Optimal Auctions with Ambiguity

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Abstract

A crucial assumption in the optimal auction literature has been that each bidder's valuation is known to be drawn from a single unique distribution. In this paper we relax this assumption and study the optimal auction problem when there is ambiguity about the distribution from which these valuations are drawn and where the seller or the bidder may display ambiguity aversion. We model ambiguity aversion using the maxmin expected utility model where an agent evaluates an action on the basis of the minimum expected utility over the set of priors, and then chooses the best action amongst them. We first consider the case where the bidders are ambiguity averse (and the seller is ambiguity neutral). Our first result shows that the optimal incentive compatible and individually rational mechanism must be such that for each type of bidder the minimum expected utility is attained by using the seller's prior. Using this result we show that an auction that provides full insurance to all types of bidders is always in the set of optimal auctions. In particular, when the bidders' set of priors is the ϵ- contamination of the seller's prior the *unique* optimal auction provides full insurance to bidders of all types. We also show that in general, many classical auctions, including first and second price are not the optimal mechanism (even with suitably chosen reserve prices). We next consider the case when the seller is ambiguity averse (and the bidders are ambiguity neutral). Now, the optimal auction involves the seller being perfectly insured. Hence, as long as bidders are risk and ambiguity neutral, ambiguity aversion on the part of the seller seems to play a similar role to that of risk aversion.

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1 Introduction

Optimal auctions for an indivisible object with risk neutral buyers and independently distributed valuations have been studied by, among others, Vickrey [20], Myerson [16], Harris and Raviv [7], and Riley and Samuelson [18]. These papers show that the set of optimal mechanisms or auctions is quite large. The set contains both the first and second price auctions with reserve prices. One of the assumptions in this literature is that each bidder's valuation is drawn from a unique distribution. In this paper we relax this assumption.

The unique prior assumption is based on the subjective expected utility model, which has been criticized among others by Ellsberg [3]. Ellsberg shows that lack of knowledge about the distribution over states can effect choices in a fundamental way that can not be captured within the subjective expected utility framework. In one version of Ellsberg's experiment, a decision maker is offered two urns, one that has 50 black and 50 red balls, and one that has 100 black and red balls in unknown proportions. Faced with these two urns, most decision makers bet on drawing either color from the first urn, rather than on drawing the same color from the second urn. It is easy to show that such behavior is inconsistent with the expected utility model. Intuitively, decision makers do not like betting on the second urn because they do not have enough information or, put differently, there is too much ambiguity. Being averse to ambiguity, they prefer to bet on the first urn. Ellsberg and many subsequent studies have[1] demonstrated that ambiguity aversion is common and incompatible with the standard expected utility theory.

Following Gilboa and Schmeidler [6], we model ambiguity aversion using the maxmin expected utility (MMEU) model. In this model bidders have a set of priors, (instead of a single prior), on the underlying state space. Bidders compute their utility as the minimum expected utility over the set of priors. The MMEU model is a generalization of the SEU model, and provides a natural and tractable framework to study ambiguity aversion.

Under MMEU, when an ambiguity averse buyer is confronted with a selling mechanism, he evaluates each action on the basis of the minimum expected utility over the set of priors, and then chooses the best among them. An ambiguity averse seller on the other hand evaluates a mechanism on the basis of its minimum expected revenue over the set of priors and chooses the best mechanism. In order to better contrast our results with the

[1] See, for example, Camerer and Weber [2] for a survey.

risk case, we assume that the buyers and the seller are risk neutral (i.e. have linear utility functions).

We consider two cases, one where the bidders are ambiguity averse (and the seller is ambiguity neutral) and the other where the seller is ambiguity averse (and the buyers are ambiguity neutral)[2].

In the first case, we show that the optimal mechanism is such that for each type of the buyer the minimum expected utility is attained by the seller's prior. We also show that an auction that provides complete insurance to the bidders (i.e., it keeps the bidders' payoffs constant for all reports of the other bidders and consequently keeps them indifferent between winning or losing the object) is always in the set of optimal mechanisms. We use our first result to obtain further insights to the optimal auction problem. First we show that, unlike in the standard situation even when a small amount of ambiguity is introduced the complete insurance auction may be the *unique* optimal auction. To show this, we study an interesting example, ϵ-contamination. Suppose that the seller's prior is denoted by F, and the buyer's set of priors is given by $\Delta_B = \{G : G = (1-\epsilon)\, F + \epsilon H\}$ where H is any distribution and ϵ is a small positive number. That is, in the case of ϵ-contamination the buyers' set of beliefs contains all small perturbation of the seller's belief. The intuition is that the seller's prior is a focal point, but the buyers allow for an ϵ-order amount of noise. With ϵ-contamination we show that the complete insurance auction is the unique optimal auction. Then we use this result to show that in the often studied case of Choquet expected utility with convex capacity complete insurance auction is the unique optimal auction as well. One practical interpretation of this result is that the complete insurance auction is the only auction within the traditional set of optimal auctions that is robust to the introduction of a small amount of ambiguity. We also show that in general, neither the first price nor the second price auction is optimal even with suitably chosen reserve prices.

To obtain some intuition for the first, suppose that the optimal mechanism is such that the minimizing set of distributions for some type of the buyer does not include the seller's prior. In this case, the particular type, say θ, of the buyer and the seller will be willing to bet against each other; essentially the seller recognizes that they have different beliefs about the underlying state space and offers "side bets" using transfers. These additional

[2]See section 7 for a discussion of the case where both the buyers and the seller are ambiguity averse.

transfers (to the seller) can be chosen so that type θ under truth telling gets the minimum expected utility that he gets in the original mechanism in every state, and thus is completely insured against the ambiguity in the new mechanism. Obviously type θ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under truth telling. More interestingly, no other type wants to imitate type θ in the new mechanism. This is because the additional transfers in the new mechanism have zero expected value under the minimizing set of distributions in the original mechanism, and has strictly positive expected value under any other distribution. Therefore, if type θ' imitates type θ in the new mechanism, he gets at best what he would get by imitating type θ in the original mechanism. Since the original mechanism is incentive compatible, the new mechanism must also be incentive compatible. Moreover, by assumption the seller's distribution is not in the minimizing set for the original mechanism, the additional transfers (to the seller) must have strictly positive expected value under the seller's distribution and the seller is better off in the new mechanism. Note that the mechanism that is constructed here gives complete insurance to the buyers and is always weakly preferred by the seller. Therefore, a complete insurance auction must always be in the set of optimal auctions. To see why in the case of ϵ-contamination the complete insurance auction is the optimal auction, suppose that the optimal auction does not provide complete insurance for some type of the buyer. Then that buyer must evaluate this auction by a distribution that moves the ϵ weight to the "unfavorable" states. But this contradicts our earlier result that the set of distributions that the buyer uses to evaluate the optimal auction must include the seller's distribution. A very similar argument proves that the first and second price auctions are in general not optimal.

The second case we consider is when the seller is ambiguity averse (and the bidders are ambiguity neutral). Within the risk framework, Eso and Futo [5] consider auctions (in IPV environments) with a risk averse seller and risk (and ambiguity) neutral buyers. They show that for every incentive compatible selling mechanism there exists a mechanism which provides deterministically the same (expected) revenue. From this it follows that the optimal selling mechanism must provide complete insurance to the seller. We show that the mechanism in Eso and Futo is the optimal mechanism in our case as well. Hence, as long as bidders are risk and ambiguity neutral, ambiguity aversion on the part of the seller plays a similar role to that of risk aversion.

1.1 Related Literature

Matthews [14] and Maskin and Riley [13] relax the assumption of risk neutrality replacing it with risk aversion. They show that the classic auctions (high bid, English) are no longer optimal. In order to contrast our results, where buyers are ambiguity averse, with the results obtained when buyers are risk averse, we will summarize the results of Maskin-Riley.

Maskin-Riley show that, in the setting with risk, the central problem is preventing high valuation buyers from bidding too low. Suppose that the seller devises an auction where bidders who bid low face risk, but high valuation buyers who bid low face greater risk. The seller would derive less revenue from the low valuation buyers than if he offered them complete insurance, but this loss would be compensated by the high valuation buyers' higher bids. Thus even though for a particular type of buyer, removal of risk can be done in such a way that the buyer type's utility remains unchanged while the seller's payoff increases, the usual screening condition dictates that risk should not be completely eliminated for types other than the one with the highest valuation.

