Individually Rational, Balanced-Budget Bayesian Mechanisms and the Informed Principal Problem.

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Abstract

We investigate the issue of implementation via individually rational ex-post budget-balanced Bayesian mechanisms. We demonstrate that all social choice rules that generate a nonnegative ex-ante surplus, including ex-post efficient ones, can generically be implemented via such mechanisms. The aggregate expected surplus in these mechanisms can be distributed in an arbitrary way. Also generically, any ex-post efficient social choice rule can be implemented in an informed principal framework, i.e. when the mechanism is offered by one of the informed parties. Only ex-post efficient social choice rules that allocate all surplus to the party designing the mechanism are both sequential equilibria and *neutral optima*, i.e. outcomes that can never be blocked.
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1 Introduction.

This paper focuses on three issues. First, we explore the existence of individually rational ex-post budget-balanced Bayesian mechanisms for implementing a broad class of social choice rules including ex-post efficient ones. Then we study the related issue of surplus allocation in such mechanisms. We also explore the issue of implementation in the informed principal context when the mechanism is designed not by an outsider, but by one of the participants after she has learned her private information.

The theory of Bayesian mechanism design provides a universally accepted implementation tool which is used in a variety of environments, including contracting, auctions, bargaining, etc. For this reason, it is important to understand the scope and limits of Bayesian implementation. One of the well-known issues in this regard is a tension between budget balancing,

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individual rationality and efficiency which one can reasonably view as desirable properties of a mechanism. Myerson and Satterthwaite (1983) have established the impossibility of Bayesian implementation of allocation rules having these three properties when private information is independently distributed across agents.[1]

Relaxing one of these three requirements makes it possible to obtain positive results. Various sufficient conditions for Bayesian implementation of efficient social choice rules with ex-post budget balancing but without individual rationality requirement have been derived by d'Aspremont and Gérard-Varet (1979), d'Aspremont, Crémer and Gérard-Varet (1990), (1996) and (2003), Matsushima (1991), Fudenberg, Levine and Maskin (1996), Aoyagi (1998), and Chung (1999). Sufficient conditions exhibited by these authors differ in terms of their generality and ease of use. d'Aspremont, Crémer and Gérard-Varet (2003) present the most general one.[2]

Crémer and McLean (1985) and (1988) and McAfee and Reny (1992) demonstrate that an uninformed mechanism designer can implement an ex-post efficient and individually rational social choice rule and extract all expected surplus from the players under a generic condition on the probability distribution of the players' types.[3] Their mechanisms are not ex-post budget-balanced. The uninformed mechanism designer plays an important role of a budget breaker. She collects transfers from the players, and may also have to pay them in some states of the world.[4] Without imposing ex-post budget balancing, McLean and Postlewaite (2002) show that only small transfers from/to each player are needed to implement a social choice rule when each player is 'informationally small.' That is, even if a player misrepresents her private information, the state of the world can still be inferred with a high degree of accuracy provided that all other players report truthfully.

Thus, the main difference between our paper and the existing literature lies in the fact that we require our mechanisms to possess all the three properties in question- interim individual rationality, ex-post budget balancing and efficiency. As far as efficiency is concerned, we take a broad approach and focus on the class of *ex-ante socially rational* social choice rules- the ones that generate a nonnegative ex-ante social surplus. This class includes ex-post efficient social choice rules as a special case. In fact, it is easy to see that ex-ante social rationality is a necessary condition for a mechanism to be interim individually rational and ex-post budget balanced.

Our main result demonstrates that any ex-ante socially rational decision rule can be implemented via an ex-post budget-balanced and interim individually rational mechanism when there exists a pair of players i^* and j^* such that i^* is *identifiable with respect to* j^*, whereas for

[1] Myerson and Satterthwaite (1983) have focused on the case where private information is distributed continuously on its support. However, their result also extends to the case of discrete distribution of private information.

[2] In particular, budget balanced mechanisms exist when the types are distributed independently and a simple regularity condition on the utility functions hold.

[3] Essentially this condition requires private information to be correlated across players, so that compared to the prior a player's type contains additional information about other players' types. The mechanism designer exploits this to crosscheck a player's report, and thereby induces her to be truthful without leaving any informational rent to her, i.e. extracts all her surplus.

[4] Ex-ante budget balance can be attained in a Crémer-McLean mechanism if the mechanism designer pays each agent an amount equal to her ex-ante transfer in the ensuing mechanism.

players other than i^* we require the well-known condition of Crémer and McLean (1985) for belief extraction. Our identifiability condition allows to distinguish any situation where player i^* has misrepresented her type and other players have reported their types truthfully from any situation where all players other than j^*, including player i^*, tell the truth, no matter what strategy j^* follows. In our mechanism, this property can be exploited to design a lottery for player i^* that punishes her (gives a negative expected payoff) when she deviates and player j^* reports truthfully, but gives player i^* a non-negative interim expected payoff when she tells the truth no matter what player j^* does.

To understand how our mechanism works, it is natural to start from the Crémer-McLean mechanism as a benchmark. In this mechanism, the principal extracts information about a player's type by offering her a lottery which has zero expected value if the player has reported her type truthfully and a negative expected value if the player has misrepresented her type.

However, if one attempts to use the Crémer-McLean approach (their conditions are assumed to hold) in our case where the mechanism has to be ex-post budget balanced and so an outside budget-breaker is not available, then we have to resolve an additional issue of allocating the transfers from such lotteries in an incentive compatible way. In particular, designating player i to receive transfers from the lottery given to player j may generate incentives for player i to misrepresent her type in a way that makes a truthful report by j to appear untruthful which would cause j to pay positive transfers to i.

Our Identifiability Condition allows to resolve this issue and preserve the individual rationality. The key feature of our mechanism which is described in detail in Section 3, is the following. When player i^* is identifiable with respect to j^* we can make i^* act as a residual claimant, or a 'sink,' for lotteries given to all other agents and j^* act as a 'residual claimant,' or s 'sink,' for the lottery given to i^*. The fact that i^* is identifiable with respect to j^* implies that neither i^* nor j^* have the ability to exploit their roles as residual claimants and rig the outcomes of those lotteries in their favor. So, the lotteries can be constructed to satisfy individual rationality and incentive compatibility constraints of all the players.

Our Identifiability condition is related to the Pairwise Identifiability Condition (PIC) introduced by Fudenberg, Levine and Maskin (1994) in the analysis of repeated games. Fudenberg, Levine and Maskin (1996) show that Pairwise Identifiability is sufficient for ex-post budget-balancing *without* the interim individual rationality requirement. We discuss the relation between the two conditions in more detail in Section 3. Intuitively, the Pairwise Identifiability condition allows to distinguish a deviation by one player from a deviation by another player, whereas in addition to that our Identifiability Condition allows to distinguish a deviation by the first player from the situation where this player tells the truth.

Naturally, our Identifiability Condition is stronger than PIC, but the argument in Fudenberg, Levine and Maskin (1996) relies on the assumption that for each player i there is counterpart j s.t. the Pairwise Compatibility holds for the pair. In contrast, we require the existence of only *one pair* of players for whom the Identifiability Condition holds. For the other players, a weaker condition of Crémer and McLean (1985) is sufficient. We also show that our conditions are generic when there are at least three players in the mechanism and none has more types than the number of type profiles of all other players.[5]

[5]Naturally, the impossibility result of Myerson and Satterthwaite (1983) implies that the Identifiability does

Ex-post budget balancing implies that all surplus generated by the mechanism is distributed among the players and is not extracted by an outside party (mechanism designer) as in Crémer and McLean (1985) and (1988) and McAfee and Reny (1992). So, it is natural to consider how the surplus can be allocated among the participants. We show that the expected surplus in our mechanisms can be allocated in any arbitrary way. That is, generically we can construct an individually rational mechanism where the aggregate expected surplus is distributed across *player types* in any desired way.

The allocation of surplus result has important implications for the analysis of the so-called informed principal problem. This problem arises when an uninformed third party (principal, mechanism designer) is not available to design a mechanism, and so this task has to be performed by one of the participants in the mechanism who has already learned her private information. Then the choice of a mechanism by a player can serve as a signal to other players about this player's type. Thus, the optimal mechanism has to balance the interests of different types of the player designing the mechanism. Otherwise, some types of this player would deviate by offering a different mechanism. These aspects make the analysis of the informed principal problem more complex.

Mechanism design by informed principal has been studied by Myerson (1983), Maskin and Tirole (1990) and (1992) who propose several solution concepts to this problem. However, except for some cases, the issue of characterizing the outcomes of this game has not been resolved.

We contribute to the study of the informed principal problem by demonstrating that any ex-post efficient social choice rule can generically be implemented in this environment. So, the informed principal problem need not cause any loss of efficiency. This result is quite robust. It is supported in a sequential equilibrium, core mechanism and neutral optimum. The latter concept is introduced by Myerson (1983). It represents 'the smallest possible set of unblocked mechanisms.'

We show that an allocation profile implementable as a neutral optimum is generically unique. It involves an ex-post efficient social choice rule, and allocates all social surplus to the player designing the mechanism in such a way that each type of her gets all expected social surplus conditional on her type. To prove this we rely on our central result that this allocation profile can generically be implemented in the standard mechanism design environment (with an outsider acting as a mechanism designer). We then demonstrate that if the player getting all surplus acts as a mechanism designer, she will offer the same mechanism. This result holds in a sequential equilibrium, core of neutral optimum solution. Moreover, any other allocation profile (i.e. not ex-post efficient one or one where some type of the informed principal gets less then expected social surplus conditional on her type) is not a neutral optimum: it can be blocked, as the mechanism designer will have a profitable and credible deviation.

The rest of the paper is organized as follows. In section 2 we develop the model. In section 3 we establish out cental implementation result. In section 4 allocation of surplus is analyzed. Section 5 deals with the informed principal problem. All proofs are relegated to the appendix.

not hold when the types are independently distributed.

2 The Model

There are n players/agents in the economy.[6] Agent i has privately known type θ_i which belongs to the type space $\Theta_i \equiv \{\theta_i^1, ..., \theta_i^{m_i}\}$ of cardinality $m_i < \infty$. A state of the world is characterized by a profile of types $\theta = (\theta_1, ..., \theta_n)$. The set of type profiles is given by $\Theta \equiv \prod_{i=1,..n} \Theta_i$ which has cardinality $L \equiv \prod\limits_{i=1,...,n} m_i$. When focussing on agent i, we will use the notation (θ_{-i}, θ_i) for the profile of agent types, where θ_{-i} stands for the profile of types of agents other than i. Let $\Theta_{-i} = \prod_{j \neq i} \Theta_j$ and $L_{-i} = \prod_{j \neq i} m_j$. θ_{-i-j} and Θ_{-i-j} are defined similarly. In Sections 2-4 we also assume the presence of a principal who does not possess any private information and acts as a mechanism designer.[7]

Let X denote the set of public decisions controlled by the mechanism designer, and x denote typical element of X. Agent i's utility function is quasilinear in the decision x and transfer t_i that she receives from the mechanism and is given by $u_i(x, \theta) + t_i$. Without loss of generality, an agent's reservation utility is normalized to zero. Let $t_i(\theta)$ be a transfer function to agent i, and $t(\theta) = (t_1(\theta), ..., t_n(\theta))$ be a collection of transfer functions to all agents. An allocation profile is a combination of a social choice rule $x(\theta)$ with a collection of transfer functions $t(\theta)$. Finally, let $p(\theta)$ denote the probability distribution over the type profiles, and $p(\theta_{-i}|\theta_i)$ denote the probability distribution over types of agents other than i conditional on the type of agent i. We assume that $p(\theta)$ is common knowledge

In the current case with an uninformed principal, we can use the Revelation Principle to restrict the analysis to a class of direct mechanisms in which the principal offers an allocation profile to the agents. The agents then decide whether to participate in the mechanism. If they decide to stay in the mechanism, the agents report their types and the allocation corresponding to the reported type profile is implemented.

Our central goal is to provide sufficient conditions for Bayesian implementation of allocation profiles that satisfy interim individual rationality and ex-post budget balancing. Let us describe these properties formally.

We will say that the allocation profile $(x(\theta), t(\theta))$ is Bayesian implementable (i.e. implementable in Bayesian equilibrium) if the following *Interim Incentive Constraints* (*IC*) are satisfied for all i and $\theta_i, \theta_i' \in \Theta_i$[8].

$$\sum_{\theta_{-i} \in \Theta_{-i}} \big(u_i(x(\theta_{-i}, \theta_i), (\theta_{-i}, \theta_i)) + t_i(\theta_{-i}, \theta_i) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - t_i(\theta_{-i}, \theta_i')\big)\, p(\theta_{-i}, \theta_i) \geq 0 \tag{1}$$

The *Interim Individual Rationality (IIR)* constraint is given by the following:[9]

$$\sum_{\theta_{-i} \in \Theta_{-i}} \left(u_i(x(\theta_{-i}, \theta_i), (\theta_{-i}, \theta_i)) + t_i(\theta_{-i}, \theta_i)\right) p(\theta_{-i}, \theta_i) \geq 0 \quad \forall \theta_i \in \Theta_i \text{ and } \quad i \tag{2}$$

[6] In the sequel, we will use the terms 'player' and 'agent' interchangeably.

