, we illustrate these problems with the enforcer example and show how the three models are natural improvements. We end up defining FBC.

For auxiliary purposes, we distinguish somewhat arbitrarily between a "non cooperative" (section 3) and a "cooperative" extension (section 4) of the A-M game. The former is a model in itself: the NCE. The brute force and the ANC are developed in the cooperative extension.

We begin with the NCE. This model addresses partially the static nature of the Myerson value (1977) in the A-M game and deals with non credible Nash equilibria as subgame perfection is required:

The NCE consists of a multistage game with observed actions where at every history of play there is a two-agent simultaneous payoff proposal game. Thus, proposals are not binding (see discussion in 5.7.3.) and a natural extension of the game in A-M (1988 page 187). For their link to form, individual payoff proposal pairs have to match component by component and have to add up to the sum of the Myerson values of the two discussants in the prospective graph. Otherwise, the link does not form and the next pair to discuss follows according to the bridge like rule of order. A graph is final if no pair not linked yet finds it optimal not to link further (subgame perfection).

Recall the example in 3.2. Assume link 12, 23 and 13 propose in that order and the first two links have been rejected. Instead of regarding the Myerson value triplet of the prospective g^{13} (2,1,1) as fixed and thus with induced final payoffs of $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$, why doesn't agent 1 suggests to agent 3 a payoff of $1\frac{1}{3}$? With the latter payoff, she may deter agent 3 from linking with agent 2 in the event agent 1 rejects unilaterally link 12 that is next to be proposed. "At least" with this payoff, it would be a subgame perfect equilibrium outcome for agent 3 not to accept to link with agent 2. Note that we are assuming very informally that we use the A-M solution thereafter! In other words, agents 1 and 3 would get $1\frac{2}{3}$ and $1\frac{1}{3}$ respectively. Agent 2 would get her Myerson value in the one-link final graph 13 equal to 1. Of course, the argument is not a rigorous explanation, however, it gives some idea of part of model.

We have argued in 2.2.2 that the Myerson value for g^{13}, $(2, 1, 1)$, looks very appealing. Actually, the payoffs for agents 1 and 3 in our solution $(1\frac{2}{3}, 1, 1\frac{1}{3})$ add up to the Myerson values of these agents, i.e., $1\frac{2}{3} + 1\frac{1}{3} = 1 + 2$. Our solution is an improvement as we predict higher or lower payoffs depending the possibilities of future link formation given by the rule of order. Nevertheless, there is a left out ad hoc troubling feature as the Myerson value of the third agent is fixed![24]

There are two types of multiple equilibria left in the NCE equilibrium selection problem that we now illustrate. A new one is a divide the dollar like equilibrium selection problem. A second one is already present in the A-M game and has dynamic bilateral conflicts of interests. Both types are solved by adding even more communication through three key features: extra actions, substages and bargaining.

Let us go back to the history where the first two pairs rejected, link 13 is accepted and 12 is rejected. Abusing language, the last proposal match[25] on the table is $(1\frac{2}{3}, 1, 1\frac{1}{3})$. Agent 3 is indifferent between linking with agent 2 or staying just with agent 1. In both cases, agent 3 would get the same payoff of $1\frac{1}{3}$. Thus, there are two subgame perfect equilibria in the NCE: one in which agent 3 decides to form link 23 and the other one where he does not go for it. In the former, agent 2 would gain and in the latter, agent 1 would not lose.

To solve this problem with bilateral conflict, the ANC model lets pairs of agents at each relevant stage propose both payoff proposals and in addition future bilateral

[24] See Jackson (May 2004) for an axiomatic but static approach. See 5.7.3 for some discussion.

[25] As the proposal match is strictly speaking a payoff pair for each of the two agents.

coordination schemes. In the brute force model, we instead have an implicit double proposal game with future bilateral binding schemes. An *appropriate* strategic game is derived in which payoffs are required to be "credible expected equilibrium payoffs outcomes" For this game to be well defined, whenever two different future bilateral binding (coordination) schemes are chosen together with the same payoff proposal match $(1\frac{2}{3}, 1, 1\frac{1}{3})$, they have to be associated with two unique payoff outcome pairs. In the example, this is achieved as, say, two bilateral binding schemes in the brute force model would be that of agent 3 to bind himself not to discuss a link with agent 2 and the other one to do so.

However, agent 2 could request agent 3 to do something different. Thus, extra substages are added where the previous linked agent, agent 3 will decide "behind closed doors" to enter or not the next substage where link 23 discussions take place. In contrast to Myerson (1989 page 295), this would yield a first mover advantage, that is, agents 1 and 3's binding or coordination scheme will prevail. In 5.2, we explain that these substages are natural (there is an exception though, thus, we have almost natural substages) as links are indestructible, bilateral and because both agents' consent is needed for link discussions.

Nevertheless, $(1\frac{2}{3}, 1, 1\frac{1}{3})$ together with any binding (coordination) scheme (enter or not) would be still a subgame perfect equilibrium outcome. To solve this divide the dollar like multiple equilibria problem, implicit (explicit) Nash bargaining is allowed in the brute force (ANC) model. A bargaining problem is derived from the appropriate strategic form game, where the disagreement payoff is naturally set to be the "credible expected value" of double proposals not matching. We argue that our bargaining game is consistent with some notion of "sequential strategic bilateral incentive constraints".

In the ANC model, agent 1 will *remind* agent 3 in the first added substage to play according to an earlier future bilateral coordination scheme. This reminder will be in turn derived from the equilibrium outcome of some "appropriate"[26] implicit "overlapping" smoothed Nash (1950) demand game. In an "appropriate" smooth game, the pair (1,3) proposing a link makes simultaneous take it or live it offers over credible expected payoffs associated to payoff proposals and future bilateral coordination schemes. Thus, in the unique subgame perfect equilibrium outcome, agent 3 will obey the reminder not to enter link discussions with agent 2 in the second substage even though he is indifferent in any event (of course if that is consistent with the NTU NBR applied to the earlier bargaining game as it is the case in this example[27]).

In a parallel way, in the brute force model, pairs of agents bargain axiomatically over credible outcomes with the NTU NBR instead and will end up "choosing" (recall there are no explicit actions) a fixed future bilateral binding scheme that entails agent 3 not entering in the example above.

[26] Recall in the limit, appropiate smooth games yield the two-agent Nash bargaining solution (See Binmore (1998) chapter 1 and (1987)).

[27] Proof upon request, unless this example is mapped to its corresponding normalized version.

As for the latter result, we argue that the smooth game is a natural and thus realistic preliminary more non cooperative version of our ad hoc extension of the two-agent NTU NBR in a three-agent set up (the brute force model). First, as required by the smooth game, either agent in a given pair can commit simultaneously and credibly to her outside option as once the link is rejected a different pair is next to propose. We conjecture that counteroffers as in Rubinstein (1982) or Stahl (1972) would not yield the NTU NBR as the time horizon is finite (See Tirole 1992 page 114).

Multiplicity of equilibria arising from the rule of order in the NCE are solved in the same way. Let the first pair in the rule of order not form link 12. In anticipation of link 13 being the unique link to form (as for the previous argument), as pair (2,3) comes first before pair $(1,3)$ proposes the first link, the symmetric pair $(2,3)$ would form their link for identical reasons. That is, because it is strongly Pareto efficient to do that as of this even earlier history of play. Thus, in contrast to A-M, the rule of order matters for comparable cases.

As the NTU NBR yields a unique payoff prediction, any other possible equilibrium selection problem is solved automatically whenever the bargaining problem is well defined in both games. The latter is proved by construction in section 6 for all three-agent normalized games.

Predictions in ANC are always efficient. For the details on the proof of it, we refer the reader to the companion paper.

As the NCE entails a static proposal game at every history this is the *static non cooperative sense* in which we allow for FBC. As the bargaining games are derived from a strategic form game in a given history of play, we say that this is the *static cooperative sense* in which we allow for FBC. From a *dynamic cooperative perspective,* bargaining games are non myopic as the feasible payoff sets have credible expected payoffs associated to static payoff proposals together with future bilateral coordination or binding schemes. The brute force model and the ANC model allow in a hybrid way for FBC[28] in an *equivalent* way because besides the non cooperative extension, they include partially or totally the cooperative extension and they are "payoff equivalent". As we give more non cooperative foundations to the brute force model, we use the Myerson value and assume that agents understand the literal meaning of words (See Myerson 1989), our solution is an almost non cooperative (ANC) solution concept in such communication environments.

3.4 Warming up for Backward Solving

Let us assume that in the example in 3.2, we have the order $(1,2)$, $(2,3)$ and $(1,3)$ and $(1,2)$, $(2,3)$ have rejected and $(1,3)$ accepts to form the first link. Next, pair

[28] This concept is close to the one in Myerson's (1986) "Multistage Games with Communication". In Myerson's paper, agents communicate over time in the complete graph. In our case, agents may be cooperating over time in incomplete graph structures.

$(1,2)$ is called upon and rejects. So now pair $(2,3)$ has to propose and accepts g^{132}. As in bridge, pair $(1,2)$ is called upon to reject or accept the last link:

If $(1,2)$ rejects the final payoffs may be given by the Myerson value of graph g^{132} $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$. Thus, the pair $(1\frac{1}{3}, 1\frac{1}{3})$ gives outside option for agents 1 and 2. If link 12 forms the final payoffs are given by the Myerson value of the complete graph g^N $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$. Thus, the pair $(1,2)$ will be indifferent between accepting and rejecting link 12.

As of link 23 discussions, agents 2 and 3, thinking ahead, may do better with transfers among them than proposing the Myerson value of graph g^{132} so that if the latter is a final graph agents would get $(1\frac{1}{3}, 1\frac{1}{3}+x, 1\frac{1}{3}-x)$, where $|x| \geq 1\frac{1}{3}$ and x can be positive, negative or zero (in the latter zero case, the standard Myerson value is proposed). Agent 1 will still get her Myerson value for g^{132}, $1\frac{1}{3}$, independently of the value of x. Note that depending on x there would be a history contingent two-agent simultaneous last game where the last link 12 would be proposed. In contrast with the standard A-M solution, the outside option payoff for agents 1 and 2 when link 12 is proposed last is $(1\frac{1}{3}, 1\frac{1}{3}+x)$.

As of link 23 discussions, agents 2 and 3 will compare among different transfers proposals (and the induced history dependent subgames) and also the option of rejecting their link to begin with. If rejection is the case the payoff could be the Myerson value for graph g^{13}. In turn, one stage backwards, maybe agents 1 and 2 looking ahead at the unfolding effects might not even want to reject to begin with. Suppose for simplicity they reject! One more stage backwards agents 1 and 3 might not find the associated Myerson value of graph g^{13} as its best option and may want to choose transfers too.

Things get "relatively simple" if we recall that if $(1,3)$ rejects they get for sure the Myerson value for the empty graph $(1,1,0)$. At this point, there is no previous proposers that might influence the mentioned value. In the derivation of our solution, we use without loss of generality the same rule of order. We begin our analysis with the last subgames induced by the proposals chosen by agents 1 and 3 whenever they propose the first link.

4 The Non-cooperative Extension

We first develop a simultaneous payoff proposal multistage game history by history so that to address the fixed nature of the Myerson value in the A-M game. We point out in 4.2.5 to the three types of equilibrium selection problem in the NCE. Subgameperfection rules out not credible Nash equilibria and it implies that agent j, linked with i already, cannot bind herself credibly not to communicate bilaterally with l and maybe form an additional link if j gains by doing so. The other two types of equilibrium selection problems will be addressed with the brute force and the ANC model in the next section.

4.1 An Overview of the Non-cooperative Multistage Game

We normalize three-agent cooperative games so that to have the following characteristic function $v : CL \rightarrow [0, d]$:

$v1 = 0, \quad v2 = 0, \quad v3 = 0, \quad v13 = a, \quad v23 = b, \quad v12 = c, \quad v123 = d,$

where $d = 1$ and $v1$ is used instead of $v(\{1\})$. We propose a *multistage game with observed actions* $M^e_{\phi^g(v)}$ as defined in Fudemberg and Tirole (1992). These games have the characteristic that

1. In each stage every agent knows all the actions taken by any agent at any previous stage including Nature.
2. Agents move simultaneously at each stage. In nodes or information sets with only one agent moving, we could say that the others move nothing.

As an overview, we have an extensive form game $M^e_{\phi^g(v)}$, where $\phi^g(v)$ reminds us of the dependence on the Myerson value for different g given v. This game has a maximum of $K+1$ stages, $\{1, 2, ..., K+1\}$, where $K+1$ is the highest stage number associated with a terminal node. In each stage, we have bilateral link discussions as defined in 3.1. At each stage a variation of the Nash (1950) demand game is played. Pairs of forward looking agents, say (i, j), propose simultaneously at each stage a non binding non negative payoff pair each (one payoff for i and the other for j) according to the A-M bridge like rule of order. A given pair forms an indestructible communication link iff they match proposals and they are "feasible". For it to be *feasible*, the proposal pair by each agent has to add up to the sum of the two agents' Myerson values in the prospective graph. Otherwise the link is not formed and the next pair in the rule of order gets to propose unless the former was the last pair to have a last chance to form their link and did not. If the latter is the case, the pair of agents that formed the last link of g get their proposal match as payoffs. The third agent that was not involved in the bilateral discussions gets her Myerson value in graph g. The graph $g \subseteq g^N$ and the associated payoffs are a subgame perfect equilibrium outcome of this extensive form game, iff there is a strategy profile whose path has the cooperation graph g to be a graph that formed at the end of play[29]. A graph is *final* iff it is both a subgame perfect equilibrium outcome and agents find it optimal not to form additional links or, better yet, match associated payoffs thereafter. To make of this extensive game a multistage game, we assume that the third agent moves nothing whenever agents i and j are discussing their link ij at a given stage. Also, if it took $k \leq K$ for the last graph to form, then in the stages $k+1, .., K+1$ for $1 \leq k \leq K$, we have all agents moving nothing.

[29] In our results, the rule of order matters in general. Thus, it is reasonable not to use the concept of stable graph as in A-M.

4.2 The Non-Cooperative Model

4.2.1 Preliminary Definitions

Nodes in the A-M Game Tree A prospective graph is the one that would result if the link being proposed forms. Graphically it is a node in the A-M linking game tree. Let the set of pairs of combinations of agents that are used in forming a given graph be $P(g)$, where $g \subseteq g^N$. Thus, in particular, $P(g^N)$ is given by $\{(1,2),(2,3),(1,3)\}$. The rule order is given $\rho_\emptyset : P(g^N) \to \{1,2,3\}$. Wlg., this function is defined as follows: $\rho_\emptyset(12) = 1 \qquad \rho_\emptyset(23) = 2 \qquad \rho_\emptyset(13) = 3.$

The interpretation is that pair $(1,2)$ in stage 1 discusses the *first* link in the game 12. Given $\rho_\emptyset$ and $g^\emptyset$, g^{12} or equivalently $g^{\emptyset+12}$ or $g^\emptyset + 12$ is a prospective graph. If link 12 is rejected, the next pair $(2,3)$ proposes in stage 2 the first link and $g^{\emptyset+23}$ is the prospective graph. In case link 23 is rejected, the last pair in the rule of order, $(1,3)$, proposes with the prospective graph $g^{\emptyset+13}$. If link 13 is not the first link to form, a *round* of play for the first link in the game is completed and the game ends. The resulting graph is $g^\emptyset$.

If a first link is accepted, we define $\rho_{ij} : P(g^N \backslash g^{ij}) \to \{1,2,\}$ to be the rule of order of the *left out* pairs without links after graph g^{ij} is formed in the previous stage. Depending on either link 12 , 23 or 13 being the first to form, we have respectively:

$\rho_{12}(23) = 1, \rho_{12}(13) = 2 \qquad \rho_{23}(13) = 1, \rho_{23}(12) = 2 \qquad \rho_{13}(12) = 1, \rho_{13}(23) = 2$

Let link 23 be the first to be accepted. From ρ_{23}, the *second prospective link* of the game is 13, consistent with $\rho_\emptyset$. As before, if link 13 is rejected then the second link of the game 12 is proposed (we go back to the first pair according to $\rho_\emptyset$). If 12 is rejected, then the round of play for the second link gets completed and as before, the game ends with resulting graph g^{23}.

As $\rho_\emptyset$ is a fixed rule of order then there is a unique sequence of link rejections and acceptances leading to a prospective graph. Hence, when we refer to two-link prospective graphs, we use ordered triplets ijl $(i \neq j \neq l)$. For example, if the two-link graph g^{132} or equivalently $g^{13} + 32$ is a prospect, then we know that link 13 was the first link to be accepted, following ρ_{13}, link 12 was rejected and link 32 is next to be discussed. If the complete graph is prospective then we write g^{ijl+il}. The superscript il stands for the third link next to be discussed by players i and l, where the order of ijl matters. Prospective graphs are denoted in general by $g^{\theta+ij}$, where the super index $\theta \in \{\emptyset, il, ilj\}$ $i \neq j \neq l$, $i, j, l \in N$.

In general, let $g^{\theta+ij}$ be the last graph to form. If the round of play for the next link (first, second or *third* link) is completed, then the game ends and the resulting graph is $g^{\theta+ij}$.

History Classes in the A-M Game A *history class* at the end of stage h^{k+1} in the A-M game is a sequence of link rejections and acceptances. Thus, history classes can be indexed by prospective graphs: $h\left(g^\theta + ij\right)$.

4.2.2 Stage Games in the NCE

Stage 1 in the NCE In the first stage of the multi-stage game (stage 1), all players $i \in N$ simultaneously choose actions $\beta_i$from their choice sets $B(i, h^1)$, where the initial history h^1 is set equal to the initial history class $h^1\left(g^{\emptyset}+12\right)$. As *histories* are defined to be sequences of past actions, this only means that that the sequence of past actions is just "nature's past action": It gave the move to pair $(1,2)$ and we are in the initial node where pair (1,2) is first to discuss link 12.