Indeed Maskin-Riley show that the optimal way to confront a buyer with risk is by using the transfers. For example, low valuation buyers may be penalized if they lose and high valuation buyers may receive a subsidy if they win. Moreover, only the buyer with the highest valuation gets perfectly insured. In other words, all buyers except the most eager buyer are better off winning than losing.

These results contrast with ours in the ambiguity setting. As we argued in the previous section, it turns out that the optimal mechanism may provide full insurance to all buyers, and, more generally, *all* types of buyers are presented with some insurance. These results not only differ from those under pure risk, but they are also driven by very different considerations. To gain a better intuition on the difference between risk aversion and ambiguity aversion, let's consider the ϵ-contamination case where perfect insurance is provided to all types of buyers, and contrast that with the situation when buyers are risk averse but face no ambiguity. The problem faced by the seller in the risk aversion case is that if risk is reduced for a particular type of buyer then the expected utility of all types when they report this particular type goes up. Reduction of risk for a type thus affects the IC constraints adversely. With ambiguity averse, but risk-neutral, buyers that is not so. Starting from a situation where a buyer type faces variable ex post utility in the mechanism, the seller can modify the mechanism in such a way as to make that type's ex post utility constant, keep the expected utility of this type the same as before, and increase his own expected payoff. More importantly, this can be done in such a way that the *other* type's expected utility, when they report the type whose payment scheme is being changed, is either unaffected or actually goes down. In other words, unlike the risk aversion situation, here, the seller can provide full insurance to a type in a way that benefits the seller's expected payoff that does not create any adverse effect on the IC constraints.

There is a small but growing literature on auction theory with non-expected utility starting with Karni and Safra ([9], [11], [10]) and Karni [8]. The papers that look at auctions with ambiguity averse bidders, and thus are closer to this paper are by Salo and Weber [19], Lo [12], Volij [21] and Ozdenoren [17]. These papers look at specific auction mechanisms, such as the first and second price auctions, and not the optimal auction problem.

Billot, Chateauneuf, Gilboa and Tallon [1] analyze the question of when it is optimal to take bets for agents with MMEU preferences in a pure exchange economy. They show that if the intersection of the set of priors for all agents is non-empty, then any Pareto optimal allocation is a full insurance allocation. This result is in the same spirit as our results. Another interpretation of our results is that we show that Bilot et. al. [1] result is robust to the introduction of incentive constraints. Another related paper is Mukerji [15] that shows that in the investment hold-up model ambiguity aversion can explain the existence of incomplete contracts. Incomplete contracts can be thought of as providing full insurance to the agents since any variation in ex-post payoffs makes the agents pessimistic.

2 Maxmin Expected Utility Model

In this section we introduce the MMEU model. Let Θ be the state space representing the agent's uncertainty. Let Σ be an algebra on Θ. Let $\mathcal{M}$ be the space of all probability measures on (Θ, Σ). Let X denote the set of outcomes. Suppose the decision maker's Von Neumann-Mongenstern utility function is given by $u : X \to \mathbb{R}$ and prior on Θ is given by a probability measure $\mu \in \mathcal{M}$. Let $\mathcal{A}$ be the set of all acts where an act is a Σ-measurable function $a : \Theta \to X$. In the standard expected utility model, utility of an act

$a \in \mathcal{A}$ is,

$$U(a) = \int_{\Theta} u(a(\omega))\, d\mu(\omega).$$

In contrast, in the MMEU model, the decision maker's prior is given by a (weak*) closed and convex set of probability measures $\Delta^m \subseteq \mathcal{M}$, and the utility of an act $h \in \mathcal{F}$ is,

$$U(a) = \min_{\mu \in \Delta^m} \int_{\Theta} u(a(\omega))\, d\mu(\omega).$$

The interpretation of the set of priors is, even if the information of the decision maker is too vague to be represented by an additive prior, it may be represented by a set of priors.

In this paper, the state space will be the possible valuations of the other bidder: this is the domain of uncertainty. For simplicity we assume in the following that $\Theta = [0, 1]$ and Σ is the Borel algebra on Θ. We assume risk neutrality (linear utility function) throughout the paper.

3 The Optimal Auction Problem

There are two bidders and a seller. We assume that both the bidders and the seller have linear VNM utility functions. Bidders have one of a continuum of valuations $\theta \in \Theta$. Each bidder knows his true valuation but not that of the other. The set Δ_B is a set of distribution functions corresponding to a (weak*) closed, convex subset of the set of probability measures over Θ, and this set represents each buyer's belief about the other bidder's valuation. Buyers believe that valuations are generated independently, but they are not confident about the probabilistic process that generates the valuations. This is reflected by the buyers having a set of priors rather than a single prior in this model.

The seller is also allowed to be uncertainty averse. The set Δ_S is a set of distribution functions corresponding to a (weak*) closed, convex subset of the set of probability measures over Θ, and it represents the seller's belief about the bidders' valuations. That is, the seller believes that buyers' valuations are generated independently from some distribution in Δ_S. We assume that $\Delta_S \subseteq \Delta_B$. Therefore the model covers two interesting cases. If Δ_S is a singleton set with the unique element F, then the seller is ambiguity neutral and believes that buyers' valuations are independently generated from the distribution F. On the other hand, if $\Delta = \Delta_S = \Delta_B$, then both the seller and the buyers are ambiguity averse with the common set of distributions Δ. We assume that the set of measures that generate Δ_B and Δ_S agree on zero probability events. Therefore, if an event has zero probability under one distribution in Δ_B, it also has zero probability under all other distributions.

Each bidder's reservation utility is 0. As is standard, we assume that all of the above is common knowledge.

We focus on the direct revelation game. In the direct revelation game, each bidder is asked to report his type, where a report is some $\theta \in \Theta$. The mechanism stipulates a probability for assigning the item and a transfer rule as a function of reported types. Let $x(\theta, \theta')$ be the item assignment probability function and $t(\theta, \theta')$ the transfer rule. The convention is that the first entry is one's own report, the second entry is the report of the other bidder.

Now, let's write the seller's problem of finding the optimal direct incentive compatible and individually rational mechanism:

$$\max_{(x,t)} \left[\min_{F \in \Delta_S} \iint \left[t\left(\theta, \theta'\right) + t\left(\theta', \theta\right) \right] dF\left(\theta\right) dF\left(\theta'\right) \right]$$

subject to

$$\begin{array}{ll} \text{(IC)} & \min_{G \in \Delta_B} \int \left(x\left(\theta, \theta'\right)\theta - t\left(\theta, \theta'\right) \right) dG\left(\theta'\right) \\ \geq & \min_{G \in \Delta_B} \int \left(x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta') \right) dG(\theta') \text{ for all } \theta, \tilde{\theta} \in \Theta \\ \text{(IR)} & \min_{G \in \Delta_B} \int \left(x\left(\theta, \theta'\right)\theta - t\left(\theta, \theta'\right) \right) dG\left(\theta'\right) \geq 0 \text{ for all } \theta \in \Theta. \end{array}$$

The first inequality gives the incentive compatibility (IC) constraints, and the second inequality gives the individual rationality (IR) or participation constraint. These are the usual constraints except that the bidders compute their utility in the mechanism using the MMEU rule. For example, the IC constraint requires that the minimum expected utility a bidder of type θ gets reporting his type truthfully is at least as much as the minimum expected utility that he gets under reporting any other type θ'.

4 Ambiguity averse buyers

The central result in this framework is that buyers and sellers "use the same distribution" to evaluate an optimal mechanism. That is, if a mechanism is optimal, then it must be the case that the intersection of the minimizing set of distributions for a given type of the buyer and the minimizing set of distributions for the seller must be non-empty for almost all types. Otherwise, the seller can offer a lottery to each type for which this intersection is empty, and be better off and keep the resulting mechanism incentive compatible and individually rational.

