Optimal Bidding in Multi-Unit Discriminatory Auctions: Two Bidders [¤]

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D44: multi-unit auctions, discriminatory auctions, Treasury auctions

This Version: March 7, 2003

[¤]Comments welcome. Comments by Jim Anton, Sudipto Bhattacharya, Vijay Krishna, Pete Kyle, John Coleman, Gopal Dasvarma, Guiseppe Lopomo, Phil Reny, Jeroen Swienkels and seminar participants at the London School of Economics, the Duke/UNC Microeconomics workshop, and the Stonybrook Conference on multi-unit auctions have been helpful.

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ABSTRACT

We analyze the two-bidder discriminatory auction with downward sloping marginal valuations and a continuous, variable award. We allow for a common component in marginal valuations and affiliation. We focus on problems that admit solutions with strictly downward sloping bidding schedules. Using the method of characteristics, we reduce the first order conditions to a pair of ordinary differential equations where we fix the equilibrium quantities that are allocated to the two bidders. These ordinary differential equations are extensions of the Milgrom-Weber equation for the first price unit auction. A new ordinary differential equation that characterizes the relation between signals yielding the fixed quantity allocation is obtained. The equilibrium solution of the discriminatory auction is given by the solution of an initial value problem for these two coupled ODEs that generalize the Milgrom-Weber equation. We show the conditions for existence of solutions to this systems of ODEs. Some examples are analytically and numerically solved using our approach.

1 Introduction

A large theory has developed to explain auctions in which bidders bid only for one unit. The work of Myerson (1981), Riley and Samuelson (1982) and Milgrom and Weber (1982) provides a characterization of the bidding strategies and revenue trade-offs across different auction formats. This theory has been extended to environments where the seller sells multiple units of the good but the bidders each only demand one unit (Harris and Raviv (1981), Weber (1982) and others). Much less is known about share auctions in which bidders bid for fractions of the good by submitting demand functions. In a classic paper, Wilson (1979) showed that share auctions have multiple equilibria and uniform price share auctions can lead to equilibrium outcomes that are inferior to the seller. This insight has been reinforced in papers by Back and Zender (1994), Engelbrecht-Wiggans and Kahn (1999) and Ausubel and Cramton (1996). Engelbrecht-Wiggans and Kahn (1999) and Ausubel and Cramton (1996) show that uniform price auctions will not result in efficient allocations. Maskin and Riley (1989) solve for the optimal mechanism with downward sloping valuation curves and private valuations when the seller is not required to sell all of the good (i.e. reserve prices exists).

While there has been progress in uniform price auctions and in optimal auctions, very little is known about the multi-unit discriminatory auction with variable awards. Recently, Engelbrecht-Wiggans and Kahn (1998) numerically solve a two-unit discriminatory auction with private values and show that the bidder will bid the same price on both units when his valuation is high. Tenorio (1999) solves a two-bidder two unit discrete example with private values. Reny (1999) shows the existence of equilibrium with discrete units, private values and downward sloping marginal valuations. Jackson and Swienkels (1999) present a different approach to existence that is related to work by Simon and Zame (1999). In work that is related to our paper, Viswanathan and Wang (1999) obtain the first order condition using a slightly different variational approach and provide an example using the limiting uniform distribution.

The main focus of this paper is to provide a detailed analysis of the two bidder discriminatory auction with variable awards and continuous units. We consider a model where two bidders submit demand functions for continuous quantities up to one unit of the good. We consider marginal valuation for each bidder that are decreasing in quantity and increasing in the bidder's signal. We allow the marginal valuation function of one bidder to depend on the signal observed by the other bidder. Further, we allow the distributions to be affiliated as in Milgrom and Weber (1982). Given the bidder's demand functions, the seller determines the stop-out price (the marginal price for the last unit) and the bidders then pay the seller the marginal prices they bid for each unit of the good (with continuous quantities, this is the area under the demand curve). Such a procedure is the dominant mechanism used by governments to raise capital all over the world.[1]

We present a characterization of the multi-unit discriminatory auction. First, we formulate our problem in a way that reduces it to a point by point maximization. This directly provides a first order condition and shows that the second order condition (pseudo-concavity) is satisfied when the other bidder uses a strictly downward sloping demand curve. A key aspect of the first order conditions is dealing with cut-off prices that determine when the other bidder will get zero units.

We focus on symmetric equilibria that involve strategies that are strictly monotone increasing in the signal and strictly monotone decreasing in the price. This is a stronger requirement that the result in Reny (1999) in his Corollary 5.3 (and Example 5.3) that the symmetric K unit pay-your-bid auction with private values has a symmetric equilibrium with non-decreasing bid functions.[2] A different approach to existence in discriminatory multi-unit auctions is the work of Simon and Zame (1999) and Jackson and Swienkels (1999). Both these papers provide existence arguments that imply the existence of equilibrium with endogenous tie-breaking rules. In the context of the discriminatory multi-unit first price auctions, their results imply that the probability of ties in equilibrium is zero. Thus, taken together, these three papers imply that the symmetric K unit pay-your-bid auction with private values has a symmetric equilibrium with non-decreasing bid functions and zero probabilities of ties. However, this is not enough to guarantee that equilibrium strategies will be strictly monotone decreasing in price.

[1] Though, the U.S. Treasury has recently shifted to the uniform-price auction.

[2] Reny's (1999) result applies to discrete units while we consider continuous quantities. It seems as if a limiting argument yields an existence result with continuous quantities.

We show that the existence of such an equilibrium depends on solving a pair of coupled first order conditions. Further, each of the first order conditions that is central to our characterization reduces to an ordinary differential equation when we use the method of characteristics, i.e., we restrict ourselves to the set of prices and signals of one bidder that yield that bidder a fixed quantity in equilibrium. These ordinary differential equations are of the same form as the ordinary differential equation that characterizes the unit auction in Milgrom and Weber (1982). If we knew the equilibrium mapping relating the two signals that yielded these fixed allocations and the appropriate boundary conditions, we could directly solve for equilibrium. We obtain a new differential equation for the unknown equilibrium mapping between the signals. The solution to this ordinary differential equation plus the Milgrom-Weber type equation plus boundary conditions yield the solution to the equilibrium mapping.

We analyze the existence of solutions to the system of ordinary differential equations and show that the symmetric equilibrium in strategies that are strictly monotone increasing in the signal and strictly monotone decreasing in the price is unique (if it exists). We also show that the restriction that the demand curves be downward sloping imposes a restriction on the slope of the equilibrium mapping between signals at the lowest signal. Surprisingly, the restriction requires that this slope be steep enough but not too steep.

Our characterization of the equilibrium in the multi-unit discriminatory auction yields the following results. First, we show that the solution to the variational problem in the multi-unit discriminatory auction requires the transversality condition that any signal must get a zero quantity when playing against the highest signal. A direct implication of this is that the probability of not getting the good is greater than zero for all signals other than the highest signal (as in a unit auction). Further, if the signal space is bounded, this transversality condition forces the highest signal to submit a flat bidding curve even if the highest signal has a downward sloping marginal valuation. These transversality conditions provide the key boundary conditions for the optimization problem. Thus, we solve the system of ordinary differential equations given the initial conditions at the highest signal.

Second, we show that on the marginal unit (the unit at the stop-out price), the bidder with the higher signal often shades his bid less than the bidder with the lower signal. The intuition for this results is as follows. When a bidder plays against the lowest signal, the bidder (here the high signal) must bid her marginal value because the cumulative density function at the lowest signal is zero. This re°ects the fact that for the high signal this is the lowest price in equilibrium. In contrast, the lowest signal while bidding against a higher signal will not bid his marginal value (because the cumulative density of the higher bidder at the stop-out price is not zero). Again, this is because this is not the lowest price in equilibrium for the lowest signal (he has to consider the e®ect of his bid on higher quantities that he obtains against lower signals than the signal he is playing against). Similarly, the transversality condition implies that when the highest signal plays against a signal arbitrarily close to the highest signal the highest signal obtains everything. In this case, we can again show that the highest signal shades his bid less than the signal arbitrarily close to the highest signal. Further, we provide a su±cient condition for such behavior to occur for all possible signal pairs. Thus, this shows that in equilibrium we can have the high signal shade his bid less than the low signal for the marginal unit.

Third, we show that our solution agrees with the Milgrom-Weber unit-auction solution when we consider the °at marginal valuation function, i.e., the unique symmetric solution when we consider °at demands and divisible goods is the Milgrom-Weber unit auction solution. Further we prove convergence to the Milgrom-Weber unit-auction solution as we make the marginal valuations functions °atter and °atter.

We show by example that equilibrium can involve \°ats", i.e., the requirement that the equilibrium involve strictly decreasing strategies (in quantity) is too strong. The existence theorems in Reny (1999) and Jackson and Swienkels (1999) consider non-decreasing bid functions and thus allow for such \°ats". We point out the di±culties involved in computing equilibria that involve \°ats".

Our paper is organized as follows. Section 2 presents the basic model while Section 3 provides the ¯rst order conditions and the su±ciency argument. Section 4 provides the reduction to ordinary di®erential equations and shows the relationship to Milgrom and Weber's (1982) work. Section 5 provides a complete solution to the system of ordinary di®erential equations. Section 6 considers numerical examples using our approach. Section 7 concludes.

2 The Basic Model

We consider a model with two bidders i and j. Bidder i observes signal s_i and bidder j observes signal s_j. The conditional distribution of s_j given s_i is a distribution over the interval $[0, \bar{s}]$ with the PDF $g(s_j|s_i)$ and the corresponding CDF denoted $G(s_j|s_i)$ (here $\bar{s}$ may be ∞). The valuation of the object to bidder i is given by $V(s_i, s_j, q)$ where q is the quantity of the good. Notice that we allow for dependence on the other bidder's signal. The marginal valuation for unit q is defined by

$$M(s_i, s_j, q) = \frac{\partial V(s_i, s_j, q)}{\partial q} \tag{2.1}$$

We assume that:

Assumption A1: $M(s_i, s_j, q)$ is strictly decreasing in q, strictly increasing in s_i and non-decreasing in s_j,

$$\frac{\partial M}{\partial q}(s_i, s_j, q) < 0; \qquad \frac{\partial M}{\partial s_i}(s_i, s_j, q) > 0; \qquad \frac{\partial M}{\partial s_j}(s_i, 0, q) \geq 0. \tag{2.2}$$

Assumption A2: s_i and s_j are affiliated. The ratio of their distributions, $G(s_j|s_i)/g(s_j|s_i)$ is decreasing in s_i (see Milgrom and Weber (1982)),

$$\frac{\partial}{\partial s_i}\left(\frac{G(s_j|s_i)}{g(s_j|s_i)}\right) < 0; \qquad 0 < s < \bar{s}. \tag{2.3}$$

Assumption A3: $M(0, 0, 1) > 0$.

Assumption A4: (greater dependence of marginal valuation in own signal) $M(x, s, 1-q)$-$M(s, x, q)$ is strictly increasing in x and is decreasing in s.

Assumption A1 is an assumption that implies that the marginal value of a larger quantity is lower. In the context of Treasury auction, this is consistent with inventory costs that make bidders (dealers) unwilling to take large positions. The assumption that $\partial M/\partial s_j(s, 0, q) \geq 0$ allows both private values ($\partial M/\partial s_j(s, 0, q) = 0$) and a common component in marginal valuations ($\partial M/\partial s_j(s, 0, q) > 0$). Assumption A2 ensures that Lemma 1 in Milgrom and Weber (1982) is valid. We will use this as in Milgrom and Weber (1982) to prove pseudo-concavity of first order conditions (when the other bidder submits a strictly downward sloping demand curve). Assumption A4 basically implies that a small change in a bidder's signal has a greater impact on his own marginal valuation than on the other bidder's valuation. It is clearly true with private values and is a restriction when there is a common component.

Examples include:

Example 1:

$$V(s_i; s_j; q) = v + (s_i + \zeta s_j)q - \frac{\rho}{2}q^2 \tag{2.4}$$

where $\zeta < 1$. Here

$$M(s_i; s_j; q) = v + (s_i + \zeta s_j) - \rho q \tag{2.5}$$

This is a reasonable representation of a model where Treasury dealers have different inventories and different information and thus there is both a private value and a common value component. We will work with Example 1 when the signal space is unbounded.