For agent $i=3$, the choice set is the singleton "do nothing", which we denote as moving "0". As $g^{\emptyset}+12$ is the prospective graph, the actions for agent i, are two payoffs proposal, one for herself, agent i, and one for agent j $(i, j=1,2, i \neq j)$. Proposals are restricted to be non-negative. Payoffs proposals by agent i will be denoted by $\beta_i^{g^{\emptyset}+12}(i)$ and $\beta_j^{g^{\emptyset}+12}(i)$ respectively. Agent i's proposals are *feasible,* iff $\beta_i^{g^{\emptyset}+12}(i)+\beta_j^{g^{\emptyset}+12}(i)=\phi_i^{g^{\emptyset}+12}+\phi_j^{g^{\emptyset}+12}$, where $i \neq j$ and $i, j \in\{1,2\}$. In words, proposal pairs by agent i are feasible, iff they add up to the sum of both agents' (1 and 2) Myerson values in the prospective graph $g^{\emptyset}+12$.

Formally, the stage 1 action set for agent i is
$B\left(i, h^1\left(g^{\emptyset}+12\right)\right)=\left\{\left(\beta_i^{g^{\emptyset}+12}(i), \beta_j^{g^{\emptyset}+12}(i)\right) \mid \beta_i^{g^{\emptyset}+12}(i) \geq 0, \beta_j^{g^{\emptyset}+12}(i) \geq 0\right\}$.

Proposals *match* for agent's i and j, iff both are feasible and "consistent". Proposals are *consistent* iff $\beta_1^{g^{\emptyset}+12}(1)=\beta_1^{g^{\emptyset}+12}(2)$. If proposals match, link 12 together with the associated graph g^{12} is formed. If proposals don't match, link 12 is rejected and the next pair in the rule of order $\rho_{\emptyset}$ follows. A proposal by agent i is called an *unilateral rejection* iff $\left(\beta_i^{g^{\emptyset}+12}(i), \beta_j^{g^{\emptyset}+12}(i)\right)$ is not feasible. As the reader may have recalled already, we have a variation of Nash (1950) demand game.

Stage 2 in the NCE In multistage games, at the end of each stage, all agents observe the stage's action profile. In particular, let
$\beta^{g^{\emptyset}+12}=\left(\beta^{g^{\emptyset}+12}(1), \beta^{g^{\emptyset}+12}(2), \beta^{g^{\emptyset}+12}(3)=0\right)$,

be the stage-1 action profile. At the beginning of stage 2, agents know history h^2, which can be identified with $\beta^{g^{\emptyset}+12}$, given that h^1 is trivial ("nature's move").

We distinguished 2 history classes in the A-M game contingent on link 12 being formed or not. Following ρ_{12}, we have history class $h^2(g^{12}+23)$, and following $\rho_{\emptyset}$, we have $h^2(g^{\emptyset}+23)$. Actions in stage 1 in the NCE will lead to a specific prospective graph and thus, induced histories in the NCE could be classified as belonging to the associated history class in the A-M game if the induced sequence of link rejections and acceptances is identical. Proposals that (don't) match in $h^1\left(g^{\emptyset}+12\right)$ lead to $\left(g^{\emptyset}+23\right)$ $g^{12}+23$ and histories belonging to history class $\left(h^2(g^{\emptyset}+23)\right)$ $h^2(g^{12}+23)$.

It will be useful to index history classes in addition by the *last proposal match* associated with the last pair to form a link in the last graph formed. This subsets are *history subclasses.* For example, let the history class be $h^2(g^{12}+23)$. The last proposal match is given by $\left(\beta^{g^{\emptyset}+12}(1), \beta^{g^{\emptyset}+12}(2)\right)$ and denoted equivalently (abusing notation) by $\beta^{g^{\emptyset}+12}$, $\left(\beta_1^{g^{\emptyset}+12}, \beta_2^{g^{\emptyset}+12}, \phi_3^{g^{\emptyset}+12}\right)$ or simply $\beta_1^{g^{\emptyset}+12}$. We include the Myerson value for agent 3 for convenience later on. This latter proposal match $\beta^{g^{\emptyset}+12}$ led to $h^2(g^{12}+23)$ that has a subclass $h^2(g^{12}+23, \beta^{g^{\emptyset}+12})$ that consists of only one history. To clarify the latter point, let us look in contrast at history class $h^2(g^{\emptyset}+23)$. The last proposal match is defined to be the Myerson values in the empty graph, i.e., $\beta^{\emptyset} = \left(\phi_1^{\emptyset}, \phi_2^{\emptyset}, \phi_3^{\emptyset}\right)$ .As there are infinite combinations of proposals not matching in $h^1\left(g^{\emptyset}+12, \beta^{\emptyset}\right)$ that lead to $h^2(g^{\emptyset}+23)$, $h^2(g^{\emptyset}+23, \beta^{\emptyset})$ stands for a history subclass with infinitely many history elements. All history subclasses at the second stage of the game are completely characterized by $h^2(g^{\theta}+23, \beta^{g^{\theta}})$, where $\theta = \{\emptyset, 12\}$. In the paper, we often use $h^2(g^{\theta}+23)$ and $h^2(g^{\theta}+23, \beta^{g^{\theta}})$ to refer to a subset of histories or a history in that subset!

With these notation, we let all agents $i \in N$ in stage 2 of the NCE choose actions simultaneously from the sets

$B\left(i, h^2(g^{\theta}+23, \beta^{g^{\theta}})\right) = B\left(i, h^2(g^{\theta}+23)\right)$

for all histories in all history classes or subclasses, where action sets are defined in an analogous way as in stage 1 with the obvious change in notation.

Stage k in the NCE In general, let $g^{\theta}+ij$ be the prospective graph and $\beta^{g^{\theta}}$ the associated last proposal match where $\theta \in \{\emptyset, ij, ijl\}$ $i \neq j \neq l$, $i, j, l \in N$.. For saving on notation, let $g = g^{\theta}$. The actions for agent i in histories belonging to history class $h\left(g+ij\right)$ are payoff proposals for agents i and j denoted $\beta_i^{g+ij}(i) \geq 0$ and $\beta_j^{g+ij}(i) \geq 0$ respectively. For $g+ij \neq g^N$, agent i's proposals are feasible, iff $\beta_i^{g+ij}(i) + \beta_j^{g+ij}(i) = \phi_i^{g+ij} + \phi_j^{g+ij}$. Let the associated stage k to a history in subclass $h\left(g+ij, \beta^g\right)$ be denoted by $k\left(g+ij, \beta^g\right) \in \{1, ..., K\}$. Formally, the stage $k\left(g+ij, \beta^g\right)$ action set with history[30] $h\left(g+ij, \beta^g\right)$ for agent i is:

$B(i, h^{k(g+ij,\beta^g)}\left(g+ij, \beta^g\right)) = B(i, h^{k(g+ij)}\left(g+ij\right)) =$
$\left\{\left(\beta_i^{g+ij}(i), \beta_j^{g+ij}(i)\right) | \beta_i^{g+ij}(i) \geq 0, \beta_j^{g+ij}(i) \geq 0\right\}$.

Note that the action sets don't depend on last proposals β^g. If proposals match, that is, iff both are feasible and consistent then the link ij is formed. Recall they are consistent iff $\beta_i^{g+ij}(i) = \beta_i^{g+ij}(j)$.Otherwise, the next pair in the rule of order gets to discuss their link, unless a given round of play is complete. In the latter case, all agents move "nothing" thereafter (we disregard the specific notation). A proposal by agent i is called an unilateral rejection iff $\left(\beta_i^{g+ij}(i), \beta_j^{g+ij}(i)\right)$ is not feasible.

If $g+ij = g^N$ then the agents not linked have just a two-action set, i.e., to form a link ($f \equiv 1$) or reject ($r \equiv 0$) it. A proposal match occurs if both play f. Thus,

$B(i, h^{k(g+ij,\beta^g)}\left(g+ij, \beta^g\right)) = \left\{\beta_i^{g+ij}(i) : \beta_i^{g+ij}(i) \in \{f, r\}\right\}$, where $g+ij = g^N$.

[30]Note that we use $h\left(g+ij, \beta^g\right)$ to denote a history subclass or a history in that subclass.

4.2.3 Strategy Sets

A pure strategy for agent i is a contingent plan of how to play in each stage k for possible history h^k, where we disregard wlg the stages with histories where all agents move nothing. If we let H^k denote the set of all stage-k histories, and let

$B\left(i, H^k\right) = \cup_{h^k \in H^k} B\left(i, h^k\right),$

a pure strategy for player i is a sequence of maps $\{s_i^k\}_{k=1}^K$, where each s_i^k maps H^k to the set of player i's feasible actions $B\left(i, H^k\right)$ (i.e., satisfies $s_i^k(h^k) \in B\left(i, h^k\right)$ for all h^k).We denote by S_i as the set of all pure strategies for player i in our extensive form game M^e.

A sequence of actions for a profile for such strategies $s \in S$ is called the *path* of the strategy profile, where S is the set of all strategy profiles: the stage one actions are $\beta^1 = s^1(h^1)$. The stage 2 actions are $\beta^2 = s^2\left(\beta^1\right)$. The stage 3 actions are $\beta^3 = s^2\left(\beta^1, \beta^2\right)$ and so on. Since the terminal histories represent an entire sequence of play or path associated with a given strategy profile, we can represent each agents' corresponding *overall's* payoff as a function $u_i : H^{K+1} \to \mathbb{R}$.

The function $u = (u_1, u_2, u_3)$ will be constructed from *stage payoff functions* μ as follows:

If the prospective graph is the complete graph, that is, if $g + ij = g^N$ and link ij forms in stage k, then the three agents get their Myerson Value in the complete graph $\mu(h^{k\left(g^\theta + ij, \beta^{g^\theta}\right)}) = \left(\phi_1^N, \phi_2^N, \phi_3^N\right)$ (Shapley values).

Let $\theta \in \{\emptyset, ij, lij\}$ $i \neq j \neq l$, $i, j, l \in N$. If at history $h^{k\left(g^\theta + ij, \beta^{g^\theta}\right)}$, a first, second or third round is completed and thus ij does not form, then the stage payoffs at k are given by $\mu(h^{k\left(g^\theta + ij, \beta^{g^\theta}\right)}) = \left(\beta_i^{g^\theta}, \beta_j^{g^\theta}, \phi_l^{g^\theta}\right)$. If $\theta = \emptyset$ payoffs are $\left(\phi_i^\emptyset, \phi_j^\emptyset, \phi_l^\emptyset\right) = (0, 0, 0)$.

Otherwise the stage payoffs are zero.

We assume no discounting. Thus, agents overall's unique payoff at the h^{K+1} terminal history associated to the outcome where the game "ends" at stage k, $k \in \{3, ..., K\}$ is given by

$u\left(h^{K+1}\right) = \mu\left(h^{k\left(g^\theta + ij, \beta^{g^\theta}\right)}\right) = \left(\beta_i^{g^\theta}, \beta_j^{g^\theta}, \phi_l^{g^\theta}\right) = \beta^{g^\theta},$

for $g + ij \neq g^N$. If $g + ij = g^N$ and ij is accepted then

$u\left(h^{K+1}\right) = u\left(h^{k(g+ij, \beta^g)}\right) = \left(\phi_i^N, \phi_j^N, \phi_l^N\right).$

Abusing notation, we will denote the payoff vector to profile $s \in S$ as $u(s) = u\left(h^{K+1}\right)$, as we can assign an outcome in H^{K+1} to each strategy profile $s \in S$.

4.2.4 Definition of Equilibrium

A pure-strategy *Nash equilibrium* in this context is a strategy profile s such that no agent i can do better with a different strategy or, using standard Fudemberg and Tirole's (1992) notation, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

Since all agents know the history h^k, of moves before stage k, we can view the game from stage k on with history h^k as a game in its own, which we will denote $M^e\left(h^k\right)$. This is always an extensive form game. If we use the same indexing with other variables, we will make it explicit that we are referring instead to a stage game at $k\left(h^k\right)$ if that is the case. To define the payoff functions in this game, note that if the sequence of actions or path in stages k through K are β^k through β^K, the final history will be $h^{K+1}=(h^k,\beta^k,\beta^{k+1},...,\beta^K)$, and so the payoffs for agent i will be $u_i(h^{K+1})$.

Strategies in $M^e\left(h^k\right)$ are defined in a way where the only histories we need consider are those consistent with h^k. Precisely, any strategy profile s of the whole game induces a strategy profile $s|h^k$ on any $M^e\left(h^k\right)$. For each i, $s_i|h^k$ is the *restriction* of s_i to the histories consistent with h^k. We denote the restriction set by $S|h^k$. Such a restriction is also denoted by a restricted strategy profile.

Let h^{K+1} be such that $h^{K+1}=(h^k,\beta^k,\beta^{k+1},...,\beta^K)$ and the associated subset of H^{K+1} be denoted by $H^{K+1}(h^k)$. As we can assign an outcome in $H^{K+1}(h^k)$ to each restriction profile $s|h^k$ where $s\in S$, the overall payoff vector to the restriction $s|h^k$, will be denoted abusing notation by $u(s|h^k)$. Thus, we can speak of Nash equilibria of $M^e\left(h^k\right)$.

As strategy profile s of a multi-stage game with observed actions is a *subgame-perfect equilibrium* if, for every h^k, the restriction $s|h^k$ to $M^e\left(h^k\right)$ is a Nash equilibrium of $M^e\left(h^k\right)$.

Given that the individual action sets are continuous there is no assurance there will be subgameperfect equilibria. In our case, we will show existence by construction. Wlg., we can restrict our search to pure action stages as mixed action stages (mixed strategies as defined in Myerson (1991)), mixed proposals, would have zero probability of inducing any proposal match.

4.2.5 Nash Equilibria, Subgameperfection and Equilibrium Selection

We distinguish three types of equilibrium selection problem. A first and a third one were illustrated in section 3.3. The first one is related to the NCE being a divide the dollar like game: There will be infinite Nash equilibrium with different proposal matches as of each simultaneous proposal game. The second and third type are dynamic in nature. The second type is related to some Nash equilibria being not credible (actually we haven't checked for this type of equilibria). The third type consists of credible Nash equilibria (already present in the A-M game) with dynamic bilateral conflicts of interest. Within the NCE, we can only solve the second type by using a refinement of Nash equilibrium:

We require any Nash equilibrium of our game M^e to be subgameperfect. As a way of illustrating this requirement, imagine agent i "agrees" to form a communication link with j so that to form graph g, that is, $ij\in g$. The latter "agreement" doesn't imply that the next pair in the rule of order to propose after ij has formed, say (j,l) in history $h^{k(g+jl,\beta^g)}$, cannot communicate separately, i.e. without the presence of agent i or, informally, in a room behind closed doors (See 5.2). When prospective graph $f + ij = g$ (where $f \in G$, $ij \notin f$) is being discussed, agent i and j expect in the case link ij is formed no more than two things: their link not to be broken by assumption and the proposal match to be *honoured realistically*, that is, to be remembered as long as it is relevant or self-enforcing, i.e. as long as another link has not been formed and so g is still the same and β^g is still the last proposal match on the table. Note that β^g is just a last proposal match and no payoffs are paid yet!, unless a round of play has been completed in which case transfers are realized. In other words, it will be honoured as long as it is attractive enough so that agents i and j don't have an incentive to link any more. In a related way, the requirement of subgameperfection implies that agent j cannot bind herself credibly not to communicate bilaterally with l and maybe form an additional link. The latter may happen even though agent j can still communicate with i.

In the next section, the other two types of equilibrium selection problem will be solved simultaneously by applying implicitly (explicitly) the NTU NBR in the brute force model (the ANC model) and allowing pairs of agents in extra substages to have implicitly (explicitly) more stage actions: future bilateral binding schemes (future bilateral coordination schemes).

5 Cooperative Extension: Overlapping Games

As for the first and third types of equilibrium selection problem in the NCE (See 4.2.6), we want agents to solve an overlapping bargaining game in two payoff equivalent hybrid cooperative transformations of the A-M game. Using the NTU NBR, pairs would bargain axiomatically (explicitly by using the smooth Nash (1953) demand game) in the brute force model (ANC model). They would bargain restricted by some notion of "sequential strategic bilateral incentive constraints". For these purposes, we have 5 steps. The first two steps (the last three steps) present (complete) the brute force (the ANC) model. The precise necessary condition for obtaining unique payoff predictions is that the "appropriate overlapping strategic form game" from which the axiomatic (explicit) two-agent bargaining game is derived has unique payoffs associated to each combination of payoff proposals together with future bilateral binding (coordination) schemes, i.e., the strategic form game has to exhibit the *uniqueness property.* To satisfy this condition, we propose three extra formal features: extra substages, more actions and bargaining with the NTU NBR in all histories.

5.1 Subsection Indexing and Summary

In section 5.2, we allow for natural extra substages so that previous pairs linked will be able "eventually" to communicate bilaterally behind closed doors. Only one of them moves to enter or not the next "meaningful substage". These extra substages makes earlier bilateral communication prevail. For these purposes, a cooperative transformation of M^e is derived, the multistage game M'^e, where the potential communication possibilities are nevertheless not realized.