More intuition can be obtained for the case where the seller is ambiguity neutral, that is $\Delta_S = \{F\}$. Then our result says that the minimum payoff for the buyers will be attained by F. To see why, first fix a mechanism (x, t) and note that for each mechanism the minimum (under truthtelling) is achieved by a possibly different set of distributions for each type. Let's call this set $\Delta(\theta)$. Now suppose that (x, t) is optimal for the seller but $F \notin \Delta(\theta)$ for some type θ. This means that the buyer of type θ and the seller have different beliefs about the value distribution of the other buyer. Therefore the seller can find a lottery over the valuations of the other bidder that will have zero expected utility under any distribution in $\Delta(\theta)$, but has strictly positive expectation under any distribution in the complement of $\Delta(\theta)$, i.e. $\Delta_B - \Delta(\theta)$. Since F is in the complement, under this new mechanism the seller is better off. Also the buyer of type θ is indifferent under truthtelling, since the lottery has zero expected utility under any distribution in $\Delta(\theta)$. Moreover, under truthtelling all other types of buyers are trivially unaffected by this change, since both mechanisms are identical for all the other types. Moreover, any type who does not tell the truth in the new mechanism has a weakly lower utility in the new mechanism by construction of the lottery. Therefore, the new mechanism is also incentive compatible.

This argument shows that, if there is a positive measure of types θ for which $F \notin \Delta(\theta)$, then the original mechanism can not be optimal in the first place since a lottery for each type can be constructed as outlined above.

This result imposes quite a lot of structure on the set of optimal mechanisms, because many mechanisms will not be evaluated with the same distribution by the buyer and seller, and are hence not optimal. Next, we give the formal statement of the proposition. The formal proof also shows how to construct the lotteries described above.

Proposition 1 *Suppose that the seller is ambiguity neutral with distribution F and the buyers are ambiguity averse with the set of priors Δ_B. Let (x,t) be an arbitrary incentive compatible mechanism. For any $\theta \in \Theta$, let $\Delta(\theta) \subseteq \Delta_B$ be the set of minimizing distributions for θ under (x,t). That is,*

$$\Delta(\theta) = \arg\min_{G \in \Delta_B} \int [x(\theta,\theta')\theta - t(\theta,\theta')]\, dG(\theta').$$

If there exists some positive measure $\widetilde{\Theta} \subseteq \Theta$ such that, $F \notin \Delta(\tilde{\theta})$ for all $\tilde{\theta} \in \widetilde{\Theta}$ then (x,t) is not optimal.

To understand this result, suppose that the optimal mechanism is such that the minimizing set of distributions for some type of the buyer does not include the seller's prior. In this case, the seller can offer additional transfers to the particular type, say θ, of the buyer . These additional transfers (to the seller) can be chosen so that type θ under truth telling gets the minimum expected utility that he gets in the original mechanism in every state, and thus is completely insured against the ambiguity in the new mechanism. Obviously type θ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under truth telling. More interestingly, no other type wants to imitate type θ in the new mechanism. This is because the additional transfers in the new mechanism have zero expected value under the minimizing set of distributions in the original mechanism, and has strictly positive expected value under any other distribution. Therefore, if type θ' imitates type θ in the new mechanism, he gets at best what he would get by imitating type θ in the original mechanism. Since the original mechanism is incentive compatible, the new mechanism must also be incentive compatible. Moreover, by assumption the seller's distribution is not in the minimizing set for the original mechanism, the additional transfers (to the seller) must have strictly positive expected value under the seller's distribution and the seller is better off in the new mechanism, contradicting the optimality of the original mechanism.

Mechanisms where the payoff of any bidder for any report of the competing bidder is constant are called *perfect (or full) insurance mechanisms*. Our next result shows that there is always a perfect insurance mechanism within the set optimal mechanisms.

Proposition 2 *There exists an auction, (x^*,t^*), that maximizes the seller's revenue such that the payoff of any type of a bidder in his auction is constant* *as a function of the other bidder's report. That is for all $\theta \in \Theta$, $x^*(\theta, \theta')\theta - t^*(\theta, \theta')$ is constant in θ'.*

To see how (x^*, t^*) is constructed, suppose that (x, t) is an auction that maximizes the seller's revenue and let $q(\theta, \theta') = x(\theta, \theta')\theta - t(\theta, \theta')$. Let,

$$K(\theta) = \min_{G \in \Delta_B} \int q(\theta, \theta') dG(\theta')$$

so that $K(\theta)$ is buyer θ's expected payoff.

Define the function $\delta : \Theta \to \mathbb{R}$ as follows:

$$\delta(\theta, \theta') = [q(\theta, \theta') - K(\theta)], \text{ for all } \theta \in \Theta.$$

Now consider the mechanism (x^*, t^*) such that $x^*(\theta, \theta') = x(\theta, \theta')$ and $t^*(\theta, \theta') = t(\theta, \theta') + \delta(\theta, \theta')$ for all $\theta \in \Theta$. Now note that $x^*(\theta, \theta')\theta - t^*(\theta, \theta') = x(\theta, \theta')\theta - t(\theta, \theta') - \delta(\theta, \theta') = K(\theta)$, which does not depend on θ'.

We can prove using simple variations of the proofs of claims 1,2 and 3 in the proof of proposition 1 that (x^*, t^*) gives the seller at least as much revenue as (x, t) and satisfies IC and IR constraints for the buyers. Thus (x^*, t^*) must be in the set of optimal auctions as well.

For complete insurance mechanisms any distribution in Δ_B gives the minimum expected utility and proposition (1) is trivially satisfied. In general though there may be other selling mechanisms that are optimal. On the other hand, if the set Δ_B is sufficiently rich and if a bidder's ex post payoffs vary enough with the report of the other bidder, then typically the set of distributions that give the minimum expected utility will not include F. In section 5.1, we give an example where the set of priors include all perturbations of the seller's prior F and show that in fact in this case the *unique* optimal mechanism is the perfect insurance mechanism.

In the next section we provide some applications of these results.

5 Applications

For the examples in this section we again look at the case where the seller is ambiguity neutral with $\Delta_S = \{F\}$. The strength of results **??** and 1 are best seen through examples. We consider a very natural Δ_B which results in perfect insurance for the buyers. We also establish that, under some broad conditions on Δ_B, first and second price auctions with reserve prices are not optimal.

5.1 Perfect insurance under ϵ-contamination

5.1.1 The optimal mechanism

A natural Δ_B to investigate is one of "small" ambiguity.[3] One such Δ_B is an ϵ-contamination of the seller's prior F.[4] In this setting, the seller's distribution F is a focal point, and buyers allow for an ϵ-order amount of noise around this focal distribution. We assume F has a strictly positive density f, and, we construct the buyers' Δ_B as follows:

$$\Delta_B = \{G : G = (1-\epsilon)\, F + \epsilon H\}$$

where H is any distribution on Θ and $\epsilon \in (0,1]$. We also make the standard assumption that

$$L(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$$

is strictly increasing in θ.

We will argue that under ϵ-contamination there is perfect insurance—that is, for a measure 1 set of buyers (with respect to the distribution F), their payoff is constant with probability one. The intuition is as follows; if there is any deviation from a constant payoff, then one of the elements of $\Delta_B \backslash \{F\}$ will yield a lower payoff, since an H can be found which weighs the discrepancies appropriately; namely inflates the low payoff states and understates the high payoff states relative to F. It follows that since the mechanism involving non-constant payoffs are being evaluated by a distribution other than F, the results of the previous section show that the mechanism cannot be the optimal one.

Next we provide a formal statement of this claim. The proof is in the appendix.

Proposition 3 *Suppose that (x, t) solves the seller's problem and let Δ_B be the ϵ-contamination of the seller's belief F:*

$$\Delta_B = \{G : G = (1-\epsilon)\, F + \epsilon H\}$$

[3] This is a generalization of the two type example given in section 1.

[4] See Epstein and Wang [4].

Then there is a measure 1 *subset* $\bar{\Theta} \subseteq \Theta$*, for which, for each* $\bar{\theta} \in \bar{\Theta}$*,* $q(\bar{\theta}, \theta)$ *is constant for almost all* $\theta \in \Theta$.