Example 2:

$$V(s_i; q) = (v + s_i)q - \frac{1}{2}(k - s_i)\rho q^2 \tag{2.6}$$

where v is a constant. Here

$$M(s_i; q) = (v + s_i) - (k - s_i)\rho q \tag{2.7}$$

where the underlying distribution is over $[0; \bar{s}]$ where $\bar{s} < k$. This is a model where dealers have heterogeneous beliefs and heterogeneous willingness to take positions in the auction. It is generally believed that some dealers in Treasury auctions can take bigger positions than others. Here one aspect of the underlying unknown variable is the ability to take this kind of position risk. This is the proto-typical example that we will work with numerically to illustrate our approach.

There is one unit of the object being auctioned. Each bidder submits a demand curve $x_i(p; s_i)$ that is downward sloping. Let the inverse demand function be given as $x_i^{-1}(q; s_i)$, i.e., this is the marginal price bid by s_i for quantity q. The total payment made by bidder i is given by

$$TP(p; s_i) = \int_0^{x_i(p;s_i)} x^{-1}(q; s_i)\, dq \tag{2.8}$$

under the discriminatory auction. Since demand has to be equal to supply

$$x_i(p, s_i) + x_i(p, s_j) = 1. \tag{2.9}$$

Given a demand curve $x(p, s_i)$ and a stop-price p, the “utility” to bidder s_i is given by:

$$V(s_i, s_j, x(p, s_i)) - TP_i(p, s_i) \tag{2.10}$$

The separability of valuation and payment seems critical to our analysis.

3 The First-Order Conditions and Their Sufficiency

The discriminatory auction involves a variational problem as the price that one bids for quantity q affects the payment at all quantities greater than q. In formulating the control problem from the perspective of bidder i, the underlying independent-variable space is bidder j's signal, s_j. In the discriminatory auction, bidder i takes bidder j's bidding curve $x_j(p, s_j)$ as given. Hence, the quantity he receives at price $p(s_j)$ is given by

$$h(p(s_j), s_j) = 1 - x_j(p, s_j) \tag{3.1}$$

Hence we are viewing $p(s_j)$ as the control variable. We can do so because the market clearing constraint implies that bidder i only gets the residual amount. Hence choosing a demand curve is the same as choosing a price.[3]

Rather than considering the total payment made by bidder i directly, we consider it as the difference between the total payment made by all bidders, $A(s_j)$ defined next, and the payment made by bidder j, $B(p(s_j))$ defined below. This substitution allows for a simpler formulation of the problem that can be integrated to obtain point by point maximization. This approach eliminates the derivative of the price with respect to s_j and makes checking the sufficient conditions easy.

We define the total payment from all bidders

$$A(s_j) = \int_0^{x_j(p, s_j)} x_j^{-1}(q, s_j)\, dq + \int_0^{h(p(s_j), s_j)} x_i^{-1}(q, s_i)\, dq \tag{3.2}$$

[3]This is subject to the assumption that both i and j submit strictly downward sloping demand curves. We will see that this is intimately tied to the sufficiency of the first order conditions.

The total derivative of this function with respect to s_j is

$$\frac{dA(s_j)}{ds_j} = p(s_j)\left[p'(s_j)\frac{\partial x_j(p;s_j)}{\partial p} + \frac{\partial x_j(p;s_j)}{\partial s_j}\right] + p(s_j)\frac{dh(p;s_j)}{ds_j}$$
$$- \int_0^{x_j(p;s_j)} \left(\frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial s_j}\middle/\frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial p}\right) dq \tag{3.3}$$

Since

$$\frac{dh(p;s_j)}{ds_j} = -\left[p'(s_j)\frac{\partial x_j(p;s_j)}{\partial p} + \frac{\partial x_j(p;s_j)}{\partial s_j}\right] \tag{3.4}$$

we obtain that

$$\frac{dA(s_j)}{ds_j} = -\int_0^{x_j(p;s_j)} \left(\frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial s_j}\middle/\frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial p}\right) dq \tag{3.5}$$

The ratio of the partial derivatives under the integral is obtained by recognizing that $d[x^{-1}(q;s_j)]/ds_j$ is the derivative of the price with respect to the signal s_j when the quantity q submitted by j is fixed, i.e., along a iso-contour line. This implies that

$$\frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial p}\frac{d[x^{-1}(q;s_i)]}{ds_j} + \frac{\partial x_j(x_j^{-1}(q;s_j);s_j)}{\partial s_j} = 0 \tag{3.6}$$

leading to the ratio of partial derivatives.

We also define:

$$B(p(s_j);s_j) = \int_0^{x_j(p;s_j)} x^{-1}(q;s_j)\,dq \tag{3.7}$$

This is correct as bidder i takes the bidding curve of the other bidder as given and thus, when he knows $p(s_j)$, j's total payment is determined. Note that

$$\frac{dB(p(s_j);s_j)}{dp} = -p(s_j)\frac{\partial h(p(s_j);s_j)}{\partial p} \tag{3.8}$$

With this, we formulate the problem for the bidder i as follows:

$$\max_{p(\cdot)} \int_0^{\bar{s}} \left[V(s_i;s_j;h(p(s_j);s_j)) - \int_0^{h(p(s_j);s_j)} x_i^{-1}(q;s_i)dq\right] g(s_j|s_i)\,ds_j$$
$$= \max_{p(\cdot)} \int_0^{\bar{s}} [V(s_i;s_j;h(p(s_j);s_j)) - A(s_j) + B(p(s_j);s_j)]\,g(s_j|s_i)\,ds_j \tag{3.9}$$

subject to the constraint that

$$h(p(s_j);s_j) \geq 0 \tag{3.10}$$

Note that the bidder j's bidding curve may have a region where $x_j(p; s_j) = 0$. In the first order condition, we will have to pay special attention to this region even though it does not impose a direct constraint. Further, we assume that $x_j(p; s_j)$ is strictly monotone increasing in s_j and strictly decreasing in p. This implies that the region with $x_j(p; s_j) = 0$ is of the form $[0; s_j^{\omega}(s_i)]$.

An integration by parts calculation shows that

$$
\begin{aligned}
\int_0^{\bar{s}} A(s_j) g(s_j|s_i)\, ds_j &= A(s_j)G(s_j|s_i)\Big|_0^{\bar{s}} - \int_0^{\bar{s}} G(s_j|s_i) \frac{dA(s_j)}{ds_j} ds_j \\
&= A(\bar{s}) - \int_0^{\bar{s}} G(s_j|s_i) \frac{dA(s_j)}{ds_j} ds_j \qquad (3.11)
\end{aligned}
$$

Hence, we can rewrite the problem for bidder i as:

$$
\max_{p(\cdot)} \int_0^{\bar{s}} \left[V(s_i; s_j; h(p(s_j); s_j)) + \frac{G(s_j|s_i)}{g(s_j|s_i)} \frac{dA(s_j)}{ds_j} + B(p(s_j); s_j) \right] g(s_j|s_i)\, ds_j - A(\bar{s}) \quad (3.12)
$$

subject to the constraint that $h(p(s_j); s_j) \geq 0$. Here $dA(s_j)/ds_j$ is given by Equation (3.5) above.[4] Since we have reduced our problem to point by point maximization, we can use Luenberger Theorem 1 page 249 and maximize

$$
\begin{aligned}
\max_{p(\cdot)} & \int_0^{\bar{s}} \left[V(s_i; s_j; h(p(s_j); s_j)) + \frac{G(s_j|s_i)}{g(s_j|s_i)} \frac{dA(s_j)}{ds_j} + B(p(s_j); s_j) \right] g(s_j|s_i)\, ds_j \\
& - \int_0^{\bar{s}} \eta_1(s_j) h(p(s_j); s_j)\, ds_j - A(\bar{s}) \qquad (3.13)
\end{aligned}
$$

The first order conditions with respect to $p(s_j)$ are that

$$
\begin{aligned}
& M(s_i; s_j; h(p(s_j); s_j)) \frac{\partial h(p; s_j)}{\partial p} - \frac{G(s_j|s_i)}{g(s_j|s_i)} \frac{\frac{\partial x_j(p; s_j)}{\partial s_j}}{\frac{\partial x_j(p; s_j)}{\partial p}} \frac{\partial x_j(p; s_j)}{\partial p} \\
& \quad - p(s_j) \frac{\partial h(p; s_j)}{\partial p} - \eta_1(s_j) \frac{\partial h(p; s_j)}{\partial p} = 0 \qquad (3.14) \\
& A(\bar{s}) - B(p(\bar{s}); \bar{s}) = 0 \qquad \text{(transversality condition)} \\
& \eta_1(s_j) h(p(s_j); s_j) = 0
\end{aligned}
$$

[4] We ignore the constraints imposed by the fact that the equilibrium bidding curve obtained by solving the optimization problem has to be non-decreasing in q. If the equilibrium bidding curves we obtain are strictly downward sloping in q, we are justified in ignoring this constraint. However, it may be the case that there is no strictly decreasing bidding curve (in q) that solves the above first order conditions. Then, the optimization would have to consider these additional constraints.

We note that solving the maximization problem requires $A(\bar{s})$ to be minimized. Since, $A(\bar{s})$ is bounded below by $B(p(\bar{s});\bar{s})$, we obtain the above condition. Directly, we can show:

Lemma 1 The minimum value for $A(\bar{s})$ is attained when $x_j(p(\bar{s});\bar{s}) = 1$ and $A(\bar{s}) = B(p(\bar{s});\bar{s})$.

Proof: Suppose not. If $x_i(p(\bar{s});\bar{s}) > 0$, then we must have bidder i paying $x_i(p;\bar{s})p$ where p is the price against the highest signal $\bar{s}$ while bidder j pays

$$B(p;\bar{s}) = \int_0^{x_j(p;\bar{s})} x_i^{-1}(q;\bar{s})\,dq \tag{3.15}$$

If we allocate the whole quantity to bidder j, we lower the price to p' and obtain

$$\begin{aligned}
A(\bar{s}) &= B(p';\bar{s}) \\
&= \int_0^{x_j(p';\bar{s})} x_i^{-1}(q;\bar{s})\,dq \\
&= \int_0^{x_j(p;\bar{s})} x_i^{-1}(q;\bar{s})\,dq + \int_{x_j(p;\bar{s})}^{x_j(p';\bar{s})} x_i^{-1}(q;\bar{s})\,dq \\
&< \int_0^{x_j(p;\bar{s})} x_i^{-1}(q;\bar{s})\,dq + x_i(p;\bar{s})p
\end{aligned}$$

because we go down bidder j's demand curve, which contradicts the assumption that we are at minimum value for $A(\bar{s})$. Thus we must have that the total payment from bidder i is zero and the total quantity allocated to bidder i is zero when he is bidding against the highest signal.

We further specialize Lemma 1 for problems with bounded signal space.

Lemma 2 With bounded signal space, $[0;\bar{s}]$, the highest signal $\bar{s}$ must submit a flat bidding curve in equilibrium. All other signals s will submit bidding curves below the flat bidding curve submitted by the highest signal.

Proof: With any downward sloping curve, the argument in Lemma 1 implies that the probability of the highest signal $\bar{s}$ winning the good is 1. Therefore, signal $\bar{s}$ will submit a flat bidding curve. With a flat bidding curve for $\bar{s}$, the argument in Lemma 1 is not valid for signal $s < \bar{s}$. But if any signal s submits a bidding curve that is flat at the same price as $\bar{s}$, then by monotonicity in the signal, there is an interval of signals that is flat at this price. But this is inconsistent with equilibrium as any signal can now deviate by raising the highest price by a very small amount (the probability of ties in equilibrium is zero). Thus, the highest signal must submit a flat demand curve and all other signals $s < \bar{s}$ must submit a bidding curve strictly less than this flat demand curve.

The transversality condition implies that there is a strictly positive probability that bidder i will not receive the good in equilibrium. This is obviously true in an unit auction. In a multi-unit auction with downward sloping marginal valuations this is more surprising. In problems with bounded support, this implies that the highest signal must submit a flat bidding schedule (even though the highest signal may have a downward sloping marginal valuation).[5] This transversality condition provides the critical end-point condition that allows us to solve the first order conditions.[6]

Rearranging the first order condition and focusing on the region where $\eta_1(s_j) = 0$ and $x_j(p; s_j) > 0$ we obtain:

$$[M(s_i; s_j; h(p(s_j); s_j)) - p(s_j)]\frac{\partial h(p; s_j)}{\partial p} - \frac{G(s_j|s_i)}{g(s_j|s_i)}\frac{\partial x_j(p; s_j)}{\partial s_j} = 0 \tag{3.16}$$

or substituting for $\partial h(p; s_j)/\partial p$,

$$p(s_j) \quad = \quad M(s_i; s_j; h(p(s_j); s_j)) + \frac{G(s_j|s_i)}{g(s_j|s_i)}\frac{\frac{\partial x_j(p;s_j)}{\partial s_j}}{\frac{\partial x(p;s_j)}{\partial p}}. \tag{3.17}$$

To understand what happens when the constraint $h(p(s_j); s)j) \geq 0$ is binding, consider the critical value of the other bidder's signal $s_j^u(s_i)$ where the constraint does not bind and $h(p(s_j); s_j) = 0$. At $s_j^u(s_i)$, the value of the objective is zero and the payment by bidder i is zero, i.e, $A(p(s_j); s_j) - B(p(s_j) = 0$. If the objective is pseudo-concave in p, we would obtain that the derivative with respect to p is zero at $s_j^u(s_i)$ and negative at p for $s_j > s_j^u(s_i)$, then the optimal solution given the constraint is binding is to set $h(p(s_j); s_j) = 0$. In a moment, we will write down the condition for pseudo-concavity in p.