[7] In section 5 we analyze the informed principal problem where an outside principal is not available and the mechanism has to be designed by one of the informed agents.

[8] We also use the notation $IC_i(\theta_i^k, \theta_i^{k'})$ and $IC_i(k, k')$ to denote an incentive constraint preventing type θ_i^k from imitating $\theta_i^{k'}$.

[9] We will use the notation $IIR_i(\theta_i)$ to denote the interim individual rationality constraint of player type θ_i.

The *Ex-post Budget Balancing (BB)* constraint can be written as follows:

$$\sum_{i=1}^{n} t_i(\theta) = 0 \quad \forall \theta \in \Theta \tag{3}$$

IIR and BB together imply the following *Ex-Ante Social Rationality (EASR)* condition[10]:

$$\sum_{\theta \in \Theta} \sum_{i=1}^{n} u_i(x(\theta), \theta) p(\theta) \geq 0 \tag{4}$$

$EASR$ must hold because an allocation profile satisfying IIR and BB must necessarily generate a nonnegative ex ante surplus. Clearly, EASR is a very weak condition. It encompasses a large variety of social choice rules including the ex-post efficient ones.

With EASR as a necessary condition on the social choice rule, we turn to the issue of sufficient conditions for Bayesian implementation of IIR and BB allocation profiles in the next section.

3 Main Results.

Let us start by introducing some notation. Let W_i be an $m_i(m_i - 1) \times L$[11] matrix the rows of which represent the vectors of the probability distributions of *reported* types when agent i misrepresents her *realized* type and all other agents report their types truthfully. Each row of W_i corresponds to one of $m_i - 1$ pure misrepresentation strategies of one of m_i types of player i, and each column corresponds to one of L possible type profiles (states of the world) in the natural order induced by the ordering of players and their types. The entries in the row corresponding to agent i of type k reporting type k' are equal to $p(\theta_{-i}, \theta_i^k)$ in the column corresponding to type profile $(\theta_{-i}, \theta_i^{k'})$ for all $\theta_{-i} \in \Theta_{-i}$[12] and zero in all other columns. This row is $(m_i - 1)k + k'$-th if $k' < k$ and $(m_i - 1)k + k' - 1)$ if $k' > k$.

Similarly, let P_i be an $m_i \times L$ matrix the rows of which represent the vectors of the true probability distributions over type profiles for each possible type of player i. Thus, the entries in the k-th row of P_i are equal to $p(\theta_{-i}, \theta_i^k)$ in the columns corresponding to type profiles $(\theta_{-i}, \theta_i^k)$ for all $\theta_{-i} \in \Theta_{-i}$ and zero in all other columns.

Also, for any collection of matrices $A_1, ..., A_t$ where A_i is of size $c_i \times d$, let $\left\| \begin{matrix} A_1 \\ ... \\ A_t \end{matrix} \right\|$ be a matrix formed by stacking matrices A_1,...,A_t one on top the other in order of their indices. Finally, let the operator $rank(.)$ denote the row rank of a matrix, i.e. the dimension of the space spanned by its rows.

[10] To the best our knowledge, in the mechanism design context this condition was first used by d'Aspremont and Gérard-Varet (1979).

[11] Recall that $L \equiv \prod_{l=1,\ldots,n} m_l$

[12] Strictly speaking, we need to use conditional, rather than marginal, probabilities as entries in this matrix. However, the difference between these two probability vectors is a matter of simple normalization, i.e. dividing all entries by $p_i(\theta_i^k)$, which can be omitted.

Definition 1 ***Identifiability Condition.*** *Say that agent i is identifiable with respect to agent j if the following condition holds:*

$$rank\left(\left\|\begin{array}{c} W_i \\ P_i \\ W_j \\ P_j \end{array}\right\|\right) = rank\left(W_i\right) + rank\left(\left\|\begin{array}{c} P_i \\ W_j \\ P_j \end{array}\right\|\right) \tag{5}$$

The main result of this paper is presented in the following theorem:

Theorem 1 *Any ex-ante socially rational choice rule $x(\theta)$ can be implemented via a Bayesian mechanism satisfying IIR and BB if:*

(i) There exists a pair of players i^* and j^* such that i^* is identifiable with respect to j^*.

(ii) There exists a player $k^* \neq i^*$ and $\theta_{i^*}^u \in \Theta_{i^*}$ s.t. $p_{-i^*-k^*}(.|\theta_{i^*}^u, \theta_{k^*}) \neq p_{-i^*-k^*}(.|\theta_{i^*}^u, \theta'_{k^*})$ for all $\theta_{k^*}, \theta'_{k^*} \in \Theta_{k^*}$.

(iii) For any player $h \neq i^*$ and any type $\theta_h^k \in \Theta_h$, there does not exist a collection of nonnegative multipliers $\{\zeta_h^{kk'}\}$, s.t.

$$p_{-h}(\theta_{-h}|\theta_h^k) = \sum_{k' \in \{1,\ldots,m_h\}, k' \neq k} \zeta_h^{kk'} p_{-h}(\theta_{-h}|\theta_h^{k'}) \text{ for all } \theta_{-h} \in \Theta_{-h}$$

(iv) For any pair of types $\theta_d \in \Theta_d$ and $\theta_l \in \Theta_l$ of any two players d and l, there exists $\theta_{-d-l} \in \Theta_{-d-l}$ s.t. $p(\theta_{-d-l}, \theta_d, \theta_l) > 0$.

Before we describe the mechanism and provide the intuition for Theorem 1, a few comments regarding conditions (i)-(iv) are in order. Condition (iv) is a simple regularity property. As shown by Crémer and McLean (1988), Condition (iii) is necessary and sufficient for the existence of an efficient, individually rational mechanism with full surplus extraction by the mechanism designer, and hence without ex-post budget balance. We refer to it as Crémer-McLean condition. Intuitively, it allows to elicit private information of an agent without any cost to the mechanism, i.e. without leaving an informational rent to the agent. Condition (ii) is a further refinement of Condition (iii). It serves to ensure that Crémer-McLean conditions (iii) also work in our context and allow to elicit private information of agents other than i^* when their reports are compared to i^*'s reports.

The Identifiability Condition (i) is a new condition introduced in this paper. It plays an important role in our analysis. Formally, it is easily recognizable as a spanning condition. To understand it, consider a revelation mechanism in which all players report their types. The agents' reporting strategies together with the underlying distribution of types induce a probability distribution of the reported type profiles. Consider two subspaces in the space of vectors characterizing all such probability distributions.

The first subspace is spanned by the vectors of probability distributions of reported types induced by all pure non-truthful reporting strategies of each type of player i^* and truthful reporting strategies of the other players. The second subspace is spanned by the vectors of probability distributions of reported type profiles induced by both: (i) truthful reporting strategies of all players given any realized type of player i^*; (ii) all possible pure reporting strategies (truthful or non-truthful) of different types of player j^* and truthful reporting strategies of other players. Then i^* is identifiable with respect to j if these two linear subspaces have an empty intersection.

Intuitively, this condition implies that a combination of *any non-truthful reporting strategy*[13] (possibly mixed) of any type of player i^* and truthful reporting strategies of players other than i^* always induces a probability distribution of the reported types that is different from the probability distribution of the reported types when i^* and all agents other than j^* report truthfully, no matter whether j^* reports her type truthfully or not. Thus, any misrepresentation by any type of player i^* combined with truth-telling by other agents can be distinguished from the situation where she and agents other than j^* report truthfully while agent j^* follows an arbitrary strategy.

So, it is possible to punish agent i^* for a misrepresentation by means of a lottery that gives her a negative expected payoff when she has, in fact, deviated, without giving agent j^* an opportunity to imitate a deviation by i^* and cause a punishment upon i^* when the latter reports truthfully. This feature turns out to be quite important for our mechanism to the description of which we now proceed.

To guarantee truthful revelation of types our mechanism can rely on lotteries in which the transfer to/from an agent is a function of other agents' reported types. Crémer and McLean (1985) have shown that this method works successfully in the presence of a budget-breaker principal who runs the lottery: a lottery has zero expected value for the player if she reports her type truthfully and a negative expected value if the player misrepresents her type. However, ex-post budget balancing requirement in our mechanism implies that, unlike in Crémer and McLean (1985), all transfers and payments have to be distributed among other players. Consequently, we must designate residual claimants for a lottery given to each player to elicit that player's information, and deal with the issue of the incentives of the players who act as such residual claimants. For example, designating player j to receive transfers from player i may generate incentives for j to 'rig the lottery:' misrepresent her type in a way that makes a truthful report by i and so would cause i to pay large transfers that go to j.

To highlight the role of the individual rationality and contrast our mechanism from the mechanisms that ensure ex-post budget balancing without individual rationality, consider the following benchmark mechanism which allows to avoid the issue of the incentives of residual claimants. Suppose that we divide all players into groups and run a separate Crémer-McLean mechanism in each group always using an agent outside the group as a residual claimant for the balance of transfers from/to this group. The incentive compatibility of this mechanism is easy to ensure by making the lotteries in each group independent of the residual claimant's reported type. Clearly, such mechanism will also be budget balanced. However, individual rationality may fail for the following reason. Indeed, the aggregate transfer from/to each group to its 'residual claimant' is determined independently of the 'residual claimant's reported type. But, given statistical interdependence between the agents' types, the probability distribution of these aggregate transfers and hence the expected utility of the 'residual claimant' will be dependent on her type. If on the basis of her private information the residual claimant will

[13] i.e. a strategy where i^* misrepresents her type at least with some probability and does so in any conceivable way.

expect a negative payoff, she would refuse to participate in this mechanisms.

Thus, all lotteries have to depend of the whole reported type profile, and one has to deal carefully with the incentives of the residual claimants. Our identifiability condition allows to resolve this issue. Note that Theorem 1 requires the existence of *only one pair of agents* i^* and j^* such that i^* is identifiable with respect to j^*. These agents are given special roles in our mechanism.

Specifically, our mechanism works as follows. Each player is given a lottery that punishes her if she deviates. i^* act as a residual claimant, or a 'sink,' for lotteries given to all other agents and j^* acts as a 'residual claimant,' or s 'sink,' for the lottery given to i^*. The fact that i^* is identifiable with respect to j^* implies that neither i^* nor j^* have the ability to exploit their roles as residual claimants and the lotteries can be constructed to satisfy individual rationality constraints of all the players.

Indeed, by identifiability of i^* with respect to j^*: (i) If i^* reports truthfully, then j^* cannot imitate a deviation by player i^* and thus trigger the transfers from i^* to j^*. So, j^* cannot exploit her position as a sink for i^*. (ii) When all players other than i^* report truthfully, any deviation by i^* can be statistically detected and punished. Thus, player i^* cannot imitate a deviation by other player and trigger payments from that player to i^* without incurring even bigger penalties payable to j^*. Finally, the Crémer-McLean conditions for players other than i^* imply that a unilateral deviation by any such player can be punished by making this player to pay transfers to i^*.

The next result establishes that the conditions of Theorem 1 are almost always satisfied and hence and $EASR$ social choice rule can almost always be implemented via an IIR and BB mechanism.

Lemma 1 *Suppose that there are at least three players ($n \geq 3$) and $\Pi_{j \neq i} m_j \geq m_i$ for all i. Also, if $n = 3$ then at least one of players has at least three types. Then the conditions (i)-(iv) of Theorem 1 are generic, i.e. hold for almost all probability distributions $p(.)$.*

Two conditions studied in the literature are related to our Identifiability Condition: the *Pairwise Identifiability Condition (PI)* of Fudenberg, Levine and Maskin (1994) and (1996) and the Compatibility Condition of d'Aspremont and Gérard-Varet (1979) and d'Aspremont, Crémer and Gérard-Varet (1990).

The PI Condition, which is a close relative of our condition, is defined as follows. Consider all pure strategy profiles of agent i in a direct mechanism. There are $m_i^{m_i}$ of them, as the set of these strategies is isomorphic to the set of functions from the set of i's possible types into itself. Fix a numbering scheme for these strategies induced by the natural order of agent i's types. (Any arbitrary numbering scheme would do as well). Let Π_i be an $(m_i^{m_i}) \times L$ matrix whose k-th row is the probability distribution over the reported type profiles when agent i uses her k-th strategy and all other agents report their types truthfully. Further, let $\left\| \begin{array}{c} \Pi_i \\ \Pi_j \end{array} \right\|$ be a matrix formed by stacking Π_i on top of Π_j. Then the *Pairwise Identifiability Condition (PI)* for players i and j holds if:

$$rank\left(\left\| \begin{array}{c} \Pi_i \\ \Pi_j \end{array} \right\| \right) = rank(\Pi_i) + rank(\Pi_j) - 1$$

Fudenberg, Levine and Maskin (1996) show that PI is a sufficient condition for the ex-post budget-balanced implementation of an implementable social choice rule.[14]

PI holds for players i and j if and only if the linear space spanned by the vectors of probabilities of type reports induced by all possible non-truthful reporting strategies of player i does not intersect the linear space spanned by the vectors of probabilities of type reports induced by all non-truthful reporting strategies of player j. Comparing this description with the description of our Identifiability Condition, one can conclude that the Identifiability Condition is a stronger one. This is formally established in the following lemma:

Lemma 2 *The Identifiability Condition implies PI.*

Proof: See the Appendix.