Next, we explicitly solve for the optimal mechanism. For simplicity we assume that $x(\theta, \theta')\theta - t(\theta, \theta')$ is constant for all θ', rather than for a set of measure one of θ'. Obviously, this does not effect our results in any fundamental way.

Next we define some useful notation. Let

$$\begin{aligned}
u(\theta) &= x(\theta, \theta')\theta - t(\theta, \theta'), \\
X(\theta) &= \int x(\theta, \theta')dF(\theta') \\
X^{\min}(\theta) &= \min_{G \in \Delta_b} \int x(\theta, \theta')dG(\theta'), \\
X^{\max}(\theta) &= \max_{G \in \Delta_b} \int x(\theta, \theta')dG(\theta').
\end{aligned}$$

Using the IC constraint we obtain,

$$\begin{aligned}
u(\theta) &= \min_{G \in \Delta_b} \int (x(\theta, \theta')\theta - t(\theta, \theta'))\, dG(\theta') \\
&\geq \min_{G \in \Delta_b} \int \left(x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta')\right) dG(\theta') \\
&\geq u\left(\tilde{\theta}\right) + \min_{G \in \Delta_b} \int \left(\theta - \tilde{\theta}\right) x(\tilde{\theta}, \theta')dG(\theta').
\end{aligned} \tag{1}$$

If $\theta > \tilde{\theta}$,we obtain,

$$u(\theta) \geq u\left(\tilde{\theta}\right) + \left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right). \tag{2}$$

Exchanging the roles of θ and $\tilde{\theta}$ in (1) we obtain.

$$u\left(\tilde{\theta}\right) \geq u(\theta) + \min_{G \in \Delta_b} \int \left(\tilde{\theta} - \theta\right) x(\theta, \theta')dG(\theta')$$

Again for $\theta > \tilde{\theta}$,we obtain,

$$u\left(\tilde{\theta}\right) \geq u(\theta) + \left(\tilde{\theta} - \theta\right) X^{\max}(\theta). \tag{3}$$

First observe that u is non-decreasing since, for $\theta > \tilde{\theta}$ by the IC constraint we have,

$$u(\theta) \geq u\left(\tilde{\theta}\right) + \left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \geq u\left(\tilde{\theta}\right).$$

Now, we are ready to prove a lemma that is useful in characterizing the optimal auction.

Lemma 4 *The function u is Lipschitz.*

Since u is Lipschitz, it is absolutely continuous and therefore is differentiable almost everywhere. For $\theta > \tilde{\theta}$ we use (2) and (3) to obtain,

$$X^{\max}(\theta) \geq \frac{u(\theta) - u\left(\tilde{\theta}\right)}{\theta - \tilde{\theta}} \geq X^{\min}\left(\tilde{\theta}\right).$$

We can take the limit as $\tilde{\theta}$ goes to θ to obtain for almost all θ that,

$$X^{\max}(\theta) \geq \frac{\partial u}{\partial \theta} \geq X^{\min}(\theta).$$

Since an absolutely continuous function is the definite integral of its derivative, we obtain,

$$\int_0^\theta X^{\max}(y)\,dy \geq u(\theta) - u(0) \geq \int_0^\theta X^{\min}(y)\,dy. \tag{5}$$

Equation (5) suggests that the auctioneer may set,

$$u(\theta) = \int_0^\theta X^{\min}(y)\,dy \tag{6}$$

and

$$t(\theta, \theta') = x(\theta, \theta')\,\theta - \int_0^\theta X^{\min}(y)\,dy, \tag{7}$$

since for a given allocation rule x, transfers as in (7) are the highest transfers the auctioneer can set without violating (5). Of course, (5) is only a necessary condition and for a given allocation rule x, the resulting mechanism (x, t) may not be incentive compatible. Fortunately, this difficulty does not arise if the allocation rule x is chosen optimally for transfers given as in (7). In other words, our strategy is to find the optimal allocation rule x, assuming that the transfers are given by (7), and then show that the resulting mechanism, (x, t) is incentive compatible.

For transfer function given by (7), we can rewrite the seller's revenue as,

$$R = 2\int_0^1 \int_0^1 \left(\theta x(\theta, \theta') - \int_0^{\theta} X^{\min}(y)\, dy \right) dF(\theta')\, dF(\theta).$$

Using integration by parts we obtain,

$$R = 2\int_0^1 \theta X(\theta) f(\theta)\, d\theta - \int_0^1 (1 - F(\theta)) X^{\min}(\theta)\, d\theta. \tag{8}$$

Define,

$$L^{\epsilon}(\theta) = \theta - (1-\epsilon)\frac{1 - F(\theta)}{f(\theta)},$$

and let $r \in (0,1)$ be such that $L^{\epsilon}(r) = 0$.

The following proposition characterizes the optimal allocation when transfer function is given by (7).

Proposition 5 *For any θ and θ', the allocation rule given by $x(\theta, \theta') = 1$ if $\theta > \theta'$ and $\theta \geq r$, $x(\theta, \theta') = \frac{1}{2}$ if $\theta = \theta'$ and $\theta \geq r$, and $x(\theta, \theta') = 0$ otherwise, is optimal for the seller if the transfer function is given by (7).*

Proof. First note that L^{ϵ} is increasing in θ, if L is increasing in θ. To seee this note that,

$$\begin{aligned}
\theta - \frac{1 - F(\theta)}{f(\theta)} &> \theta' - \frac{1 - F(\theta')}{f(\theta')} \\
&\Rightarrow \theta - \theta' > \frac{1 - F(\theta)}{f(\theta)} - \frac{1 - F(\theta')}{f(\theta')} \\
&\Rightarrow \theta - \theta' > (1-\epsilon)\left(\frac{1 - F(\theta)}{f(\theta)} - \frac{1 - F(\theta')}{f(\theta')}\right) \\
&\Rightarrow \theta - (1-\epsilon)\frac{1 - F(\theta)}{f(\theta)} > \theta' - (1-\epsilon)\frac{1 - F(\theta')}{f(\theta')}.
\end{aligned}$$

Note that $X^{\min}(\theta) \leq X(\theta)$. Therefore if $X(\theta) = 0$, $X^{\min}(\theta) = 0$ as well. Letting $\frac{X^{\min}(\theta)}{X(\theta)} = 1$ whenever $X(\theta) = 0$, we define $M(\theta) = \theta - \frac{X^{\min}(\theta)}{X(\theta)}\frac{1-F(\theta)}{f(\theta)}$. We can rewrite R as,

$$R = 2\int_0^1 \int_0^1 M(\theta) x(\theta, \theta') f(\theta') f(\theta)\, d\theta' d\theta. \tag{9}$$

Now we can show that the optimal allocation rule is given by setting $x(\theta,\theta')=1$ if $\theta>\theta'$ and $\theta\geq r$, $x(\theta,\theta')=\frac{1}{2}$ if $\theta=\theta'$ and $\theta\geq r$, and $x(\theta,\theta')=0$ otherwise. First note that, in the ϵ-contamination case, $X^{\min}(\theta)\geq(1-\epsilon)X(\theta)$ for all θ such that $X(\theta)<1$[5]. Under the above allocation rule $X^{\min}(\theta)=(1-\epsilon)X(\theta)$ for all θ such that $X(\theta)<1$. Therefore this allocation rule maximizes $M(\theta)$. By construction $x(\theta,\theta')=1$ if and only if $M(\theta)>M(\theta')$ and $M(\theta)\geq 0$ therefore maximizing (9). ∎

The following proposition concludes our discussion of the optimal mechanism in the ϵ-contamination case.

Proposition 6 *The mechanism described in proposition 5 is incentive compatible, and thus optimal for the seller in the ϵ-contamination case.*

Proof. First we show that if $X^{\min}$ is non-decreasing selecting u as in (6) satisfies IC. We check two cases.

Case 1: If $\theta>\tilde{\theta}$,

$$\begin{aligned} u(\theta)-u\left(\tilde{\theta}\right) &= \int_{\tilde{\theta}}^{\theta} X^{\min}(y)\,dy \\ &\geq X^{\min}\left(\tilde{\theta}\right)\left(\theta-\tilde{\theta}\right) \end{aligned}$$

which is the IC constraint.