The region where $x_j(p(s_j); s_j) = 0$ is of the form $[0; s_j^l(s_i)]$. We are assuming that bidder j's demand function is strictly monotone increasing in s_j and strictly monotone decreasing in p. Hence, given s_j such that $s_j < s_j^l(s_i)$, we know that that $x(p+\epsilon; s_j) = 0$ and $x(p; s_j + \epsilon) = 0$

[5] Thus the highest signal submits a flat bidding curve while every other signal submits a strictly downward sloping bidding curve. We will refer to this a symmetric equilibrium with strictly downward sloping strategies in q even though this is not true at $\bar{s}$.

[6] In Engelbrecht-Wiggans and Kahn (1998) the transversality condition is satisfied because the highest signal bids the same bid on both of the units being sold.

for $p < p(s_j^{\omega}(s_i))$ where ε is a small positive or negative number. This implies that the first order condition reduces to

$$[M(s_i; s_j; h(p(s_j); s_j)) - p(s_j)]0 - 0 = 0 \tag{3.18}$$

and hence imposes no restriction on the price in this region. Therefore we can set the price in this region as the end-point price $p(s_j^{\omega}(s_i)$ to ensure continuity at $s_j^{\omega}(s_i)$.

Thus when both bidders i and j get allocated positive quantities, the price inverts bidder j's signal and hence we condition via the price on bidder j's signal. However, when bidder i gets 1 unit and bidder j gets zero units, the total payment does not vary with signal j for $s_j < s_j^{\omega}(s_i)$ (this is similar to a unit auction). The total payment is that determined when bidder i bids against bidder $s_j^{\omega}(s_i)$. In a unit auction, the critical $s_j^{\omega}(s_i)$ is just s_i and hence the payment by bidder i in the unit auction does not vary with bidder j for $s_j < s_i$.

To complete the equilibrium we need an extension of the prices below $p(s_j^{\omega}(s_i))$ if $s_j^{\omega}(s_i)$ is well defined. For low values of s_i, it may be the case that in equilibrium the maximum quantity is less that one unit. For such s_i, we take $s_j^{\omega}(s_i) = 0$. Hence the off-equilibrium extension with this added restriction is that the inverse demand curve is flat at the price $p(s_j^{\omega}(s_i))$, i.e. bidder s_i will take infinite quantities at the price $p(s_j^{\omega}(s_i))$.

We prove the required pseudo-concavity of the maximand in p as follows. Consider s_i's optimization against s_j where s_j receives an interior allocation between 0 and 1. Pseudo-concavity requires that for $p' < p(s_j)$ that

$$\frac{\partial h(p'; s_j)}{\partial p}\left([M(s_i; s_j; h(p'; s_j)) - p'] - \frac{G(s_j|s_i)}{g(s_j|s_i)}\frac{\frac{\partial x_j(p';s_j)}{\partial s_j}}{\frac{\partial x_j(p';s_j)}{\partial p}}\right) > 0 \tag{3.19}$$

and vice versa for $p' > p(s_j)$. We note that $\partial h(p'; s_j)/\partial p$ is positive if $x_j(p'; s_j)$ is an interior allocation. Since $p' < p$, we need only consider p' such that p' corresponds to a point on j' demand curve. If the solution to the first order conditions in Equation (3.17) is strictly downward sloping in q and strictly increasing in s, than against j a price p' corresponds to an equilibrium price only if there exists a s_i' such $s_i' < s_i$ and

$$\frac{\partial h(p'; s_j)}{\partial p}\left([M(s_i'; s_j; h(p'; s_j)) - p'] - \frac{G(s_j|s_i')}{g(s_j|s_i')}\frac{\frac{\partial x_j(p';s_j)}{\partial s_j}}{\frac{\partial x_j(p';s_j)}{\partial p}}\right) = 0 \tag{3.20}$$

Since $M(s_i; s_j; q)$ is monotone increasing in s_i and by Lemma 1 in Milgrom and Weber (1982), a±liation implies that $G(s_j|z)/g(s_j|z)$ is decreasing in z, see (2.3), and the conjectured monotonicity on $x_j(p; s_j)$ the second order condition follows.

For p' less than $p(s_j)$, the second order condition holds for all prices until we reach $p(s^l_{i'}(s_j))$. Here $s_{i'}$ is the signal who receives zero allocation against s_j (if such a $s_{i'}$ exists, otherwise we set $s^l_{i'}(s_j)=0$). If $x_j(p(s^l_{i'}(s_j)); s^l_{i'}) = 0$, then $V(s_i; s_j; h(p'; s_j)) = 0$ for $p' < p(s^l_{i'}(s_j))$ and we are done. If $s^l_{i'}(s_j) = 0$, then for $p' < p(s^l_{i'}(s_j))$, $x_i(p'; s_j) = 0$ as j's demand curve is °at at this price.

For prices such that $p' > p(s_j)$, an argument similar to that in the prior paragraph holds if p' corresponds to a price where $x_j(p; s_j) > 0$. Let $s^u_{i'}(s_j)$ be such that for $p' > p(s^u_{i'}(s_j))$ we have $x_j(p; s_j) = 0$. At this price $p(s^u_{i'}(s_j))$ we know that the ¯rst order condition is negative for s_i and is zero for higher prices p' from which pseudo-concavity follows. In problems where the signal has bounded support, for p' higher than the price that corresponds to $s'_i = \bar{s}$ we note that s_i is worse o® at such prices than at the price that corresponds to the highest equilibrium price as he receives the same quantity (one unit) but pays a higher price (we are using the transversality condition here).

Now consider a s_i who receives one unit against s_j, i.e. $s_j < s^l_j(s_i)$ and $s_i > s^u_{i'}(s_j)$. If bidder i were to consider price $p' < p(s^u_{i'}(s_j))$ the ¯rst order condition is positive. For any price $p' > p(s^u_{i'}(s_j))$ the ¯rst order condition is zero. Since the quantity is not changed in this region and the total payment is not changed, the objective does not change as we change the price in this region. Since the highest possible signal in a bounded support problem, $\bar{s}$, receives 1 unit against all s_j, this argument holds to show optimality for s_j.

Finally consider a s_i who receives 0 units against s_j, i.e., $s_i < s^l_{i'}(s_j)$ and $s_j > s^u_j(s_i)$. At $p(s^l_{i'}(s_j))$, signal $s^l_{i'}(s_j)$'s ¯rst order condition is zero. By monotonicity in signal and a±liation, signal s_i ¯rst order condition at this price is negative. At any price lower than this price signal s_i's gets zero quantity and thus in indi®erent across all such prices. Thus again pseudo-concavity holds.

4 The Optimal Bidding Problem and Its Relation to the Milgrom-Weber Unit Auction Solution

We next show how to solve the pair of first order conditions for the two bidders (3.17) above to obtain an optimal solution $x(p; s_i)$. In doing so, we relate our results to the seminal work due to Milgrom and Weber (1982) on unit auctions.

In Milgrom and Weber (1982), it is shown that the equilibrium in a single-unit discriminatory auction solves the differential equation that:

$$\frac{db}{ds_i} = [v(s_i; s_i) - b(s_i)]\frac{g(s_i|s_i)}{G(s_i|s_i)}, \tag{4.1}$$

which is an ordinary differential equation. Milgrom and Weber provide a solution to this ordinary differential equation, which is

$$b(s_i) = \int_0^{s_i} v(\alpha; \alpha)\, dL(\alpha|s_i) \tag{4.2}$$

$$L(\alpha|s_i) = \exp\left(-\int_\alpha^{s_i} \frac{g(t|t)}{G(t|t)} dt\right) = \frac{G(\alpha|\alpha)}{G(s_i|s_i)} \tag{4.3}$$

where the end-point condition is that $b(0) = v(0; 0)$

In the two bidder case, we obtain the first order condition that

$$p(s_j) = M(s_i; s_j; h(p(s_j); s_j)) + \frac{G(s_j|s_i)}{g(s_j|s_i)} \frac{\frac{\partial x_j(p;s_j)}{\partial s_j}}{\frac{\partial x_j(p;s_j)}{\partial p}}. \tag{4.4}$$

We use the method of characteristics to convert the first order condition to an ODE. Suppose $x_j(p; s_j) = q$. Along the isocontour-line

$$\frac{\partial x_j(p; s_j)}{\partial p}\frac{dp(s_j)}{ds_j} + \frac{\partial x_j(p; s_j)}{\partial s_j} = 0 \tag{4.5}$$

Substituting this back to the first order condition and rearranging implies

$$\left.\frac{dp(s_j)}{ds_j}\right|_{x(p;s_j)=q} = [M(s_i; s_j; 1 - q) - p(s_j)]\frac{g(s_j|s_i)}{G(s_j|s_i)} \tag{4.6}$$

which is very similar to the ODE in Milgrom and Weber (1982) except that we have both s_i and s_j in the equation.

The above suggests the following approach to solving the pair of first order conditions for s_i and s_j (and the demand clearing condition). Fix q and define

$$s_i = H(s_j; q) \tag{4.7}$$

as the endogenous mapping that maps all s_j to s_i such that $x_j(p; s_j) = q$, where $q < 1/2$. We solve for the mapping $H(q; s_j)$ as follows. Fix $q < 1/2$. The two first order conditions can be rewritten as

$$\frac{dp}{ds}(s_j; q) = [M(H(s_j; q); s_j; 1 - q) - p(s_j; q)]\frac{g(s_j|H(s_j; q))}{G(s_j|H(s_j; q))} \tag{4.8}$$

$$\frac{dp}{ds}(s_i; 1 - q) = [M(H^{-1}(s_i; q); s_i; q) - p(s_i; 1 - q)]\frac{g(s_i|H^{-1}(s_i; q))}{G(s_i|H^{-1}(s_i; q)} \tag{4.9}$$

If we knew this mapping and the price bid by the lowest signal that receives allocation q (we denote this by $s_*(q)$ and the corresponding price by $p(s_*(q)) = M(H(s_*(q); q); s_*(q); 1 - q)$), we could directly solve for Equation (4.8) for the price by a variation of the Milgrom-Weber solution:[7]

$$p(s_j) = \int_{s_*(q)}^{s_j} M(H(\alpha; q); \alpha; 1 - q)\, dL^1(\alpha|s_j) + p(s_*(q))L^1(s_*(q)|s_j)) \tag{4.10}$$

where $L^1(\alpha|s_j)$ is defined as

$$L^1(\alpha|s_j) = \exp\left(-\int_{\alpha}^{s_j} \frac{g(t|H(t; q))}{G(t|H(t; q))}dt\right). \tag{4.11}$$

We obtain the differential equation for the mapping $H(s; q)$ by noting that

$$p(s; q) = p(H(s; q); 1 - q) \tag{4.12}$$

and by total differentiation of this equation we obtain

$$\frac{dp(s; q)}{ds} = \frac{dp(H(s; q); 1 - q)}{ds}\frac{dH(s; q)}{ds} \tag{4.13}$$

which leads to (using (4.8) and (4.9))

$$\frac{dH(s; q)}{ds} = \frac{[M(H(s; q); s; 1 - q) - p(s; q)]\frac{g(s|H(s;q))}{G(s|H(s;q))}}{[M(s; H(s; q); q) - p(s; q)]\frac{g(H(s;q)|s)}{G(H(s;q)|s)}} \tag{4.14}$$

[7] If we had $s_*(q) = 0$, then the fact that $L^1(0|s_j) = 0$ would simplify the solution further as in Milgrom and Weber (1982). We note we can similarly solve the other equation as

$$p(H(s_j; q)) = \int_{H(s_*(q); q)}^{H(s_j; q)} M(H^{-1}(\beta; q); \beta; q)\, dL^2(\beta|H(s_j; q)) + M(H(s_*(q); q); s_*(q); 1 - q)L^2(H(s_*(q); q)|H(s_j; q))$$

where $L^2(\beta|s_i)$ is defined as

$$L^2(\beta|s_i) = \exp\left(-\int_{\beta}^{s_i} \frac{g(t|H^{-1}(t; q))}{G(t|H^{-1}(t; q))}dt\right).$$

If marginal valuations and densities are Lipschitz, the system of ordinary differential equations given by equations (4.8) and (4.14) has a unique solution given the boundary conditions. However if $s_*(q) = 0$, then at 0, the left hand side of equation (4.14) above has both its numerator and denominator equal to zero (because $G(0|s_j) = 0$ and $p(0;q) = M(H(q;0);0;1-q)$) (see section 5.1). This singularity is also present in the price equation (equation (4.8)) which is similar to the Milgrom-Weber equation (this is well known from the unit auction literature). Unfortunately, the singularity in the dH/ds equation implies that we need more information to pin down the derivative. Further, we do not know for the given q the lowest signal $s_*(q)$ that receives that allocation.