In section 5.3.1, we derive the strategic form substage game $M'(h^{k_m})$ associated to $M'^e(h^{k_m})$ at each relevant history h^{k_m} and define formally credible expected payoffs associated to a subgame perfect restriction profile in future histories induced by a substage action profile in h^{k_m}, a payoff proposal. If the associated restriction profile is subgame perfect in h^{k_m}, it is a credible Nash equilibrium restriction profile of $M'(h^{k_m})$. It will be clear that the M'^e is "equivalent" to the NCE. Thus, we use $M'(h^{k_m})$ to point out in 5.3.2 to infinite Nash equilibria in the NCE as in a divide the cake like equilibrium selection problem (type 1 problem). We do the same with the third type of equilibrium selection problem (bilateral dynamic conflicts of interest) whenever there are two different credible expected payoffs associated with a unique proposal match. In 5.3.3, we define the credible expected payoff matrix $M'(h^{k_m})$ that will be useful to solve the model by brute force.

With the definitions so far, the reader should be able to try the proof in the companion paper by brute force by using the NTU NBR axiomatically. The brute force model has implicit substage double proposal (payoffs and binding schemes) and bargaining games. Each bargaining game is derived from a credible expected payoff matrix $M'(h^{k_m})$ that has a fixed overall future bilateral binding scheme so that to exhibit the uniqueness property. It is in the latter sense, that we say it is an *overlapping* bargaining *game* as agents would have to compute first that matrix. As we use backward induction, this is sketched in section 5.3.4.

In section 5.4, we begin giving microeconomic foundations to the brute force model. We do so by developing a multistage game M''^e with a simultaneous double proposal substage game at each relevant history $P''^e\left(h^{k_m}\right)$, i.e. at each relevant substage, agents try to match both pairs of payoff proposals and future bilateral coordination schemes. This latter formal modification is one among three that helps yield the uniqueness property in this more non cooperative set up. It deals partially and specifically with the third type of equilibrium selection problem (bilateral conflicts). Preliminaries for the payoff equivalence result between the brute force and the ANC model are given.

In section 5.5, we derive at each relevant substage a two-agent overlapping bargaining problem $F\left(h^{k_m}, \psi^{h^{k_m}}\right)$ "excluding" the third agent, from an appropriate two-agent strategic substage overlapping game $T''^{\breve{\eta}}\left(h^{k_m}\right)$, in turn derived from $M''^e\left(h^{k_m}\right)$. Payoffs associated to double proposals in this strategic substage also overlapping game are credible expected payoffs. For $T''^{\eta'}\left(h^{k_m}\right)$ to be well defined it is necessary for it to have the uniqueness property. Thus, besides future bilateral coordination schemes and substages, a third formal requirement is setting a fixed future bargaining rule $\breve{\eta}$ (used as superscript). This last feature will solve finally both types of equilibrium selection problem. The overlapping bargaining game derived this way is consistent with some notion of "sequential strategic bilateral incentive constraints". And because of "the payoff equivalence" result, so is the overlapping bargaining game in the brute force model M''^e.

In section 5.6, we define at all relevant histories an implicit "two-agent" simultaneous take it or live it offer overlapping stage game $J''^\eta\left(h^{k_m}(g+ij,...)\right)$ after John Nash's (1950) smoothed demand game. This extension will set automatically the fixed future bargaining rule $\breve{\eta}$ to the NTU NBR η. The final multistage game, the ANC model is denoted by $M^{\eta e}$.

We conclude by emphasizing the key features of our extension and the limitations in section in 5.7. With all the formal tools in hand, a proof by construction of existence and uniqueness for our solution is given in section 6.

5.2 From M^e to M'^e : "Meeting Behind Closed Doors"

As two agents's consent is needed to form a prospective link, it is natural to assume that each one of them can unilaterally decide not to discuss the associated link "before hand". Additionally, as links are bilateral and indestructible, earlier pairs of discussants should be able to coordinate on future actions and remind themselves of earlier discussions. As there are three agents, there is the possibility of simultaneous conflicting communication (See Myerson (1989), page 296). However, it will turn out that pairs of earlier linked agents may look "almost" like their are communicating bilaterally in future histories without being influenced by the third agent, that is *behind closed doors*. Thus, the following extra substages are sufficient for adequate modelling and they will imply a hierarchy where earlier communication will prevail:

Whenever one (two links) link has (have) already formed, we allow in M^e at each relevant k-stage history h^k two (three) *meaningful* substages k_1 and k_2.(k_1 and k_2 and k_3). In order to have a total of three, we add when necessary *not meaningful* substages where agents move nothing.

Suppose link ij is the only link formed and jl is a prospective link. In the first substage k_1, proposer j, chooses to enter ($e \equiv 1$) or not ($ne \equiv 0$) link discussions with l. The latter chooses nothing and i can only communicate with j in this first substage. In the second substage k_2, we reach history h^{k_2} if agent i decided to enter ($e \equiv 1$) and the discussion game described before takes place. There is nothing agent l can do to influence agent j as for discussions to happen it is necessary that agent l has j's consent and thus the second substage is reached. In contrast to Myerson[31] (1989 page 295), our model has a first mover advantage.

If say ij and jl have formed *in that order*, and pair (i, l) is next to propose then proposer i chooses (wlg. as (j, l) could come first too) in the first substage k_1 to e or ne a second substage k_2 with associated history $h^{k_2}(g+il)$, where agent l chooses to e or ne link discussions with i in substage k_3 with associated history $h^{k_3}(g+il)$. We allow communication only between i and j in the first substage $k_1$and only between j and

[31] In that paper, a later agent needs no consent of the previous agent to request some curse of action (the action ne is inexistent), and thus there is the possibility of overlapping conflictive requests. In our case, that is not possible.

l in the second substage k_2. Another modelling possibility would have been to have just one substage. However, we think it is more clear to visualize the inexistence of bilateral communication and thus of "direct" discussion between agents i and l with two substages instead. Note that we have excluded the possibility of agent l to suggest the "intermediary" j in graph $g^{ijl}$of some course of action for i in substage k_1 and thus overlapping contradictory requests. As it will turn out in equilibrium, reminders can only be credible ("requests are tenable" in Myerson (1989)) if agent i is indifferent between e or ne a second substage k_2 (See 5.3.2). In this situation, either j looses and l gains or vice versa or payoffs are the same.

Hence, in all cases but one-the constant case-it will not be in the interest of agent j to listen to l, and thus she won't. When payoffs are the same we **assume** that there is as first mover advantage in contrast to Myerson's (1989 page 296). He raises the question as to which agent should j obey, whenever there are contradictory simultaneous requests by agents i and l to j, and the latter is willing to obey both. It is because of the latter case that we argue that it is *almost* like earlier agents are communicating bilaterally behind closed doors first.

The new multistage game, will be denoted by M'^{e}. For indexing purposes, we renumber substages as $1_1 = 1, 1_2 = 2, 1_3 = 3, 2_1 = 4, 2_2 = 5$ and so on. Strategy sets and equilibrium definitions are renamed in the obvious way.

5.3 The Strategic Form Game $M'(h^{k_m})$ and Credible Nash equilibria

5.3.1 Credible Nash Equilibrium Strategy Profiles

For now, let us shut down communication possibilities in the extra substages but for k_m , where k_m is a redefined *relevant* substage where the simultaneous proposal game described before takes place. Also $g = g^{\theta}$, where θ is any history class. Within this history class, let an element of a history subclass be $h^{k_m(g+ij,\beta^g)}$ and the associated subgame be $M'^{e}\left(h^{k_m}\right)$.

Let $s|h^{k_m}$ be a restriction profile of $M'^{e}\left(h^{k_m}\right)$, where $s \in S$. If $s|h^{k_m+1}$ is a subgame perfect equilibrium restriction profile of $M'^{e}\left(h^{k_m+1}\right)$, we define the *credible expected payoff* triplet associated to $s|h^{k_m}$ to be $u\left(s|h^{k_m}\right)$ and denote it by $v\left(s|h^{k_m}\right)$. If $s|h^{k_m}$ is subgameperfect on $M'^{e}\left(h^{k_m}\right)$, then $v\left(s|h^{k_m}\right)$ is a *credible Nash equilibrium expected payoff of* the associated strategic form game $M'^{e}(h^{k_m})$. From now on, when using $v\left(s|h^{k_m}\right)$, it is always assumed that $s|h^{k_m}$ has one characteristic or the other. The set of strategies of the strategic form of $M'^{e}(h^{k_m})$, $M'(h^{k_m})$, are identical to the ones of $M'^{e}(h^{k_m})$. Thus, the set of credible Nash equilibrium restriction profiles of $M'(h^{k_m})$ is the same as the set of subgame perfect equilibrium restrictions profiles in $M'^{e}(h^{k_m})$. Unless the distinction is necessary, we will use $M'(h^{k_m})$ or $M'^{e}(h^{k_m})$ indistinctly. The same applies for the respective strategies.

Formally, we denote the contingent history substage k_m game in strategic form as

$M'(h^{k_m}) = \left(\{1,2,3\}, S_1|h^{k_m}, S_2|h^{k_m}, S_3|h^{k_m}, u_1(s|h^{k_m}), u_2(s|h^{k_m}), u_3(s|h^{k_m})\right)$,

or in a simpler form:

$M'(h^{k_m}) = \left(\{1,2,3\}, S|h^{k_m}, u(s|h^{k_m})\right)$.

Note that strategic games and thus $v\left(s|h^{k_p}\right)$ may be also defined for all substages in the obvious way for $p = 1,2,3$.

5.3.2 Multiple Credible Expected Payoffs for Stage Action Profile $s^{k_m}(h^{k_m})$

Let k_m be a relevant substage. For understanding dynamic bilateral conflicts of interest, let us define in an analogous way $u(s^{k_m}(h^{k_m}))$ as the *overall payoff* triplet *of stage action profile* $s^{k_m}(h^{k_m}) \in B\left(h^{k_m}\right)$ when $M'^e(h^{k_m})$ is played according to the restriction $s|h^{k_m}$ (where $s \in S$). If $s|h^{k_m+1}$ is a subgame perfect equilibrium strategy profile of $M'^e\left(h^{k_m+1}\right)$, we have instead the *credible expected payoff for stage action profile* $s^{k_m}(h^{k_m})$ and denote it by $v\left(s^{k_m}\left(h^{k_m}\right)\right)$. If $s|h^{k_m}$ is subgameperfect on $M'^e\left(h^{k_m}\right)$, then $v\left(s^{k_m}(h^{k_m})\right)$ is a *credible Nash equilibrium expected payoff of* $M'^e(h^{k_m})$ *for stage action profile* $s^{k_m}(h^{k_m})$. We also say $s^{k_m}(h^{k_m})$ *induces* $s|h^{k_m}$.

As we will see, when backward solving our game, there will exist $s, s' \in S$ such that their restrictions $s'|h^{k_m} \neq s|h^{k_m}$, $s^{k_m\prime}(h^{k_m}) = s^{k_m}(h^{k_m})$, $v\left(s'^{k_m}(h^{k_m})\right)$ and $v\left(s^{k_m}(h^{k_m})\right)$ are both credible expected payoffs of $s^{k_m}(h^{k_m})$ in $M'^e(h^{k_m})$, however $v\left(s'^{k_m}(h^{k_m})\right) \neq v\left(s^{k_m}(h^{k_m})\right)$.

In words, the same proposal match may induce two different restriction profiles and thus two subgameperfect equilibrium outcomes with different credible expected payoffs. As a preview, let i and j be the agents proposing in stage h^{k_m}. Assume that they decide to form ij and next to propose are i and l. We will often encounter a last common history in the path of $s'|h^{k_m}$ and $s|h^{k_m}$ induced by $s^{k_m\prime}(h^{k_m}) = s^{k_m}(h^{k_m})$ where agent i is indifferent between entering or not the next meaningful history (and thus forming or not a link with l). The two induced different subgame perfect equilibrium outcomes will lead to different credible expected payoffs in most of cases for agents other than i.

As before, $v\left(s^{k_p}(h^{k_p})\right)$ may be also defined in the obvious way for $p = 1,2,3$.

5.3.3 Construction of the Credible Expected Payoffs Matrix $M'(h^k)$

Let $s|h^{k_m} \in S|h^k$ where $s|h^{k_m+1}$is a subgame perfect equilibrium strategy profile on $M'^e\left(h^{k+1}\right)$, where $s \in S$. The credible expected payoffs matrix of the strategic game $M'(h^{k_m})$ has action set $S|h^{k_m}$ and its elements are restricted to be subgame perfect in future histories. This matrix is constructed by backward solving, i.e., in general it can be derived by finding $v\left(s|h^{k_m}\right) = v\left(s|h^{k_m+1}\right)$ for $k \in \{k, ..., K\}$.

5.3.4 Backward Solving with the Brute Force Model

Let k_m be a relevant substage. Restrictions $s|h^{k_m}$, $s'|h^{k_m} \in S|h^{k_m}$ in $M'^e\left(h^{k_m}\right)$ have the same *future bilateral binding scheme* iff $s^{k_m}\left(h^{k_m}\right)$ is a proposal match in history h^{k_m}, $s^{k_m}\left(h^{k_m}\right) = s^{k_m\prime}\left(h^{k_m}\right)$ and $s'^{k_o^+}\left(h^{k_o^+}\right) = s^{k_o^+}\left(h^{k_o^+}\right)$, where $k^+ > k$, $o \neq m$ and at substage k_o^+, agents i and j get to meet behind closed doors. If in addition $s^{k_m}\left(h^{k_m}\right)$ induces $s|h^{k_m}$, its associated future bilateral binding scheme is called *credible*. For such $s|h^{k_m}$, whenever we want to refer to all credible bilateral binding schemes for different pairs of agents associated to $s|h^{k_m}$ and restrictions of $s|h^{k_m}$ to all future histories $h^{k_m^+}$, we say the *overall credible bilateral binding scheme* associated to $s|h^{k_m}$. Also, proposal match $s^{k_m}\left(h^{k_m}\right)$ *induces* in $s|h^{k_m}$ a future bilateral binding scheme and an overall bilateral binding scheme whenever they are credible.

With the definitions so far, the reader should be able to try the proof in the companion paper. One should go to all furthest subgames $M'^e\left(h^{k_m^+}(g^{ij}+jl)\right)$, where $M'^e\left(h^{k_m^+}\right)$ should be such that (j,l) is the last pair that can discuss coordination in the future. Literally, fix earlier future bilateral coordination binding schemes, that of i and j as of h^{k_m}, in cases of bilateral conflict (whenever agent i is indifferent between to e or ne the next meaningful substage) and use the NTU NBR to select over a feasible set with credible expected payoffs v_{jl} derived from the triplets $v(s|h^{k_m^+}(g^{ij}+jl))$. Each of these payoffs will be associated "uniquely" (as it is proved by construction or using the payoff equivalent results of the brute force and the ANC model later on in 5.5.3) to both a proposal match together with a future bilateral binding scheme by agents j and l.[32] The associated optimal future bilateral binding scheme $s|h^{k_m}$ or *implicit double proposal* are referred as *consistent* ones. The outside options are just $v_{jl}(s|h^{k_m^+})$ such that $s^{k_m^+}\left(h^{k_m^+}\right)$ is not a proposal match. With this result computed for all furthest histories, move on to earlier ones, say $h^{k_m}(g^{\emptyset}+ij)$, as earlier linked agents i and j, will now know the "unique" effect (unique induced payoffs) of all their implicit double proposals on optimal implicit future double proposals by say j and l (or i and l). Let agents i and j bargain with the NTU NBR.

In general, at any relevant history h^{k_m}, we will be able to come up with $s|h^{k_m}$ that is subgameperfect in future histories and that has an overall credible bilateral binding scheme consistent with the NTU NBR. The associated payoff will be $v\left(s|h^{k_m}\right)$, where $s^{k_m}\left(h^{k_m}\right)$ *induces consistently* $s|h^{k_m}$ in $M'^e\left(h^{k+1}\right)$.

Finally, note that with the extra substages and binding schemes, we have eliminated dynamic bilateral conflicts of interest or conflictive reminders. The NBR can be applied in a coherent way or without conflict once subgame perfect equilibria can be distinguished in the NCE according to different binding schemes and substages

[32] This bargaining game should have been derived from an "appropiate" implicit strategic double proposal game where agents propose both payoffs and future bilateral binding schemes (See ANC model).

enable a hierarchy over "credible" binding agreements over time. Earlier ones will be *enforced* first. An analogous argument is given in the ANC model

5.4 From M'^e to M''^e : The Double Simultaneous Proposal Multistage Game

When backward solving the brute force model, we fixed future actions by a pair of agents linked earlier as they may be wondering about the induced effects of them doing that. We argued implicitly that a binding scheme over future actions will be "credible" if the agreement is over subgame perfect equilibria in the multistage game $M'^e\left(h^{k_m}\right)$ or equivalently in the NCE, $M^e\left(h^{k_m}\right)$. How would credible binding agreements be rationalized with only selfish behavior, without any binding element? For answering that question, we give some microfoundations to the brute force model by defining first a double proposal substage game where proposals are also of coordination schemes. Preliminaries for the payoff equivalence result are given. In this more non cooperative set up, this second extra key formal feature is also needed for the uniqueness property to hold and our models to be well defined.

5.4.1 Preliminary definitions.

Let $s|h^{k_m} \neq s'|h^{k_m}$ in $M'^e$and both restrictions have the same future bilateral binding scheme. Then we say that $s|h^{k_m}$ and $s'|h^{k_m}$ are *strategically equivalent as of* h^{k_m}. We define also to be strategically equivalent $s|h^{k_m} \neq s'|h^{k_m}$ such that $s^{k_m}\left(h^{k_m}\right)$ is not a proposal match. Thus $S|h^{k_m}$ can be partitioned in equivalent classes $S|h^{k_m}_{s|h^{k_m}}$ composed by strategically equivalent restrictions $s|h^{k_m}$. The *quotient* of $S|h^{k_m}$ with respect to the the obvious equivalent relationship is denoted by $S|h^{k_m}/\equiv$.