To establish that the Identifiability Condition is strictly stronger than PI, note that PI holds when types are independent, as shown by Fudenberg, Levine and Maskin (1996). Yet, we have:

Lemma 3 *If types are distributed independently, then Identifiability Condition does not hold.*

Proof: See the Appendix.

Lemma 4 *If player i is identifiable with respect to some player j then the following Compatibility*-(i) Condition holds:*

Consider any collection of vectors of coefficients $\mu \in R^L$, $\lambda_h \in R_+^{m_h}$, $\gamma_h \in R_+^{m_h(m_h-1)}$ $h = 1, ..., n$ such that for all players $h = \{1, ..., n\}$ and states $(\theta_{-h}, \theta_h^k) \in \Theta$

$$\mu(\theta_{-h}, \theta_h^k) + \lambda_h^k p(\theta_{-h}, \theta_h^k) + \sum_{k' \neq k} \gamma_h^{kk'} p(\theta_{-h}, \theta_h^k) = \sum_{k' \neq k} \gamma_i^{k'k}, p(\theta_{-h}, \theta_h^{k'}) \tag{6}$$

$$\text{Then,} \quad \mu(\theta_{-i}, \theta_i^k) + \lambda_i^k p(\theta_{-i}, \theta_i^k) = 0 \quad \text{for any } (\theta_{-i}, \theta_i^k) \in \Theta \tag{7}$$

Proof: See the Appendix.

Our Compatibility*-(i) Condition is closely related to the Compatibility conditions of d'Aspremont and Gérard-Varet (1979) which guarantees that an implementable allocation profile can be implemented via a balanced-budget mechanism. In one respect, our Compatibility* condition is stronger because it contains additional multipliers λ_i stemming from IIR constraints. However, we only require (7) to hold for player i, whereas the corresponding condition in d'Aspremont and Gérard-Varet (1979) has to hold for all players.

Using the Compatibility*-(i) condition we can establish the following Theorem.

Theorem 2 *Fix any EASR social choice rule $x(\theta)$. Suppose there is a player i such that Compatibility*-(i) Condition holds and Condition (ii)-(iv) of Theorem also hold. Then $x(\theta)$ can be implemented via a Bayesian mechanism satisfying (IIR) and (BB).*

We prefer to work with the Identifiability Condition because it is easier to interpret and verify.

[14]Recall that the social choice rule $x(\theta)$ is implementable if there exists a system of transfer functions $t(\theta)$ s.t. direct mechanism $(x(\theta), t(\theta))$ is interim incentive compatible.

4 Surplus Allocation

Theorem 3 *Suppose that Conditions (i)-(iv) of Theorem 1 hold. Fix an EASR social choice rule $x(\theta)$, some player $i' \in \{1, ..., n\}$ and her type $\theta_{i'}^k \in \Theta_{i'}$. Then there exists an IC, IIR and BB Bayesian mechanism $(x(\theta), t(\theta))$ s.t. IIR constraints of all types of all players except type $\theta_{i'}^k$ of player i' bind. So, type θ_i^k obtains all ex-ante expected surplus from the mechanism, i.e.*

$$\sum_{\theta_{-i} \in \Theta_{-i}} (u_i(x(\theta_{-i}, \theta_i), \theta) + t_i(\theta))p(\theta_{-i}, \theta_i) = 0 \quad \textit{for all} \quad \theta_i \in \Theta_i, \ \ i \neq i', \quad \textit{and} \quad \theta_{i'} \neq \theta_{i'}^k$$

$$\sum_{\theta_{-i'} \in \Theta_{-i'}} (u_{i'}(x(\theta_{-i'}, \theta_{i'}^k), (\theta_{-i'}, \theta_{i'}^k)) + t_i(\theta_{-i'}, \theta_{i'}^k))p(\theta_{-i'}, \theta_{i'}^k) = \sum_{\theta \in \Theta, i \in \{1, \ldots, n\}} u_i(x(\theta), \theta)p(\theta)$$

Proof: See the Appendix.

The proof of Theorem 3 requires only some modification of the proof of Theorem 1. However, it has considerable economic significance. In particular, Theorem 3 implies that the mechanism designer can allocate the ex-ante social surplus generated by the mechanism between agent types in a completely arbitrary way.

Corollary 1 *Suppose that Conditions of Theorem 1 hold. Fix any EASR social choice rule $x(\theta)$. Then for any collection of nonnegative constants $v_i^{k_i}$, $i \in \{1, ..., n\}$, $k_i \in \{1, ...m_i\}$ satisfying:*

$$\sum_{i \in \{1, \ldots, m\}, k_i \in \{1, \ldots, m_i\}} v_i^{k_i} p(\theta_i^{k_i}) = \sum_{\theta \in \Theta, i \in \{1, \ldots, n\}} u_i(x(\theta), \theta)p(\theta)$$

there exists an (IC), (BB), and (IR) Bayesian mechanism $(x(\theta), t(\theta))$ s.t. type θ_i^k of agent i earns a surplus equal to $v_i^{k_i}$, i.e.

$$\sum_{\theta_{-i} \in \Theta_{-i}} (u_i(x(\theta_{-i}, \theta_i^{k_i}), \theta) + t_i(\theta_{-i}, \theta_i^{k_i}))p(\theta_{-i} | \theta_i^{k_i}) = v_i^{k_i}$$

Proof: Let $(x(\theta), t^{i(k_i)}(\theta)$ be an IC, IIR, BB direct mechanism which implements social choice rule $x(\theta)$ and allocates all surplus to type θ_i^k of agent i. By Theorem 3 such mechanism exists. Also, let $\alpha_i^{k_i} = \frac{v_i^{k_i} p(\theta_i^{k_i})}{\sum_{\theta \in \Theta, i \in \{1, \ldots, n\}} u_i(x(\theta), \theta)p(\theta)}$.

Now consider direct mechanism $\bar{M}$ which implements social choice rule $x(\theta)$ and offers transfers $\bar{t}(\theta)$ which can be represented as convex combinations of transfers in mechanisms $(x(\theta), t^{i(k_i)}(\theta))$ with nonnegative weights $\alpha_i^{k_i}$, i.e.

$$\bar{t}(\theta) = \sum_{i=1, \ldots, n, k_i = 1, \ldots, m_i} \alpha_i^{k_i} t_j^{i(k)}(\theta)$$

Since BB, IC and IIR constraints are linear in transfers, all weights $\alpha_i^{k_i}$ sum and to one, and the allocation rule $x(\theta)$ is the same in all mechanisms, we conclude that $\bar{M}$ is also IC, BB, IIR mechanism. It is also easy to see that the expected surplus of type θ_i^k is equal to $v_i^{k_i}$, i.e. $\sum_{\theta_{-i} \in \Theta_{-i}} (u_i(x(\theta_{-i}, \theta_i^{k_i}), \theta) + \bar{t}_i(\theta_{-i}, \theta_i^{k_i}))p(\theta_{-i} | \theta_i^{k_i}) = v_i^{k_i}$. *Q.E.D.*

5 Informed Principal Problem.

The analysis in the previous sections relied on the existence of a mechanism designer who did not possess any private information. In this section we consider the situation arising when such mechanism designer is not available[15], and so the mechanism has to be designed by one of the agents (referred to as the primary agent in the sequel) after all the agents have already learned their private information.[16] In the literature this is known as an 'informed principal problem.' Since different types of the primary agent may decide to offer different mechanisms, the choice of the mechanism could serve as a signal which the other agents would use to update their beliefs about the primary agent's type. Naturally, the outcome of this inference process would affect the agents' incentive constraints. As a result, the informed principal problem is more complex to analyze and solve than the standard mechanism design situation.

In this section we advance the investigation of the informed principal problem by showing that generically it possesses an ex-post efficient solution with all social surplus allocated to the primary agent.

Before describing the solution concepts, let us introduce the necessary notation and formally describe the informed principal game which we denote by Γ. Without loss of generality assume that agent 1 is the primary agent with the authority to propose the mechanism to the agents and implement it. The timeline of the game Γ is as follows:

- Stage 1. All agents learn their types.
- Stage 2. Agent 1 (the primary agent) proposes a mechanism M.
- Stage 3. Agents 2 to n simultaneously decide whether to participate in the mechanism.
- Stage 4. If all agents have agreed to participate, the mechanism M is implemented.[17] The agents' strategy choices determine the outcome of M.

The strategy space of agent 1 from which mechanism M is drawn is a class of mechanisms Z. We will require Z to be such that the continuation game after a mechanism from Z has been offered possesses a sequential equilibrium for arbitrary agents' beliefs. This can be ensured simply by assuming that all mechanisms in Z are finite, i.e. have a finite set of terminal nodes.

By the Revelation Principle, we assume without loss of generality that all mechanisms in Z are direct and incentive compatible. Thus, a mechanism from Z can be represented as an outcome function $(x(\hat{\theta}), t(\hat{\theta}))$ mapping the agents' reported types into the allocation profiles. The incentive compatibility of a mechanism is defined in a standard way relative to the agents'

[15] The absence of a mechanism designer is natural in many situations. For example, in the collusion context it appears likely that colluding parties will have to attain an agreement on the mechanism without any outside participation.

[16] If a mechanism designer is not available, but the mechanism can be designed at an ex-ante stage when no agent has yet received her private information, then we would expect an efficient social choice rule to be implemented and the allocation of surplus to be determined by the ex-ante distribution of the bargaining power.

[17] For simplicity, we assume that the mechanism is not implemented and all agents get their outside options if at least one of them has refused to participate in stage 3. It will be easy to see that the outcomes yielded by our solution concepts can also be obtained under the same solution concepts if one makes alternative assumptions regarding what happens when some subset of agents refuses to participate in the mechanism.

beliefs in stage 4. So, the incentive constraints of agent $i \in \{1, ..., n\}$ are given by the same inequalities as in (1), with the only difference that for $i \in \{2, ..., n\}$ her prior beliefs given by the conditional probabilities $p(\theta_{-i}|\theta_i)$ have to be replaced by the posterior beliefs $b_i(\theta_{-i}|\theta_i, M)$ that she holds at stage 4.

We will need to consider situations in which some subset of types of player 1 offer a mechanism different from the one offered by the other types. Therefore, the following concept is useful in the analysis. Say that a mechanism $(x(\theta), t(\theta))$ is *incentive compatible given a subset R of* Θ_1 if $(x(\theta), t(\theta))$ satisfies the standard interim incentive constraints of agent 1 given by (1), while for any agent $i \in \{2, ..., n\}$ the following incentive constraints hold $\forall \theta_i \in \Theta_i$:

$$\sum_{\theta_{-i}\colon \theta_{-1-i} \in \Theta_{-1-i}, \theta_1 \in R} \left(u_i(x(\theta_{-i}, \theta_i), (\theta_{-i}, \theta_i)) + t_i(\theta_{-i}, \theta_i) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - t_i(\theta_{-i}, \theta_i') \right) p(\theta_{-i}, \theta_i) \geq 0 \tag{8}$$

Let $U_1(M|\theta_1)$ be the expected payoff that agent 1 of type θ_1 gets in mechanism M when all other agents, including agent 1- the primary agent- report their types truthfully in M.

Myerson's Inscrutability Principle (see Myerson (1983)) says that without loss of generality all types of the primary agent would offer the same mechanism, so that the other agents will not update their prior beliefs after the mechanism is offered at stage 2. Such a mechanism is called inscrutable. This is so, because for any mechanism in which the primary agent reveals some information about her type in the mechanism-proposal stage 2, there is an equivalent inscrutable mechanism in which the primary agent reveals her private information only through her type announcement in stage 4.

The Inscrutability Principle is useful for characterizing the solutions. However, it is less useful for the analysis of possible deviations from the solution mechanism. In particular, such deviations arise when *some, but not all* types of the primary agent choose to offer a mechanism different from the candidate solution causing the other agents to update their priors in a non-trivial manner.[18]

Let us now introduce the solution concepts. The weakest solution concept that we will use is sequential equilibrium. It is well-known that sequential equilibrium allows too much freedom in the specification of the posterior beliefs after a deviation. Therefore, we strengthen our analysis by relying on the additional solution concepts proposed by Myerson (1983): the core mechanism and the neutral optimum.

The neutral optimum solution represent the smallest possible set of unblocked mechanism (Myerson 1983). To define the notion of blocking, note that an allocation profile $(x(\theta), t(\theta))$ implemented in mechanism M uniquely determines the vector of expected payoffs of agent 1 denoted by $\{U(M|\theta_1)\}_{\theta_1 \in \Theta_1}$. Let $B(\Gamma)$ be the set of blocked expected payoff vectors of agent 1 in Γ (i.e. the expected payoffs vectors corresponding to blocked mechanisms). $B(\Gamma)$ is assumed to satisfy the following axioms:

Axiom 1 *(Domination) For any vectors $w(.)$ and $z(.)$ in $\mathbb{R}^{\#\Theta_1}$, if $w(.) \in B(\Gamma)$, and $z(\theta_1) \leq w(\theta_1)$ for every $\theta_1 \in \Theta_1$, then $z(.) \in B(\Gamma)$.*

[18] A mechanism offered as a result of such deviation is not an inscrutable one because the other agents recognize that it is offered by a type from the set of the deviating types of agent 1.