Case 2: If $\theta<\tilde{\theta}$,

$$\begin{aligned} u\left(\tilde{\theta}\right)-u(\theta) &= \int_{\theta}^{\tilde{\theta}} X^{\min}(y)\,dy \\ &\leq X^{\min}\left(\tilde{\theta}\right)\left(\tilde{\theta}-\theta\right) \end{aligned}$$

again giving us the IC constraint.

[5] This is true since:

$$\begin{aligned} X^{\min}(\theta) &= \min_{G\in\Delta_b}\int x(\theta,\theta')dG(\theta') \\ &= \min_{H\in\mathcal{D}}\int x(\theta,\theta')d\left((1-\epsilon)F+\epsilon H\right)(\theta') \\ &\geq (1-\epsilon)\int x(\theta,\theta')dF(\theta'). \end{aligned}$$

Now, note that for the allocation rule in proposition 5, $X^{\min}$ is non-decreasing, and thus the mechanism (x, t) is incentive compatible. ■

Remark 7 *A natural question is how to implement the optimal mechanism described above. There are several ways to implement the mechanism. We will describe one such auction here. Consider an auction where bidders bid for the object, the highest bidder who bids above the reservation value r obtains the object, and any bidder (regardless of winning or losing) who bids b receives a gift $S(b) = (1-\varepsilon)\int_r^b F(y)\,dy$ from the seller. In this auction, the equilibrium strategy of a bidder with valuation θ is to bid his valuation. To see this note that the allocation rule is the same as the one in proposition 5. Moreover, a bidder who bids θ pays $\theta - (1-\varepsilon)\int_r^b F(y)\,dy$ if he wins the auction and $-(1-\varepsilon)\int_r^b F(y)\,dy$ if he loses the auction, and these transfers are also the ones in proposition 5. Since reporting your value is incentive compatible in the optimal mechanism, it is also optimal to bid your value in this auction.*

5.2 First and Second Price auctions are not optimal

It has been established in Lo [12] that the second price auction is not optimal since the first price auction may generate more revenue than the first price auction when the bidders are ambiguity averse. Yet, it was an open question whether the first price auction with an optimally chosen reserve price is the optimal auction. Our result gives a negative answer to this question. Under rather general conditions, the first and second price auctions, as well as many other natural auction types, will not be optimal in this setting. The intuition is as follows. In most auctions, sellers obtains more revenue the higher the valuations of the buyers. Thus under ambiguity aversion, they consider *high valuations unlikely*. Similarly, buyers hope that their opponents' valuations are low; hence they consider *low valuations unlikely*. Quite naturally, then, one would expect that seller and buyer's beliefs would depart; and hence these auctions are not optimal. Even, as we assumed above, F is fixed, unless F is the lower envelope of Δ_B, these auctions will not be optimal.

We propose the following corollary which describes one possible restriction on Δ_B which would result in the non-optimality of first- and second-price auctions.

Formally, consider the following corollary of the theorems presented in section 4:

Corollary 8 *Suppose there exists some distribution $G \in \Delta_B$ such that G first-order stochastically dominates F. Now, if under some mechanism (x, t), there exists a positive measure subset $\widetilde{\Theta} \subseteq \Theta$ such that for all $\tilde{\theta} \in \widetilde{\Theta}$,*

$$q(\tilde{\theta}, \theta') = x(\tilde{\theta}, \theta')\tilde{\theta} - t(\tilde{\theta}, \theta')$$

is weakly decreasing in θ' and over some interval strictly decreasing, then (x, t) is not optimal.

Proof. First-order stochastic dominance implies:

$$\int q(\tilde{\theta}, \theta')dG(\theta') < \int q(\tilde{\theta}, \theta')dF(\theta')$$

By Proposition 1, (x, t) is not optimal. ■

Now, to show that the first and second price auctions are not optimal, we need only establish that under first and second price auctions, there exists some $\widetilde{\Theta}$ which meets this criterion.

Consider a positive measure set $\widetilde{\Theta}$ of Θ such that, for all $\tilde{\theta} \in \widetilde{\Theta}$, $\tilde{\theta}$ is strictly above the reserve price (if there is one) and $\tilde{\theta}$ is strictly below the highest in Θ. We will show that the first and second price auctions are not optimal for any $\tilde{\theta} \in \widetilde{\Theta}$.

Now, consider an arbitrary $\tilde{\theta} \in \widetilde{\Theta}$. The following describes the first-price auction for type $\tilde{\theta}$:

$$\begin{aligned} x(\tilde{\theta}, \theta') &= \begin{cases} 1, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \\ t(\tilde{\theta}, \theta') &= \begin{cases} b(\tilde{\theta}), & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}b(\tilde{\theta}), & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

for a fixed optimal bid $b(\tilde{\theta}) < \tilde{\theta}$. Hence,

$$q(\tilde{\theta}, \theta') = \begin{cases} \tilde{\theta} - b(\tilde{\theta}) > 0, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}\left(\tilde{\theta} - b(\tilde{\theta})\right) > 0, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases}$$

So as θ' increases, $q(\tilde{\theta}, \theta')$ falls monotonically from $\left(\tilde{\theta} - b(\tilde{\theta})\right)$ to zero. So the first price auction is not optimal as long as there is some distribution in Δ_B which first-order stochastically dominates F.

Similarly, for second price auctions,

$$\begin{aligned} x(\tilde{\theta}, \theta') &= \begin{cases} 1, & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}, & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \\ t(\tilde{\theta}, \theta') &= \begin{cases} \theta', & \text{if } \tilde{\theta} > \theta' \\ \frac{1}{2}\theta', & \text{if } \tilde{\theta} = \theta' \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

So, for second price auctions,

$$q(\tilde{\theta}, \theta') = \begin{cases} \tilde{\theta} - \theta' > 0, & \text{if } \tilde{\theta} > \theta' \\ 0, & \text{otherwise} \end{cases}$$

So as θ' increases, $q(\tilde{\theta}, \theta')$ falls monotonically from $\tilde{\theta}$ to zero. So the second price auction is also not optimal as long as there is some distribution in Δ_B which first-order stochastically dominates F.

5.3 Choquet Expected Utility with Convex Capacity

Choquet expected utility (CEU) model is an alternative to MMEU model that is used to represent ambiguity averse preferences. In the CEU model an ambiguity averse agent's subjective belief is represented by a convex capacity μ satisfying the following properties: $0 \le \mu(A) \le 1$ for all $A \in \Sigma$, $\mu(\emptyset) = 0$, $\mu(\Theta) = 1$ and $\mu(A) + \mu(B) \le \mu(A \cap B) + \mu(A \cup B)$ for $A, B \in \Sigma$. It is this last property that captures ambiguity aversion and says that the union of disjoint events may be assigned a larger weight than the sum of the weights assigned to each event, since these events may be more ambigous than their union. The core of a convex capacity μ, denoted by $\Pi(\mu)$, is given by:

$$\Pi(\mu) = \{p \in \mathcal{M} : p(A) \ge \mu(A) \text{ for all } A \in \Sigma\}.$$

The Choquet expected utility (with respect to the convex capacity μ) of an agent who evaluates an act $f : \Theta \to \mathbb{R}$ is given by:

$$CE(f) = \int f d\mu = \min_{p \in \Pi(\mu)} \int f dp.$$

Here the first integral is the Choquet integral of f with respect to capacity μ, and the second integral is the usual Lebesgue integral of f with respect to probability measure p where the minimum is taken over all probability meaures in the core of capacity μ.

In the case of CEU with convex capacity we can apply proposition 1 to show optimality of the complete insurance auction. To see this let's first introduce some additional notation. For $p \in \mathcal{M}$, let G_p denote the distribution function associated with p. Let $\Delta(\mu)$ be the set of all distribution functions associated with measures in $\Pi(\mu)$. Now suppose that $\bar{p} \in \mathcal{M}$ is associated with the seller's distribution F. Suppose that $\bar{p}(A) \geq \mu(A) + \varepsilon$ for all $A \in \Sigma$ and some $\varepsilon > 0$. In this case we can show that the ε-contamination set of F is a subset of the core and by proposition ?? complete insurance auction is the unique optimal auction.