Let the history of play be $h^{k_m}(g+ij)$ and k^+, k_o^+ be as defined in 5.3.4. As we want to analyze convergent sequences in a real product space, we want $S|h^{k_m}/\equiv$ to be a subset of $\mathbb{R}^{3\left(\#h^{k_m}\right)}$, where $\#h^{k_m}$ is the number of histories including and following h^{k_m}. For this purpose, we define $I : S|h^{k_m} \to \mathbb{R}^{3\left(\#h^{k_m}\right)}$. If $s^{k_m}\left(h^{k_m}\right)$ is a proposal match, $I(s\left(h^{k_o^+}\right)) = (1,1,1)$, $I(s^{k_o^+}\left(h^{k_o^+}\right)) = (0,0,0)$ whenever $s^{k_o^+}(h^{k_o^+}) = e$ or ne respectively[33]. For other histories, we have $I(s^{k_m}\left(h^{k_m}\right)) = s^{k_m}(h^{k_m})$ and $I(s^{k_m^+}\left(h^{k_m^+}\right)) = (0,0,0)$ otherwise. If h^{k_m} is such that the prospective graph is the complete graph, $I(s^{k_m}\left(h^{k_m}\right)) = (1,1,1)$ or $(0,0,0)$ whenever $s^{k_m}\left(h^{k_m}\right) = f$ or r respectively (See end of 4.2.3). Now, $S|h^{k_m}_{s|h^{k_m}} \equiv I\left(s|h^{k_m}\right)$. Finally, if $s^{k_m}\left(h^{k_m}\right)$ is not a proposal match $S|h^{k_m}_{s|h^{k_m}} \equiv I\left(s|h^{k_m}\right) = (0,0,0)^{\left(\#h^{k_m}\right)}$.

[33] Recall $e \equiv 1$ and $ne \equiv 0$.

5.4.2 The Simultaneous Double Proposal Game $P''^{e}\left(h^{k_m}\right)$: The Key 2nd Formal Modification

We define a two-agent simultaneous *explicit* double proposal stage game $P''^{e}\left(h^{k_m}\right)$ at all relevant histories h^{k_m}, where an action for agent i is $S_i|h^{k_m}_{s|h^{k_m}} \in S|h^{k_m}/\equiv$.

Links form iff they match equivalent classes or double proposals. That is, iff the equivalent classes they propose are identical component by component and in addition $s^{k_m}\left(h^{k_m}\right)$ in $s|h^{k_m}$ of $S|h^{k_m}_{s|h^{k_m}}$is a proposal match. As confusion is possible, we write instead $s^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$. An agent proposing $S|h^{k_m}_{s|h^{k_m}}$ where $s^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$ is not a proposal match acts like one proposing a not feasible proposal match or unilateral rejecting in section 4.2.3. As before, the third agent moves nothing.

In other meaningful substages, the action set for the only agent moving non trivially, say i, is identical as before in M'^{e}: $e \equiv 1$ or $ne \equiv 2$. In contrast to M'^{e}, in M''^{e}, earlier linked agents, say i and j, do communicate in these substages! For simplicity, we could think of agent j *reminding* agent i to behave according to an earlier double proposal match $S_{ij}|h^{k_m}_{s|h^{k_m}}$, or simply $S|h^{k_m}_{s|h^{k_m}}$. We leave the notation about reminding implicit!

Double proposal matches are observed. Whenever we index histories h^{k_m} in addition by double proposal matches, we are referring to M''^{e} (or its further cooperative transformations). Formally, history subclasses or histories in them! are denoted by:

$$h^{k_m}\left(g^{\theta}+ij, S|h^{k_m^-}_{s|h^{k_m^-}}, ..., S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}\right),$$

where $k^- < k$ and g^{θ} is the last graph formed and $s^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}\left(h^{k_m^-(\theta)}\right) = \beta^{g^{\theta}}$ is the last proposal match. The initial history is $h^1(g^{\emptyset}+ij, S|h^{k_m^-}_{\emptyset})$. A strategy profile in M''^{e} is defined in the obvious way and will be denoted by $\widehat{s} \in \widehat{S}$.

Note that substages have more histories in the multistage game M''^{e} than in M'^{e}. Formally, for each history $h^{k_m}\left(g^{\theta}+ij, \beta^{g^{\theta}}\right)$ its *image set* is composed by several histories $h^{k_m}\left(g^{\theta}+ij, ..., S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}\right)$, where $s^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}\left(h^{k_m^-(\theta)}\right) = \beta^{g^{\theta}}$. In words, in link discussions as of $h^{k_m^-(\theta)}\left(g^{\theta}\right)$ in M'^{e}, a pair has matched only a given $\beta^{g^{\theta}}$, but in M''^{e} they have matched in addition to the same given $\beta^{g^{\theta}}$ future bilateral coordination schemes, i.e., they have matched double proposal $S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}$. And so on backwards!

Payoffs for the restriction $\widehat{s}|h^{k_m}$ in $M''^{e}\left(h^{k_m}\right)$ are the ones of a *payoff equivalent* restriction profile $s^*|h^{k_m}$ in $M'^{e}\left(h^{k_m}\right)$, i.e., $\widehat{u}\left(\widehat{s}|h^{k_m}\right) = u\left(s^*|h^{k_m}\right)$, where the history h^{k_m} in $M''^{e}\left(h^{k_m}\right)$ is in the image of the one in $M'^{e}\left(h^{k_m}\right)$ and $s^*|h^{k_m}$ has a path with the same actions in each substage as that ones implied by $\widehat{s}^{p_o}\left(h^{p_o}\right)$ for $p \geq k$, where h^{p_o} is a history in the path of $\widehat{s}|h^{k_m}$ and $o \neq m$. In histories h^{p_m} in the path of $\widehat{s}|h^{k_m}$such that $\widehat{s}^{p_m}\left(h^{p_m}\right) = S_{ij}|h^{p_m}_{s|h^{p_m}}$is a double proposal match, we have that $s^{*p_m}\left(h^{p_m}\right) = s^{p_m}_{s|h^{p_m}}\left(h^{p_m}\right)$. Whenever $\widehat{s}^{p_m}\left(h^{p_m}\right) = S|h^{p_m}_{s|h^{p_m}}$ is not a double proposal match then $s^{*p_m}(h^{p_m})$ is not a proposal match. Note that $\widehat{u}\left(\widehat{s}|h^{k_p}\right)$ can also be defined in the obvious way for any $p \in \{1,2,3\}$.

The following definitions will be used intensively later on:

A double proposal $S_{ij}|h^{k_m}_{s|h^{k_m}}$ *induces* $\widehat{s}|h^{k_m}$ iff $\widehat{s}^{k_m}\left(h^{k_m}\right) = S_{ij}|h^{k_m}_{s|h^{k_m}}$ and $\widehat{s}|h^{k_m}$ is subgameperfect in subgames in future histories, that is in $M''^e\left(h^{k_m+1}\right)$. A double proposal match $S|h^{k_m}_{s|h^{k_m}}$ has a *credible future bilateral coordination scheme* iff it induces $\widehat{s}|h^{k_m}$ and $\widehat{s}^{k_o^+}\left(h^{k_o^+}\right) = s^{k_o^+}_{s|h^{k_m}}\left(h^{k_o^+}\right)$ for all $h^{k_o^+}$, where k_o^+ is as defined in 5.3.4. In words, agents i and j find it optimal to obey the corresponding reminder in $S|h^{k_m}_{s|h^{k_m}}$ by j and i respectively in future substages where they communicate again. It will be useful to refer to several credible bilateral coordination schemes by different pairs of agents associated to $\widehat{s}|h^{k_m}$ induced by a double proposal match $S|h^{k_m}_{s|h^{k_m}}$. Let us have such a $\widehat{s}|h^{k_m}$ with all $h^{k_m^+}$ that consist of double proposal matches with associated credible future bilateral coordination schemes. The induced $\widehat{s}|h^{k_m}$ has the *conditional bilateral coordination behavior property* (CBCBP). The associated schemes are called the *credible overall bilateral coordination schemes (or the induced ones by)* of $\widehat{s}|h^{k_m}$. In words, in equilibrium, any agent that moves e or ne has no incentive to deviate from the reminder to abide by future bilateral coordination scheme associated to earlier double proposal matches, i.e., that ones associated with $S|h^{k_m^-}_{s|h^{k_m^-}}, ..., S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}$. In this situation, later linking pairs of agents whenever choosing their future bilateral coordination schemes would be acting conditioned on observable credible future bilateral coordination schemes by earlier linked pairs. They would regard them as "fixed" (as in the brute force model) because they are credible. Thus, there is a 1-1 correspondence between histories in the image of $h^{k_m}\left(g^\theta + ij, \beta^{g^\theta}\right)$, i.e., histories of the type $h^{k_m}\left(g^\theta + ij, S|h^{k_m^-}_{s|h^{k_m^-}}, ..., S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}\right)$and *implicit expanded!* histories $h^{k_m}\left(g^\theta + ij, \beta^{g^\theta}\right)$ where agents i and j act as having unique earlier credible future bilateral binding schemes derived in the obvious way from the credible future bilateral coordination schemes in $S|h^{k_m^-}_{s|h^{k_m^-}}, ..., S|h^{k_m^-(\theta)}_{s|h^{k_m^-}(\theta)}$. It is in this sense that we say that histories in M''^e indexed by earlier future coordination schemes are *equivalent* to expanded histories in M'^e indexed implicitly by the uniquely implied earlier future binding schemes.

As for the above discussion, if $S|h^{k_m}_{s|h^{k_m}}$ induces $\widehat{s}|h^{k_m}$ in $M''^e\left(h^{k_m}\right)$ that has the CBCBP, the payoff equivalent restriction profiles at each history h^{p_m} can be used to derive a subgame perfect payoff equivalent $s^*|h^{k_m}$ in $M'^e\left(h^{k_m}\right)$ whose restrictions to future histories are subgame perfect equivalents of $\widehat{s}|h^{p_m}$ for all p_m too (with the p_m *property*). The converse is also true provided we restrict ourselves to induced $\widehat{s}|h^{k_m}$ with the CBCBP. It is not complicated to imagine a subgame perfect equilibrium with double proposal matches without credible coordination schemes. However, its equilibrium payoff outcome is the same as one with a credible one. Just coordinate on what the other agent will do anyway in the associated added substages. In other words, there is a 1-1 correspondence between subgame perfect restriction profiles in $M''^e\left(h^{k_m}\right)$ and the ones in $M'^e\left(h^{k_m}\right)$ with the p_m property provided induced $\widehat{s}|h^{k_m}$ in $M''^e\left(h^{k_m}\right)$ have the CBCBP.

5.5 Overlapping Bargaining with "Strategic Constraints"

First extra substages were added. A second extra key feature needed to solve the two equilibrium selection problems in a more non cooperative set up was introduced in 5.4. The relevant substage action sets were extended by defining $P''^e\left(h^{k_m}\right)$ in $M''^e\left(h^{k_m}\right)$ that adds proposals of future bilateral coordination schemes. The latter let pairs distinguish over time between subgame perfect equilibria of the NCE as these schemes are history of play. Extra substages enables a hierarchy among conflicting bilateral communication by different pairs as it occurs "behind closed doors". Earlier communication prevails. Thus, everyone *focus* as in Schelling (1960), however endogenously, on one equilibrium in the NCE. Also, earlier "credible" future bilateral binding schemes in $M'^e\left(h^{k_m}\right)$ are rationalized through earlier credible future bilateral coordination schemes. However, there are still multiple equilibrium double proposal matches that induce subgame perfect behavior. To finally solve such problem, a third formal feature is added so that to yield the uniqueness property in an appropriate strategic form game: future overlapping bargaining. We also show the equivalence between overlapping bargaining games in the brute force model at the ANC model.

5.5.1 Preliminary Definitions

At relevant histories of M''^e, we add more communication by allowing a NTU two-agent overlapping bargaining problem for player i and j *excluding* l. The overlapping bargaining problem at h^{k_m} is derived in 5.5.2 from a cooperative transformation of $M''^e\left(h^{k_m}\right)$ in strategic form at the same substage with history h^{k_m} and denoted by $T''^{\breve{\eta}e}\left(h^{k_m}\right)$, where $\breve{\eta}$ is a future bargaining rule.

For now let us assume that such a bargaining game can be derived so that to add some definitions where the dependence on a future bargaining rule $\breve{\eta}$ is implicit. We define for any two vectors x and y in $\mathbb{R}^2$

$x \geq y$ iff and $x_i \geq y_i$ and $x_j \geq y_j$, and

$x > y$ iff and $x_i > y_i$ and $x_j > y_j$.

Let $h^{k_m} = h^{k_m}(g+ij)$ and $i \neq j \neq l$. The bargaining problem for agents i and j excluding l consists of a pair $(F(h^{k_m}), \psi^{h^{k_m}})$, where $F(h^{k_m})$ is a closed convex subset of $\mathbb{R}^2$, $\left(\psi_i^{h^{k_m}}, \psi_j^{h^{k_m}}, \psi_l^{h^{k_m}}\right)$ is a vector in $\mathbb{R}^3$ and the set of *individually rational feasible allocations (IRF set)*

$$F(h^{k_m})\cap\left\{\left(x_i^{h^{k_m}}, x_j^{h^{k_m}}\right) | x_i^{h^{k_m}} \geq \psi_i^{h^{k_m}} \text{ and } x_j^{h^{k_m}} \geq \psi_j^{h^{k_m}} \text{ or equivalently } x_{ij}^{h^{k_m}} \geq \psi_{ij}^{h^{k_m}}\right\}$$

is nonempty and bounded. Here $F(h^{k_m})$ represents the *set of feasible payoff allocations* or the *feasible set*, and $\psi_{ij}^{h^{k_m}}$ represents the *disagreement payoff allocation* or the *disagreement point*.

A bargaining game $(F(h^{k_m}), \psi^{h^{k_m}})$ is *essential* iff there exists at least one allocation $x^{h^{k_m}}$ in $F(h^{k_m})$ that is strictly better for agents than the disagreement allocation $\psi_{ij}^{h^{k_m}}$, i.e., $x^{h^{k_m}} > \psi_{ij}^{h^{k_m}}$.

A point x in F is *strongly (Pareto) efficient* iff there is no other point y in F such that $y \geq x$ and $x_w > y_w$ for at least one player $w \in \{i, j\}$. A point x in F is *weakly (Pareto) efficient* iff there is no other point y in F such that $y > x$. The *feasible frontier* is the set of feasible payoffs allocations that are strongly Pareto efficient in F. The *IRF frontier* is the set of points in F that are strongly Pareto efficient in the IRF set.

In standard two-agent bargaining theory there are two possibilities for deriving F. The feasible set F consists of the set of payoffs associated to correlated strategies (correlated equilibriums) of a strategic form game, iff strategies are contractible (iff there is moral hazard). The set of *correlated strategies* is the set of probability distributions over strategy profiles of a strategic form game. The set of *correlated equilibriums* is the subset of correlated strategies that satisfy *strategic incentive constraints* (See Myerson 1991).

We don't have a complete theory for some notion of *bilateral correlated strategies* in the strategic form game of $M^e\left(h^k\right)$, $M'\left(h^k\right)$. Second, as in Myerson (1986), it is likely that this strategic form might not be a good departure point to define such a notion. In this second respect and if we would allow for moral hazard, neither we have a definition for something like *sequential strategic bilateral incentives constraints.* The latter constraints would enable us to know what are those *bilateral correlated equilibria* over time that may be defined so as to be immune to deviation by any of the agents in the links formed earlier from their earlier future bilateral coordination schemes (or recommended by a *link specific mediator* at each relevant substage). That is also the case for some notion for *sequential individually rational constraints* that would be important when defining disagreement payoffs at some history.

Nevertheless, we conjecture that the way we derive $F\left(h^{k_m}\right)$ in 5.5.3 from the strategic form game $T''^{\eta' e}\left(h^{k_m}\right)$ described in 5.5.2 will be consistent with some implicit notion of sequential strategic bilateral incentives constraints Moreover, it will be strong Pareto efficient. We plan to address even more general related notions in the near future.

5.5.2 The Overlapping $T''^{\check{\eta} e}\left(h^{k_m}\right)$: The Key 3rd Formal Modification

An *appropriate* cooperative transformation of M''^e is proposed by modelling an overlapping and *nested*−as it will be used to derive an overlapping bargaining game−substage game in strategic form in substages with histories h^{k_m}. Such transformation consists of a simultaneous "two-agent" double proposal substage game $T''^e\left(h^{k_m}\right)$ with the same action set $\left(S|h^{k_m}/\equiv\right)^2$ as the one in substage h^{k_m} of M''^e.

There is some similarity between $T''^e\left(h^{k_m}\right)$ and the strategic form game in 5.3. However, expected payoffs are required to be credible−thus, it is an overlapping strategic game−for an action profile $S_{ij}|h^{k_m}_{s|h^{k_m}} \in \left(S|h^{k_m}/\equiv\right)^2$ in the strategic form game $T''^e\left(h^{k_m}\right)$ are equal to $\widehat{u}_{ij}\left(\widehat{s}|h^{k_m}\right) = u_{ij}\left(s^*|h^{k_m}\right)$, where $\widehat{s}|h^{k_m}$, is induced by $S_{ij}|h^{k_m}_{s|h^{k_m}}$. Following 5.4.2, there is a unique subgameperfect $s^*|h^{k_m}$ in $M'^e\left(h^{k+1}\right)$. As in section 5.3, the credible expected payoff of $\widehat{s}|h^{k_m}$ is denoted by $\widehat{v}\left(\widehat{s}|h^{k_m}\right)$ where $\widehat{v}\left(\widehat{s}|h^{k_m}\right) = v\left(s^*|h^k\right)$. Also, the credible expected payoff of action profile $\widehat{s}^{k_m}(h^{k_m})$ is $\widehat{v}\left(\widehat{s}^{k_m}(h^{k_m})\right) = v\left(\widehat{s}|h^{k_m}\right)$. Wlg (See end of 5.4.2), we will restrict $T''^e\left(h^{k_m}\right)$ to be played with restriction profiles $S|h^{k_m}_{s|h^{k_m}}$ that induce credible future bilateral coordination schemes.