Axiom 2 *(Openness) The set of blocked allocations $B(\Gamma)$ is open in $\mathbb{R}^{\#\Theta_1}$.*

Axiom 3 *(Extension) Let $\bar{\Gamma}$ be an informed principal problem that differs from Γ only because its set of enforceable actions $\bar{X}$ is larger than the set of enforceable actions X in Γ, i.e. $X \subset \bar{X}$. Then $B(\Gamma) \subset B(\bar{\Gamma})$.*

Axiom 4 *(Strong Solutions) If mechanism M is incentive compatible given any $\theta_1 \in \Theta_1$ and there does not exist another incentive compatible mechanism M' satisfying $U_1(M|\theta_1) \leq U_1(M'|\theta_1)$ with strict inequality for at least one θ_1, then $U_1(M|\theta_1) \notin B(\Gamma)$. Such mechanism is called a strong solution.*

These axioms do not define the set of blocked payoff vectors $B(\Gamma)$ uniquely. Rather, there may be several sets of blocked allocations $B_k(\Gamma)$ for $k \in \mathcal{I}$ where $\mathcal{I}$ is some index set. To avoid ambiguity, let $B^*(\Gamma)$ denote the union of all sets of blocked payoff vectors, i.e. $B^*(\Gamma) = \cup_{k\in\mathcal{I}} B_k(\Gamma)$.

Definition 2 *(Myerson 1983) A mechanism $\tilde{M}$ is a neutral optimum if it is incentive compatible*[19] *and the vector $\{U(\tilde{M}|\theta_1)\}$ of the expected payoffs of agent 1 does not belong to $B^*(\Gamma)$.*

The core mechanism is defined as follows:

Definition 3 *(Myerson 1983) A mechanism $\tilde{M}$ is a core mechanism if it is incentive compatible (with respect to prior beliefs) and there does not exist any other mechanism $\hat{M}$ such that*

$$\left\{\theta_1 \in \Theta_1 | U_1(\hat{M}|\theta_1) > U_1(\tilde{M}|\theta_1)\right\} \neq \emptyset$$

and $\hat{M}$ is incentive compatible given set S for any S that satisfies:

$$\left\{\theta_1 \in \Theta_1 | U_1(\hat{M}|\theta_1) > U_1(\tilde{M}|\theta_1)\right\} \subset S \subset \Theta_1$$

The attractiveness of the neutral optimum as a solution concept stems from the fact that a number of other solution concepts, including sequential equilibrium and core mechanisms, give rise to sets of outcomes that can be described via some concept of blocking satisfying the Axioms 1-4. Since neutral optima correspond to the smallest set of unblocked outcomes satisfying these axioms, it follows that a neutral optimum also constitutes a solution according to those other solution concepts. In particular, the following theorem is established in Myerson (1983) (see Theorem 5):

Theorem 4 *Any neutral optimum is also a sequential equilibrium and a core mechanism.*

The main result of this section provides a generic characterization of the set of neutral optima. Specifically, we have:

[19] Since $\tilde{M}$ is taken to be inscrutable, the incentive compatibility is defined with respect to prior beliefs (Myerson 1983).

Theorem 5 *Suppose that conditions (i)-(iv) of Theorem 1 hold. Then the set of neutral optima consists of mechanisms $(x(\theta), t(\theta))$ satisfying (1)-(2) and such that the following conditions hold:*[20]

$$(i)\quad x(\theta) \in \arg\max_{x\in X} \sum_i u_i(x,\theta) \quad \forall\theta \in \Theta$$

$$(ii)\ \forall\theta_1 \in \Theta_1 \qquad \sum_{\theta=(\theta_{-1},\theta_1):\theta_{-1}\in\Theta_{-1}} (u_1(x(\theta),\theta) + t_1(\theta))\, p(\theta_{-1}|\theta_1) = \sum_{i\in\{1,...,n\},\theta=(\theta_{-1},\theta_1):\theta_{-1}\in\Theta_{-1}} u_i(x(\theta),\theta)p(\theta_{-1}|\theta_1)$$

Theorem 5 says that generically the neutral optimum allocation profile is unique. It implements an ex-post efficient social choice rule and allocates all surplus to the primary agent in such a way that each type of the primary agent gets all the expected social surplus conditional on her type.

In the appendix we also demonstrate how neutral optimum can be supported as a sequential equilibrium.

6 Appendix

Proof of Theorem 1: Fix some EASR social choice rule $x(.)$. The theorem will be proved in a number of steps.

Step 1. Restatement of the Problem. To begin with, let us rewrite IC, IIR and BB constraints given by (1), (2) and (3) respectively in a matrix form. Let B_i be a matrix of size $m_i(m_i-1)\times L$ each row of which corresponds to a different IC constraint of agent i. All rows from $k(m_i-1)+1$ to $(k+1)(m_i-1)$ of B_i correspond to incentive constraints of type $k\in\{1,...,m_i\}$ of player i, and each column corresponds to one of L possible type profiles in the natural order induced by the ordering of players and their types. Thus, in the row corresponding to $IC_i(k,k')$, an entry in the column corresponding to the type profile (θ_{-i},θ_i^k) for some $\theta_{-i}\in\Theta_{-i}$ is equal to $p(\theta_{-i},\theta_i^k)$, and entry in the column corresponding to the type profile $(\theta_{-i},\theta_i^{k'})$ is equal to $-p(\theta_{-i},\theta_i^k)$, while entries in all other columns are zero.

To deal with the IIR constraints we will use the $m_i\times L$ matrix P_i defined in Section 3. Note that the k-th row of this matrix corresponds to $IIR_i(\theta_i^k)$, since its entry is equal to $p(\theta_{-i},\theta_i^k)$ in the column corresponding to the type profile (θ_{-i},θ_i^k) for some $\theta_{-i}\in\Theta_{-i}$, and zero in all other columns. Finally, let I be an identity matrix of size $L\times L$.

For all $i\in\{1,...,n\}$ and $k,k'\in\{1,...,m_i\}$ s.t. $k\neq k'$, let

$$\begin{aligned} \bar{u}_{ik} &= \sum_{\theta_{-i}\in\Theta_{-i}} u_i(x(\theta_{-i},\theta_i^k),(\theta_{-i},\theta_i^k))p(\theta_{-i},\theta_i^k) \\ \hat{u}_{ikk'} &= \sum_{\theta_{-i}\in\Theta_{-i}} \Big(u_i(x(\theta_{-i},\theta_i^k),(\theta_{-i},\theta_i^k)) - u_i(x(\theta_{-i},\theta_i^{k'}),(\theta_{-i},\theta_i^k))\Big)\, p(\theta_{-i},\theta_i^k) \qquad (9) \end{aligned}$$

Finally, construct vectors $\bar{u}_i$ and $\hat{u}_i$ by concatenating $\bar{u}_{ik}$ and $\hat{u}_{ikk'}$, i.e. $\bar{u}_i = (\bar{u}_{i1},...,\bar{u}_{im_i})'$ and $\hat{u}_i = (\hat{u}_{i12},...,\hat{u}_{i1m_i},...,\hat{u}_{im_i1},...,\hat{u}_{im_i(m_i-1)})'$ where prime denotes a transpose. Then (1),

[20]By Theorem 1 and Corollary 1 such mechanisms exist.

(2) and (3) can respectively be rewritten as follows:

$$(IC) \quad \left\| \begin{array}{cccc} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & B_n \end{array} \right\| \times \begin{bmatrix} t_1 \\ t_2 \\ ... \\ t_n \end{bmatrix} \geq \begin{bmatrix} -\hat{u}_1 \\ -\hat{u}_2 \\ ... \\ -\hat{u}_n \end{bmatrix}$$

$$(IIR) \quad \left\| \begin{array}{cccc} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & P_n \end{array} \right\| \times \begin{bmatrix} t_1 \\ t_2 \\ ... \\ t_n \end{bmatrix} \geq \begin{bmatrix} -\bar{u}_1 \\ -\bar{u}_2 \\ ... \\ -\bar{u}_n \end{bmatrix} \tag{10}$$

$$(BB) \quad \left\| \begin{array}{cccc} I & I & ... & I \end{array} \right\| \times \begin{bmatrix} t_1 \\ t_2 \\ ... \\ t_n \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Thus, a social choice rule $x(\theta)$ can be implemented via a BB and IIR mechanism if and only if there exists a solution $(t_1, ..., t_n)$ to the system (10).

Step 2. *Necessary and Sufficient Conditions for the Existence of a solution to (10).*

Consider the following system of linear inequalities:

$$\begin{array}{l} Ax \geq a \\ Bx = b \end{array} \tag{11}$$

where a and b are fixed vectors of size l_1 and l_2 correspondingly, while A and B are fixed matrices of size $l_1 \times l_3$ and $l_2 \times l_3$ correspondingly.

The Theorem of The Alternative: System (11) has a solution x^* if and only if for any row vector λ_B of size l_2 with nonnegative components and any row vector λ_A of size l_1 the following condition holds:

$$\lambda_A A + \lambda_B B = \mathbf{0} \;\; \Rightarrow \;\; \lambda_B b + \lambda_A a \leq 0.$$

For the proof of the Theorem see Mangasarian (1969) (page 34).

Using the Theorem of The Alternative and exploiting the block structure of matrices in (10), we conclude that system (10) has a solution if and only if the following property holds:

Property D. *Consider any two collections of row vectors $\{\gamma_1, ..., \gamma_n\}$ and $\{\lambda_1, ..., \lambda_n\}$ such that γ_i is of size $m_i(m_i - 1)$, λ_i is of size m_i, $\gamma_i \geq 0$, $\lambda_i \geq 0$ for all $i = 1, ..., n$, and a row vector μ of size L* satisfying

$$\gamma_i B_i + \lambda_i P_i + \mu = \mathbf{0} \quad \text{for all} \quad i = 1, ..., n, \tag{12}$$

$$\textit{Then,} \qquad \textstyle\sum_{i=1,...,n} \gamma_i \hat{u}_i + \sum_{i=1,...,n} \lambda_i \bar{u}_i \geq 0$$

In the rest of the proof we will show that **Property D** holds under the conditions of the Theorem. So, *let γ_i, λ_i, μ be an arbitrary collection of the vectors of coefficients that satisfies the conditions of Property D.* Let $\gamma_i^{k,k'}$ denote the entry of vector γ_i corresponding to the incentive constraint $IC_i(k, k')$. Note that this entry is $(k-1)m_{i-1} + k'$-th in the vector if $k' < k$ and $(k-1)m_{i-1} + k' - 1$-th if $k' > k$. Also, let λ_i^k denote the k-th entry of vector λ_i corresponding to the individual rationality constraint $IR_i(k)$.

Step 3. $\gamma_{i^*} B_{i^*} = \mathbf{0}$.
Proof: By (12), we have:

$$\gamma_{i^*} B_{i^*} + \lambda_{i^*} P_{i^*} = \gamma_{j^*} B_{j^*} + \lambda_{j^*} P_{j^*} \tag{13}$$

Note that for all i

$$B_i = R_i - W_i$$

where W_i was defined in Section 3, and R_i is an $m_i(m_i - 1) \times L$ matrix such that the entry in its $(k-1)(m_i-1) + k'$-th row for $k \in \{1, ..., m_i\}$ and $k' \in \{1, ..., m_i - 1\}$ and the column corresponding to type profile $(\theta_{-i}, \theta_i^k)$ for some $\theta_{-i} \in \Theta_{-i}$ is equal to $p(\theta_{-i}, \theta_i^k)$, while all other entries are zero.

Since the $k(m_i - 1) + k'$-th row of matrix R_i is equivalent to the k-th row of matrix P_i for all $k \in \{1, ..., m_i\}$ and $k' \leq m_i - 1$, we have:

$$\gamma R_i + \lambda P_i = \tilde{\lambda}_i P_i$$

where $\tilde{\lambda}_i^k = \lambda_i^k + \sum_{k' \in \{1,...,m_i\}, k' \neq k} \gamma_i^{kk'}$, and λ_i^k and $\tilde{\lambda}_i^k$ denote the k-th entries in the vectors λ_i and $\tilde{\lambda}_i$ respectively, while $\gamma_i^{kk'}$ denotes the $(m_i - 1)k + k'$-th entry in the vector γ_i if $k' < k$ and $(m_i - 1)k + k' - 1$-th entry if $k' > k$. So, (13) can be rewritten as:

$$-\gamma_{i^*} W_{i^*} + \tilde{\lambda}_{i^*} P_{i^*} = -\gamma_{j^*} W_{j^*} + \tilde{\lambda}_{j^*} P_{j^*}$$

The identifiability of i^* with respect to j^* implies that the space spanned by the rows of matrix W_{i^*} and the space spanned by the rows of matrices P_{i^*}, W_{j^*} and P_{j^*} do not intersect except at point $\mathbf{0}$. Therefore, $\gamma_{i^*} W_{i^*} = 0$.