Now suppose that for some $\bar{\theta} \in \bar{\Theta}$, $q(\bar{\theta}, \theta)$ is a simple function that is not constant for a positive measure set of θ. Therefore there exists $K_1 > \cdots > K_m$ and Θ_j for $j = 1, ..., m$ such that $q(\bar{\theta}, \theta) = K_j$ for all $\theta \in \Theta_j$ and $\cup_{j=1}^m \Theta_j = \Theta$.

$$\begin{aligned} CE\left(q(\bar{\theta}, \cdot)\right) &= \int q(\bar{\theta}, \cdot) d\mu \\ &= \sum_{i=1}^{m} (K_i - K_{i+1}) \, \mu\left(\cup_{j=1}^{i} \Theta_j\right) \\ &< \int q(\bar{\theta}, \cdot) dF. \end{aligned}$$

6 Ambiguity Averse Seller

In this section we first provide a counterpart of proposition 1 when the seller is ambiguity averse. The proof of the result is in the appendix.

Proposition 9 *Suppose that the seller is ambiguity averse, with a set of priors Δ_S and the buyers are ambiguity neutral with a prior $F \in \Delta_S$. Let (x, t) be an arbitrary incentive compatible mechanism. Let $\Delta_S^{\min} \subseteq \Delta_S$ be the set of minimizing distributions for the seller under (x, t). That is,*

$$\Delta_S^{\min} = \arg\min_{G \in \Delta_S} \iint t(\theta, \theta') + t(\theta', \theta) dG(\theta) dG(\theta').$$

If $F \notin \Delta_S^{\min}$, then (x, t) is not optimal.

The idea of the proof is based on Eso and Futo [5] who consider auctions with a risk averse seller in independent private values environments with risk (and ambiguity) neutral buyers. They show that for every incentive compatible selling mechanism there exists a mechanism which provides deterministically the same (expected) revenue. From this it follows that the optimal selling mechanism must provide complete insurance to the seller. We next show that the mechanism in Eso and Futo is the optimal mechanism in our case as well.

Proposition 10 *Then there exists an incentive compatible and individually rational selling mechanism, (x^*, t^*), that maximizes minimum expected revenue of the seller over the set of priors 's Δ_S and provides the same revenue to the seller no matter what the buyer's types are. That is $t(\theta,\theta')+t(\theta',\theta)$ is constant for all $\theta,\theta'\in\Theta$.*

The proof is omitted since it is a straightforward extension of the proof of proposition 9. The basic idea is very simple. For any individually rational and incentive compatible mechanism (x,t), one can define a new mechanism (x,t') with the same allocation rule, but with the following transfers:

$$t'(\theta,\theta') = T(\theta) - T(\theta') + \int T(i)dF(i).$$

Note that in the new mechanism $t'(\theta,\theta')+t'(\theta',\theta)$ is always $\int T(i)dF(i)$ which is constant. It is straightforward to check that this mechanism incentive compatible as well. The reason this mechanism works in both risk and ambiguity settings is that, since the buyers are risk and ambiguity neutral (x,t') is incentive compatible in both settings and provides perfect insurance to the seller against both ambiguity and risk.

7 Conclusion

In this paper we analyzed selling mechanisms (or auctions) from the seller's point of view when either the buyers or the seller is ambiguity averse. We have showed that selling mechanisms that provide full insurance to the buyers when the buyers are ambiguity averse and to the seller when the seller is ambiguity averse are in the set of optimal mechanisms for the seller. We have also showed that a necassary condition for the optimality of a mechanism is that the buyers and the seller use the same subjective beliefs to evaluate the mechanism.

There are at least two directions these results may be extended. The first one is to allow the buyers and the seller both be ambiguity averse. Our results easily extend to the case when the buyers are more ambiguity averse (i.e., have a larger set of priors) then the seller. On the other hand, when the seller is more ambiguity averse than the buyers the above results may break down. This point is left for further research.

The second is to use the methods developed here in mechanism design problems in quasilinear environments with incomplete information where the agent's are ambiguity averse. We believe that the results in this paper will naturally extend to these environments. For example in a bargaining problem (see Myerson) we conjecture that efficiency (from the mechanism designer's point of view) would require that the agent's are completely insured against the ambiguity. This extension is also left for further research.

8 Appendix

First we prove a proposition that is more general than proposition 1. In the following proposition the seller is allowed to be ambiguity averse with a set of priors $\Delta_S \subseteq \Delta_B$. For the case where $\Delta_S = \{F\}$, this proposition implies proposition 1 in the text.

Proposition 11 *Suppose that the seller and the buyers are ambiguity averse. Suppose that the seller's set of priors is Δ_S and the buyers' set of priors is Δ_B. Suppose further that $\Delta_S \subseteq \Delta_B$. Let (x, t) be an arbitrary incentive compatible mechanism. Let $\Delta_S^{\min} \subseteq \Delta_S$ be the set of minimizing distributions for the seller under (x, t). That is,*

$$\Delta_S^{\min} = \arg\min_{G \in \Delta_S} \iint 2t(\theta, \theta')dG(\theta)\, dG(\theta').$$

For any $\theta \in \Theta$, let $\Delta(\theta) \subseteq \Delta_B$ be the set of minimizing distributions for θ under (x, t). That is,

$$\Delta(\theta) = \arg\min_{G \in \Delta_B} \int [x(\theta, \theta')\theta - t(\theta, \theta')]\, dG(\theta').$$

If there exists some positive measure $\widetilde{\Theta} \subseteq \Theta$ such that, for all $\tilde{\theta} \in \widetilde{\Theta}$,

$$\Delta_S^{\min} \cap \Delta(\tilde{\theta}) = \emptyset$$

then (x,t) *is not optimal.*

Proof. Define,

$$q(\theta,\theta') = x(\theta,\theta')\theta - t(\theta,\theta').$$

Now let,

$$K(\theta) = \min_{G\in\Delta_B} \int q(\theta,\theta')dG(\theta')$$

so that $K(\theta)$ is buyer θ's expected payoff.

Now we will find a set of transfers for types in $\widetilde{\Theta}$ which will make the seller better off and keep the mechanism incentive compatible. Define the function $\delta : \widetilde{\Theta} \to \mathbb{R}$ as follows:

$$\delta(\tilde{\theta},\theta') = \left[q(\tilde{\theta},\theta') - K(\tilde{\theta})\right], \text{ for all } \tilde{\theta} \in \widetilde{\Theta}$$

Now consider the mechanism (x,t') such that:

$$t'(\theta,\theta') = \begin{cases} t(\theta,\theta') + \delta(\theta,\theta'), & \text{for all } \theta \in \widetilde{\Theta} \\ t(\theta,\theta'), & \text{otherwise} \end{cases}.$$

We show that the mechanism (x,t') makes the seller strictly better off, leaves the buyers' payoffs unchanged, and is incentive compatible.

Claim 1: (x,t') **makes the seller strictly better off**
The seller's payoff for a general symmetric t is:

$$\begin{aligned} & \min_{G\in\Delta_S} \iint \left[t(\theta,\theta') + t(\theta',\theta)\right] dG(\theta)dG(\theta') \\ = \; & 2\min_{G\in\Delta_S} \iint t(\theta,\theta')dG(\theta)dG(\theta') \end{aligned}$$

Hence the seller's payoff for the mechanism (x,t') is:

$$\begin{aligned} & \min_{G\in\Delta_S} \iint 2t'(\theta,\theta')dG(\theta)dG(\theta') \qquad (10) \\ = \; & \min_{G\in\Delta_S} \left[\iint 2t(\theta,\theta')dG(\theta)dG(\theta') + \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta')dG(\theta')dG(\tilde{\theta})\right] \end{aligned}$$

Note that,

$$
\begin{aligned}
& \min_{G\in\Delta_S} \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta')dG(\theta')dG(\tilde{\theta}) && (11)\\
= \;& 2\min_{G\in\Delta_S} \int_{\widetilde{\Theta}} \left[\int_{\Theta} q(\tilde{\theta},\theta')dG(\theta') - K(\tilde{\theta})\right] dG(\tilde{\theta}) \\
= \;& 2\min_{G\in\Delta_S} \int_{\widetilde{\Theta}} \left[\int_{\Theta} q(\tilde{\theta},\theta')dG(\theta') - \min_{H\in\Delta_B}\int q(\tilde{\theta},\theta')dH(\theta')\right] dG(\tilde{\theta}).
\end{aligned}
$$