The strategic substage form game $T''^e\left(h^{k_m}\right)$ is still not well defined. To see this, first let $h^{k_m} = h^{k_m}(g+ij,...)$ and $p \geq k$. As expected, any future $T''^e\left(h^{p_m}\right)$ will in general contain infinitely many double proposal matches $S|h^{p_m}_{s|h^{p_m}}$ that are credible Nash equilibria restriction profiles even though the pair playing in h^{p_m} has observed earlier coordination schemes that maybe even credible in future subgames. As it will become more clear when we construct our solution, this multiplicity of equilibria, makes it impossible for earlier linked pairs to associate a unique credible expected payoff to their own double proposal match.

For deriving an overlapping bargaining game from a well defined nested strategic form substage game, we propose our key second formal modification and allow for a fixed rule $\breve{\eta}\left(h^{k_m^+}\right)$ (recall $k^+ > k$) according to which future discussants select a unique Nash equilibrium among multiple ones in any future $T''^e\left(h^{k_m^+}\right)$. It will follow by construction that $T''^{\breve{\eta}e}\left(h^{k_m}\right)$ exhibits the uniqueness property.

5.5.3 Derivation of $F\left(h^{k_m}, \psi^{h^{k_m}}\right)$ from $T''^{\breve{\eta}e}\left(h^{k_m}\right)$

Provided that we proof (by construction) that $T''^{\breve{\eta}e}\left(h^{k_m}\right)$ has the *uniqueness property*, each $\left(x_i^{h^{k_m}}, x_j^{h^{k_m}}\right) \in F(h^{k_m})$ is such that

$$\left(x_i^{h^{k_m}}, x_j^{h^{k_m}}\right) = u_{ij}\left(s^*|h^{k_m}\right) = \widehat{v}_{ij}\left(\widehat{s}|h^{k_m}\right),$$

where $s^*|h^{k_m}$ is a payoff equivalent strategy profile of $\widehat{s}|h^{k_m}$ which in turn is induced by $S_{ij}|h^{k_m}_{s|h^{k_m}} \in \left(S|h^{k_m}/\equiv\right)^2$ in $M''^e\left(h^{k_m}\right)$. From 5.4.2, there is a unique subgame perfect payoff equivalent $s^*|h^{k_m}$. Hence, we can write $\left(x_i^{h^{k_m}}, x_j^{h^{k_m}}\right) = v_{ij}\left(s^*|h^{k_m}\right) = v_{ij}\left(s^*\left(h^{k_m}\right)\right)$. In this part of the paper, we will use $\widehat{v}\left(\widehat{s}\left(h^{k_m}\right)\right)$ for the credible expected payoff associated to $\widehat{s}\left(h^{k_m}\right) = S_{ij}|h^{k_m}_{s|h^{k_m}}$. We assume that it is possible to convexify over payoffs in F so derived!

The disagreement payoff $\psi_{ij}^{h^k}$ is derived from ψ^{h^k} set equal to $\widehat{v}\left(\widehat{s}\left(h^{k_m}\right)\right)$, where $\widehat{s}\left(h^{k_m}\right) = S_{ij}|h^{k_m}_{s|h^{k_m}}$ is not a proposal match.

From 5.4.2, there is a 1-1 correspondence between subgame perfect restriction profiles with the CBCBP in $M''^{e}\left(h^{k_m}\right)$ and subgame perfect payoff equivalent restrictions in $M'^{e}\left(h^{k_m}\right)$ with the p_m *property* and thus $\widehat{v}_{ij}\left(\widehat{s}|h^{k_m}\right) = v_{ij}\left(s^*|h^{k_m}\right)$. It follows that the furthest bargaining games with equivalent histories (as defined in 5.4.2) $h^{k_m^+}$ are equivalent (feasible sets are identical and so on). Note that in $M''^{e}\left(h^{k_m^+}\right)$ this bargaining games are indexed by earlier credible bilateral future coordination schemes. As for the arguments in 5.4.2, they are regarded as fixed, thus, they imply respective unique earlier future bilateral binding schemes in an equivalent history in $M'^{e}\left(h^{k_m^+}\right)$. It follows that if the same future bargaining rule $\breve{\eta}$ is used, earlier respective bargaining games will be equivalent in equivalent histories as they would have identical outside options. In other words, allowing bargaining with $\breve{\eta}$ yields bargaining games "indexed" by earlier credible bilateral coordination schemes in $M''^{e}\left(h^{k_m}\right)$ that would correspond 1-1 with bargaining games "indexed" by earlier credible bilateral binding schemes in $M'^{e}\left(h^{k_m}\right)$. This equivalence is illustrated partially in the proof of theorem s1.

We distinguish two types of strategic form games $T''^{\breve{\eta}e}\left(h^{k_m}(g+ij,...)\right)$ as we will often encounter them when backward solving. They will induce respective types of bargaining games:

1. In type 1, $\exists$ a better double proposal match $S_{ij}|h^{k_m}_{s|h^{k_m}} \in \left(S|h^{k_m}/\equiv\right)^2$.

2. In type 2, $\nexists$ a better double proposal match $S_{ij}|h^{k_m}_{s|h^{k_m}} \in \left(S|h^{k_m}/\equiv\right)^2$.

where a *better* double proposal match $S_{ij}|h^{k_m}_{s|h^{k_m}}$ is such that:

If $S_{ij}|h^{k_m}_{s'|h^{k_m}} \neq S_{ij}|h^{k_m}_{s|h^{k_m}}$ and $S_{ij}|h^{k_m}_{s'|h^{k_m}}$ is not a double proposal match then $\widehat{v}_{ij}\left(\widehat{s}\left(h^{k_m}\right)\right) \geq \widehat{v}_{ij}\left(\widehat{s}'\left(h^{k_m}\right)\right)$.

Wlg., we are assuming implicitly that F is derived from $T''^{\breve{\eta}e}\left(h^{k_m}\right)$ with contracts even though it matters if we assume binding contracts or if there is moral hazard in h^{k_m}. However, this is not the case for the IRF set and the outside options. As the NTU NBR depends only on the latter two variables our assumption is wlg.[34]

[34] In the associated payoff matrix type 1, proposing $S_{ij}|h^{k_m}_{s|h^{k_m}}$ such that it is a double proposal match is a Nash equilibrium of $T''^{\eta' e}\left(h^{k_m}(g+ij,...)\right)$. So is both agents choosing $S|h^{k_m}_{s|h^{k_m}}$ such that $s^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$ is not a proposal match. Proposing different equivalent classes $S'|h^{k_m}_{s|h^{k_m}} \neq S|h^{k_m}_{s|h^{k_m}}$ where $s'^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$ or $s^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$ is a proposal match is not a Nash equilibrium of $T''^{\eta' e}\left(h^{k_m}(g+ij,...)\right)$. Note that the last case has associated payoffs that coincide with the ones corresponding to both agents choosing $S|h^{k_m}_{s|h^{k_m}}$ such that $s^{k_m}_{s|h^{k_m}}\left(h^{k_m}\right)$ is not a proposal match (credible expected payoff of not forming a link). Thus, it doesn't matter if we assume the existence of binding contracts or if there is moral hazard for the derivation of the feasible set. The reason is that any payoff outcome of $T''^{\eta' e}\left(h^{k_m}(g+ij,...)\right)$ can be obtained by payoffs of Nash equilibria of $T''^{\eta' e}\left(h^{k_m}(g+ij,...)\right)$.

The outside options $\psi_{ij}^{h^{k_m}}$ coincide with the ones derived from min max values and the ones derived from a focal Nash equilibrium of the game as it is always a Nash equilibrium to unilateral reject.[35]Deriving the outside options from a Nash's (1953) rational threats game would be possible too. However, our objective is not to define a solution for network games where threats can be enforced but where they are credible without external enforcement device.

Note that if $\widehat{v}_{ij}\left(\widehat{s}\left(h^{k_m}\right)\right)$ is a continuous function in $S|h^{k_m}/\equiv$ (*closure condition* in h^{k_m}), then $T''^{\check{\eta}e}\left(h^{k_m}\right)$ has the uniqueness property! Moreover $F(h^{k_m})$ is non empty and closed. As we assume that is possible to convexify over credible expected payoffs, $F(h^{k_m})$ is well defined. Finally, a bargaining game could be defined trivially in any substage different than k_m.

5.5.4 Feasibility and Link (P) Feasibility

The outcome of the bargaining game may end up in disagreement and thus in different bargaining game. To emphasize that while remaining in the same strategic form game something better that the outside options is feasible by forming a link, we define the *credible expected proposal match payoff feasible set* (P). This is the set of credible expected payoff outcomes associated to double proposal matches $S|h_{s|h^{k_m}}^{k_m}$ in $T''^{\check{\eta}e}\left(h^{k_m}(g+ij,...)\right)$. The set of strong Pareto efficient points of P or the *frontier of* P is denoted by PF. In the type 2 bargaining game, the outside options are feasible but not P *feasible.*

5.6 The Overlapping Smoothed Nash Demand Game $J''^{\eta e}\left(h^{k_m}\right)$

We want all agents to bargain with only one rule. First, the nested bargaining game $F\left(h^{k_m}, \psi^{h^{k_m}}\right)$ is assumed to be solved with an implicit (for simplicity) take it or

In the associated payoff matrix type 2, choosing only different $S|h_{s|h^{k_m}}^{k_m}$ is a Nash equilibrium of $M^{\eta' c}\left(h^{k_m}(g+ij)\right)$. The associated payoffs are $\widehat{v}_{ij}(\widehat{s}\left(h^{k_m}\right)) = \psi_{ij}^{h^{k_m}}$ where $s_{s|h^{k_m}}^{k_m}\left(h^{k_m}\right)$ is not a proposal match. As there is no better double proposal match by definition, then at least one agent would deviate if both agents would match $S|h_{s|h^{k_m}}^{k_m}$. Because not all payoff outcomes can be obtained payoffs of Nash equilibria (recall both agents matching double proposals $S|h_{s|h^{k_m}}^{k_m}$ in h^{k_m} would yield payoffs lower at least for one agent) it matters if we assume binding contracts or moral hazard for the derivation of F. However, that does not matter for the derivation of the IRF set. Note that whatever the assumption is, $\psi_{ij}^{h^{k_m}}$ is the only individual rational feasible payoff pair. Hence, our assumption is inoquous.

[35]If $\psi_{ij}^{h^k} = \left(\psi_i^{h^{k_m}}, \psi_j^{h^{k_m}}\right)$ are the min max payoffs for players i and j in $T''^{\eta' e}\left(h^{k_m}(g+ij,...)\right)$, then

$\psi_i^{h^{k_m}} = \min_{S_j|h_{s|h^{k_m}}^{k_m} \in S|h^{k_m}/\equiv} \max_{S_i|h_{s|h^{k_m}}^{k_m} \in S|h^{k_m}/\equiv} \widehat{v}_i\left(\widehat{s}\left(h^{k_m}\right)\right)$

$\psi_j^{h^{k_m}} = \min_{S_i|h_{s|h^{k_m}}^{k_m} \in S|h^{k_m}/\equiv} \max_{S_j|h_{s|h^{k_m}}^{k_m} \in S|h^{k_m}/\equiv} \widehat{v}_j\left(\widehat{s}\left(h^{k_m}\right)\right)$

where $s \in S$ and $S|h^{k_m}/\equiv$ is the individual action set and $\widehat{s}\left(h^{k_m}\right) = S_{ij}|h_{s|h^{k_m}}^{k_m}$.

leave it *simultaneous offer* overlapping game that yields the unique non transferable utility (NTU) Nash bargaining rule (NBR) payoff pair, where our extension of the *symmetric* NTU NBR payoff $\eta(F(h^{k_m}), \psi^{h^{k_m}}))$ is defined as follows:

$\eta_{ij}(F(h^{k_m}), \psi^{h^{k_m}})) \in \arg\max_{x \in F(h^k),\, x \geq \psi^{h^k}} \left(x_i^{h^{k_m}} - \psi_i^{h^{k_m}}\right)\left(x_j^{h^{k_m}} - \psi_j^{h^{k_m}}\right),$

where $h^{k_m} \in H$. If $F(h^{k_m}), \psi^{h^{k_m}})$ is of type 1, the third agent gets $\eta_l(F(h^{k_m}), \psi^{h^{k_m}}) = \phi_l^{g+ij}$ a constant!! If $F(h^{k_m}), \psi^{h^{k_m}})$ is of type 2, then $\eta(F(h^{k_m}), \psi^{h^{k_m}}) = \psi^{h^{k_m}}$, also a constant as of history h^{k_m}.

The implicit final overlapping game is the *smoothed* Nash (1950) demand game applied to $F\left(h^{k_m}, \psi^{h^{k_m}}\right)$ and it is denoted by $J''^{\breve{\eta}e}\left(h^{k_m}(g+ij, ...)\right)$. In our game, pairs of agents can credibly threat with their outside options and not discuss a link further as if they reject the next pair proposes according to the rule of order. Thus, there is commitment as required by Nash's game (See Binmore 1998 chapter 1). As links can be formed only if both agree, take it or leave it simultaneous offers are natural.

Second, we want all future pairs of agents bargaining with the smoothed Nash demand game. So we set $\breve{\eta} = \eta$ and $T''^{\breve{\eta}e}\left(h^{k_m}(g+ij, ...)\right)$ becomes $T''^{\eta e}\left(h^{k_m}(g+ij, ...)\right)$ and we write instead $J''^{\eta e}\left(h^{k_m}(g+ij, ...)\right)$. To derive the associated multistage game $M^{\eta e}$ (the ANC model), we assume that the third agent moves nothing in h^{k_m}. Note that in substages different than h^{k_m}, $J''^{\eta e}$ may defined in a trivial way.

Let strategies $\widehat{s}^{\eta e} \in \widehat{S}^{\eta e}$ of $M^{\eta e}$ and other related concepts be implicitly defined in an analogous way as we did explicitly already for M''^{e} in section 5.4. As for the possibility of multiple $\widehat{s}^{\eta}$ equilibrium strategy profiles, the solution correspondence applied to $(F(h^{k_m}), \psi^{h^{k_m}})$ yields a vector $\left(\widehat{s}^{\eta e} | h^{k_m}, \eta(F(h^{k_m}), \psi^{h^{k_m}})\right)$ for all $h^{k_m} \in H$.

Our payoff solution for the game as a whole will be denoted by the function

$\Re(v, \rho_{\emptyset}) = \left(\eta(F(h^1), \psi^{h^1})\right).$

For strictly superadditive games, we predict graphs structures uniquely thus:

$\Re(v, \rho_{\emptyset}) = \left(\eta(F(h^1), \psi^{h^1}), g\right),$

where the only exogenous argument will be the characteristic function and the rule of order of a three-agent game.

We proof existence of our payoff prediction by showing that in relevant substages, the bargaining game is well defined. In particular, we check the uniqueness property in the feasible sets. As we assume that convexifying is possible, uniqueness of our payoff solution follows from the NTU NBR.

Also, from 5.4.2 and 5.5.3 it follows that $M^{\eta e}$ and the history expanded M'^{e} (brute force model) are *payoff equivalent*. The equivalence with the NCE is left to the reader (hint: assume a cutoff rule in cases of dynamic bilateral conflicts of interest). Thus, analytical results for the key subgames of the brute force model (See companion paper for details) in any normalized game are stated in section 7.

Finally, note that we are assuming that agents compromise to play nested or overlapping games and do the required computations. For a model with long cheap talk where this assumption could be explicitly modelled see Aumann-Hart (2003).

5.7 Conclusions

5.7.1 Three Key Features and General Results

The ANC adds full bilateral communication possibilities in the A-M game and payoffs exist and more importantly they are unique. Besides, we predict graphs and coalition structures for all three-agent normalized games.

In the NCE, at each relevant stage there is a Nash demand like game where pairs of agents propose simultaneously payoffs out of their Myerson values in the prospective link. Subgame perfection eliminates non credible Nash equilibria. In the ANC model, natural extra substages behind closed doors and proposals of future bilateral coordination schemes are added. Coordination will enable pairs of agents to distinguish as of earlier link discussions over subgame perfect equilibrium payoff outcome pairs of the NCE whenever credible, i.e., in cases of indifference of one of the earlier linked agent linking or not with a third one (dynamic bilateral conflicts of interests). Extras substages will make earlier communication prevail: "As a third agent needs my consent to begin discussing our link!!, I would help you and not her whenever I'm indifferent by doing so and provided we talked about it before and you remind me not to discuss with her right before she gets a chance to". Both consequences, will yield a "more" certain curse of induced actions and thus more certain expected payoffs as the third agent will literally take that earlier coordination scheme as given and would even play and coordinate herself optimally with another agent and so on. However, there would be still in equilibrium multiple double proposal matches that can be chosen as of earlier and future double proposal games (divide the dollar like problems). Nevertheless, it is possible to derive a well defined two agent overlapping and nested appropriate strategic form substage game and thus solve a two agent bargaining game in all histories simultaneously, if every pair in all relevant histories is using axiomatic (brute force) or explicit (ANC) bargaining (third feature) consistent with the NTU NBR. Hence, agents focus endogenously on a unique equilibrium payoff out of the several in the NCE.