The transversality condition provides the additional information that allows us to solve the system of ordinary differential equations. We use that fact that in a bounded problem the transversality condition implies that the highest signal submits a flat bidding curve and that every signal receives zero units against the highest signal. With our smoothness assumptions on the bidding schedule, this implies that we must have $H(\bar{s}) = \bar{s}$. Further we will prove that $H(s;q) > s$ (this is implied by the monotonicity of the bidding schedule in s and our transversality condition). Finally, we will show that whenever the lowest signal obtains quantity q (i.e. $s_*(q) = 0$), we must have $p(0;q) = M(H(q;0);0;1-q)$ to ensure that prices are positive.[8]

Further restrictions are imposed by the requirement that the bidding curves have to be strictly downward sloping. In particular, the bidding curve for the lowest signal is restricted in the following way. Suppose $1/2 > q' > q$ where q' and q are allocation that the lowest signal receives in equilibrium. Then we need $H(0;q') < H(0;q)$ and $M(H(0;q');0;1-q') < M(H(0;q);0;1-q)$, i.e., the reduction in the signal has to more than compensate for the increase in marginal valuation due to the lower quantity that is obtained by equilibrium by the signal $H(0;q)$ so that the price falls. Hence the monotone mapping $H(0;q)$ is restricted.

[8] If the lowest signal obtains quantity q and we have $M(H(q;0);0;1-q) > p(0;q)$, the prices that we obtain from solving the differential equation will go to $-\infty$. The constraint that prices be positive creates flat demands at 0 and thus the possibility of rationing when the price is 0. This is ruled out since we know from Jackson and Swienkels (1999) that ties are a zero probability event.

5 Solution of the optimal bidding problem

The solution of the optimal bidding problem, for $0 < s < \bar{s}$ and $0 < q < 1/2$, is given by the demand curve $p(s;q)$ obtained from the equations

$$\frac{dp(s;q)}{ds} = [M(H(s;q);s;1-q) - p(s;q)]\frac{g(s|H(s;q))}{G(s|H(s;q))} \tag{5.1}$$

$$\frac{dH(s;q)}{ds} = \frac{[M(H(s;q);s;1-q) - p(s;q)]\frac{g(s|H(s;q))}{G(s|H(s;q))}}{[M(s;H(s;q);q) - p(s;q)]\frac{g(H(s;q)|s)}{G(H(s;q)|s)}} \tag{5.2}$$

subject to the endpoint condition on $H(s;q)$ that

$$H(\bar{s};q) = \bar{s}. \tag{5.3}$$

Equation (5.3) will imply the transversality condition that $H(\bar{s};0) < \bar{s}$.

For $1 \geq q > 1/2$, $p(s;q)$ is defined by the relation

$$p(s;q) = p(H^{-1}(s;1-q);1-q); \tag{5.4}$$

where $H^{-1}(s;q)$ is the inverse function of $H(s;q)$, that is

$$H(H^{-1}(s;q);q) = s. \tag{5.5}$$

In this section we describe how to solve this model to obtain the demand curve $p(s;q)$. We will establish necessary conditions for the equilibrium solution $p(s;q)$ to be strictly increasing in s, and strictly decreasing in q,

$$\frac{\partial p}{\partial q}(s;q) < 0 \qquad \text{for } 0 < q < 1. \tag{5.6}$$

We begin by establishing some results on the properties of the solutions of the governing equations (5.1, 5.2).

Theorem 1 (<u>H-solution for $q = 1/2$</u>) $H(s;1/2) = s$ is the exact solution of (5.2) for $q = 1/2$.

Direct substitution of $H = s$ into the right-hand side of (5.2) yields

$$\frac{dH}{ds} = \frac{[M(s;s;\frac{1}{2}) - p]\frac{g(s|s)}{G(s|s)}}{[M(s;s;\frac{1}{2}) - p]\frac{g(s|s)}{G(s|s)}} = 1; \tag{5.7}$$

consistent with H being linear in s with slope one. Since (5.2) satisfies a Lipschitz condition at $s = s^\infty$, we can conclude that $H(s;1/2) = s$ is the unique solution satisfying boundary condition (5.3).

Theorem 2 (<u>p-solution for $q = 1/2$</u>) For $q = 1/2$ there is a unique smooth, bounded positive solution $p(s;1/2)$. This solution is given by the integral (5.9).

Using theorem 1, we may substitute $H = s$ into equation (5.1) to obtain a first-order linear differential equation for $p(s;1/2)$,

$$\frac{dp(s;1/2)}{ds} = [M(s;s;1/2) - p(s;1/2)]\frac{g(s|s)}{G(s|s)}. \tag{5.8}$$

This equation has the same form as the Milgrom-Weber problem (4.1) and the solution is given by

$$p(s;\tfrac{1}{2}) = \int_0^s \frac{g(s'|s')}{G(s'|s')} M(s';s';\tfrac{1}{2}) e^{\int_s^{s'} g(t|t)/G(t|t)\,dt} ds'; \tag{5.9}$$

where the constant of integration was selected to eliminate any potential singularities as $s \to 0$. As will be discussed later, the limit of (5.9) as $s \to 0$ is $p(0;1/2) = M(0;0;1/2)$. Equation (5.9) gives the unique smooth bounded solution of (5.1) on $0 < s < s^\infty$ for $q = 1/2$.

We define P^∞ to be the value of the demand curve determined from (5.9), $P^\infty = p(s^\infty;1/2)$,

$$P^\infty = \int_0^{s^\infty} \frac{g(s'|s')}{G(s'|s')} M(s';s';\tfrac{1}{2}) e^{\int_s^{s'} g(t|t)/G(t|t)\,dt} ds'. \tag{5.10}$$

For the case of unaffiliated distributions, we can use the relation between the PDF and the CDF, $g(s) = G'(s)$, to simplify (5.9) to obtain

$$p(s;\tfrac{1}{2}) = \frac{1}{G(s)} \int_0^s M(s';s';\tfrac{1}{2}) g(s')\,ds'; \tag{5.11}$$

and similarly from (5.10), $P^\infty = p(s^\infty;1/2)$ is given by,

$$P^\infty = \int_0^{s^\infty} M(s';s';\tfrac{1}{2}) g(s')\,ds'. \tag{5.12}$$

We will show that this value of the end-point condition $p(s^\infty;q) = P^\infty$ determines the unique solution (if it exists) defined on the whole interval $0 < s < s^\infty$ for $0 < q < 1/2$.

Theorem 3 (<u>Boundary condition for $p(s;q)$</u>) To satisfy the monotonicity requirement (5.6), at $s = s^\infty$, $p(s;q)$ must be a constant independent of q, for $0 < q < 1$,

$$p(s^\infty;q) = P^\infty. \tag{5.13}$$

Since H satisfies (5.3) at $s = \hat{s}$, from (5.5), we have $H^{-1}(\hat{s}; q) = \hat{s}$ also. Therefore, (5.4) yields

$$p(\hat{s}; q) = p(\hat{s}; 1 - q). \tag{5.14}$$

Hence $p(\hat{s}; q)$ is symmetric about $q = 1/2$, and upon differentiation with respect to q,

$$\frac{\partial p}{\partial q}(\hat{s}; q) = -\frac{\partial p}{\partial q}(\hat{s}; 1 - q). \tag{5.15}$$

Condition (5.6) can be satisfied on $0 < q < 1$ only if $p(\hat{s}; q)$ is independent of q. From (5.10) we know the value of $p(\hat{s}; q) = \hat{P}$ at $q = 1/2$, and therefore it must be equal to $\hat{P}$ for all values of q, (5.13).

Theorem 4 (Uniqueness) The bounded positive equilibrium solution of the optimal bidding problem, if it exists, is unique.

Given the assumption that equations (5.1, 5.2) satisfy a local Lipschitz condition at $s = \hat{s}$ for $0 < q < 1/2$, then equations (5.1, 5.2) along with conditions (5.3, 5.13) define a well-posed initial value problem in s, starting at $s = \hat{s}$, with a unique solution for $p(s; q)$ and $H(s; q)$ at each fixed value of q. This formulation of the optimal bidding problem as an initial value problem can also be applied to unbounded intervals, where the initial conditions are applied in the limit $s \to \hat{s} = \infty$. The properties of the solution for this case are discussed in Appendix A.

Theorem 5 (Relation to the Milgrom-Weber model) If the marginal valuation function $M(s_i; s_j)$ is independent of q, then (5.1, 5.2) reduce to the Milgrom-Weber equation. Further, in the limit that $\|\partial M/\partial q\|_\infty \to 0$, the Milgrom-Weber solution is the limiting behavior of $p(s; q)$.

If $M(s_i; s_j)$ is independent of q, then it follows from theorem 1 that $H(s) = s$ is the solution of (5.2) for all q. Consequently, equation (5.1) reduces to the Milgrom-Weber equation (4.1), written in the form

$$\frac{dp}{ds} = [M(s; s) - p(s)]\frac{g(s|s)}{G(s|s)}. \tag{5.16}$$

Further, the end-point condition is that $p(0; q) = M(0; 0; q)$ for all q. This follows from the fact that $M(0; 0; q)$ is independent of q and for $q = 1/2$ the end-point condition holds. This observation shows that equations (5.1, 5.2) generalize the original Milgrom-Weber model and can recover the previous results in the case that $\partial M/\partial q \equiv 0$. Theorem 4 represents a modest generalization of Milgrom and Weber (1982) in that it shows that with flat marginal valuations and the ability to divide the good; the Milgrom-Weber solution is still the unique symmetric equilibrium in strategies that strictly increase in s.

More generally, the theorem on continuous dependence of solutions of differential equations (Walter (1998)) can be applied to show that for problems where the marginal valuation depends weakly on q, the Milgrom-Weber solution is the limiting behavior of $p(s; q)$. We write the marginal valuation in the form $M(s_i; s_j; q) = M_0(s_i; s_j) + \epsilon M_1(s_i; s_j; q)$, to explicitly separate out the q-dependence, with $\partial M/\partial q = \epsilon \partial M_1/\partial q$. If ϵ is a small perturbation parameter, and $\epsilon ||M_1||_\infty \ll ||M_0||_\infty$, then the influence of q in (5.1, 5.2) enters as a regular perturbation to the q-independent Milgrom-Weber solution for $\epsilon = 0$. Consequently the solution can be expressed as $p(s; q) = p_0(s) + \epsilon p_1(s; q) + \cdots$ as $\epsilon \to 0$, where $p_0(s)$ is the solution of the Milgrom-Weber equation.

Having established local uniqueness of the solution by expressing the optimal bidding problem as a backward initial value problem in s, we now must show that the solution exists and is well-defined on the whole interval $0 < s < \hat{s}$. We will derive lower and upper bounds on the set of possible solution, to assist in the construction of the argument for the existence of the solution. Later we will go on to consider the local structure of solutions at $s = 0$, where the equations do not satisfy a Lipschitz condition.

Theorem 6 (<u>Lower bounds for H</u>) Lower bounds are given by $H(s; q) > s$ for $0 < s < \hat{s}$ and $0 < q < 1/2$.