Finally, to see that $\gamma_{i^*} W_{i^*} = 0$ implies $\gamma_{i^*} B_{i^*} = 0$, suppose otherwise. Then $\gamma_{i^*} R_{i^*} \neq 0$, and so some entries of the vector γ_{i^*} are strictly positive and the rest are zero (recall that all entries of γ_{i^*} are nonnegative). But since all entries of matrices W_{i^*} and R_{i^*} are nonnegative, and each row of the matrix W_{i^*} contains the same non-zero elements as the corresponding row of matrix R_{i^*}, we must have $\gamma_{i^*} W_{i^*} > 0$. Contradiction.

Step 4.

Lemma 5 *Consider a collection $\{v_1, ..., v_r\}$ of vectors of size l with nonnegative entries. Suppose that for any $k, k' \in \{1, ..., r\}$ there is no $z \in \mathbb{R}$ s.t. $v_k \neq z v_{k'}$. Then there does not exist a collection of coefficients $\zeta_k^h \geq 0$ s.t. $v_k = \sum_{h \neq k, h \in \{1,..,r\}} \zeta_k^h v_h$ for all $k \in \{1, .., r\}$.*

Proof: Let s be the maximal number of linearly independent vectors in $\{v_1, ..., v_r\}$. By assumption of the Lemma $s \geq 2$. If $s = r$, then the result is immediate. If $s < r$, then, without loss of generality, let the vectors $\{v_1, ..., v_s\}$ be linearly independent.

Suppose that there exists a collection of coefficients $\hat{\zeta}_k^h \geq 0$ s.t. $v_k = \sum_{h \neq k, h \in \{1,..,s+1\}} \zeta_h^k v_h$ for all $k, h \in \{1, .., s+1\}$. Since $\{v_1, ..., v_s\}$ are linearly independent, there exist a unique collection of coefficients a_1, a_{s+1} not all of which are equal to zero such that $\sum_{h \in \{1,..,s+1\}} a_h v_h = 0$. Since all entries of a vector v_h are positive and there is no $z \in \mathbb{R}$ s.t. $v_k \neq z v_{k'}$, there exist *at least three* vectors v_{h_1}, v_{h_2}, v_{h_3} s.t. $a_{h_1} \neq 0$, $a_{h_2} \neq 0$, $a_{h_3} \neq 0$. Suppose without loss of generality that a_{h_1}, a_{h_2} are of the same sign. Then, $\zeta_{h_2}^{h_1} = -\frac{a_{h_2}}{a_{h_1}} < 0$.

Now let us proceed by induction. Suppose that for $\{v_1, ..., v_t\}$ where t is s.t. $s \leq t < r$ there is no collection of coefficients ζ_k^h with the desired properties. To show that such a collection does not exist for $\{v_1, ..., v_{t+1}\}$ assume otherwise. Let $\tilde{\zeta}_k^h$ be such collection. Then $v_{t+1} = \sum_{h \in \{1,..,t\}} \tilde{\zeta}_h^{t+1} v_h$. Using this expression to substitute v_{t+1} out, we obtain that for all $k \in \{1, ..., t\}$, $v_k(1 - \tilde{\zeta}_{t+1}^k \tilde{\zeta}_k^{t+1}) = \sum_{h \neq k, h \in \{1,..,t\}} \left(\tilde{\zeta}_h^k + \tilde{\zeta}_{t+1}^k \tilde{\zeta}_h^{t+1} \right) v_h$. All coefficients on the right-hand side of this inequality are nonnegative and some of $\tilde{\zeta}_h^k$ must be strictly positive. Therefore, $1 - \tilde{\zeta}_{t+1}^k \tilde{\zeta}_k^{t+1} > 0$. So, dividing by it we obtain a collection of coefficients for $\{v_1, ..., v_t\}$ which contradicts the inductive assumption. *Q.E.D.*

Step 5. *There exists $\bar{\lambda}$, s.t. $\lambda_{i^*}^1 = ... = \lambda_{i^*}^{m_{i^*}} = \bar{\lambda}$.*
Proof: Consider the system of equations

$$\gamma_{k^*} B_{k^*} + \lambda_{k^*} P_{k^*} = \lambda_{i^*} P_{i^*} \tag{14}$$

Pick the columns of the matrices P_{i^*}, B_{k^*} and P_{k^*} corresponding to some type $\theta_{i^*}^l$ of player i^* and some type $\theta_{k^*}^r$ of player k^*. Let $\lambda_{i^*}^l$ be the entry of vector λ_i corresponding to $IR_{i^*}(\theta_{i^*}^l)$. Equation (14) implies that

$$\left(\sum_{s=1,...,m_{k^*}, s \neq r} \gamma_{k^*}^{r,s} + \lambda_{k^*}^r \right) \vec{p}_{-i^*-k^*}(\theta_{i^*}^l, \theta_{k^*}^r) - \sum_{s=1,...,m_{k^*}, s \neq r} \gamma_{k^*}^{s,r} \vec{p}_{-i^*-k^*}(\theta_{i^*}^l, \theta_{k^*}^s) = \lambda_{i^*}^l \vec{p}_{-i^*-k^*}(\theta_{i^*}^l, \theta_{k^*}^r) \tag{15}$$

Let $\hat{\lambda}_{l,r} = -\lambda_{i^*}^l + \sum_{s=1,...,m_{k^*}, s \neq r} \gamma_{k^*}^{r,s} + \lambda_{k^*}^r$. Then 15 can be rewritten as

$$\sum_{s=1,...,m_{k^*}, s \neq r} \gamma_{k^*}^{s,r} \vec{p}_{-i^*-k^*}(\theta_{i^*}^l, \theta_{k^*}^s) = \hat{\lambda}_{l,r} \vec{p}_{-i^*-k^*}(\theta_{i^*}^l, \theta_{k^*}^r) \tag{16}$$

Since $\gamma_{k^*}^{s,r} \geq 0$, we must have $\hat{\lambda}_{l,r} \geq 0$. Recall that both l and r were chosen arbitrarily, so (16) holds for all $l \in \{1, ..., m_{i^*}\}$ and $r \in \{1, ..., m_{k^*}\}$. In particular, it holds for $l = u$. By assumption (ii) of the Theorem, for any $s, s' \in \{1, ..., m_{k^*}\}$ there is no $z \in \mathbb{R}$ s.t. $\vec{p}_{-i^*-k^*}(\theta_{i^*}^u, \theta_{k^*}^s) = z \vec{p}_{-i^*-k^*}(\theta_{i^*}^u, \theta_{k^*}^s)$. So, by Lemma 5 there exists $r' \in \{1, ..., m_{k^*}\}$ s.t. $\hat{\lambda}_{u,r'} = 0$. From (16) it follows that $\gamma_{k^*}^{s,r'} = 0$ for all $s \in \{1, ..., m_{k^*}\}$, $s \neq r'$, which in turn implies that $\hat{\lambda}_{l,r'} = 0$, and so $\lambda_{i^*}^l = \sum_{s=1,...,m_{k^*}, s \neq r'} \gamma_{k^*}^{r',s} + \lambda_{k^*}^{r'} \equiv \bar{\lambda}$ for all $l \in \{1, ..., m_{i^*}\}$.

Step 6. For any $h \neq i^*$, $\gamma_h = \mathbf{0}$ and $\lambda_h^1 = ... = \lambda_h^{m_h} = \bar{\lambda}$.
Proof: By (12) and Step 4, we have:

$$\gamma_h B_h + \lambda_h P_h = \lambda_{i^*} P_{i^*} \tag{17}$$

Since $\lambda_{i^*}^l = \bar{\lambda}$ for all $l \in \{1, ..., m_{i^*}\}$, (17) is equivalent to the following:

$$\text{For all } r \in \{1, ..., m_h\}, \quad \left(\sum_{s \in \{1,...,m_h\}, s \neq r} \gamma_h^{r,s} + \lambda_h^r \right) \vec{p}_{-h}(\theta_h^r) - \sum_{s \in \{1,...,m_h\}, s \neq r} \gamma_h^{s,r} \vec{p}_{-h}(\theta_h^s) = \bar{\lambda} \vec{p}_{-h}(\theta_h)$$

which can be rewritten as follows:

$$\text{For all } r \in \{1, ..., m_h\}, \quad \sum_{s \in \{1,...,m_h\}, s \neq r} \gamma_h^{s,r} \vec{p}_{-h}(\theta_h^s) = \left(\sum_{s \in \{1,...,m_h\}, s \neq r} \gamma_h^{r,s} + \lambda_h^r - \bar{\lambda} \right) \vec{p}_{-h}(\theta_h^r) \tag{18}$$

Condition (iii) of the Theorem implies that $\gamma_h^{s,r} = 0$ for all $s, r \in \{1, ..., m_h\}$. So, $\lambda_h^r = \bar{\lambda}$ for all $r \in \{1, ..., m_h\}$.

Step 7. Implementability. There exists a collection of transfer functions $\tilde{t}(\theta)$ such that the allocation profile $(x(\theta), \tilde{t}(\theta))$ satisfies interim incentive constraints IC .

Proof: Part 1. Conditions (i)-(iii) of Theorem 1 imply that $p_{-i}(.|\theta_i) \neq p_{-i}(.|\theta_i')$ *for all* $i = 1, ..., n$ *and all* $\theta_i, \theta_i' \in \Theta_i$.

By conditions (ii) and (iii), this is immediate for any player $i \neq i^*$. Thus, it remains to establish the following: If i^* is identifiable with respect to j^*, then $p_{-i^*}(.|\theta_{i^*}^k) \neq p_{-i^*}(.|\theta_{i^*}^{k'})$ for all $\theta_{i^*}^k, \theta_{i^*}^{k'} \in \Theta_{i^*}$. To see this suppose otherwise, i.e. there exist $\theta_{i^*}^k$ and $\theta_{i^*}^{k'} \in \Theta_{i^*}$ s.t. $p_{-i^*}(.|\theta_{i^*}^k) = p_{-i^*}(.|\theta_{i^*}^{k'})$. Then the row of W_{i^*} corresponding to $IC_{i^*}(\theta_{i^*}^k, \theta_{i^*}^{k'})$ is equal to the k'-th row of matrix P_{i^*} (corresponding to $IR_{i^*}(\theta_{i^*}^{k'})$) multiplied by $\frac{p_{i^*}(\theta_{i^*}^k)}{p_{i^*}(\theta_{i^*}^{k'})}$.

Let the rank of matrix W_{i^*} be equal to τ, and $\{r_1, ..., r_\tau\}$ be a collection of its τ linearly independent rows. Also, let the rank of matrix $\left\| \begin{matrix} P_{i^*} \\ W_{j^*} \\ P_{j^*} \end{matrix} \right\|$ be equal to q, and $\{v_1, ..., v_q\}$ be a collection of its q linearly independent rows. Since i^* is identifiable with respect to j^*, all vectors in the collection $\{r_1, ..., r_\tau, v_1, ..., v_q\}$ must be linearly independent. However, this contradicts the fact that the row of W_{i^*} corresponding to $IC_{i^*}(\theta_{i^*}^k, \theta_{i^*}^{k'})$ is some linear combination of $\{r_1, ..., r_\tau\}$, while the k'-th row of matrix P_{i^*} is some linear combination of $\{v_1, ..., v_q\}$.

Part 2. A social choice rule $x(\theta)$ *is implementable if* $p_{-i}(.|\theta_i) \neq p_{-i}(.|\theta_i')$ *for all* $i \in \{1, ..., n\}$, θ_i *and* $\theta_i' \in \Theta_i$.

This result is established in Part (ii) of Lemma 2 in Aoyagi (1998). For convenience of the reader, we provide a proof here.

Consider a direct mechanism where transfers to player i are given by:

$$\tilde{t}_i(\theta_{-i}, \theta_i)) = \frac{p(\theta_{-i}, \theta_i)}{\sqrt{\sum_{\theta_{-i} \in \Theta_{-i}} p^2(\theta_{-i}, \theta_i)}}. \tag{19}$$

Compare the expected transfer to agent i of type θ_i when she reports her type truthfully to her expected transfer when she reports a different type θ_i'. We have:

$$\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}, \theta_i) \frac{p(\theta_{-i}, \theta_i)}{\sqrt{\sum_{\theta_{-i} \in \Theta_{-i}} p^2(\theta_{-i}, \theta_i)}} - \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}, \theta_i) \frac{p(\theta_{-i}, \theta_i')}{\sqrt{\sum_{\theta_{-i} \in \Theta_{-i}} p^2(\theta_{-i}, \theta_i')}} > 0 \tag{20}$$

The strict inequality follows from *Fact 1* and Cauchy-Schwartz inequality. Since $u_i(x,\theta)$ is bounded on X, it follows that for any social choice rule $x(\theta)$ there is a constant A large enough that mechanism $(x(\theta), A\tilde{t}(\theta))$ satisfies interim incentive constraint (1) for all i.