For $G \in \Delta(\tilde{\theta})$,

$$
\int_{\Theta} q(\tilde{\theta},\theta')dG(\theta') - \min_{H\in\Delta_B}\int q(\tilde{\theta},\theta')dH(\theta') = 0
$$

by definition of $\Delta(\tilde{\theta})$. On the other hand for $G \notin \Delta(\tilde{\theta})$, the difference above is strictly positive. From this we can conclude that,

$$
\begin{aligned}
& \min_{G\in\Delta_S} \left[\iint 2t(\theta,\theta')dG(\theta)dG(\theta') + \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta')dG(\theta')dG(\tilde{\theta})\right] && (12)\\
> \;& \min_{G\in\Delta_S} \iint 2t(\theta,\theta')dG(\theta)dG(\theta') + \min_{G\in\Delta_B} \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta')dG(\theta')dG(\tilde{\theta})
\end{aligned}
$$

To see this first suppose that the min of the l.h.s. of equation (12) is achieved by $G \in \Delta_s^{\min}$. Since $\Delta(\tilde{\theta}) \cap \Delta_s^{\min} = \emptyset$, we know that the min of the second term on the r.h.s. can not be achieved by this G. On the other hand, suppose that the min of the l.h.s. of equation (12) is achieved by $G \notin \Delta_s^{\min}$, this time the min of the first term on the r.h.s. can not be achieved by this G. So in either case, we obtain the strict inequality.

But, by the IR constraint

$$
\min_{G\in\Delta_B} \int_{\widetilde{\Theta}} \int_{\Theta} 2\delta(\tilde{\theta},\theta')dG(\theta')dG(\tilde{\theta}) \geq 0,
$$

and putting equations (10) and (12) together delivers the result.

Claim 2: (x, t') leaves the buyers' payoffs unchanged under truth-telling

For all buyers $\theta \notin \widetilde{\Theta}$, the mechanisms (x, t) and (x, t') are identical under truth-telling. So their payoffs are trivially identical. Now, consider an arbitrary buyer $\tilde{\theta} \in \widetilde{\Theta}$. Under truth-telling, the payoff for this buyer is:

$$
\begin{aligned}
& \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\tilde\theta - t'(\tilde\theta,\theta')\right] dG(\theta') \\
= & \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\tilde\theta - t(\tilde\theta,\theta') - \delta(\tilde\theta,\theta')\right] dG(\theta') \\
= & \min_{G\in\Delta_B} \int \left[q(\tilde\theta,\theta') - q(\tilde\theta,\theta') + K(\tilde\theta)\right] dG(\theta') = K(\tilde\theta)
\end{aligned}
$$

where $K(\tilde\theta)$ is buyer $\tilde\theta$'s original payoff.

Claim 3: (x, t') is incentive compatible

For $\theta \notin \widetilde\Theta$, $(x(\theta,\theta'), t'(\theta,\theta'))$ is identical to $(x(\theta,\theta'), t(\theta,\theta'))$. Furthermore, truth-telling payoffs for all types are unchanged. This immediately implies that, since (x,t) was incentive compatible, no θ can profitably deviate to any $\hat\theta \notin \widetilde\Theta$. Hence we only check if any type has an incentive to deviate to some $\tilde\theta \in \widetilde\Theta$.

The payoff for $\theta \in \Theta$ to deviate to an arbitrary $\tilde\theta \in \widetilde\Theta$, $\theta \neq \tilde\theta$, is:

$$
\begin{aligned}
& \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\theta - t'(\tilde\theta,\theta')\right] dG(\theta') \\
= & \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\theta - t(\tilde\theta,\theta') - \delta(\tilde\theta,\theta')\right] dG(\theta') \\
\leq & \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\theta - t(\tilde\theta,\theta')\right] dG(\theta') - \min_{G\in\Delta_B} \int \delta(\tilde\theta,\theta') dG(\theta')
\end{aligned}
$$

Because

$$
\begin{aligned}
& \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\theta - t(\tilde\theta,\theta') - \delta(\tilde\theta,\theta') + \delta(\tilde\theta,\theta')\right] dG(\theta') \\
\geq & \min_{G\in\Delta_B} \int \left[x(\tilde\theta,\theta')\theta - t(\tilde\theta,\theta') - \delta(\tilde\theta,\theta')\right] dG(\theta') + \min_{G\in\Delta_B} \int \delta(\tilde\theta,\theta') dG(\theta')
\end{aligned}
$$

But note that

$$
\begin{aligned}
\min_{G\in\Delta_B} \int \delta(\tilde\theta,\theta') dG(\theta') &= \min_{G\in\Delta_B} \int \left[q(\tilde\theta,\theta') - K(\tilde\theta)\right] dG(\theta') \\
&= \min_{G\in\Delta_B} \int q(\tilde\theta,\theta') dG(\theta') - K(\tilde\theta) = 0
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \min_{G\in\Delta_B}\int\left[x(\tilde{\theta},\theta')\theta-t'(\tilde{\theta},\theta')\right]dG(\theta') \\
\leq\; & \min_{G\in\Delta_B}\int\left[x(\tilde{\theta},\theta')\theta-t(\tilde{\theta},\theta')\right]dG(\theta')-\min_{G\in\Delta_B}\int\delta(\tilde{\theta},\theta')dG(\theta') \\
=\; & \min_{G\in\Delta_B}\int\left[x(\tilde{\theta},\theta')\theta-t(\tilde{\theta},\theta')\right]dG(\theta')
\end{aligned}
$$

Now the payoff for type θ to truth-telling must be weakly larger than the last expression, because the mechanism (x,t) was assumed to be incentive compatible. Hence the payoff to truth-telling is weakly larger than deviating to any type $\tilde{\theta}\in\widetilde{\Theta}$. Hence the mechanism (x,t') is incentive compatible.

Since the mechanism (x,t') makes the seller strictly better off, leaves buyers' payoffs unchanged, and remains incentive compatible, the mechanism (x,t) must not have been optimal. This completes the proof. ■

Proof of Proposition 3

Proof. We know by Proposition **??** that there is a measure 1 set $\bar{\Theta}\subseteq\Theta$ such that for all $\theta\in\bar{\Theta}$ we have $F\in\Delta(\theta)$. For any $\theta\in\bar{\Theta}$ we have

$$
\min_{H\in\Delta}\int q(\theta,\theta')dH(\theta')=\int q(\theta,\theta')dF(\theta')=K(\theta).
$$

Now suppose that for some $\bar{\theta}\in\bar{\Theta}$, $q(\bar{\theta},\theta)$ is not constant for a positive measure set of θ. Let Θ_+ be the set of all $\theta'\in\Theta$ such that $q(\bar{\theta},\theta')>K(\bar{\theta})$ and let Θ_- be the set of all $\theta'\in\Theta$ such that $q(\bar{\theta},\theta')<K(\bar{\theta})$. Clearly both Θ_+ and Θ_- have positive measure. Let $\Theta_+\bigcup\Theta_-=\hat{\Theta}$ and note that $\Theta_+\bigcap\Theta_-=\emptyset$.

Now construct the following measure dH (corresponding to the distribution H) which increases the probability weight on the low-payoff states:

$$
dH(\theta')=\begin{cases}=\frac{1}{2}dF(\theta'), & \text{if } \theta'\in\Theta_+\\ =(1+\frac{1}{2}\gamma)dF(\theta'), & \text{if } \theta'\in\Theta_-\\ =dF(\theta'), & \text{if } \theta'\in\Theta-\hat{\Theta}\end{cases}
$$

where $\gamma=\frac{\int_{\Theta_+}dF}{\int_{\Theta_-}dF}$[6]. Note that $\Theta-\hat{\Theta}$ could be the empty set. This would not influence our results.