We prove that $H(s; q) > s$ for $0 < s < \hat{s}$ and $q < 1/2$ by contradiction. Let $D(s) = H(s; q) - s$. Assume that s' and $\bar{s}$ are successive points where $H(s; q)$ crosses the line $H = s$ and hence $D(s') = D(\bar{s}) = 0$. From assumption A1, for $q < 1/2$, $M(s; s; q) > M(s; s; 1 - q)$, therefore at a crossing point s', we have

$$\left.\frac{dH}{ds}\right|_{s=s'} = \frac{[M(s'; s'; 1 - q) - p(s'; q)]}{[M(s'; s'; q) - p(s'; q)]} < 1; \tag{5.17}$$

and therefore $D'(s') < 0$ and $D'(\bar{s}) < 0$. Then $D(s)$ is locally decreasing at each crossing point. In a positive neighborhood of s', D is negative, $D(s' + \epsilon) < 0$, while in a neighborhood below $\bar{s}$, D is positive, $D(\bar{s} - \epsilon) > 0$. By the intermediate value theorem, D must have another zero between s' and $\bar{s}$, but this contradicts our assumption that s' and $\bar{s}$ are successive crossings. Therefore,

$$H(s;q) > s; \qquad 0 < s < \bar{s}; \qquad q < 1/2. \tag{5.18}$$

Theorem 7 (<u>Upper bounds for monotonicity in s</u>) *In private values problems, $H(s;q)$ and $p(s;q)$ are monotone increasing functions of s on $0 < s < \hat{s}$ if they lie below the upper bounds given by $H^*(s;q)$, $p^*(s;q)$.*

In private values problems, where the marginal valuation function is independent of s_j, $M = M(s_i;q)$, and (5.1, 5.2) reduce to

$$\begin{aligned} \frac{dp}{ds} &= [M(H;1-q) - p]\frac{g(s|H)}{G(s|H)}; & (5.19)\\ \frac{dH}{ds} &= \frac{[M(H;1-q) - p]\frac{g(s|H)}{G(s|H)}}{[M(s;q) - p]\frac{g(H|s)}{G(H|s)}}. & (5.20)\end{aligned}$$

If $p = M(H;1-q)$ and $M(s;q) \neq M(H;1-q)$ then $dp/ds = dH/ds = 0$, and only trivial constant solutions are possible. Nontrivial solutions with $dp/ds = 0$ at some s are possible only if $M(s;q) = M(H;1-q)$. These considerations define the p-nullcline curve, given by the equations,

$$p^*(s;q) = M(H^*(s;q);1-q); \qquad M(H^*(s;q);1-q) = M(s;q). \tag{5.21}$$

If both marginal valuations, $M(H;1-q)$ and $M(s;q)$, are greater than the price $p(s;q)$, then from equation (5.19, 5.20) both $p(s)$ and $H(s)$ have positive slopes. If either function is to be non-monotone at some point, then there must be a value of s where its derivative is zero. If the solutions do not intersect the p-nullcline, they must be monotone functions. Therefore solutions that lie below the p-nullcline,

$$p(s;q) < p^*(s;q) \tag{5.22}$$

are monotone increasing functions,

$$\frac{dp}{ds} > 0; \qquad \frac{dH}{ds} > 0. \tag{5.23}$$

To ensure that $dp(\hat{s})/ds > 0$ we must require that M and g satisfy

$$M(\hat{s};1-q) > \hat{P} = \int_0^{\hat{s}} \frac{g(s'|s')}{G(s'|s')} M(s';\tfrac{1}{2}) e^{\int_s^{s'} g(t|t)/G(t|t)\,dt} ds'. \tag{5.24}$$

For the more general problem of marginal valuations with common values, $M(s_i; s_j; q)$, parts of the above argument change somewhat. The factor $[M(H; s; 1-q) - p]$ appearing in the numerators of (5.1, 5.2) now has an explicit s-dependence and therefore nontrivial solutions can occur with $dp/ds = dH/ds = 0$ at points apart from the p-nullcline, which is now defined as

$$p^*(s; q) = M(H^*(s; q); s; 1-q); \qquad M(H^*(s; q); s; 1-q) = M(s; H^*(s; q); q). \tag{5.25}$$

The p-nullcline still serves as an upper bound (5.22) on the set of monotone increasing solutions. However, in common values problems, it is also possible to find non-monotone solutions below the nullcline. In general, the p-nullcline bound (5.22) is a necessary but not a sufficient condition to establish monotone increasing behavior in s. In Appendix C we will derive necessary conditions for local increasing behavior at $s = 0$. The local conditions for the solution to be increasing at $\bar{s}$ generalize directly, in particular, we must require that

$$M(\bar{s}; \bar{s}; 1-q) > \bar{P} = \int_0^{\bar{s}} \frac{g(s'|s')}{G(s'|s')} M(s'; s'; \tfrac{1}{2}) e^{\int_s^{s'} g(t|t)/G(t|t)\, dt}\, ds'. \tag{5.26}$$

Note that from the definition (5.21) and Assumption A4, it can be shown that $p^*(s; q)$ and $H^*(s; q)$ are increasing functions of s. This condition, along with assumption A1 on the monotone decreasing dependence of M on q leads to the necessary condition

$$M(\bar{s}; \bar{s}; 1) > \bar{P}. \tag{5.27}$$

We next explore the nature of bid-shading on the marginal unit in equilibrium using Theorem 7. In particular, we ask whether

$$M(H(s; q); 1-q) - p(s; q) \quad <; >; = \quad M(s; q) - p(s; q). \tag{5.28}$$

In an unit auction there is no distinction between the marginal and the average bid and thus the higher signal must shade his bid more. However, in a multi-unit discriminatory auction, we show that on the marginal bid, the high signal may shade his bid less than the low signal.

Corollary to Theorem 7 (<u>Bid shading on the marginal unit</u>) On the marginal unit, the higher signal shades his bid less than the lower signal when the low signal is $s = 0$ or $s = \bar{s}$.

Further in the private values case if

$$M_s(s;q)\frac{G(s|H^*)}{g(s|H^*)} < M_s(H^*(s);1-q)\frac{G(H^*|s)}{g(H^*|s)} \tag{5.29}$$

the higher signal shades his bid less than the lower signal for arbitrary values of the low signal.

At $s = 0$, the corollary follows directly from from considering Equation (3.17) and using the fact the CDF is zero at $s = 0$. At $s = \bar{s}$ the corollary follows from the fact that for any q, $H(\bar{s};q) = \bar{s}$ and $M(\bar{s};1-q) < M(\bar{s};q)$. Note that at $\bar{s}$ and $q = 1/2$, $H^*(\tilde{s};1/2) = \tilde{s}$ and from A4, $H^*(s;q)$ increases as q decreases. Suppose that H and H^* intersection at $\hat{s} < \tilde{s}$, i.e., $H(\hat{s};q) = H^*(\hat{s};q)$ but $p(\hat{s};q) < p^*(\hat{s};q)$. Then at $\hat{s}$

$$\frac{dH^*(\hat{s};q)}{ds} = \frac{M_s(\hat{s};q)}{M_s(H^*(\hat{s};q);1-q)} < \frac{dH(\hat{s};q)}{ds} = \frac{\frac{g(\hat{s}|H^*(\hat{s};q))}{G(\hat{s}|H^*(\hat{s};q))}}{\frac{g(H^*(\hat{s};q)|\hat{s})}{G(H^*(\hat{s};q)|\hat{s})}} \tag{5.30}$$

where we have used the fact that $M(H^*(\hat{s};q);1-q) = M(\hat{s};q) > p(\hat{s};q)$ and our earlier assumption. Condition (5.30) implies that at $\tilde{s}$, $H^*(\tilde{s};q) < H(\tilde{s};q)$, but this contradicts our earlier results. Therefore, it must be the case that $H(s;q) < H^*(s;q)$ if $p(s;q) < p^*(s;q)$ (i.e. we are below the p-nullcline). From this, we obtain

$$M(H(s;q);1-q) - p(s;q) < M(H^*(s;q);1-q) - p(s;q) = M(s;q) - p(s;q) \tag{5.31}$$

which implies that the high signal shades his bid less than the low signal on the marginal unit. Later, we will show that our numerical example satisfies the hypothesis of the corollary. For common values problem, the sufficient condition in the corollary has to be changed to account for the dependence of one bidder's marginal valuation on the other bidders.

Theorem 8 (<u>The condition for $p(s;q)$ to be monotone-in-q</u>) The upper and lower bounds on $\partial p/\partial q$ needed for the demand curve $p(s;q)$ to be monotone decreasing for all q, $0 < q < 1$, are given by equation (5.35).

For $0 < q < 1/2$, the condition for p to be decreasing in q is simply that $\partial p/\partial q < 0$. For $q > 1/2$, $p(s;q)$ is defined by (5.4) in terms of the inverse of $H(s;q)$ defined by (5.5). Differentiating (5.4) with respect to q yields

$$\frac{\partial p}{\partial q}(s;q) = -\frac{\partial p}{\partial s}(H^{-1};1-q)\frac{\partial H^{-1}}{\partial q}(s;1-q) - \frac{\partial p}{\partial q}(H^{-1};1-q) < 0. \tag{5.32}$$

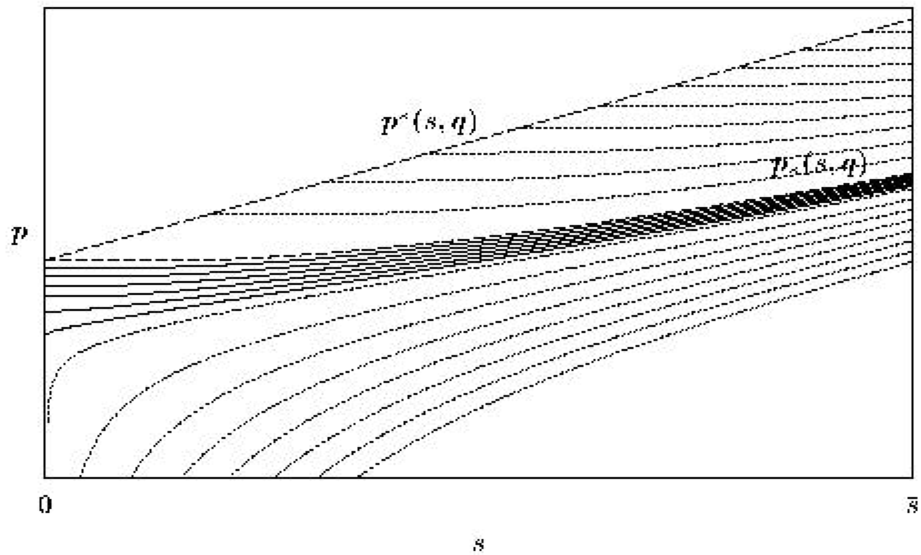

Figure 1: Illustration of the set of trial solutions and bounds on p(s) used in the existence argument. The range of bounded solutions that exists on the whole interval, $0 < s < \bar{s}$ are plotted with solid curves.

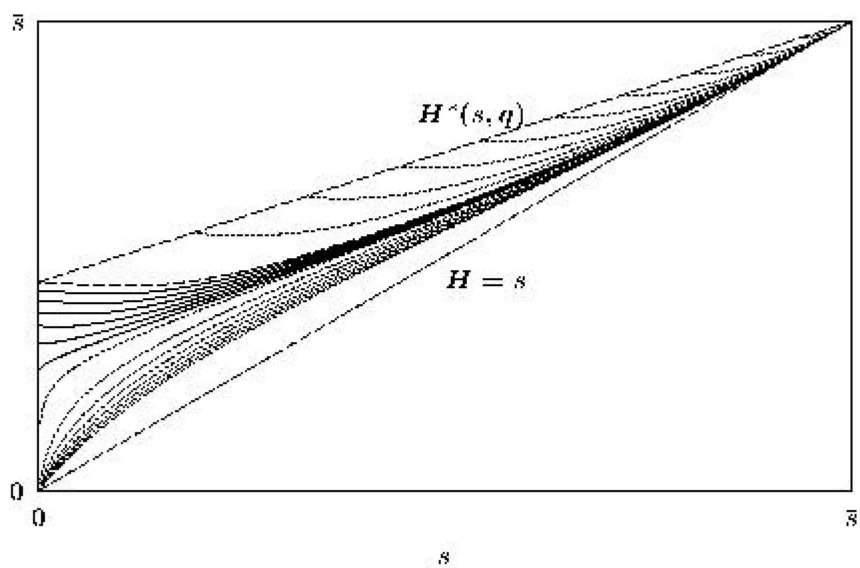

Figure 2: The corresponding plot of H(s) solutions for the existence argument.