Step 8. $\sum_{i=1,\ldots,n} \gamma_i \hat{u}_i \geq 0$.

Proof: We have shown that $\gamma_i = \mathbf{0}$ for all $i \neq i^*$, and that $\gamma_{i^*} B_{i^*} = \mathbf{0}$. By Step 7 $x(.)$, is implementable,. So there exists a solution to the sub-system of inequalities (IC) in (10) taken separately (without IIR and BB). Applying the Theorem of Alternative to this subsystem only yields the following: $\gamma_i B_i = \mathbf{0}\quad$ for all $\ i\ \ \Rightarrow\ \ \gamma_i \hat{u}_i \geq 0$ for all i.

Step 8. To establish **Property D**, we need to show that $\sum_{i=1,\ldots,n} \gamma_i \hat{u}_i + \sum_{i=1,\ldots,n} \lambda_i \bar{u}_i \geq 0$.

But by Step 7, $\sum_{i=1,\ldots,n} \gamma_i \hat{u}_i \geq 0$. By Step 5 and 6, $\sum_{i=1,\ldots,n} \lambda_i \bar{u}_i = \bar{\lambda} \sum_{i=1,\ldots,n} \sum_{k=1,\ldots,m_i} \bar{u}_i^k$, where $\bar{\lambda} \geq 0$ by assumption of Property D, while $\sum_{k=1,\ldots,m_i} \bar{u}_i^k \geq 0$ by EASR.

Proof of Lemma 1:

Step 1. *Definition of genericity.* Let us at first provide a definition of genericity and introduce a convenient transformation of the space of probability distributions. The probability distribution vector $\mathbf{p}(\theta)$ has to obey the restrictions $\sum_{\theta\in\Theta} p(\theta) = 1$ and $p(\theta) \geq 0$. So, $\mathbf{p}(\theta)$ belongs to $L-1$ dimensional simplex $\Delta^{L-1} \subset R^{L-1}$. We say that a property holds generically on the simplex if it fails on a subset $S \subset \Delta^{L-1}$ of Lebesgue measure 0.[21]

We will need to show that the set of probability distribution vectors in Δ^{L-1} s.t. the conditions of Theorem 1 fail has Lebesgue measure zero. To avoid operating on a simplex, let us introduce the following transformation. Consider an L-vector $\mathbf{q}(\theta) \in [0,1]^L$. To transform it into a probability vector $\mathbf{p}(\theta) \in \Delta^{L-1}$, let $p(q(.))(\theta) = \frac{q(\theta)}{\sum_{\theta\in\Theta} q(\theta)}$.

This transformation is a continuous open map from $[0,1]^L \setminus \mathbf{0}$ onto Δ^{L-1}. It is easy to see that if the Conditions (i)-(iv) of Theorem 1 hold for $\mathbf{q}(.) \in [0,1]^L \setminus \mathbf{0}$, they also hold for $p(q(.))$. So we can assume without loss of generality that $p(\theta) \in [0,1]^L$.

Step 2. *Genericity of Conditions (ii)-(iii).*

Obvious given that $m_i \leq \prod_{j\neq i} m_j$.

Step 3. *Genericaly, the Identifiability Condition holds for players i and j s.t. $i \in \arg\min_{h\in\{1,\ldots,n\}} m_h$ and $j \in \arg\min_{h\in\{1,\ldots,n\},h\neq i} m_j$.*

Agent i is identifiable with respect to j when the following **Condition G** holds:

$$\begin{aligned} &If \quad \psi_i W_i + \psi_j W_j + \zeta_i P_i + \zeta_j P_j = \mathbf{0} \\ &for\ some\ row\ vectors\ \psi_i, \psi_j, \zeta_i\ and\ \zeta_j,\ \ then\ \ \psi_i \equiv \mathbf{0}. \end{aligned} \tag{21}$$

Let us reorder the rows of the matrix $\left\| \begin{matrix} W_i \\ P_i \\ W_j \\ P_j \end{matrix} \right\|$. First, we reorder the rows of the 'upper part' $\left\| \begin{matrix} W_i \\ P_i \end{matrix} \right\|$ which has size $m_i^2 \times L$ in the following way. In the new matrix, let the rows from $m_i(k-$

[21] A Lebesgue measure on the simplex Δ^{L-1} is introduced by a simple renormalization of a Lebesgue measure on R^{L-1}. Specifically, let $\mu(.)$ be a Lebesgue measure on R^{L-1}. We have: $\mu(\Delta^{L-1}) > 0$. Then, the Lebesgue measure $\mu^{\Delta}(.)$ on the simplex $\Delta^{L-1})$ is defined as follows. For any $A \in \Delta^{L-1}$, let $\mu^{\Delta}(A) = \frac{\mu(A)}{\mu(\Delta^{L-1})}$.

$1) + 1$-th to $m_i k$-th correspond to $IR_i(\theta_i^k)$ from matrix P_i and $m_i - 1$ constraints $IC_i(\theta_i^k, \theta_i^{k'})$ $k' =\in \{1, ..., m_i\}$, $k' \neq k$ from matrix W_i. The rows corresponding to IC constraints are ordered by k', while the IR constraint occupies the k-th position in this block. Thus, for example, the first m_i rows correspond to $IR_i(\theta_1)$ followed by $m_i - 1$ rows corresponding to constraints $IC_i(\theta_i^1, \theta_i^k)$, $k = 2, ..., m_i$, ordered by k.

The m_j^2 rows in the 'lower' part $\left\| \begin{array}{c} W_j \\ P_j \end{array} \right\|$ are reordered in a similar fashion, so that the rows from $m_j(k-1) + 1$-th to $m_j k$-th correspond to $IR_j(\theta_j^k)$ and $m_j - 1$ constraints $IC_j(\theta_j^k, \theta_j^{k'})$ $k' =\in \{1, ..., m_j\}$, $k' \neq k$.

Also, reorder the columns of all matrices by player types in the following sequence: i, j,1,...,n, so that the first $m_i m_j$ columns correspond to types $(\theta_{-i-j}, \theta_i^1, \theta_j^1)$ and so on. This reordering results in the following matrix Z:

$$Z = \left\| \begin{array}{cccccccccc} A_1 & ... & A_{m_j} & 0 & 0 & 0 & .... & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & ... & A_{m_j} & .... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & .... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & .... & A_1 & ... & A_{m_j} \\ C_1 & 0 & 0 & C_2 & 0 & 0 & .... & C_{m_j} & 0 & 0 \\ ... & ... & ... & ... & ... & ... & .... & ... & ... & ... \\ 0 & 0 & C_1 & 0 & 0 & C_2 & .... & 0 & 0 & C_{m_j} \end{array} \right\| \tag{22}$$

where

$$A_k = \left\| \begin{array}{c} \mathbf{p}(\theta_{-i-j}, \theta_i^1, \theta_j^k) \\ \mathbf{p}(\theta_{-i-j}, \theta_i^2, \theta_j^k) \\ ... \\ \mathbf{p}(\theta_{-i-j}, \theta_i^{m_i}, \theta_j^k) \end{array} \right\| \quad C_h = \left\| \begin{array}{c} \mathbf{p}(\theta_{-i-j}, \theta_i^h, \theta_j^1) \\ \mathbf{p}(\theta_{-i-j}, \theta_i^h, \theta_j^2) \\ ... \\ \mathbf{p}(\theta_{-i-j}, \theta_i^h, \theta_j^{m_j}) \end{array} \right\| \tag{23}$$

where $\mathbf{p}(\theta_{-i-j}, \theta_i^h, \theta_j^k)$ is an $\#\Theta_{-i-j}$-vector of probabilities of type profiles $\theta_{-i-j}, \theta_i^h, \theta_j^1)$ for fixed θ_i^h, θ_j^k and all $\theta_{-i-j} \in \Theta_{-i-j}$.

Note that $\psi_i = 0$ in (21) if matrix Z has one-dimensional kernel, i.e. the system

$$xZ = \mathbf{0} \tag{24}$$

has a unique non-zero solution. The existence of at least one non-zero solution to (24) follows from the fact that (21) has the following solution $\psi_i = \psi_j = \mathbf{0}$, $\zeta_i^k = 1$ and $\zeta_i^{k'} = -1$.

Note that any two matrices C_h and C_k do not have any common elements. The same with A_k's. Also, any pair A_k and C_h has one common row: the h-th row of matrix A_k and k-th row of matrix C_h.[22] Also, since $m_i \leq \prod_{l \neq i} m_i$ and $m_i \leq \prod_{l \neq i} m_i$

Lemma 6 *Generically, square matrices C_1,...,C_m and A_1,...,A_m are nonsingular.*

The rest of the proof is given separately for two cases: **Case 1.** $n = 3$ and $m_1 = m_2 = m_3 = m$. **Case 2.** The complement of Case 1. In particular, there exists $k \neq i, j$ s.t. $m_k > m_i$.

[22] This implies the existence role of a solution to (24) of the form $x = (vec(I_m)', vec(I_m)')$, where I_m identity matrix of size m and the matrix operator $vec(I_m)$ creates vector of size m^2 by putting all columns of I_m below each other.

Case 1. Let $x \equiv \left(\delta^1, ..., \delta^m, -\eta, ..., -\eta^m\right)$. Then (24) is equivalent to:

$$\delta^h A_k = \eta^k C_h \ \ for \ \ all \ \ k \in \{1, ..., m\} \ \ and \ \ h \in \{1, ..., m\}$$

Thus, $\eta^k = \delta^1 A_k C_1^{-1}, \quad \delta^h = \eta^k C_h A_k^{-1} \quad for \quad all \quad k \in \{1, ..., m\} \quad and \quad h \in \{1, ..., m\}$. Substituting η^k out, we obtain that δ^1 has to satisfy the following equations:

$$\delta^1 A_k C_1^{-1} C_h A_k^{-1} = \delta^1 A_{k'} C_1^{-1} C_h A_{k'}^{-1} \ \ for \ \ all \ \ k, k' \in \{1, ..., m\} \ \ and \ \ h \in \{1, ..., m\} \tag{25}$$

Now let us show that generically for $m \geq 3$ (25) has a unique solution $\delta^1 = (1, 0, ..., 0)$. For any solution $\delta^1 = (0, x_2, ..., x_m) = (0, x)$ for $x \neq 0$ (25) implies that:

$$(0, x) A_k C_1^{-1} C_h A_1^{-1} = (0, x) A_2 C_1^{-1} C_h A_2^{-1}. \tag{26}$$

For each $h \in \{2, ..., m\}$, the dimensionality of the space $\Lambda_h \equiv \{x | (0, x) \text{ is a solution to (26)}\}$ is at most $m - 2$. For, if $\dim(\Lambda_h) = m - 1$, then

$$A_1 C_1^{-1} C_2 A_1^{-1} = A_2 C_1^{-1} C_2 A_2^{-1}$$

which is not generic.

Any element in each space Λ_h depends upon the third row of the matrix C_h in a non-degenerate way. So, the dimension of intersection of is no more than Λ_h $(m-2)(m-1) - (m-2)(m-1) = 0$. Hence generically there in no solution for δ^1 of form $(0, x)$. An application of similar argument for all δ's and η's shows that the kernel of system $xZ = \mathbf{0}$ is of dimension one.

Case 2. $m_i < m_l$ for some l. Fix some $k \in \{1, ..., m_i\}$, and let $X_i(k)$ be an $(m_i + m_j^2) \times L_{-i}$ matrix defined as follows:

$$X_i(k) = \left\| \begin{array}{ccc} A_1 & ... & A_{m_j} \\ C_k & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & C_k \end{array} \right\| \tag{27}$$

Note that the system (21) has no solution such that $\psi_i \neq 0$ if the system $x \ X_i(k) = 0$ has a unique solution x.

In the rest of the proof we will show that all rows of matrix $X_i(k)$ are linearly independent for generic probability distribution vectors $\mathbf{p}(\theta)$. Thus, the following lemma suffices to complete the proof:

Lemma 7 *If* $L_{-i-j} \geq m_j + \frac{m_i - 1}{m_j}$, *then* $rank(X_i(k)) < m_i + m_j^2 - 1$ *on a subset of* $[0,1]^L$ *of measure 0.*

Proof: Note that any element of $\mathbf{p}(\theta)$ is present in at most one entry of matrix $X_i(k)$ and the condition of the lemma implies that the number of columns of $X_i(k)$ is greater than its number of rows.

The lemma will be proved in two steps.

Step 1. Consider a sequence of minors (square submatrices) $\{Z_1, Z_2, ..., Z_{m_j}\}$ of $C_(k)$, s.t. Z_l is an $l \times l$ matrix consisting of the elements of the first l rows and l columns of $C_(k)$. Then almost everywhere on $[0,1]^L$, $\det(Z_l) \neq 0$ for all $l = 1, ..., m_j$.