Also, bargaining with future bilateral coordination schemes without external enforcement is equivalent to bargaining with future credible bilateral binding schemes.

Note that we are assuming that agents compromise and play nested or overlapping games and do the required computations.

Finally, as the only thing needed for the proof are fixed payoff allocation rules, an ANC analytical payoff function exists for any A-M-like game.

5.7.2 A Key Ad hoc Limitation

Let $g^{li} + ij$ be the prospective graph. According to the Myerson value, if graph g^{lij} forms, there may be transfers of utility among all linked agents (directly or indirectly) l, i and j. However, in our models the payoff of the third agent in the prospective graph is fixed. Thus, our solution should be applied to situations where once the proposal match and the link forms and it is the last to form, agents i and j are not able to renegotiate neither bilaterally nor with the third agent l. In this sense, the ANC model is similar to the sequential and simultaneous proposal games in the Networks and Bargaining literature (See Jackson 2003 page 23). The key difference is that we don't use a valuation function but an allocation rule: the Myerson value.[36] Possible real applications may include the ones in Currarini and Morelli's (2000 page 230) paper to formation of economic unions, in which negotiations are multilateral in nature, and each player (country) makes an absolute claim on the total surplus from cooperation. In our case, negotiations would have to occur also sequentially but bilaterally. "The key issue is that the bargaining games have the demand of a payoff for participation in the formation of the network (the link with the implied graph in our case) as a crucial variable". We would add to this quote: with no ex post renegotiation! Our bargaining model is consistent with that feature, as links can be formed only if proposals match.

Nevertheless, as payoffs cannot be renegotiated, there is a binding element. On the other hand, as the decision of forming a link or not involves simultaneous credible take it or leave it offers there is no binding element. Without such a binding element discussed and still assuming i and j can not transfer from their Myerson values to l or vice versa, the game would have a flavor that of a centipede game as once the link is formed agent i would be tempted to give agent j just enough so that j does not link with l. Thus, agent j would not get the payoff we predict.[37] The nature of the game changes completely as agent j may not have wanted to form link ij to begin with. [38]

In any case, we postpone for another paper a deserved axiomatic and noncooperative foundation to deal with this ad hoc feature in our solution concept so that not to rely on any type of binding element.

5.7.3 Testable Hypotheses and Extensions

1. As for the analytics in the companion paper, straightforward experiments should be testable for any three-agent A-M-like game.

2. Efficiency analysis should be straightforward.

3. The n-agent case looks promising

[36] Actually, we could have used other continuous allocation rule functions.

[37] Recall that once a link is formed it cannot be destroyed so outside options are different than when links are discussed

[38] For a discussion of how this might be solved though in the standard centipede game see Binmore (1998 page 113)

6 Existence and Uniqueness in $\Re(v, \rho_\emptyset)$

Most of the footnotes in the proof of theorem s1 are not necessary for the logic of the proof but may make it more accessible. Also it may help if the reader is interested in understanding the analytics in the companion paper.

Theorem s1: A unique payoff solution exists for three-agent normalized games with the Myerson value as a fixed allocation rule. It is the solution to $M^{\eta e}$, the ANC model.

Proof:

Overview

As for 5.5.3, it suffices to show that $\widehat{v}_{ij}\left(\widehat{s}\left(h^{k_m}\right)\right)$ is a continuous function in $S|h^{k_m}/\equiv$ (closure condition).

Abusing notation, let $h^3 = h^{3_m}$. Wlg., the rule of order is $(1,2)$, $(2,3)$ and $(1,3)$, the first two pairs rejected and we are in a subgame in $h^3(g^{\emptyset+13})$. Agents 1 and 3 wonder about the induced effects of $S|h^3_{s|h^3}$ in $T''^{\eta e}\left(h^3\right)$ nested in $M^{\eta e}$, and thus wonder about P. Existence is proved by constructing well defined bargaining games in future histories after h^3 in $M^{\eta e}$. After solving the almost trivial bargaining games in the furthest histories, its results are used to proof that in the *preceding* bargaining game in $M^{\eta e}$, P is well defined and thus F. The outside options are continuous in $S|h^3/\equiv$ while P is independent. Thus, the associated bargaining solution is a continuous composite function in $S|h^3/\equiv$ which is composed by *bilateral coordination scheme subset types*. It follows that the outside options of the game *before the preceding bargaining game* are continuous and thus, its solution exist and is continuous composite functions of $S|h^3_{s|h^3}$, composed now by *bilateral coordination scheme subset subtypes*. The sufficient condition follows in h^3 and thus, F is well defined.

Formally, as we want to backward solve partially with the brute force model, we would need to check any of the closure conditions below, provided, in particular, $\widehat{v}_{13}\left(\widehat{s}|h^3\right)$ exists and it is a composite function of $S|h^3_{s_n|h^3}$, where $\widehat{s}|h^3$ is constructed assuming that η is the fixed future bargaining rule $\breve{\eta}$ so that to have $T''^{\eta e}\left(h^3\right)$ nested in $M^{\eta e}\left(h^3\right)$!! (See section 5.5.2):

$\lim_{S|h^3_{s_n|h^3} \to S|h^3_{\overline{s}|h^3} \text{ as } n\to\infty} \widehat{v}_{13}\left(\widehat{s}_n|h^3\right) = \widehat{v}_{13}\left(\overline{\widehat{s}}|h^3\right)$for all convergent sequences (subtypes) $\{S|h^3_{s_n|h^3}\}$ in $S|h^3/\equiv$ of $T''^{\eta e}\left(h^3\right)$ nested in $M^{\eta e}\left(h^3\right)$ and induced constructed $\{\widehat{s}_n|h^3\}$ and thus $\{\widehat{v}_{13}\left(\widehat{s}_n|h^3\right)\}$

$\lim_{s_n|h^3 \to \overline{s}|h^3 \text{ as } n\to\infty} v_{13}\left(s_n|h^3\right) = v_{13}\left(\overline{s}|h^k\right)$ for all convergent sequences $\{s_n\left(h^3\right)\}$in $S|h^3$ of $M'^{e}\left(h^3\right)$ and consistently induced constructed $\{s_n|h^3\}$ and thus $\{v_{13}\left(s_n|h^3\right)\}$.

Note that for convergence, the elements in the constructed sequences $\{S|h^3_{s_n|h^3}\}$ must have the same future bilateral coordination scheme. Also, we require the same overall future bilateral binding scheme[39] for constructed $\{s_n|h^3\}$.

[39] Wlg, one may fix the actions where payoff proposals don't match so that to have convergent sequences.

As for the results for F in h^3, all first prospective link bargaining games can be solved for arbitrary outside options. We proof the claim for the game as a whole by referring to the short argument in theorem S1 (section 7.1).

1. Agents 1 and 3's furthest Coordination Horizon as of h^3

When evaluating coordination on double proposal match $S|h^3_{s|h^3}$ with $s^3_{s|h^{3m}}(h^3) = \beta^{13}$ in $T''^{\eta e}(h^3)$ nested in $M^{\eta e}(h^3)$ or proposal match $s^3(h^3) = \beta^{13}$ in $M'^{e}(h^3)$, we need to look as far as in history class[40] $h^{6_1}(g^{132+12})$. In particular for $M^{\eta e}(h^3)$, as histories[41] $h^{6_1}\left(g^{132+12}, S|h^3_{s|h^3}, S|h^{5_2}_{s|h^{5_2}}\right)$ in M''^{e} depend also on $S|h^{5_2}_{s|h^{5_2}}$, we move backwards to $h^{5_2}(g^{13+32}, S|h^3_{s|h^3})$. There, given $S|h^3_{s|h^3}$, we can evaluate $S|h^{5_2}_{s|h^{5_2}}$ after it is chosen optimally by agents 2 and 3 that in turn care about induced equilibrium payoff outcomes even further on as of histories h^{6_3}.

2. The Well Defined Bargaining Game in $h^{6_3}(g^{132}+12)$

2.1 The Trivial P Set in $(F(h^{6_3}), \psi^{h^{6_3}})$

Note that looking ahead as of h^{5_2}, $P-$that coincides with the PF in both bargaining games[42] $(F(h^{6_3}), \psi^{h^{6_3}})-$ has always the same payoff pair in the plane $\left(\beta_1^{132}, \beta_2^{132}\right)$: those of the complete graph $\left(\phi_1^N, \phi_2^N\right)$ implied by link 12 forming.

2.2 Exogeneity of $F(h^{6_3})$ as for Agent 1's Exogenous Outside Option

The other component of $F(h^{6_3})$ is agents 1 and 2's outside options[43]. Agent 1's exogenous outside option is her Myerson value (See diagram 2) in g^{132}, that is, $\psi_1^{h^{6_3}} = \frac{2(d-b)+a}{6} = \phi_1^{132} \leq \phi_1^N = \frac{c+2(d-b)+a}{6} \Leftrightarrow 0 \leq c$.

Different Parameter Cases: If $c = 0$, agent 1 will get the same payoff if forming or not the third link 12 in both multistage games. In $M^{\eta e}$, agents 1 and 3 may wonder about the induced effect on coordinating on $S|h^3_{s|h^3}$ such that $s^{6_1}_{s|h^3}(h^{6_1}) = e$ or ne if histories like $h^{6_1}\left(g^{132+12}, S|h^3_{s|h^3}, S|h^{5_2}_{s|h^{5_2}}\right)$ in $M''^{e}$are reached. Recall that this entails agent3 reminding either e or ne respectively in such a contingency. Thus, in any subgameperfect equilibrium of $M^{\eta e}$ or M''^{e}, it is optimal for agent 1 to obey the reminder, whatever this is. This coordination equilibrium outcome will be referred as *credible and non trivial.* In M'^{e}, we just say that there is an equilibrium selection problem with bilateral conflict.[44]If $c > 0$ they would coordinate *credibly but trivially* if $s^{6_1}_{s|h^3}(h^{6_1}) = e$ because in any subgame perfect equilibrium of $M^{\eta e}$ or M''^{e}, agent 1 would enter. They would *coordinate not credibly* if $s^{6_1}_{s|h^3}(h^{6_1}) = ne$, i.e. in any subgameperfect equilibrium outcome of $M^{\eta e}$, agent 1 would not follow such a

[40] When there is no extra indexing we are referring to histories in the two games.

[41] Note that we denote history classes and elements in them identically.

[42] Bargaining games can be defined in the obvious way in the brute force model by using the credible expected payoff matrix $M'(h^3)$ in 5.3.3. to derive a feasible set.

[43] The complete derivation would include convex combinations of the ouside options and the elements on the P feasible set. Implicitly, we are assuming that convexifying over payoff outcomes is possible.

[44] There is perfect recall but pairs of earlier linked agents did not discussed future actions in cases of indifference of one of the agents!

Diagram 2: Myerson Values for Normalized Games

One-link Graphs

$$
\begin{array}{ccc}
 & \left(\frac{a}{2}\right)3 & \\
 & / & \\
\left(\frac{a}{2}\right)\ 1 & & 2\ (0)
\end{array}
$$

graph g^{13}

$$
\begin{array}{ccc}
 & 3(0) & \\
\left(\frac{c}{2}\right)\ 1 & \text{———} & 2\ \left(\frac{c}{2}\right)
\end{array}
$$

graph g^{12}

$$
\begin{array}{ccc}
 & 3\ \left(\frac{b}{2}\right) & \\
 & \backslash & \\
(0)1 & & 2\ \left(\frac{b}{2}\right)
\end{array}
$$

graph g^{23}

Two-Link Graphs

$$
\begin{array}{ccc}
 & \left(\frac{2d+b+a}{6}\right)\ 3 & \\
 & / \qquad \backslash & \\
\left(\frac{2(d-b)+a}{6}\right)1 & & 2\ \left(\frac{2(d-a)+b}{6}\right)
\end{array}
$$

graph g^{132}

$$
\begin{array}{ccc}
 & \left(\frac{2(d-c)+a}{6}\right)\ 3 & \\
 & / & \\
\left(\frac{2d+a+c}{6}\right)\ 1 & \text{———} & 2\ \left(\frac{2(d-a)+c}{6}\right)
\end{array}
$$

graph g^{312}

$$
\begin{array}{ccc}
 & 3\ \left(\frac{2(d-c)+b}{6}\right) & \\
 & \backslash & \\
\left(\frac{2(d-b)+c}{6}\right)\ 1 & \text{———} & 2\ \left(\frac{2d+b+c}{6}\right)
\end{array}
$$

graph g^{123}

Complete graph

$$
\begin{array}{ccc}
 & 3\ \left(\frac{a+2(d-c)+b}{6}\right) & \\
 & / \qquad \backslash & \\
\left(\frac{c+2(d-b)+a}{6}\right)1 & \text{———} & 2\ \left(\frac{b+2(d-a)+c}{6}\right)
\end{array}
$$

graph g^N

reminder. In M'^e, agent 1 would always enter in equilibrium.

We assume for now that $F(h^{6_3})$ is derived assuming $c = 0$ and that e is coordinated. In $M^{\eta e}$, we say that the history of play is such that, $S|h^3_{s|h^3}$ has $s^{6_1}_{s|h^3}\left(h^{6_1}\right) = e$. In the equivalent![45] history in M'^e, we fix the obvious binding scheme[46] where $s^{6_1}\left(h^{6_1}\right) = e$.

2.3 Endogeneity of $F(h^{6_3})$ as for Agent 2's Endogenous Outside Option

Let any given double proposal match $S|h^{5_2}_{s|h^{5_2}}$ be such that its payoff proposal match $s^{5_2}\left(h^{5_2}\right) = s^{5_2}_{s|h^{5_2}}\left(h^{5_2}\right) = \left(\beta_2^{132}, \beta_3^{132}, \phi_1^{132}\right)$ (where h^{5_2}, or any history from now on, whenever used indistinctly for both games are to be understood as equivalent histories as for 5.4.2) $\beta_2^{132} \in \left[0, \phi_2^{132} + \phi_3^{132}\right]$ and $\phi_3^{132} + \phi_2^{132} = \frac{4d+2b-a}{6} > 0$ (See diagram 2 for Myerson values). Then, the endogenous outside option pair in $(F(h^{6_3}), \psi^{h^{6_3}})$ is $\psi_{12}^{h^{6_3}} = \left(\phi_1^{132}, \beta_2^{132}\right)$. Let us denote $\phi_3^{132} + \phi_2^{132} - \phi_2^N = \frac{2d+a+b-c}{6}$ as $\widetilde{\beta}_3^{132}$ and the associated proposal match by $\widetilde{\beta}^{132}$.

2.4 Trivial Continuity of $\eta(F(h^{6_3}), \psi^{h^{6_3}})$ as a function of $S|h^{5_2}_{s|h^{5_2}}$

As $\widetilde{\beta}_3^{132} > 0$, there exist double proposal matches $S|h^{5_2}_{s|h^{5_2}}$ with proposal matches in h^{5_2} that give agent 2 $\beta_2^{132} > \phi_2^N$. If so, agent 2 would be strictly better off if the complete graph would not form: the bargaining game $(F(h^{6_3}), \psi^{h^{6_3}})$ would be of type 2 (See section 5.5.3). If $S|h^{5_2}_{s|h^{5_2}}$ has $s^{5_2}_{s|h^3}\left(h^{5_2}\right) = \widetilde{\beta}^{132}$in $T''^{\eta e}\left(h^{5_2}\right)$ nested in $M^{\eta e}\left(h^{5_2}\right)$, agent 2 is indifferent between $\widehat{s}^{\eta e}\left(h^{6_2}\right) = e$ or ne (or $\widehat{s}\left(h^{6_2}\right) = e$ or ne if thinking of $M''^e\left(h^{5_2}\right)$) in h^{6_2}. In M'^e, there is an equilibrium selection problem with a *degenerate* bilateral conflict of interest as in this case everyone would get the same payoff independently of link 12 forming or not.[47]

As $\phi_2^{132} + \phi_3^{132} \geq \widetilde{\beta}_3^{132} > 0$ and $\beta_2^{132} \in \left[0, \phi_2^{132} + \phi_3^{132}\right]$, which is a compact set, there exists in the preceding strategic form game $T''^{\eta e}\left(h^{5_2}\right)$ nested in $M^{\eta e}\left(h^{5_2}\right)$ two types of $S|h^{5_2}_{s'|h^{5_2}}$ and $S|h^{5_2}_{s''|h^{5_2}}$,and two respective subsets of $S|h^{5_2}/\equiv$. These types of $S|h^{5_2}_{s|h^{5_2}}$ are such that $e = s^{6_2}_{s'|h^{5_2}}\left(h^{6_2}\right) \neq s^{6_2}_{s''|h^{5_2}}\left(h^{6_2}\right) = ne$. Also, $s^{5_2}_{s'|h^{5_2}}\left(h^{5_2}\right)$ has $\beta_2^{132} \leq \phi_2^N$ while $s^{5_2}_{s''|h^{5_2}2}\left(h^{5_2}\right)$ has $\beta_2^{132} \geq \phi_2^N$. Both induce[48] two types of $\widehat{s}^{\eta e}|h^{5_2}$ with the CBCBP in $M^{\eta e}\left(h^{6_1}\right)$ (and two types of $\widehat{s}|h^{5_2}$ with the CBCBP in $M''^e\left(h^{6_1}\right)$)

[45] Note how given $S|h^{5_2}_{s|h^{5_2}}$ and the associated last proposal match $s^{5_2}_{s|h^{5_2}}\left(h^{5_2}\right) = \beta^{132}$ histories in both games are equivalent as of h^{6_3} in the sense defined in 5.4.2.