[6]Since Θ_- and Θ_+ are of positive measure, $0<\gamma<\infty$. If one of these sets were not of positive measure, then there would not be positive probability mass that could be moved relative to F, so no such H could be constructed.

Claim: H is a well-defined distribution, in that $H(\theta)$ is increasing in θ, and $\int dH(\theta') = 1$.

Proof: $H(\theta)$ increasing trivially, since dF is a measure and dH is constructed as a positive multiple of dF.

Now

$$\begin{aligned}\int dH(\theta') &= \frac{1}{2}\int_{\Theta_+} dF(\theta') + (1+\frac{1}{2}\gamma)\int_{\Theta_-} dF(\theta') + \int_{\Theta-\hat{\Theta}} dF(\theta') \\ &= \frac{1}{2}\int_{\Theta_+} dF(\theta') + \int_{\Theta_-} dF(\theta') + \frac{1}{2}\frac{\int_{\Theta_+} dF}{\int_{\Theta_-} dF}\int_{\Theta_-} dF(\theta') + \int_{\Theta-\hat{\Theta}} dF(\theta') \\ &= \int_{\Theta_+} dF(\theta') + \int_{\Theta_-} dF(\theta') + \int_{\Theta-\hat{\Theta}} dF(\theta') = \int dF(\theta') = 1.\end{aligned}$$

So H is indeed a well-defined distribution.

Now, we will show that, if $q(\bar{\theta},\theta)$ is not constant for a positive measure set of θ, then the existence of $(1-\epsilon)F + \epsilon H$ in Δ_B implies that $F \notin \Delta(\theta)$, which leads to a contradiction. First note that,

$$\int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta') > K(\bar{\theta})\int_{\Theta_+} dF,$$

since for all $\theta' \in \Theta_+$, $q(\bar{\theta},\theta') > K(\bar{\theta})$.Similarly,

$$\int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') < K(\bar{\theta})\int_{\Theta_-} dF,$$

since for all $\theta' \in \Theta_-$, $q(\bar{\theta},\theta') < K(\bar{\theta})$. Putting these two inequalities together we obtain,

$$\begin{aligned}\frac{\int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta')}{\int_{\Theta_-} dF} &< \frac{\int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta')}{\int_{\Theta_+} dF} \qquad (13)\\ &\Leftrightarrow \frac{\int_{\Theta_+} dF}{\int_{\Theta_-} dF}\int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') < \int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta') \\ &\Leftrightarrow \gamma\int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') < \int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta')\end{aligned}$$

Now we show that,

$$\int q(\bar{\theta},\theta')\left[(1-\epsilon)dF(\theta') + \epsilon dH(\theta')\right] < \int q(\bar{\theta},\theta')dF(\theta')$$

which holds if and only if

$$\int q(\bar{\theta},\theta')dH(\theta') < \int q(\bar{\theta},\theta')dF(\theta').$$

But,

$$\int q(\bar{\theta},\theta')dH(\theta') = \frac{1}{2}\int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta') + (1+\frac{1}{2}\gamma)\int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') + \int_{\Theta-\hat{\Theta}} q(\bar{\theta},\theta')dF(\theta')$$

And,

$$\int q(\bar{\theta},\theta')dF(\theta') = \int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta') + \int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') + \int_{\Theta-\hat{\Theta}} q(\bar{\theta},\theta')dF(\theta').$$

So this implies that

$$\int q(\bar{\theta},\theta')dH(\theta') < \int q(\bar{\theta},\theta')dF(\theta')$$

if and only if

$$\gamma \int_{\Theta_-} q(\bar{\theta},\theta')dF(\theta') < \int_{\Theta_+} q(\bar{\theta},\theta')dF(\theta'),$$

which holds by (13). ■

Proof of Lemma 4

Proof. We need to show that there exists $M > 0$ such that

$$\left| u(\theta) - u\left(\tilde{\theta}\right) \right| \leq M \left| \theta - \tilde{\theta} \right|.$$

We know that,

$$\left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \leq u(\theta) - u\left(\tilde{\theta}\right) \leq \left(\theta - \tilde{\theta}\right) X^{\max}\left(\tilde{\theta}\right).$$

So if $\theta > \tilde{\theta}$, using the fact that u is increasing we can conclude that,

$$u(\theta) - u\left(\tilde{\theta}\right) \leq \left(\theta - \tilde{\theta}\right) X^{\max}\left(\tilde{\theta}\right) \leq \left|\theta - \tilde{\theta}\right|.$$

Similarly if $\theta < \tilde{\theta}$, then

$$-\left(u(\theta) - u\left(\tilde{\theta}\right)\right) \leq -\left(\theta - \tilde{\theta}\right) X^{\min}\left(\tilde{\theta}\right) \leq \left|\theta - \tilde{\theta}\right|.$$

Together these imply that Lipschitz condition holds with $M = 1$. ■

Proof of Proposition 9

Proof. Define $T(\theta)$ as buyer θ's expected transfer under F, that is,

$$T(\theta) = \int t(\theta, \theta')dF(\theta') \tag{14}$$

Now let

$$\tilde{t}(\theta, \theta') = T(\theta) - T(\theta') + \int T(i)dF(i) \tag{15}$$

We show that the mechanism $(x, \tilde{t})$ makes the seller strictly better off, leaves the buyers' payoffs unchanged, and is incentive compatible.

Claim 1: $(x, \tilde{t})$ **makes the seller strictly better off**

TThe seller's payoff for the mechanism (x, t') is:

$$\begin{aligned} & \min_{G \in \Delta_S} \iint \left[\tilde{t}(\theta, \theta') + \tilde{t}(\theta', \theta)\right] dG(\theta)dG(\theta') \\ = \; & \min_{G \in \Delta_S} \iint [T(\theta) - T(\theta') + \int T(i)dF(i) + T(\theta') - T(\theta) + \int T(j)dF(j)]dG(\theta)dG(\theta') \\ = \; & \min_{G \in \Delta_S} \iint \left[2 \int T(i)dF(i)\right] dG(\theta)dG(\theta') \\ = \; & 2 \int T(i)dF(i) \\ = \; & 2 \iint t(\theta, \theta')dF(\theta)dF(\theta') \end{aligned}$$

However, $F \notin \Delta_S^{\min}$ implies:

$$\min_{G \in \Delta_S} \iint \left[t(\theta, \theta') + t(\theta', \theta)\right] dG(\theta)dG(\theta') < 2 \iint t(\theta, \theta')dF(\theta)dF(\theta')$$

Hence the seller is made strictly better off by this change.

Claim 2: $(x, \tilde{t})$ **leaves the buyers' payoffs unchanged under truth-telling**

By construction:

$$\begin{aligned}
\int \tilde{t}(\theta,\theta')dF(\theta') &= \int \left[T(\theta) - T(\theta') + \int T(i)dF(i)\right] dF(\theta') \\
&= T(\theta) - \int T(\theta')dF(\theta') + \int T(i)dF(i) \\
&= T(\theta)
\end{aligned}$$

Since $T(\theta) = \int t(\theta,\theta')dF(\theta')$, we are finished.

Claim 3: (x, t') **is incentive compatible**
The payoff for type θ to pretend to be $\tilde{\theta}$ remains unchanged. Namely, θ's deviation payoff under (x, t) is:

$$\begin{aligned}
& \int \left[\theta x(\tilde{\theta},\theta') - t(\tilde{\theta},\theta')\right] dF(\theta') \\
= & \int \theta x(\tilde{\theta},\theta')dF(\theta') - \int t(\tilde{\theta},\theta')dF(\theta') \\
= & \int \theta x(\tilde{\theta},\theta')dF(\theta') - T(\tilde{\theta}) \\
= & \int \theta x(\tilde{\theta},\theta')dF(\theta') - \int \tilde{t}(\tilde{\theta},\theta')dF(\theta')
\end{aligned}$$

as per the above proof. Hence the payoff to deviation is unchanged for all types $\theta \in \Theta$, so, since (x, t) was incentive compatible, $(x, \tilde{t})$ must be as well.

Since the mechanism (x, t') makes the seller strictly better off, leaves buyers' payoffs unchanged, and remains incentive compatible, the mechanism (x, t) must not have been optimal. This completes the proof. ■

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