Differentiating (5.5) with respect to q, we obtain

$$\frac{\partial H^{-1}}{\partial q} = -\left(\frac{\partial H}{\partial q}\right)\Big/\frac{\partial H}{\partial s}. \tag{5.33}$$

Substituting this into (5.32) yields the condition to yield the condition

$$\frac{\partial p}{\partial s}(H^{-1}; 1-q)\frac{\frac{\partial H}{\partial q}(H^{-1}; 1-q)}{\frac{\partial H}{\partial s}(H^{-1}; 1-q)} < \frac{\partial p}{\partial q}(H^{-1}; 1-q). \tag{5.34}$$

Using the definition of the inverse, (5.5), we can map $(H^{-1}; 1-q) \to (s; q)$ back to obtain the overall bounds

$$\frac{\partial p}{\partial s}(s; q)\,\frac{\frac{\partial H}{\partial q}(s; q)}{\frac{\partial H}{\partial s}(s; q)} < \frac{\partial p}{\partial q}(s; q) < 0 \quad \text{at each point in } \{(s; q)\,|\,0 < s < \hat{s}; 0 < q < 1/2\}. \tag{5.35}$$

Theorem 9 (Monotone structure of the solution $p(s; q)$) A monotone increasing solution of the optimal bidding problem exists on the interval $0 < s < \hat{s}$ corresponding to sufficiently low end-point values of the price, $P^1 < P_*(q)$.

The Peano theorem of existence for solutions of initial value problems, see Walter (1998) for example, ensures the existence of a solution within the domain in $(s; H; p)$ where the differential equations are not degenerate. For the optimal bidding problem, equation (5.2) becomes degenerate at the p-nullcline. Therefore, if the solution $p(s; q)$, $H(s; q)$ does not intersect the nullcline at any point within the range $0 < s < \hat{s}$, then the existence of the strong solution on $0 < s < \hat{s}$ is guaranteed.

Consider solving the initial value problem for (5.1, 5.2) starting from $H(\hat{s}; q) = \hat{s}$ and an arbitrary value for $p(\hat{s}; q) = P(q)$. For $p(s)$ to be monotone increasing, $P(q)$ must lie below the nullcline,

$$P(q) < p^*(\hat{s}; q) = M(\hat{s}; \hat{s}; 1-q). \tag{5.36}$$

In Appendix B we demonstrate that a comparison theorem shows that if two solutions $p(s; q)$ and $\tilde{p}(s; q)$ start from initial conditions, P and $\tilde{P}$ respectively, with $P < \tilde{P}$, then the solutions do not intersect,

$$p(s; q) < \tilde{p}(s; q). \tag{5.37}$$

There can exist a range of values $P < M(\hat{s}; \hat{s}; 1-q)$, for which the trial solution intersects the p-nullcline at a point s_*, $p = p^*(s_*; q)$. From the results of the comparison theorem, (5.37), and the fact that $p^*(s;q)$ is an increasing function, as P decreases, the value of the intersection point s_* also decreases. There exists a value $P_*(q)$ such that its corresponding solution $p_*(s;q)$ intersects the nullcline at $s_* = 0$. All solutions starting with initial conditions satisfying $P < P_*(q)$ are bounded away from the nullcline and are covered by the Peano existence theorem. The value of $P_*(q)$ must be calculated using numerical quadrature, but once this has been determined, the condition for existence of the solution of the optimal bidding problem is

$$\int_0^{s_*} M(s'; s'; \tfrac{1}{2}) g(s'|s')\, ds' < P_*(q) \qquad 0 < q < \tfrac{1}{2}. \tag{5.38}$$

Note that this condition involves both the marginal valuation function M and the conditional distribution g. While the above condition ensures the existence of a solution of the backward initial value problem, the solution may not be bounded, $p \to -\infty$ as $s \to 0$, with $H \to 0$ as $s \to 0$. In the following section we examine the structure of the solution at the singular end-point $s = 0$.

5.1 Local existence at $s = 0$

Complete characterization of the solutions of (5.1, 5.2) requires an analysis of their local structure near the end-point $s = 0$, which is a regular singular point where the existence theorems fail (see Ince (1956) for example). We now establish the necessary conditions at $s = 0$ for the existence of locally smooth, bounded solutions of (5.1, 5.2). We will show that there exists a finite range of values for $H = H_0(q)$ at $s = 0$ possible for any solutions of the optimal bidding problem. One condition follows from the upper bound (5.21); at $s = 0$, the upper bound $H^*(s;q)$ imposes the constraint that

$$H_0(q) < H^*(0;q). \tag{5.39}$$

To describe the local structure of the solutions at $s = 0$, we assume that the ratio of the CDF/PDF (which is also the inverse of the hazard ratio) can be written as

$$K(s|H) = \frac{G(s|H)}{g(s|H)} = sJ(s|H); \qquad J(0|H) > 0; \tag{5.40}$$

this is a valid assumption for any regular distribution on $0 < s < \hat{s}$. Then equation (5.1) takes the form

$$\frac{dp}{ds} = \frac{[M(H; s; 1 - q) - p]}{sJ(s|H)} \tag{5.41}$$

We require $p(s; q)$ to be bounded and differentiable on $0 < s < \bar{s}$. In order that the derivative dp/ds be finite at $s = 0$, (5.41) must be degenerate, $0/0$, at the singular point $s = 0$. This condition forces the relation,

$$p_0(q) = p(0; q) = M(H_0(q); 0; 1 - q) > 0; \tag{5.42}$$

where we have assumed that $H_0(q) > 0$. It can be shown that the case $H(0; q) = 0$ forces $p(s \to 0; q) \to -\infty$. Due to the singularity at $s = 0$, equations (5.1, 5.2) do not satisfy a Lipschitz condition at $s = 0$. In addition to the nontrivial smooth solutions that we seek, the equations also admit trivial constant solutions, $p(s; q) = p_0(q)$, $H(s; q) = H_0(q)$. We summarize our results as

Theorem 10 (Local existence at $s = 0$) Any bounded solution of the optimal bidding problem must satisfy the following initial conditions at $s = 0$,

$$p(0; q) = p_0(q) = M(H_0(q); 0; 1 - q) > 0 \tag{5.43}$$

where

$$H(0; q) = H_0(q); \qquad \text{for some } H_0(q) \text{ in } 0 < H_0(q) < H^*(0; q): \tag{5.44}$$

For $q = 1/2$, the solution satisfying these conditions is the strong solution, with the endpoint condition $p(\hat{s}; 1/2) = \hat{p}$ given by (5.10).

Further, if the marginal valuation function depends on private values, $M = M(s_i; q)$, then an improved lower bound on $H_0(q)$ can be obtained. For common value problems, there is an improved upper bound on $H_0(q)$.

We leave the details of the algebra supporting these results to the Appendix C. We note that unbounded solutions of the form

$$p(s) \sim -Cs^{-\gamma} \to -\infty; \qquad H(s) \sim Ds; \qquad s \to 0; \tag{5.45}$$

with $\gamma > 0$, exist when the condition for bounded solutions (5.43) is not satisfied.

Theorem 11 (Conditions for monotone behavior of $p(0;q)$) The requirement of monotone decreasing behavior of $p(s;q)$ for $s = 0$, yields additional bounds on the possible values for $H_0(q)$,

$$H_0^-(q) < H_0(q) < H_0^+(q) \tag{5.46}$$

From (5.42), the requirement that $p_0(q)$ is a decreasing function of q sets the following condition on $H_0(q)$,

$$\frac{dH_0}{dq} < \frac{\partial M}{\partial q}(H_0;0;1-q) \Big/ \frac{\partial M}{\partial s_-}(H_0;0;1-q). \tag{5.47}$$

From assumption A1, $\partial M/\partial q < 0$ and $\partial M/\partial s_- > 0$, therefore $H_0(q)$ must be a strictly decreasing function of q, $dH_0/dq < 0$. Similar comments have to be applied for $q > 1/2$. For $q > 1/2$, $p(s;q)$ is defined by (5.4) in terms of the inverse of $H(s;q)$ defined by (5.5). Recalling equation (5.34), let $\bar{q} = 1 - q < 1/2$ and evaluate this equation at $s = H_0(\bar{q})$, so that $H^{-1}(s;\bar{q}) = 0$, and $H(0;\bar{q}) = H_0(\bar{q})$ then

$$\frac{\partial p}{\partial s}(0;\bar{q})\frac{\frac{\partial H}{\partial q}(0;\bar{q})}{\frac{\partial H}{\partial s}(0;\bar{q})} < \frac{\partial p}{\partial q}(0;\bar{q}) \tag{5.48}$$

expanding each term,

$$[M(0;H_0;\bar{q}) - M(H_0;0;1-\bar{q})]\frac{g(H_0|0)}{G(H_0|0)}\frac{dH_0}{dq} < \frac{\partial M}{\partial s_-}(H_0;0;1-\bar{q})\frac{dH_0}{dq} - \frac{\partial M}{\partial q}(H_0;0;1-\bar{q}). \tag{5.49}$$

And finally, we conclude that for each $0 < q < 1/2$,

$$\frac{dH_0}{dq} > \frac{\frac{\partial M}{\partial q}(H_0;0;1-q)}{\frac{\partial M}{\partial s_-}(H_0;0;1-q) + [M(H_0;0;1-q) - M(0;H_0;q)]\frac{g(H_0|0)}{G(H_0|0)}}. \tag{5.50}$$

Equation (5.50) always gives a well-defined lower bound to sandwich the derivative of $H_0(q)$,

$$(5.50) < \frac{dH_0}{dq} < (5.47) < 0; \tag{5.51}$$

in particular we note that since the derivative is absolutely bounded, $H_0(q)$ must be a continuous, differentiable function. From theorem 1, we know that every $H_0(q)$ must satisfy $H_0(1/2) = 0$ at $q = 1/2$, therefore, we can use this initial condition to integrate first order differential equations for an upper bound $H_0^+(q)$, from (5.47), and a lower bound $H_0^-(q)$, from (5.50), for the set of allowable end-point functions $H_0(q)$. In fact $H_0^+(q)$ is a tighter upper bound than our previous estimate, $H^*(0;q)$, equation (5.39), (see Figure 5).

Equation (5.51) is a necessary local condition at $s = 0$ to ensure that $p(s;q)$ is monotone decreasing in q. This condition can serve to verify or disprove the acceptability of a proposed solution $p(s;q)$. A condition for the global monotone property of $p(s;q)$ is described in Appendix A.

6 Examples

In this section we apply our work to several fundamental examples to illustrate our approach. Due to the complexity of equations (5.1, 5.2), in general, numerical methods are necessary to obtain solutions. To establish the reliability of these calculations, computations were performed with two independent schemes; first, using a standard explicit fourth-order Runge-Kutta method, and also with an implicit second-order trapezoidal method (Press et al 1992). The latter method is particularly well-suited for stiff differential equations, such as the behavior of equations (5.1, 5.2) near the singular point $s = 0$. Our case studies of the examples will also draw upon comparisons of the numerical results with the analytic theory given above.

6.1 A private values problem

We consider a simple unaffiliated private values example with

$$M(s;q) = v + s - (k - s)\rho q \tag{6.1}$$

where s is uniformly distributed over $[0;\bar{s}]$, i.e., $g(s) = 1/\bar{s}$, and $G(s) = s/\bar{s}$. Here $k; v; \bar{s}; \rho$ are fixed positive parameters where $\bar{s} < k$. In this case the first order conditions are

$$\frac{dp(s;q)}{ds} = [v + H - (k - H)\rho(1 - q) - p]\frac{1}{s}; \tag{6.2}$$

$$\frac{dH(s;q)}{ds} = \frac{[v + H - (k - H)\rho(1 - q) - p]}{[v + s - (k - s)\rho q - p]}\frac{H}{s}$$

with the boundary condition that

$$H(\bar{s};q) = \bar{s}; \tag{6.3}$$

and from (5.10) we obtain

$$p(\bar{s};q) = \bar{P} = v + \tfrac{1}{2}\bar{s} - \tfrac{1}{2}\rho(k - \tfrac{1}{2}\bar{s}). \tag{6.4}$$

The upper bounds on the set of monotone increasing feasible trial solutions are given by the p-nullcline,

$$H^*(s;q) = \frac{(1+\rho q)s + k\rho(1-2q)}{1+\rho(1-q)}, \qquad p^*(s;q) = (1+\rho q)s + (v - \rho q k) \tag{6.5}$$

We verify that the assumptions for the Corollary to Theorem 7 holds, i.e.,

$$[1+\rho q]s < [1+\rho(1-q)]H^*(s;q) \tag{6.6}$$

which is true. Thus in this example, on the marginal unit, the higher signal shades his bid less than the lower signal in equilibrium, as described by the Corollary to Theorem 7.