[46] We leave to the reader the case of $s^{6_1}_{s|h^3}\left(h^{6_1}\right) = ne$ being coordinated and how this will influence the construction of the PF, and thus play by agents 2 and 3 in $h^{5_2}(g^{13+32}, S|h^3_{s|h^3})$. Or in a parallel way, if we fix future play by setting $s\left(h^{6_1}\right) = ne$ and move backwards to history subclass $h^{5_2}(g^{13+32}, s^3\left(h^3\right))$ and construct an analogous P.

[47] Using 5.3.2, $s^{5_2}(h^{5_2}) = \widetilde{\beta}^{132}$ induces 2 subgame perfect strategy profiles $s^{5_2}|h^{5_2}$ that depend on fixing $s^{6_2}\left(h^{6_2}\right) = e$ or ne. In an ad hoc way, we build the analogous of P in the brute force model in the equivalent history $h^{5_2}(g^{13+32}, s\left(h^3\right))$, by including the two latter credible expected payoff outcomes that actually coincide in this case. This case is crucial for the possibility of the complete graph forming (See companion paper).

[48] Also, both induce two types of restriction profiles $\widehat{s}|h^{5_2}$in M''^e where the future bargaining rule

such that $\widehat{s}'^{\eta e}(h^{6_1}) = \widehat{s}''^{\eta e}(h^{6_1}) = e$ (and $\widehat{s}'(h^{6_1}) = \widehat{s}''(h^{6_1}) = e$ and so on in what follows) but $e = \widehat{s}'^{\eta e}(h^{6_2}) \neq \widehat{s}''^{\eta e}(h^{6_2}) = ne$. Also[49], $(1,1,1) = \widehat{s}'^{\eta e}(h^{6_3}) \neq \widehat{s}''^{\eta e}(h^{6_3}) = (0,0,0)$, where $\widehat{s}'^{\eta e}(h^{6_3})$ is a proposal match and $\widehat{s}''^{\eta e}(h^{6_3})$ not. There is a third trivial type but it is payoff irrelevant (See 4.3 below). It follows that there are two types of non trivial convergent sequences. Formally:

$\exists\{S|h^{5_2}_{s'_n|h^{5_2}}\}$ type 1 sequences such that $s^{5_2}_{s'_n|h^{5_2}}(h^{5_2})_2 \to s^{5_2}_{\overline{s}'|h^{5_2}}(h^{5_2})_2$ as $n \to \infty$, where $s^{5_2}_{s'_n|h^{5_2}}(h^{5_2})$ has $\beta_2^{132} \leq \overline{\beta}_2^{132} \leq \phi_2^N$. Hence, $S|h^{5_2}_{s'_n|h^{5_2}} \to S|h^{5_2}_{\overline{s}'|h^{5_2}}$ as $n \to \infty$. Recall other components in the vectors $S|h^{5_2}_{s'_n|h^{5_2}}$ are fixed as representing a given future bilateral coordination scheme!!! We want to show that $\eta'(F(h^{6_3}), \psi^{h^{6_3}}) \to \bar{\eta}'(F(h^{6_3}), \psi^{h^{6_3}}) = \phi^N$as $n \to \infty$, where η' is the *restriction* of η to type 1 subset $S'|h^{5_2}/\equiv$.

$\exists\{S|h^{5_2}_{s''_n|h^{5_2}}\}$ type 2 sequences such that $s^{5_2}_{s''_n|h^{5_2}}(h^{5_2})_2 \to s^{5_2}_{\overline{s}''_n|h^{5_2}}(h^{5_2})_2$ as $n \to \infty$, where $s^{5_2}_{s''_n|h^{5_2}}(h^{5_2})$ has $\beta_2^{132} \geq \overline{\beta}_2^{132} \geq \phi_2^N$. Hence $S|h^{5_2}_{s''_n|h^{5_2}} \to \overline{S}|h^{5_2}_{\overline{s}''|h^{5_2}}$ as $n \to \infty$. We want to show that $\eta''(F(h^{6_3}), \psi^{h^{6_3}}) \to \bar{\eta}''(F(h^{6_3}), \psi^{h^{6_3}}) = \overline{\beta}^{132} = \left(\overline{\beta}_3^{132}, \overline{\beta}_2^{132}, \phi_1^{132}\right)$as $n \to \infty$, where η'' is the restriction of η to type 2 subset $S''|h^{5_2}/\equiv$.

The singleton P set is independent of any $S|h^{5_2}_{s|h^{5_2}}$. Recall that $\psi^{h^{6_3}} = \left(\phi_1^{132}, \beta_2^{132}, \beta_3^{132}\right)$. Because the identity function is continuous so is $\psi^{h^{6_3}}$ in $S|h^{5_2}/\equiv$ thus, the restriction $\psi'^{h^{6_3}}_{12}$ is continuous on $S'|h^{5_2}/\equiv$. Note that $\eta_{12}(F(h^{6_3}), \psi^{h^{6_3}}) = \phi_{12}^N$ (constant function) whenever defined on the image of $\psi'^{h^{6_3}}_{12}$. Thus, as a composite restriction, $\eta'_{12}(F(h^{6_3}), \psi^{h^{6_3}})$ is continuous on $S'|h^{5_2}/\equiv$. In turn, $\eta_3\left(F(h^{6_3}), \psi^{h^{6_3}}\right) = \phi_3^N$ (a constant as for section 5.6)) whenever defined on the image of $\psi'^{h^{6_3}}_{12}$, thus $\eta'(F(h^{6_3}), \psi^{h^{6_3}})$ is continuous in $S'|h^{5_2}/\equiv$. On the other hand, $\eta(F(h^{6_3}), \psi^{h^{6_3}}) = \psi^{h^{6_3}}$ (the identity function), when defined on the image of $\psi''^{h^{6_3}}_{12}$. In an analogous way, $\eta''(F(h^{6_3}), \psi^{h^{6_3}})$ is continuous in $S''|h^{5_2}/\equiv$. Thus, as a composite function (See first footnote in 4.4 below), $\eta(F(h^{6_3}), \psi^{h^{6_3}})$ is continuous in $S|h^{5_2}/\equiv$.

3. "Partial" Construction and Equivalence between $M^{\eta e}$ and M'^{e} :

As for the previous results, it is clear that the two types of induced $\widehat{s}|h^{5_2}$ have as unique payoff equivalent consistently induced $s^{5_2}|h^{5_2}$ where $s^{5_2}(h^{5_2}) = s^{5_2}_{s|h^{5_2}}(h^{5_2})$ (recall histories are equivalent in the sense of 5.4.2). So wlg.[50], we proceed only

is set to η.

[49]Recall $f \equiv 1$ and $r \equiv 0$, when the complete graph is prospective. See section 5.4.1 and 4.2.3.

[50]In particular, if $s^{5_2}_{s|h^{5_2}}\left(h^{5_2}\right) = \widetilde{\beta}^{132}$ then the fixed future behavior, set at $s^{6_2}\left(h^{6_2}\right) = e$ or ne corresponds uniquely to the credible bilateral coordination scheme of the induced $\widehat{s}'^{\eta e}|h^{5_2}$ and $\widehat{s}''^{\eta e}|h^{5_2}$ just derived. Thus, provided $s^{6_2}\left(h^{6_2}\right) = e$, $v\left(s^{5_2}|h^{5_2}\right) = \widehat{v}\left(\widehat{s}'^{\eta e}|h^{5_2}\right)$.

The converse is also true. We would just need to analyze not credible future bilateral coordination schemes in equilibrium. In particular, let $S|h^{5_2}_{s'''|h^{5_2}}$ be such that $s^{5_2}_{s'''|h^{5_2}}\left(h^{5_2}\right)$ has $\beta_2^{132} < \phi_2^N$ and $s^{6_2}_{s'''|h^{5_2}}\left(h^{6_2}\right) = ne$. It is not a subgame perfect equilibrium for agent 2 to follow such a reminder. Thus, the payoffs of the induced $\widehat{s}'''^{\eta e}|h^{5_2}$ are identical to the ones of $\widehat{s}'^{\eta e}|h^{5_2}$ where $s^{5_2}_{s'''|h^{5_2}}\left(h^{5_2}\right) =$

working with $M^{\eta e}$ (See footnote 41).

4. The Well Defined Bargaining Game in h^{5_2}

4.1. Deriving the P Set in $(F(h^{5_2}),\psi^{h^{5_2}})$ and its Independence from $S|h^3_{s|h^3}$

4.1.1. Case $c=0$

It is clear from 2.4 above and section 5.5.3 that the closure condition in h^{5_2} holds provided $S|h^3_{s|h^3}$ is such that $s^{6_1}_{s|h^3}(h^{6_1})=e$ as $\widehat{v}(\widehat{s}|h^{5_2})$ in M''^{e} that equals $\eta(F(h^{6_3}),\psi^{h^{6_3}})$ are both continuous composite functions in $S|h^{5_2}/\equiv$. It is "easy" to see that the set P $(=PF)$[51] in $(F(h^{5_2}),\psi^{h^{5_2}})$ with set of feasible proposal matches such that $0\leq\beta_3^{132}\leq\phi_3^{132}+\phi_2^{132}$ is given by the segment connecting and including $\left(0,\phi_3^{132}+\phi_2^{132}\right)$ and $\widetilde{\beta}_{32}^{132}$ in the plane $\left(\beta_3^{132},\beta_2^{132}\right)$, where $\widetilde{\beta}_3^{132}>0$ and $\phi_2^N=\frac{b+2(d-a)+c}{6}\geq 0$. Thus, for the parameter values under analysis (including $c>0$), all possible P[52] have an associated $\widetilde{\beta}_{32}^{132}$ that lies either on the interior of that segment or on its intersection with the horizontal axis at $\left(\phi_3^{132}+\phi_2^{132},0\right)$! Note that the slope of the PF is negative 1. The NTU NBR payoff pair $\eta_{32}(F(h^{5_2}),\psi^{h^{5_2}})$ in $(F(h^{5_2}),\psi^{h^{5_2}})$ is a continuous function in the image of the TU NBR pair $\eta_{32}^{TU}(F(h^{5_2}),\psi^{h^{5_2}})$ under some set of non negative P feasible outside options.[53] In turn, the TU NBR is continuous in that set. Hence, as a composite function $\eta_{32}(F(h^{5_2}),\psi^{h^{5_2}})$ is also continuous in the same set. This claim will be used below.

4.1.2. Case $c>0$: The Necessity of Double Proposals

From the sequence analysis of 2.4 and the payoff equivalence results in 5.5.3, if $s^{5_2}_{\overline{s}''_n|h^{5_2}}(h^{5_2})$ has $\beta_2^{132}=\phi_2^N\Rightarrow v(\overline{s}''|h^{5_2})=v(\overline{s}'|h^{5_2})$, but this it is not necessarily the case! If instead $\phi_1^{132}<\phi_1^N\Leftrightarrow 0<c$ then there is no credible coordination

---

$s^{5_2}_{s'|h^{5_2}}(h^{5_2})$ and instead $s^{6_2}_{s'|h^{5_2}}(h^{6_2})=e$! Hence, we have disregarded the analysis of not credible future bilateral coordination schemes in equilibrium and their induced restriction profiles as both will not have any effect on the derivation of the P set in the histories associated with $h^{5_2}(g^{13+32})$ in both games.

So far, we would have shown, in particular, that the credible expected payoffs of any equivalent pair of action profiles in the following two histories, corresponding two different multistage games yield identical results: in $M^{\eta e}$ we would say that we are in history $h^{5_2}(g^{13+32},S|h^3_{s|h^3})$ where agents 13, as of earlier link discussions, coordinated $S|h^3_{s|h^3}$ such that agent 1 would enter in any of the subsequent histories to $h^{5_2},h^{6_1}\left(g^{132+12},S|h^3_{s|h^3},S|h^{5_2}_{s|h^{5_2}}\right)$. In M'^{e}, we would just say that we are in history $h^{5_2}(g^{13+32},s\left(h^3\right))$ where we have fixed future behavior at $s^{6_1}\left(h^{6_1}\right)=e$ arbitrarily. In some sense, agents 2 and 3 think of choosing $s^3\left(h^3\right)$ taking as fixed future actions at h^{6_1}.

[51] In lemma PF in Appendix A of the companion paper, we get this result too for the brute force model .

[52] Of course this is not the case if ne instead coordinated in $h^{6_1}(g^{132+12})$. In that case the PF is identical to the set of proposal matches.

[53] To see this, fix $\widetilde{\beta}_{32}^{132}$. The TU NBR solution lies on the segment from $\left(0,\phi_3^{132}+\phi_2^{132}\right)$ to $\left(\phi_3^{132}+\phi_2^{132},0\right))$. The PF is always a fixed (segment) subset, in most cases a proper subset of the TU frontier. Thus the NTU NBR is either the identity, and thus equals the TU NBR, or the constant $\widetilde{\beta}_{32}^{132}$. Moreover the NTU is continuous at $\widetilde{\beta}_{32}^{132}$. See companion paper for different parameter cases in lemma 1.

between agent 3 and 1 on agent 1 playing ne in h^{6_1}. Agent 1 will always play e in equilibrium! If $s^{5_2}_{\overline{s}''_n|h^{5_2}}(h^{5_2})$ has $\beta_2^{132}=\phi_2^N \Rightarrow v_{32}(\overline{s}''|h^{5_2}) \neq v_{32}(\overline{s}'|h^{5_2})$, more precisely $v_{32}(\overline{s}''|h^{5_2}) \geq v_{32}(\overline{s}'|h^{5_2})$ as even though agent 2 would break even, agent 3 would be strictly worse off by agent 2 entering in h^{6_2}. Of course agent 1 would be better off (dynamic bilateral conflicts of interest).

This might look like we have a correspondence and even a discontinuity if we assume a cutoff rule and assume that agent 2 say would always enter as when proposal matches tend to the same proposal match, that is when $s^{5_2}_{s''_n|h^{5_2}}(h^{5_2}) \to s^{5_2}_{\overline{s}''_n|h^{5_2}}(h^{5_2})$ and $s^{5_2}_{s'_n|h^{5_2}}(h^{5_2}) \to s^{5_2}_{\overline{s}'_n|h^{5_2}}(h^{5_2})$ as $n \to \infty$, the induced payoffs for agents 2 and 3 differ. However, this is not the correct argument in $M^{\eta e}$. In $M^{\eta e}$, bilateral coordination of actions eliminates multiple equilibria or discontinuity associated with such a given proposal match in M'^{e}. The same can be done if we distinguish proposal matches according to fix future binding schemes (as we do in the brute force model). Thus, instead of a correspondence in M'^{e}, we get in $M^{\eta e}$ a function that depends on both proposal matches and future bilateral coordination schemes.

From the discussion above and repeating the sequence analysis as in 2.4 above, it follows that if $c>0$, the P $(=PF)$ set in $(F(h^{5_2}),\psi^{h^{5_2}})$ is closed, this time *unconditionally* as it does not depend on any history of play $S|h^3_{s|h^3}$ leading to h^{5_2}.

In both cases, $c>0$ and $c=0$ (in the latter case, provided e is coordinated in $h^{6_1}(g^{132+12})$) P is independent of any sequence of convergent $S|h^3_{s_n|h^3}$. Loosely speaking, once we take as given a credible bilateral coordination scheme, P can be thought as being fixed when agents 1 and 3 are coordinating in h^3 "along" a given convergent sequence.

4.2. Endogenizing $F(h^{5_2})$ with Endogenous Outside Options $\psi^{h^{5_2}}$

For completing the derivation of the feasible set of $(F(h^{5_2}),\psi^{h^{5_2}})$ in $h^{5_2}(g^{13+32},S|h^3_{s|h^3})$, we derive endogenously $\psi^{h^{5_2}}$ from the last proposal match in the last double proposal match $S|h^3_{s|h^3}$, i.e.,$\psi^{h^{5_2}}=s^3_{s_n|h^3}(h^3)=\left(\beta_1^{13},\beta_3^{13},\phi_2^{13}=0\right)$. Recall, that we are assuming $c=0$ and agents 1 and 3 coordinate on equivalent classes $S|h^3_{s_n|h^3}$ in $T''^{\eta e}(h^3)$ such that $s^{6_1}_{s_n|h^3}(h^{6_1})=e$ in $h^{6_1}(g^{132+12},S|h^3_{s|h^3},S|h^{5_2}_{s|h^{5_2}})$ and are "still" wondering what would be the whole effect on this way of coordinating, in particular, by "fine tuning" outside options pair $\psi^{h^{5_2}}_{32}$ in the earlier bargaining game in h^{5_2}.

4.3. Types of $S|h^3_{s|h^3}$, $\psi^{h^{5_2}}$ and Bargaining Games

As $\widetilde{\beta}_3^{132}>0$, the PF of $(F(h^{5_2}),\psi^{h^{5_2}})$ (in the plane $\left(\beta_3^{132},\beta_2^{132}\right)$) in $h^{5_2}(g^{13+32},S|h^3_{s|h^3})$ has at least one payoff pair that gives agent 3 a positive payoff. Thus, there are two possibilities:

Either $s^3_{s_n|h^3}(h^3)$ induces $\psi^{h^{5_2}}_{32}=\left(\beta_3^{13},0\right)$ that maybe P feasible or not. As $\beta_1^{13}+\beta_3^{13}=\phi_1^{13}+\phi_3^{13}=a$, there may be (in case a is big enough) a threshold $\widetilde{\psi}_3^{h^{5_2}}(a)$ such that if $s^3_{s_n|h^3}(h^3)$ has $\beta_3^{13}=\widetilde{\psi}_3^{h^{5_2}}(a)$, agent 3 will be indifferent between forming or not link 23. It is clear that $\widetilde{\psi}_3^{h^{5_2}}(a)=\widetilde{\beta}_3^{132}$. If so, agents 1 and 3 may wonder about the induced effect on coordinating on $S|h^3_{s|h^3}$ such that agent 3 plays $s^{5_1}_{s_n|h^3}(h^{5_1}) = e$ or ne in $h^{5_1}(g^{13+32}, S|h^3_{s|h^3})$. If $s^3_{s_n|h^3}(h^3)$ is such that $\beta^{13}_3 > \widetilde{\beta}^{132}_3$, agent 3 would be strictly better off if link 23 does not form. If instead $\beta^{13}_3 < \widetilde{\beta}^{132}_3$, agent 3 would be strictly better off if link 23 forms.