First, note that the assumption of the Lemma implies that $L_{-i-j} \geq m_j$, i.e. $C_(k)$ has more columns than rows. Further, Z_1 is equivalent to $p(\theta')$ for some $\theta' \in \Theta$ and its determinant is equal to $p(\theta')$. The set of vectors $\mathbf{p}(\theta) \in [0,1]^L$ s.t. $p(\theta') = 0$ has measure zero. To proceed by induction, suppose that the determinant of Z_l for some $l \in \{1, ..., m_j - 1\}$ is equal to zero on a subset of $[0,1]^L$ of measure zero and consider the determinant of Z_{l+1}. We have:

$$\det(Z_{l+1}) = \sum_{t=1}^{l+1} (-1)^{l+1+t} p(\theta_{-i-j}^{(t)}, \theta_i^k, \theta_j^{l+1}) \det(Z_{(\theta_{-i-j}^{(t)}, \theta_i^k, \theta_j^{l+1})}). \tag{28}$$

where $p(\theta_{-i-j}^{(t)}, \theta_i^k, \theta_j^{l+1})$ is the t-th element of the vector $\mathbf{p}(\theta_{-i-j}, \theta_i^k, \theta_j^{l+1})$ and $Z_{(\theta_{-i-j}^{(t)}, \theta_i^k, \theta_j^{l+1})}$ is the minor of Z_{l+1} complementary to $p(\theta_{-i-j}^{(t)}, \theta_i^k, \theta_j^{l+1})$.

Note that $Z_{(\theta_{-i-j}^{(l+1)}, \theta_i^k, \theta_j^{l+1})} = Z_l$. By the inductive assumption $\det(Z_{(\theta_{-i-j}^{(l+1)}, \theta_i^k, \theta_j^{l+1})}) \neq 0$ except perhaps on a subset of $[0,1]^L$ of measure 0. Since $Z_{(\theta_{-i-j}^{(l+1)}, \theta_i^k, \theta_j^{l+1})}$ does not contain any elements of the vector $(p(\theta_{-i-j}, \theta_i^k, \theta_j)$, $\det(Z_{l+1}) = 0$ only if the vector $(p(\theta_{-i-j}^{(1)}, \theta_i^k, \theta_j^{l+1}), ..., p(\theta_{-i-j}^{(l+1)}, \theta_i^k, \theta_j^{l+1}))$ satisfies a linear constraint given by (28) where not all coefficients are not equal to zero. The set of vectors in $[0,1]^L$ satisfying a non-trivial linear constraint has measure zero. Since a finite union of sets of measure zero has measure zero, we conclude that $\det(Z_{l+1}) \neq 0$ almost everywhere. Proceeding by induction, we conclude that $\det(Z_{m_j}) \neq 0$ almost everywhere.

Step 1 implies, in particular, that $rank\,(Z_{ij}(k)) = \#\ rows(Z_{ij}(k))$ almost everywhere on $[0,1]^L$, i.e. the rows of $Z_{ij}(k)$ are not linearly independent on a set of measure 0, and so $rank\left(\left\|\begin{matrix} Z_{ij}(k) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & Z_{ij}(k) \end{matrix}\right\|\right) = m_j^2$ almost everywhere on $[0,1]^L$

Step 2. $rank(X_i(k)) = \#\ rows(X_i(k)) - 1$ almost everywhere on $[0,1]^L$.

Let $X_i^s(k)$ $s = 0, ..., m_i - 1$ be a matrix consisting of the top $m_j^2 + s$ rows of matrix $X_i(k)$ and M_s be the minor (square submatrix) of $X_i^s(k)$ consisting of the first $m_j^2 + s$ columns of $X_i^s(k)$

We will prove that $rank(X_i^s(k)) = \#\ rows(X_i^s(k))$ almost everywhere on $[0,1]^L$ by showing that $\det(M_s) \neq 0$ almost everywhere on $[0,1]^L$. The proof is by induction on s. If $s = 0$, then this property holds by Step 1, because $X_i^0(k) = \left\|\begin{matrix} Z_{ij}(k) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & Z_{ij}(k) \end{matrix}\right\|$.

Now suppose that $\det(M_s) \neq 0$ almost everywhere on $[0,1]^L$ for some $s \in \{0, ..., m_i - 2\}$. Consider M_{s+1}. The determinant of M_{s+1}can be calculated by using the decomposition on the elements of the last row which is given by the first $m_j^2 + s + 1$ elements of the row vector $\{\mathbf{p}(\theta_{-i-j}, \theta_i^{s+1}, \theta_j^1), ..., \mathbf{p}(\theta_{-i-j}, \theta_i^{s+1}, \theta_j^{m_j})\}$ which for simplicity we denote by $(x_1, ..., x_{m_j^2+s+1})$. Let B_l^{s+1} be the minor of M_{s+1} complementary to x_l for $l \in \{1, ..., m_j^2 + s + 1\}$. Then,

$$\det(M_{s+1}) = \sum_{l=1}^{m_j^2+s+1} (-1)^{m_j^2+s+1+l} x_l B_l^{s+1} \tag{29}$$

Since $B^{s+1}_{m_j^2+s+1}$ does not contain any elements of the vector $\{\mathbf{p}(\theta_{-i-j}, \theta_i^{s+1}, \theta_j^1), ..., \mathbf{p}(\theta_{-i-j}, \theta_i^{s+1}, \theta_j^{m_j})\}$ and $\det(B^{s+1}_{m_j^2+s+1}) = \det(M_s) \neq 0$, we conclude that $\det(M_{s+1}) = 0$ only if $(x_1, ..., x_{m_j^2+s+1})$ satisfies a non-trivial linear constraint given by (29). Recall that $(x_1, ..., x_{m_j^2+s+1})$ is a sub-vector of $\mathbf{p}(\theta)$, and the set of vectors in $[0,1]^L$ satisfying a non-trivial linear constraint on its elements has measure zero. Since a finite union of sets of measure zero has measure zero, we conclude that $\det(M_{s+1}) \neq 0$ almost everywhere. Proceeding by induction, we conclude that $\det(M_{m_i}) \neq 0$ almost everywhere, which completes the proof of the lemma. *Q.E.D.*

Proof of Lemma 2:

First, note that every row of matrix Π_i is a linear combination of the rows of matrices B_i and P_i. Indeed, consider an arbitrary pure strategy profile $(s_1, ..., s_{m_i})$ of player i where $s_k \in \{1, ..., m_i\}$, i.e. type 1 of player i reports type s_1,..., type m_i of player i reports type s_{m_i}. Then the corresponding row $\pi_i(s_1, ..., s_{m_i})$ of matrix Π_i can be represented as follows:

$$\pi_i(s_1, ..., s_{m_i}) = \phi(s_1, ..., s_{m_i}) B_i + \mathbf{e}_{m_i(m_i-1)} P_i$$

where $\phi_i(s_1, ..., s_{m_i})$ is a row vector of length $m_i(m_i-1)$ with the $k(m_i-1)+k'$-th entry equal to -1 if $s_k = k'$ for $k' < k$, $s_k = k'+1$ for $k' > k$ and 0 otherwise, and $\mathbf{e}_{m_i(m_i-1)}$ is a row vector of units of size $m_i(m_i-1)$. Hence,

$$\Pi_i = \Phi_i B_i + E_i P_i \tag{30}$$

where Φ_i is a $(m_i^{m_i}) \times (m_i(m_i-1))$ matrix formed by stacking all rows of the form $\phi(s_1, ..., s_{m_i})$ on top of each other in the same order according to which strategies are ordered in matrix Π_i, and E_i is $(m_i^{m_i}) \times m_i$ matrix each element of which is equal to 1. Analogously, for player j we have: $\Pi_j = \Phi_j B_j + E_j P_j$.

Now, suppose that PI does not hold. Then there are row vectors υ_i and υ_j such that

$$\upsilon_i \Pi_i = \upsilon_j \Pi_j = \kappa \neq z\pi^*$$

where κ is a non-zero row vector of size L, π^* is the row vector (of size L) of true probabilities of the agents' type profiles, and z is an arbitrary constant. Let $\tilde{\upsilon}_i = \upsilon_i \Phi_i$ and $\tilde{\upsilon}_j = \upsilon_j \Phi_j$.

Since $\upsilon_i E_i P_i = z_i \pi^*$ and $\upsilon_j E_j P_j = z_j \pi^*$ for some scalars z_i and z_j, and using (30), we obtain:

$$\tilde{\upsilon}_i B_i + z_i \pi^* = \tilde{\upsilon}_j B_j + z_j \pi^* = \kappa$$

Multiplying each part of the above inequality by $\mathbf{e}'_L$, a column vector of units of size L and noticing that $B_i \mathbf{e}'_L = B_j \mathbf{e}'_L = 0$, $\pi^* \mathbf{e}'_L = 1$, we obtain that $z_i = z_j$. Thus,

$$\tilde{\upsilon}_i B_i = \tilde{\upsilon}_j B_j \neq 0$$

So, the Identifiability Condition with respect to i and j does not hold. *Q.E.D.*

Proof of Lemma 3: If the types are independently distributed, then the $k(m_i-1)+k'$-th row of W_i is equal to the k-th row of P_i. Therefore,

$$rank\left(\left\|\begin{array}{c} W_i \\ P_i \\ W_j \\ P_j \end{array}\right\|\right) = rank\left(\left\|\begin{array}{c} W_i \\ P_j \\ W_j \end{array}\right\|\right)$$

But $rank\,(B_i) > 0$, so the Identifiability Condition fails for any i and j. *Q.E.D.*

Proof of Lemma 4: Consider a collection of row vectors μ, $\lambda_i \geq 0$ and $\gamma_i \geq 0$ such that (6) holds for all players i and all states of nature θ, i.e. in matrix notation we have:

$$\gamma_i B_i + \lambda_i P_i + \mu = \mathbf{0}. \tag{31}$$

As we have shown in the proof of Theorem 1, (31) implies that $\lambda B_i = 0$ and so $\lambda_i P_i + \mu = \mathbf{0}$ for all i. So, the Compatibility* Condition holds. *Q.E.D.*

Proof of Theorem 2: The proof is the same as that of Theorem 1, except for **Step 3**. To establish the result of Step 3, note that Compatibility* Condition implies the following: If $\gamma_i B_i + \lambda_i P_i + \mu = \mathbf{0}$, then $\lambda_i P_i + \mu = \mathbf{0}$. So, we also have $\gamma_{i^*} B_{i^*} = 0$.

Proof of Theorem 3:

There exists a BB, IC, IIR mechanism in which the type $\theta_{i'}^k$ of player i' gets all expected surplus and all other player types are held to their reservation utility levels if there exists a solution to the system of inequalities/equalities obtained from the system (10) by changing all inequalities in its subsystem IIR, except for the one corresponding to $IIR_{i'}(\theta_{i'}^k)$, to strict equalities, and leaving unchanged all other inequalities corresponding to BB and IC constraints.

Repeating Steps 1 and 2 in the proof of Theorem 1, we conclude that the modified system of inequalities/equalities has a solution, if the following **Property D'** holds:

$$\begin{aligned} &For\ all\ families\ of\ row\ vectors\ \{\gamma_i\}_{i=1,\dots,n} \geq 0,\ \ \{\lambda_i\}_{i=1,\dots,n}\ s.t.\ \lambda_{i'}^k \geq 0\ and\ vectors\ \mu: \\ &B_i + \lambda_i P_i + \mu = \mathbf{0}\ \ \forall i = 1,\dots,n \quad \Longrightarrow \quad \sum_{i=1,\dots,n} \gamma_i \hat{u}_i + \sum_{i=1,\dots,n} \lambda_i \bar{u}_i \geq 0 \end{aligned} \tag{32}$$

Note that Property D' differs from Property D (see Step 2 of Theorem 1) only insofar that the sign of all λ's except $\lambda_{i'}^k$ is now unrestricted.

The rest of the proof can be completed by repeating Steps 3-8 in the proof of Theorem 1 verbatum. Note that the argument in Step 4-6 relies on the nonnegativity of the entries of vectors γ_i, but not λ_i. Steps 5 and 6 show that all λ's are equal to each other. Since $\lambda_{i'}^k \geq 0$, we conclude that all λ's are nonnegative. So, the argument in Step 8 of Theorem 1 remains valid. *Q.E.D.*

Proof of Theorem 5: By Theorem 1, if $x(.)$ is an EASR allocation rule $x(.)$, then there exist an incentive compatible, individually rational budget balanced direct mechanism $(x(.), t(.))$ allocating the surplus in an arbitrary way (see Corollary 1). When offered by the primary agent, this mechanism is inscrutable and remains incentive compatible with respect to the prior beliefs. By the Inscrutability Principle we can restrict our analysis to such mechanisms.

If mechanism $(x(\theta), t(\theta))$ is a neutral optimum, then by Theorem 8 of Myerson (1983), there exists a collection of strictly positive multipliers $\lambda(.) \in \mathbb{R}_{++}^{\#\Theta_1}$ s.t. $(x(\theta), t(\theta))$ maximizes

$$\sum_{\theta_1 \in \Theta_1} \left(\lambda(\theta_1) \sum_{\theta_{-1} \in \Theta_{-1}} (u_1(x(\theta), \theta) + t_1(\theta))\, p(\theta_{-1}|\theta_1) \right) \tag{33}$$

subject to IC, BB and IIR constraints (1)-(2).