In this example the restrictions (C.13, C.14) on the initial values of $H(0;q)$ are that

$$\frac{1}{2}\frac{k\rho(1-2q)}{1+\rho(1-q)} < H_0(q) < \frac{k\rho(1-2q)}{1+\rho(1-q)} \tag{6.7}$$

and the derivative restriction (5.51) on $dH(0;q)/dq$ is that

$$-\frac{\rho(k-H_0)H_0}{2(1+\rho(1-q))H_0 - k\rho(1-2q)} < \frac{dH_0}{dq} < -\frac{\rho(k-H_0)}{1+\rho(1-q)} \tag{6.8}$$

As described in Appendix C, for a private values problem, at any fixed value of q, there is only a discrete set of H_0 values, corresponding to $\alpha = 1, 2, 3, \ldots$ in (C.11), that yield analytic solutions for $H(s)$ near $s = 0$. All other values produce solutions that can be represented in terms of Frobenius series (see Ince (1956)). The bounds, $\underline{H}_0(q) < H_0(q) < \overline{H}_0(q)$, given by (6.7) correspond to the bounds $0 < \alpha < \infty$. The bounds, $H_0^-(q) < H_0(q) < H_0^+(q)$, derived from the derivative bounds (6.8) are independent conditions. As was described in (C.13), the upper bound $\overline{H}_0(q)$ is given by $\overline{H}_0(q) = H^*(0;q)$. In this private values example, since M has a linear dependence on s, we find that both lower bounds are identical, $H_0^-(q) = \underline{H}_0(q)$. However, the upper bound $H_0^+(q)$ given by the nonlinear lower bound on dH_0/dq gives a tighter lower bound than $\overline{H}_0(q)$ (see Figure 5).

We consider numerical solutions of this example with the parameter values

$$\bar{s} = 1, \qquad k = 1.2, \qquad v = 3, \qquad \rho = 1.$$

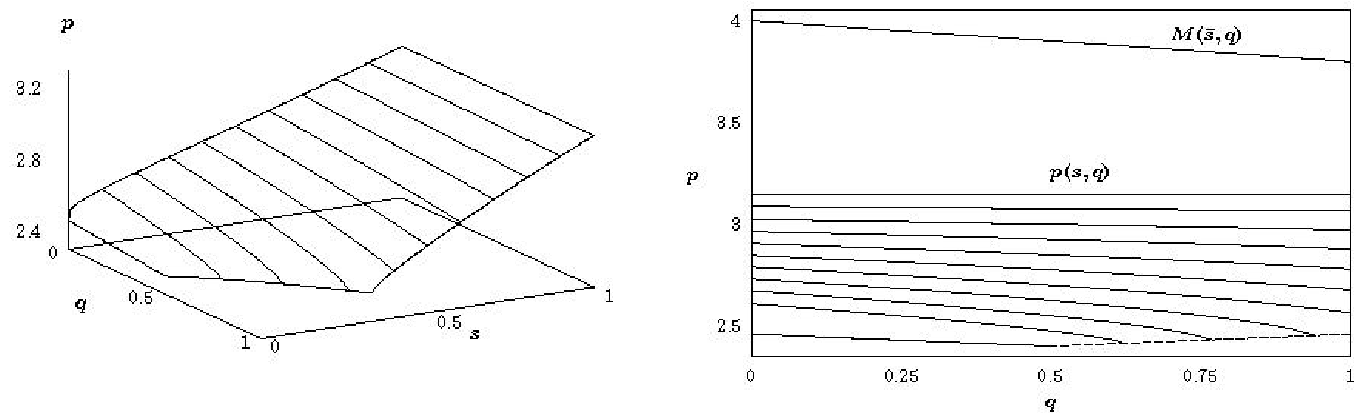

Figure 3: (Left) 3-d view of the surface $p(s;q)$ for the private values example. (Right) $p(s;q)$ for the private values example as a function of q with s held fixed at several values.

We first consider the case where the equilibrium where the lowest signal receives a maximum of $1/2$ in equilibrium. Thus $s_*(1/2) = 0$. In Figures 3, 4, and 5 we show the unique equilibrium solution for this optimal bidding problem. Figure 3 shows a 3-d view the function $p(s;q)$. The calculated price is an increasing function of s and a decreasing function of q. For low s values, $s < H_0^{-1}(1-q)$ it is undefined for high values of $q > 1/2$. For example, at $s = 0$, we will not see a value of the equilibrium for $q > 1/2$. In Figure 5, we show that the calculated constraint $H(s;q)$ at $s = 0$ indeed lies within the predicted bounds (5.51).

We note that the transversality condition forces the highest signal to submit a flat bidding curve even though his marginal valuation is downward sloping. Equation (5.24) ensures that the flat bidding curve lies below the highest signal's marginal valuation curve. The transversality condition is satisfied here because $H(s;q) < 1$ (this occurs by construction). Because $H(s;q)$ is monotone decreasing in q, that is sufficient. Figure 3 shows $p(s;q)$ as function of q for various values of s (s starts at 0 and increases in increments of 0.1). It is clear that lower signals submit more downward sloping bidding curves. Figure 4 shows constant contours $p(s;q) = p$ in the $(s;q)$ plane. These contours have to be upward sloping in s and q to be a valid equilibrium, as they are. Finally, Figure 5 shows that $H(s;q)$ is increasing in s and decreasing in q as required.

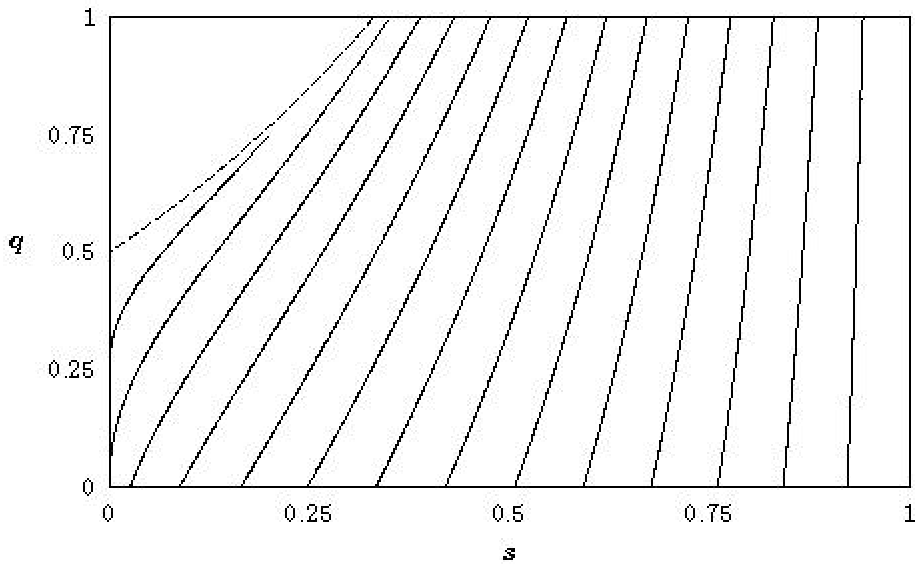

Figure 4: Contours of constant price, $p(s;q) = p$, in the $(s;q)$ plane for the private values example.

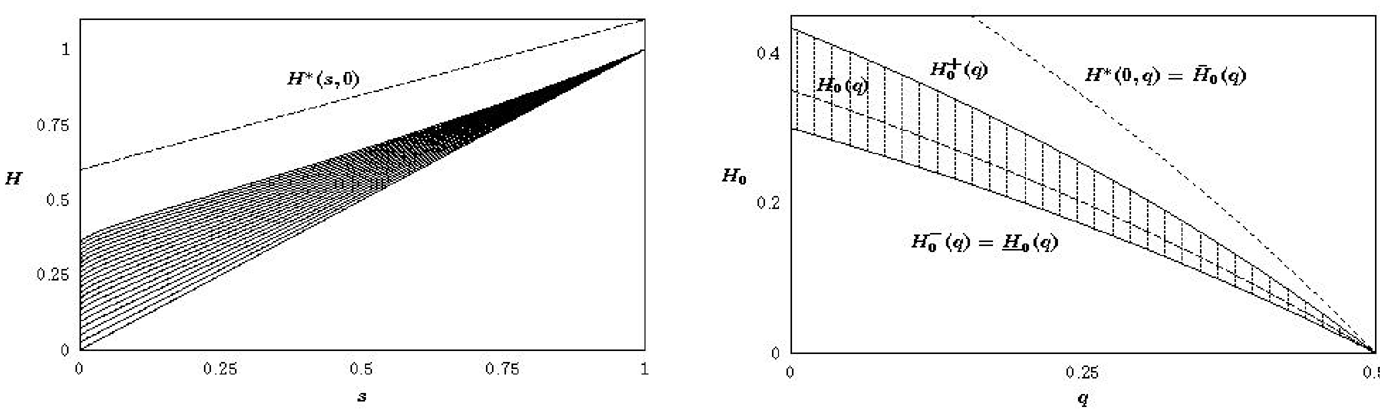

Figure 5: Cross-sections of the mapping function $H(s;q)$; (left) $H(s;q)$ as a function of s for several ¯xed values of q, $0 < q < 1/2$ for the private values example, (right) veri¯cation of the optimal bidding solution: demonstration that the computed constraint curve $H_0(q) = H(0;q)$ lies with the upper and lower bounds (the shaded region) that ensure that $p(s;q)$ is decreasing in q locally for $s = 0$.

6.2 Non-existence of strictly decreasing in p equilibria for a private values problem

In our approach, we look for symmetric equilibria that involve strategies that are strictly monotone increasing in the signal and strictly monotone decreasing in the price. As we have already stated in the introduction, the work of Reny (1999), Simon and Zame (1999) and Jackson and Swienkels (1999) collectively imply that the symmetric K-unit pay-your-bid auction with private values has a symmetric equilibrium with non-decreasing bid functions and zero probabilities of ties.[9] Thus equilibrium strategies can have "flats", i.e., prices where the quantity jumps. However, a strictly positive measure of types cannot have such "flats" any given price.

We show next via example that "flats" may be a necessary part of equilibrium strategies, i.e., for some problems there exist no equilibrium with strategies that are strictly monotone decreasing in the price. While our approach guarantees a solution to the pair of ordinary differential equations (5.1) and (5.2) we cannot guarantee that the solutions will be strictly decreasing in the price for every problem.

We consider the example where

$$M(s;q) = v + s - \rho q \tag{6.9}$$

where s is uniformly distributed over $[0;\bar{s}]$, then the first order conditions yield the ODEs,

$$\frac{dp(s;q)}{ds} = [v + H - \rho(1-q) - p]\frac{1}{s}; \tag{6.10}$$

$$\frac{dH(s;q)}{ds} = \frac{[v + H - \rho(1-q) - p]}{[v + s - \rho q - p]}\frac{H}{s}$$

with the boundary condition that

$$H(\bar{s};q) = \bar{s}; \tag{6.11}$$

and from (5.10) we obtain

$$p(\bar{s};q) = \bar{P}^1 = v + \tfrac{1}{2}\bar{s} - \tfrac{1}{2}\rho. \tag{6.12}$$

[9] Again, to be accurate, these papers deal with discrete quantities while ours considers continuous quantities.

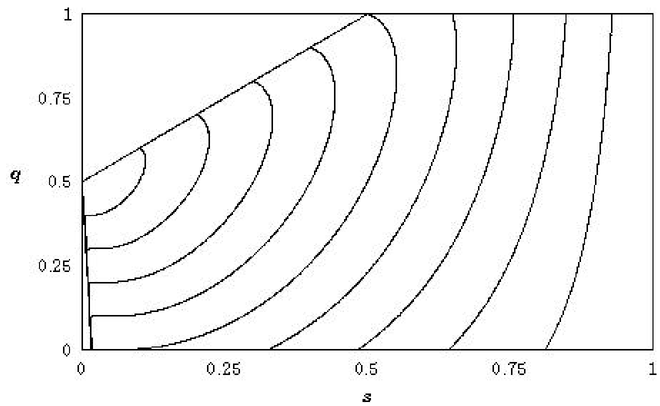

Figure 6: Non-existence of a strictly decreasing $p(s;q)$ equilibria solution for a private values values example. Contours of constant price, $p(s;q) = p$, in the $(s;q)$ plane showing non-monotone behavior.