Three implied types (and also subsets of $S|h^3/\equiv$) of double proposal matches $S|h^3_{s|h^3}$ induce two types of *payoff relevant* bargaining games:

Type $S|h^3_{s'|h^3}$ Inducing Solutions to Bargaining Games of Type 1: In this type 1 (See 5.5.3.), $s^3_{s'|h^3}(h^3)_3 \leq \widetilde{\beta}^{132}_3$. There is credible coordination as in equilibrium, $\widehat{s}'^{\eta e}(h^{5_1}) = s^{5_1}_{s'|h^3}(h^{5_1}) = e$, $\widehat{s}'^{\eta e}(h^{6_1}) = s^{6_1}_{s'|h^3}(h^{6_1}) = e$. Also, $\widehat{s}'^{\eta e}(h^{5_2}) = S|h^{5_2}_{s|h^{5_2}}$ is such that $s^{5_2}_{s|h^{5_2}}(h^{5_2})$ is a proposal match and consistent with the NTU NBR applied to $(F(h^{5_2}), \psi^{h^{5_2}})$. We refer to 4.1 for completing history contingent construction of associated induced $\widehat{s}'^{\eta e}|h^{5_2}$ that are restrictions of $\widehat{s}'^{\eta e}|h^3$ induced by $S|h^3_{s'|h^3}$.

Type $S|h^3_{s''|h^3}$ Inducing Solutions to Bargaining Games of Type 2: In this type, $s^3_{s''|h^3}(h^3)_3 \geq \widetilde{\beta}^{132}_3$. There is credible coordination as in equilibrium, $\widehat{s}''^{\eta e}(h^{5_1}) = s^{5_1}_{s''|h^3}(h^{5_1}) = ne$, $\widehat{s}''^{\eta e}(h^{6_1}) = s^{6_1}_{s''|h^3}(h^{6_1}) = e$. Also, $\widehat{s}''^{\eta e}(h^{5_2}) = S|h^{5_2}_{s|h^{5_2}}$ is such that $s^{5_2}_{s|h^{5_2}}(h^{5_2})$ is not a proposal match as we would have type 2 matrix.[54]

Payoff Irrelevant Trivial Type $S|h^3_{s'''|h^3}$: In this type, $s^3_{s'''|h^3}(h^3)_3 = \widetilde{\beta}^{132}_3$. Also $\widehat{s}'''^{\eta e}(h^{5_1}) = s^{5_1}_{s'''|h^3}(h^{5_1}) = ne$, $\widehat{s}'''^{\eta e}(h^{6_1}) = s^{6_1}_{s'''|h^3}(h^{6_1}) = e$. Also, double proposal match $\widehat{s}'''^{e}(h^{5_2}) = S|h^{5_2}_{s'''|h^{5_2}}$ where $s^{5_2}_{s|'''h^{5_2}}(h^{5_2})$ is a proposal match and consistent with the NTU NBR used in h^{5_2}. This is a one element type. It is even *payoff irrelevant*, so we disregard its further analysis.

4.4. Continuity of $\eta(F(h^{5_2}), \psi^{h^{5_2}})$ as a function of $S|h^3_{s|h^3}$

Note that $s^3_{s'|h^3}(h^3)$ has $\beta^{132}_3 \in \left[0, \widetilde{\beta}^{132}_3\right]$ and $s^3_{s''|h^3}(h^3)$ has $\beta^{132}_3 \in \left[\widetilde{\beta}^{132}_3, \phi^{132}_2 + \phi^{132}_3\right]$ where both intervals are closed, actually compact sets. Thus, from 4.3., there exists two non trivial (different than the constant sequence) types of convergent sequences of $S|h^3_{s_n|h^3}$that are relevant for checking the continuity of $\eta(F(h^{5_2}), \psi^{h^{5_2}})$ in $S|h^3/\equiv$[55]. It is crucial to have in mind that the range of that function is in $\mathbb{R}^3$! We care about the triplet because in the before to the preceding history, $\eta_{12}(F(h^{5_2}), \psi^{h^{5_2}})$ is taken as the outside option pair (See 5). Formally:

Type 1: $\exists\{S|h^3_{s'_n|h^3}\}$ such that $s^3_{s'_n|h^3}(h^3) \to s^3_{\overline{s}'|h^3}(h^3)$ (the latter has $\beta^{13}_3 = \overline{\beta}^{13}_3$) as $n \to \infty$, where $\overline{\beta}^{13}_3 \leq \widetilde{\beta}^{132}_3$. Thus, $S|h^3_{s'_n|h^3} \to S'|h^3_{\overline{s}'|h^3}$ as $n \to \infty$.

[54] The out of equilibrium path analysis in history h^{5_2} , say if $\widehat{s}''^{\eta e}\left(h^{5_2}\right)$ is instead a double proposal match would be the one implied by the smooth game. In any case, this analysis is irrelevant for the proof..

[55] This is a composite function and it is defined as $\eta\left(F(h^{5_2}), \psi^{h^{5_2}}\left(S|h^3_{s|h^3}\right)\right) : \mathbb{R}^{3\left(\#h^{k_m}\right)} \to \mathbb{R}^3$, where $\eta(F(h^{5_2}), \psi^{h^{5_2}}) : \mathbb{R}^3 \to \mathbb{R}^3$, and $\psi^{h^{5_2}} : \mathbb{R}^{3\left(\#h^{k_m}\right)} \to \mathbb{R}^3$.

Let $\eta_n'(F(h^{5_2}),\psi^{h^{5_2}})$ be such that $h^{5_2}=h^{5_2}(g^{13+32},S|h^3_{s_n'|h^3})$ and $\overline{\eta}'(F(h^{5_2}),\psi^{h^{5_2}})$ be such that $h^{5_2}=h^{5_2}(g^{13+32},S|h^3_{\overline{s}'|h^3})$. From 4.2, it is clear that the restriction $\psi'^{h^{5_2}}$ and thus $\psi_{32}'^{h^{5_2}}$ is continuous in $S'|h^3/\equiv$. As the TU NBR pair $\eta_{32}^{TU}(F(h^{5_2}),\psi^{h^{5_2}})$ is a continuous function in the image of outside option pair $\psi_{32}'^{h^{5_2}}$ and hence of $\psi'^{h^{5_2}}$ trivially, it follows that the composite restriction $\eta_{32}'^{TU}(F(h^{5_2}),\psi^{h^{5_2}})$ is continuous in $S'|h^3/\equiv$. Hence, $\eta_{32}'(F(h^{5_2}),\psi^{h^{5_2}})$ is a continuous function in the image of $\psi'^{h^{5_2}}$ and so is $\eta'(F(h^{5_2}),\psi^{h^{5_2}})$ (ϕ_1^{132} is constant!). (See 4.1.1 above). Hence, $\eta_n'(F(h^{5_2}),\psi^{h^{5_2}})\to\overline{\eta}'(F(h^{5_2}),\psi^{h^{5_2}})=\left(\overline{\beta}_3^{132},\overline{\beta}_2^{132},\phi_1^{132}\right)$ as $n\to\infty$

Type 2: Analogously $\eta_n''(F(h^{5_2}),\psi^{h^{5_2}})\to\overline{\eta}''(F(h^{5_2}),\psi^{h^{5_2}})=\left(\overline{\beta}_1^{13},\phi_2^{13}=0,\overline{\beta}_3^{13}\right)$as $n\to\infty$ with the obvious modifications in the notation.

It follows that the composite function $\eta(F(h^{5_2}),\psi^{h^{5_2}})$ in $h^{5_2}(g^{13+32},S|h^3_{s|h^3})$ is continuous in $S|h^3/\equiv$.

5. Earlier Histories: The "Before the Preceding Bargaining Game"

We move backwards to $(F(h^{4_2}),\psi^{h^{4_2}})$ in $h^{4_2}(g^{31+12},S|h^3_{s|h^3})$ where link 12 is being proposed. Assume that history of play has the future bilateral coordination scheme for $S|h^3_{s|h^3}$ just analyzed. The reader may set agent 3's play, that is, playing either e or ne in histories $h^{5_1}(g^{13+32},S|h^3_{s|h^3})$. Recall agents 1 and 3 are wondering about the effects of a given future bilateral coordination scheme in even earlier stages.

As above, depending on how agents 1 and 3 coordinate as of $h^3(g^{\emptyset+13},\beta^{\emptyset})$ in cases of indifference of agent 3 in $h^{5_1}(g^{312+32},S|h^3_{s|h^3},S|h^{4_2}_{s|h^{4_2}})$, agents 2 and 1 will play $(F(h^{4_2}),\psi^{h^{4_2}})$ conditionally! so that P in $(F(h^{4_2}),\psi^{h^{4_2}})$ can be derived again as in 4.1.It is assumed again that they coordinate on choosing $s^{5_1}_{s|h^3}(h^{5_1})=e$ rather than ne now whenever $b=0$!

From 4.4, the outside options $\psi_{12}^{h^{4_2}}$ are continuous as a function[56] of $S|h^3_{s|h^3}$ as[57] $\psi^{h^{4_2}}=\eta(F(h^{5_2}),\psi^{h^{5_2}})$. Wlg., let us have the composite restriction $\eta'(F(h^{5_2}),\psi^{h^{5_2}})$, recall $S|h^3_{s'|h^3}$ is such that $s^3_{s'|h^3}(h^3)=\beta_3^{132}\in\left[0,\widetilde{\beta}_3^{132}\right]$. There is a possibility that at $h^{4_2}(g^{31+12},S|h^3_{s|h^3})$, $S|h^3_{s'|h^3}$ is such that at $\eta_3'(F(h^{5_2}),\psi^{h^{5_2}})=\widetilde{\psi}_3^{h^{4_2}}(S|h^3_{s'|h^3})=\widetilde{\widetilde{\beta}}_3^{132}\in\left[0,\widetilde{\beta}_3^{132}\right]$, agent 1 is indifferent between linking or not with agent 2 (see lemma 1 in companion paper for different cases). Agents 1 and 3 will wonder then about coordinating on agent 1 playing $s^{4_1}_{s|h^3}(h^{4_1})=e$ or ne in $h^{4_1}(g^{31+12},S|h^3_{s|h^3})$. Given e is coordinated $h^{6_1}(g^{132+12},S|h^3_{s|h^3},S|h^{5_2}_{s|h^{5_2}})$ and now additionally letting e (assuming say $b=0$) being coordinated in $h^{5_1}(g^{312+32},S|h^3_{s|h^3},S|h^{4_2}_{s|h^{4_2}})$, we distinguish sub types of convergent sequences among $S|h^3_{s_n'|h^3}$ and so on...

[56]Recall that the range of $\psi_{12}^{h^{4_2}}$ is in $\mathbb{R}^2$ and that of $\eta(F(h^{5_2}),\psi^{h^{5_2}})$ is in $\mathbb{R}^3$! See related remark in 4.4 above.

[57]Also $\psi^{h^{4_2}}=\widehat{v}\left(\widehat{s}|h^{5_1}\right)$, where $\widehat{s}|h^{5_1}$is a restriction of $\widehat{s}|h^{4_2}$ such that $\widehat{s}\left(h^{4_2}\right)$ is not a double proposal match

6. Checking closure condition in $(F(h^3), \psi^{h^3})$

The constructed subtypes of $\widehat{s}_n|h^3$ and $\{s_n|h^3\}$ can be used to check successfully the continuity of another composite function $\eta(F(h^{4_2}), \psi^{h^{4_2}})$ in $S|h^3/\equiv$. Where $S|h^3/\equiv$ can be decomposed in subtypes subsets of double proposal matches that have payoff matches in either $\left[0, \overset{\approx 132}{\beta_3}\right]$ or $\left[\overset{\approx 132}{\beta_3}, \overset{\sim 132}{\beta_3}\right]$ which are compact subsets (following case in 5) of $\left[0, \overset{\sim 132}{\beta_3}\right]$. The other possible sub types of $\{S|h^3_{s_n|h^3}\}$ are derived analogously. As compositions of continuous function are continuous, then by using the results on $\eta(F(h^{4_2}), \psi^{h^{4_2}})$ the closure condition holds in h^3. Equivalently, $T''^{\eta e}(h^3)$ nested in $M^{\eta e}(h^3)$ has the uniqueness property and F is well defined.

7. Existence and Uniqueness in $M^{\eta e}_{\phi^g(v)}$ **and** $M'^{e}_{\phi^g(v)}$.

The proof is completed by simple backward induction following theorem S1 once three P sets are derived as we can get payoff solutions for bargaining games histories $h^1(g^{\emptyset+12}, \beta^{\emptyset})$ and $h^2(g^{\emptyset+23}, \beta^{\emptyset})$ and $h^3(g^{\emptyset+13}, \beta^{\emptyset})$ respectively with arbitrary outside options■

Corollary s1:A unique payoff solution exists for three-agent normalized games with any valuation function and any fixed payoff allocation rule. It is the solution to $M^{\eta e}$, the ANC model.

Proof: In the proof, we only used the continuity of the key composite function $\eta\left(\eta^{TU}(F(h^k), \psi^{h^k})\right)$defined in compact sets that are fixed because the valuation functions and payoff allocation rules are fixed.

7 Analytical Results

7.1 Results for Normalized Superadditive Games

Theorem S.1*:* A unique payoff exists for all normalized games. If the solution predicts different graph structures, then these are payoff equivalent.

Proof:

Without loss of generality, let us say that the rule of order is $(1,2)$, $(2,3)$ and $(1,3)$, the first two pairs rejected. Theorem S.1 follows as from lemmas, corollaries and propositions 1 to 12 in companion paper, we have unique payoff solutions for bargaining games histories $h^1(g^{\emptyset+12}, \beta^{\emptyset})$ and $h^2(g^{\emptyset+23}, \beta^{\emptyset})$ and $h^3(g^{\emptyset+13}, \beta^{\emptyset})$ with arbitrary outside options.

Given that the outside options are zero for agents 1 and 3, link 13 will always form[58] with expected payoffs given by $\eta(F(h^3), \psi^{h^3})$, where $h^3 = h^3(g^{\emptyset+13}, \beta^{\emptyset})$.In turn the latter NTU NBR payoffs are outside options for agents 2 and 3 in the bargaining game one stage earlier in $h^2 = h^2(g^{\emptyset+23}, \beta^{\emptyset}), i.e.$

$\psi^{h^2} = \eta(F(h^3), \psi^{h^3})$.

[58]That would be the case even if $\eta_{13}(F(h^3), \psi^{h^3}) = 0$.

Given that the bargaining game at h^2 is also well defined payoffs will be given by $\eta(F(h^2), \psi^{h2})$. Link 23 might form or not. Finally, the latter NTU NBR payoffs are outside options for agents 1 and 2 in the bargaining game one stage earlier in $h^1 = h^1(g^{\emptyset+12}, \beta^{\emptyset})$, *i.e.*

$\psi^{h^1} = \eta(F(h^2), \psi^{h^2})$.

Given that the bargaining game is also well defined at the initial history $h^1(g^{\emptyset+12}, \beta^{\emptyset})$, $\Re(v, \rho_{\emptyset}) = \left(\eta(F(h^1), \psi^{h^1})\right)$ exists and it is unique because the NTU NBR is unique.■

Theorem S.2: If a first link is being discussed, a necessary condition for it to be the last link to form is that the value of the two discussants is equal to what the grand coalition can achieve.

Proof:

It follows from Corollary 2 and proposition 3, propositions 6, 9 statement 2. and proposition 12 in the companion paper■

Theorem S.3: If a first link is being discussed, a necessary condition for the complete graph to form after the latter link forms is that the value of the two last pair of discussants thereafter, according to the rule of order, is equal to zero.

Proof:

It follows from lemmas, corollaries and propositions 1 to 12 in the companion paper. See irrelevant parameter cases■

7.2 Results for Normalized Strictly Superadditive Games

Theorem T.1: Only two link graphs form. If a first link forms, then the next link in the rule of order is the last to form.

Proof:

It follows form theorem S.2 and S.3■

7.3 Results for Normalized Mixed Superadditive Games

Theorem M.2: Let $d \geq a, b, c \geq 0$. Let h^k be the history where a one-link graph,wlg link 13 (also wlg 12, 23 are discussed next in that order) is the first prospective graph. If outside options are P feasible, $d = a$ and $b \neq 0$ and $c \neq 0$, then link 13 is the first and last link to form. Payoffs are $\eta(F(h^3), \psi^{h^3}) = \left(\widetilde{\beta}_1^{312}, 0, \widetilde{\beta}_3^{132}\right) = \left(\frac{2d+a+c-b}{6}, 0, \frac{2d+a+b-c}{6}\right)$ and the payoff proposal match consistent with the Nash bargaining solution is

$s\left(h^3\right) = \left(\beta_1^{13}, \beta_3^{13}\right) = \left(\widetilde{\beta}_1^{312}, \widetilde{\beta}_3^{132}\right) = \left(\frac{2d+a+c-b}{6}, \frac{2d+a+b-c}{6}\right) = \beta_1^{13} + \beta_3^{13} = \phi_1^{13} + \phi_3^{13} = a$.

Proof:

It follows from corollary 2a in the companion paper■

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