By Corollary 1 any allocation of ex-ante social can be supported by a mechanism satisfying (1)-(2). Therefore, agent 1 should obtain all surplus in a neutral optimum. Otherwise, the value of (33) can be increased by using a mechanism implementing the same social choice function but providing more surplus to all types of agent 1. In turn, Theorem 1 and Corollary 1 imply that $x(\theta)$ must be ex-post efficient.

Let us now show that $(x(\theta), t(\theta))$ satisfying condition (ii) of Theorem 5 is a neutral optimum. By Theorem 7 of (Myerson 1983) an incentive compatible mechanism $(x(\theta), t(\theta))$ is a neutral optimum if and only if for $k = 1, ..., \infty$ there exist collections of multipliers $\lambda(.)^k \in \mathbb{R}_{++}^{\#\Theta_1}$, $\alpha_i^k(.|.) \in \mathbb{R}_{+}^{(\#\Theta_i)^2}$, $\alpha_{i,0}^k(.) \in \mathbb{R}_{+}^{\#\Theta_i}$ $i = 1, ..., n$, and 'warranted claims' $\omega^k(.) \in \mathbb{R}^{\#\Theta_1}$ s.t. for all $\theta_1 \in \Theta_1$ and k

$$\begin{aligned} &\left(\lambda^k(\theta_1) + \sum_{\theta_1' \in \Theta_1} \alpha_1(\theta_1'|\theta_1) \right) \omega^k(\theta_1) - \sum_{\theta_1' \in \Theta_1} \alpha_1(\theta_1|\theta_1') \omega^k(\theta_1') = \\ &\sum_{\theta_{-1} \in \Theta_{-1}} \max_{\hat{x} \in X, \hat{t} \in \Delta^{n-1}} \left\{ \lambda^k(\theta_1) p(\theta_{-1}|\theta_1) \left\{ u_1(\hat{x}, (\theta_{-1}, \theta_1)) + \hat{t}_1 \right\} + \sum_{i=2,n} \alpha_{i,0}(\theta_i) \left(u_i(\hat{x}, (\theta_{-1-i}, \theta_1, \theta_i)) + \hat{t}_i \right) p(\theta_1, \theta_{-1-i}|\theta_i) \right. \\ &\left. + \sum_{i=1,...,n; \theta_i' \in \Theta_i} \alpha_i(\theta_i'|\theta_i) \left(u_i(\hat{x}, (\theta_{-1}, \theta_1)) + \hat{t}_i \right) p(\theta_{-i}|\theta_i) - \alpha_i(\theta_i|\theta_i') \left(u_i(\hat{x}, (\theta_{-1-i}, \theta_1, \theta_i')) + \hat{t}_i \right) p(\theta_{-i}|\theta_i') \right\} \end{aligned} \tag{34}$$

$$\lim_{k \longrightarrow \infty} \sup \omega^k(\theta_1) \leq \sum_{\theta_{-1} \in \Theta_1} (u_1(x(\theta_{-1}, \theta_1), (\theta_{-1}, \theta_1)) + t_1(\theta_{-1}, \theta_1))\, p(\theta_{-1}|\theta_1) \tag{35}$$

Let $\lambda^k(\theta_1) \equiv p_1(\theta_1)$, $\alpha_{i,0}^k(\theta_i) \equiv p_i(\theta_i)$ and $\alpha_i^k(\theta_i'|\theta_i) \equiv 0$ for all $i = 2, ..., n$ and $\theta_i, \theta_i' \in \Theta_i$. Then (34) becomes

$$p_1(\theta_1) \omega^k(\theta_1) = \sum_{\theta_{-1} \in \Theta_{-1}} \max_{\hat{x} \in X} \sum_{i=1,...,n} u_i(\hat{x}, (\theta_{-1}, \theta_1)) p(\theta_{-1}, \theta_1)$$

Then the system of (34) and (35) is solved by $(x(\theta), t(\theta))$ s.t. $x(\theta)$ is ex-post efficient and $\sum_{\theta_{-1} \in \Theta_1} (u_1(x(\theta), \theta) + t_1(\theta))\, p(\theta_{-1}|\theta_1) = \omega^k(\theta_1) = \sum_{i=1,...,n} u_i(x(\theta), \theta) p(\theta_{-1}|\theta_1)$, i.e. agent 1 (the primary agent) gets all social surplus conditional on her type.

Finally, we will show that such allocation of social surplus is unique in a neutral optimum. Let x^* be an ex-post efficient social choice rule and define a set of blocked payoff vectors $\hat{B}(\Gamma)$

as follows:

$$\hat{B}(\Gamma) \equiv \Big\{ z(.) \in \mathbb{R}_+^{\#\Theta_1} \mid \sum_{\theta_1 \in \Theta_1} z(\theta_1) p_1(\theta_1) \leq \sum_{i=1,...,n;\theta\in\Theta} u_i(x^*(\theta), \theta) p(\theta)$$

$$\exists \theta_1 \in \Theta_1 \text{ s.t. } z(\theta_1) < \sum_{i=1,...,n;\theta_{-1}\in\Theta_{-1}} u_i(x^*(\theta), \theta) p(\theta_{-1}|\theta_1) \Big\}$$

We need to prove that $\hat{B}(\Gamma)$ satisfies Axioms 1-4. It is immediate that Openness and Dominance Axioms are satisfied. The Extension Axiom also holds because adding additional actions to X may change the set of ex-post efficient actions and cause an increase in $\sum_{i=1,...,n;\theta\in\Theta} u_i(x^*(\theta), \theta)p(\theta)$ and $\sum_{i=1,...,n;\theta_{-1}\in\Theta_{-1}} u_i(x^*(\theta), \theta)p(\theta_{-1}|\theta_1)$.

Finally, let us show that $\hat{B}(\Gamma)$ satisfies The Strong Solutions Axiom. By Theorem 1 and Corollary 1, if a strong solution M' exists then it must implement an ex-post efficient social choice function $x^*(.)$ and allocate all social surplus to agent 1, so that agent 1's vector of expected payoffs $U_1(.|M')$ is such that $\sum_{\theta_1\in\Theta_1} p_1(\theta_1)U_1(\theta_1|M') = \sum_{i=1,...,n;\theta\in\Theta} u_i(x^*(\theta), \theta)p(\theta)$. Otherwise, it will be dominated by some incentive compatible mechanism that has these two properties.

Now suppose that some type θ_1 of agent 1 earns an expected payoff that is strictly less then the social surplus conditional on her type, i.e. $U_1(\theta_1|M') < \sum_{i=1,...,n;\theta_{-1}\in\Theta_{-1}} u_i(x^*(\theta), \theta)p(\theta_{-1}|\theta_1)$. Then there exists θ_1' s.t. $U(\theta_1'|M') > \sum_{i=1,...,n;\theta_{-1}\in\Theta_{-1}} u_i(x^*(\theta_{-1}, \theta_1'), (\theta_{-1}, \theta_1'))p(\theta_{-1}|\theta_1')$. This implies that some type $\hat{\theta}_i$ of a player $i \in \{2, ..., n\}$ has to earn a negative payoff conditional on $(\hat{\theta}_i, \theta_1'$. But this contradicts the fact that M' is a strong solution, i.e. it is incentive compatible given any type of player 1.

Construction of a sequential equilibrium in the informed principal problem.

Consider an inscrutable direct mechanism $(x^*(\theta), t^*(\theta))$ which is a neutral optimum. Let $U_i^*(\theta_i)$ stand for the interim expected payoff of type θ_i of agent i in this mechanism. By Theorem 5, $x^*(\theta)$ is ex-post efficient, and $U_1^*(\theta_1)$ is equal to the expected social surplus conditional on agent 1's type θ_1, while $U_i^*(\theta_i) = 0$ $\forall i \in \{2, ..., n\}$ and $\forall \theta_i \in \Theta_i$.

Let us demonstrate the existence of a sequential equilibrium on the equilibrium path of which all types of player 1 offer $(x^*(\theta), t(\theta))$, in stage 2 agents from 2 to n do not update their prior beliefs about the type of player 1 and agree to participate in the mechanism, while in stage 3 all agents, including agent 1, report their types truthfully.

Since $(x^*(\theta), t(\theta))$ is incentive compatible given prior beliefs, we only need to rule out deviations by player 1. So, suppose that in stage 1 player 1 deviates and offers some mechanism $\nu \in Z$. Let us show that after this deviation there exists an equilibrium of the continuation game (i.e. consistent beliefs and sequentially rational strategies) in which the expected payoff to type $\theta_1 \in \Theta_1$ of agent 1 does not exceed $U_1^*(\theta_1)$.

For this, consider a modified game Γ^ν which differs from Γ only in one aspect: in stage 2 agent 1's only choice is either to drop out and earn the *modified reservation payoff* $U_1^*(\theta_1)$ or to offer the mechanism ν. Consider a sequential equilibrium of this game. Let $q(\theta_1)$ denote the probability that agent 1 chooses mechanism ν in this equilibrium, $\{P_i^\nu(\theta_{-i}|\theta_i), O_i^\nu(\theta_{-i}|r, \theta_i)\}$ denote the agents' belief system and σ^ν denote the agents' strategy profiles in the continuation game after ν is offered. $P_i^\nu(\theta_{-i}|\theta_i)$ stands for the posterior beliefs of agent i in stage 3 after the mechanism ν is offered, while $O_i^{\nu}(\theta_{-i}|r,\theta_i)$ stands for the posterior beliefs of agent i in stage 4 after the agents have taken participation decisions described by vector r. Note that for agent 1 we have $P_1^{\nu}(\theta_{-1}|\theta_1) = p_1(\theta_{-1}|\theta_1)$. Also, let $W_i(\theta_i|\nu,\sigma^{\nu},P_i^{\nu}(.))$ denote the expected payoff of agent type θ_i in this equilibrium.

The sequential rationality of strategies requires that:

$$q(\theta_1) = \begin{cases} 1 & \text{if } W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) > U_1^*(\theta_1) \\ any\ x \in [0,1] & \text{if } W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) = U_1^*(\theta_1) \\ 0 & \text{if } W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) < U_1^*(\theta_1) \end{cases} \tag{36}$$

Consistency of beliefs requires that for $i = 2,...,n$:

$$P_i^{\nu}(\theta_1,\theta_{-1-i}|\theta_i) = \frac{p_i(\theta_1,\theta_{-1-i}|\theta_i)Q(\theta_1)}{\sum_{\theta'_{-1-i}\in\Theta_{-i-1},\theta'_1\in\Theta_1} p_i(\theta'_1,\theta'_{-1-i}|\theta_i)Q(\theta'_1)} \tag{37}$$

where $Q(\theta_1)$ satisfies: $Q(\theta_1)\left(\sum_{\theta'_1\in\Theta_1} q(\theta'_1)\right) = q(\theta_1)$ for all $\theta_1 \in \Theta_1$.

If $W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) \leq U_1^*(\theta_1)$, then the strategies σ^{ν} and beliefs $P_i^{\nu}(\theta{-}i|\theta_i)$ in the continuation game also support a sequential equilibrium in which all types of player 1 choose the reservation payoff with probability 1, i.e. $q(\theta_1) = 0$ for all $\theta_1 \in \Theta_1$.

Now suppose that $W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) > U_1^*(\theta_1)$ for some $\theta_1 \in \Theta_1$. Then, by (36) and (37), $P_i^{\nu}(\theta_1,\theta_{-1-i}|\theta_i) > 0$ implies that $W_1(\theta_1|\nu,\sigma^{\nu},p_1(.|\theta_1)) \geq U_1^*(\theta_1)$. But since $x^*(\theta)$ is ex-post socially optimal and $U_1^*(\theta_1) = \sum_{\theta_{-1}\in\Theta_{-1},i\in\{1,...,n\}} u_i(x^*(\theta_{-1},\theta_1),(\theta_{-1},\theta_1))p(\theta_{-1}|\theta_1)$, i.e. agent 1's payoff is at least as large as the expected social surplus conditional on her type, there must exist an agent type $\theta_i \in \Theta_i$, $i \neq 1$ s.t. $W_i(\theta_i|\nu,\sigma^{\nu},P_i^{\nu}(.)) < 0$. However, this agent can always ensure herself a reservation payoff of zero by dropping out in stage 3.

Thus, we conclude that there exists a sequential equilibrium of the game Γ^{ν} where all types of agent 1 drop out with probability 1 in stage 2, and the beliefs and strategies in the continuation game after agent 1 offers some mechanism $\nu \in Z$ are given by $\{P_i^{\nu}(.),O_i^{\nu}(.)\}$ for $i = 1,...,n$ and σ^{ν} respectively.

Recall than $\nu \in Z$ is an arbitrary mechanism. So, it follows immediately that the 'informed principal' game Γ^* possesses a sequential equilibrium in which all types of agent 1 offer the mechanism $(x^*(\theta),t(\theta))$ with probability 1 in stage 2. Off the equilibrium path, if agent 1 offers a mechanism $\nu \in Z$, then beliefs and strategies in the continuation game are the same as in the sequential equilibrium of the game Γ^{ν}, i.e. $\{P_i^{\nu}(.),O_i^{\nu}(.)\}$ for $i = 1,...,n$ and σ^{ν} respectively. *Q.E.D.*

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