Proceeding numerically, as was done in the prior example, we compute the solution of this system for the values of the parameters given by,

$$\bar{s} = 1; \qquad v = 3; \qquad \rho = 1/2. \tag{6.13}$$

Figure 6 shows the constant price contours, $p(s;q) = p$, in the $(s;q)$ plane. For an equilibrium $p(s;q)$ that is strictly increasing in s and strictly decreasing in q, all of these contours must have positive slope for changes of q with respect to s at constant p (see Figure 4 for example). As is clear from Figure 6, here the contours are non-monotone as they bend backward for low s and high q. This implies that type s receives allocation either q_1 or q_2, $q_1 < q_2$ at the same price. Thus it seems as if the equilibrium strategies must involve "flats" for low signals.[10]

6.3 An Affiliated common values example

We now consider an affiliated common values example. The marginal valuation function is given by

$$M(s_i;s_j;q) = v + s_i + \theta s_j - (k - s_i)\rho q \tag{6.14}$$

[10] Our approach can be amended to obtain first order conditions when there are flats. However, we need to be able to conjecture where flats occurs. The approach we use in this paper cannot deal with such problems as we need to be able to conjecture where the flats will occur and recursively solve the problem (the strategies of the high signal bidders will change)

where s_i is distributed over $[0,k]$. We make s_i and s_j affiliated as follows. The distribution of s_i and s_j conditional on the parameter w be independent with distribution,

$$\hat{G}(s|w) = s^w \qquad \hat{g}(s|w) = ws^{w-1} \tag{6.15}$$

over $[0,1]$ for any $w > 0$. Since $G(0|w) = 0$ and $G(1|w) = 1$ and the $g(s|w)$ is positive, this is a density. Then we let $w \in \{\frac{1}{2}, 1\}$ with probability β that $w = 1/2$. Note that $\hat{g}(s_2, s_1|w) = \hat{g}(s_2|w)\hat{g}(s_1|w)$. The conditional density and conditional distribution function of s_2 given s_1 is given by

$$\begin{aligned} g(s_2|s_1) &= \frac{(1-\beta)\hat{g}(s_2, s_1|w_1) + \beta\hat{g}(s_2, s_1|w_2)}{(1-\beta)\hat{g}(s_1|w_1) + \beta\hat{g}(s_1|w_2)} \\ G(s_2|s_1) &= \int_0^{s_2} g(t|s_1)\,dt \end{aligned} \tag{6.16}$$

Note that $G(0|s_1) = 0$ and $G(1|s_1) = 1$ as is the case for a distribution. Affiliation requires that (see footnote 16 in Milgrom and Weber) for $s' > s$ and $x' > x$

$$g(x|s)g(x'|s') \geq g(x|s')g(x'|s) \tag{6.17}$$

which is true. Hence these are affiliated distributions.

Our two differential equations are now given by

$$\begin{aligned} \frac{dp}{ds} &= [v + H + \theta s - (k - H)\rho(1-q) - p]\frac{1}{K(s|H)} \\ \frac{dH}{ds} &= \frac{[v + H + \theta s - (k - H)\rho(1-q) - p]}{[v + s + \theta H - (k - s)\rho q - p]}\frac{K(H|s)}{K(s|H)} \end{aligned} \tag{6.18}$$

where the inverse hazard ratio is now

$$K(s_2|s_1) = \frac{G(s_2|s_1)}{g(s_2|s_1)} \tag{6.19}$$

with boundary conditions

$$H(k,q) = k, \qquad p(k,q) = \hat{P} \tag{6.20}$$

where $\hat{P} = p(k, 1/2)$ is given by the solution of the initial value problem

$$\frac{dp}{ds} = [v + s + \theta s - \tfrac{1}{2}\rho(k - s) - p]\frac{1}{K(s|s)} \qquad p(0, \tfrac{1}{2}) = M(0, 0, \tfrac{1}{2}). \tag{6.21}$$

We consider the example where

$$k = 1, \qquad v = 2, \qquad \rho = 1, \qquad \theta = 0.1, \qquad \beta = 0.2 \tag{6.22}$$

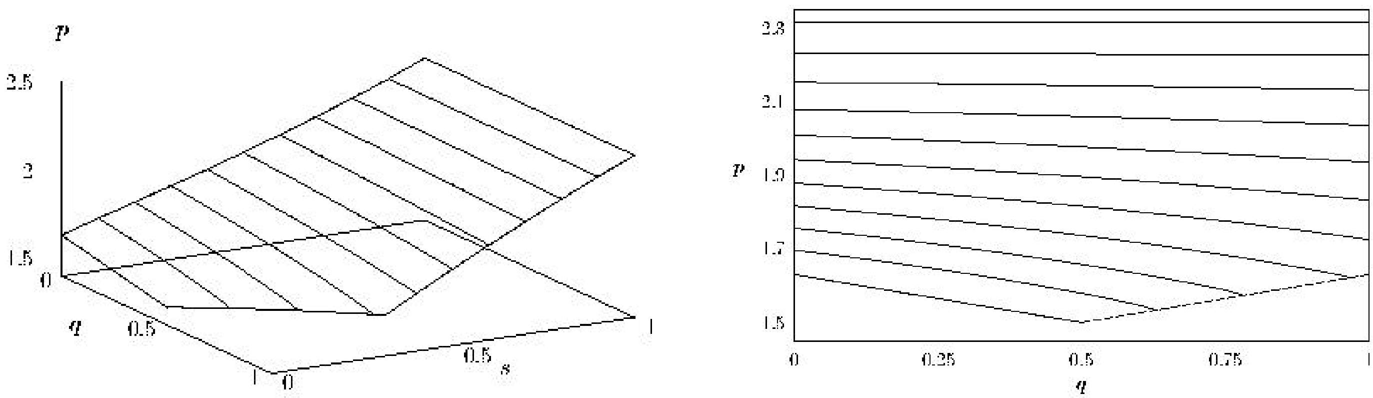

Figure 7: (Left) 3-d view of the surface $p(s; q)$ for the a±liated common values example. (Right) $p(s; q)$ for the a±liated common values example as a function of q with s held ¯xed at several values.

$$w_1 = 1; \qquad w_2 = \frac{1}{2} \tag{6.23}$$

Figures 7 and 8 show the numerical solution for the a±liated distribution common values example where the °at bid by the highest signal is given by $\overline{P}$ (see Equation (5.10)). One question of interest for this problem is whether the linkage principle holds in this example. Our numerical calculations indicate that if the auctioneer were to reveal that the distribution was drawn from $w = 1$ or $w = 1=2$, then the revenue is improved, i.e., the linkage principle holds at least in this example.

7 Conclusions and Extensions

Our paper provides a characterization theorem for the two-bidder multiple bidder auction with variable awards. We show that if a symmetric equilibrium with bidding strategies that are strictly monotone increasing in the signal and strictly monotone decreasing in the price exists, then such an equilibrium is characterized by the solution to two ordinary di®erential equations for each quantity between zero and a half. The ¯rst of these equations is very close to the celebrated Milgrom-Weber solution to the ¯rst price auction while the second characterizes the mapping between the two signals that yields ¯xed quantity. Since the transversality condition requires that the any signal must lose for sure against the highest signal, we use the transversality condition to solve the two ordinary di®erential equation given the end-point conditions at the highest signal. Using this approach, we solve a variety

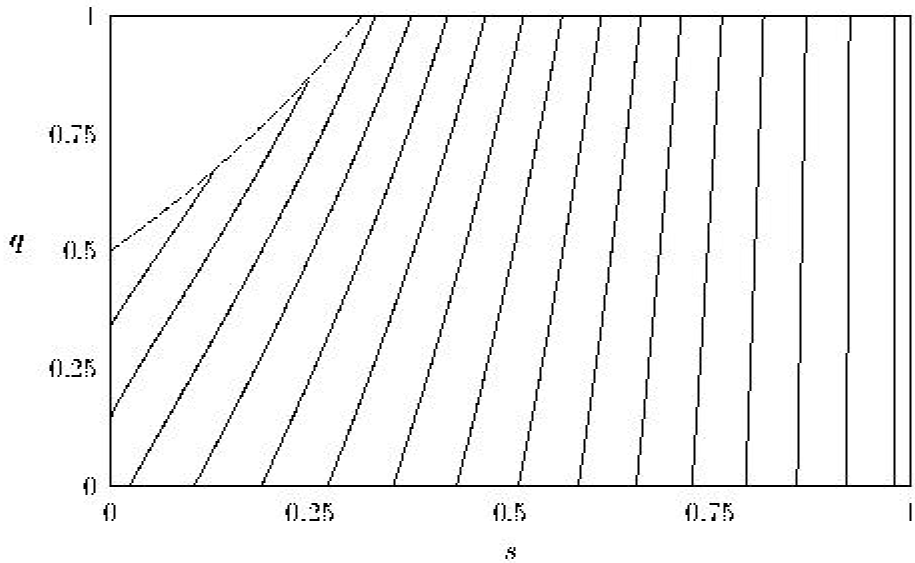

Figure 8: Contours of constant price, $p(s;q) = p$, in the $(s;q)$ plane for the affiliated common values example.

of different problems numerically and analytically and explore the nature of equilibrium in multi-unit discriminatory auctions. Further, we show convergence to the Milgrom-Weber unit auction solution as the marginal valuations become flatter. Lastly, we relate our work to the existence theorems of Reny (1999), Simon and Zame (1999) and Jackson and Swienkels (1999) and show that in some problems no strictly decreasing in price symmetric equilibrium exists.

Our focus in this paper has been exclusively on two bidder problems with one-dimensional private information. While the approach in this paper can be used to provide first order conditions for the 3 bidder problem, it is difficult to provide a solution method.[11] In the 2 bidder problem, when one bidder receives one unit, the other bidder receives zero units. This is no longer true with 3 bidders. Bidder 1 can receive zero units and bidders 2 and 3 can be in the interior, i.e. at half a unit each. Further, it seems more likely that problems with multi-dimensional information and multiple bidders are more likely to have flats in the equilibrium bidding schedule.

[11] An earlier version of this paper provided the first order conditions for problems with multi-dimensional information and multiple bidders.

A Uniqueness for unbounded problems

Uniqueness of the solution of the optimal bidding problems on unbounded domains, $0 < s < \infty$, can be established by determining the asymptotic boundary conditions satisfied by $p(s;q)$ and $H(s;q)$ as $s \to \infty$. In this discussion we present results for the marginal valuation function $M(s_i;s_j;q) = v + \eta s_i + \theta s_j - \rho q$, of the form (2.5), these results also generalize for other classes of M functions. For problems on unbounded intervals with unaffiliated distributions, we can classify the forms of the solutions in terms of the asymptotic properties of the hazard ratio, $K(s) = G(s)/g(s)$ as $s \to \infty$. Consider

$$K(s) \sim \alpha s^{\beta}; \qquad s \to \infty; \tag{A.1}$$

For example, if $G(s) = s^2/(1+s^2)$ for $0 < s < \infty$, then $K(s) = s(1+s^2)/2 \sim s^3/2$. The behavior of the price for $s \to \infty$ breaks down to four cases:

1. If $\beta = 1$, then $p(s)$ approaches a linear function of s as $s \to \infty$.
2. If $1 < \beta < 2$, then $p(s)$ grows like $O(s^{\gamma})$ as $s \to \infty$ with $0 < \gamma < 1$ — slower than linearly.
3. If $\beta = 2$, then $p(s)$ grows logarithmically $p = O(\ln(s))$ as $s \to \infty$.
4. If $\beta > 2$, then $p(s;q)$ is finite and bounded as $s \to \infty$, $p(s;q) < \bar{P}_\infty < \infty$.

In the first three cases given above, $\bar{P}_\infty$ from (5.10) is infinite. However, asymptotic analysis of the differential equations can be used to derive asymptotic boundary conditions on $p(s;q)$. In particular, for (A.1), we find that the asymptotic behavior of $H(s;q)$ for $\beta > 1$ is

$$H(s;q) \sim s + \frac{\rho(1-2q)}{\eta(\beta+1)+\theta(\beta-1)}; \qquad s \to \infty; \tag{A.2}$$

note that we recover the exact solution $H = s$ for $q = 1/2$. Asymptotic boundary conditions of form (A.2) can be used in numerical computations, on large but finite domains, to obtain the solution on the unbounded domain as a regular limit for $\bar{s} \to \infty$. Similarly, it can be shown that as $s \to \infty$, $p(s \to \infty;q) \to p(s \to \infty;1/2)$ for all values of q. Therefore, uniqueness of the solution can also be established for problems on unbounded domains, with distributions that satisfy (A.1).

B Comparison theorems

We will demonstrate that comparison theorems for second order ordinary differential equations can be applied to equations (5.1, 5.2) to help describe solutions of the optimal bidding problem. Here, for simplicity, we consider unaffiliated problems and define $K(s) = G(s)/g(s)$. Since $M(s;q)$ is strictly monotone in s, we can invert (5.1) to obtain

$$H(s;q) = M^{-1}\left(K(s)\frac{dp}{ds} + p(s); 1-q\right). \tag{B.1}$$

Differentiating this result with respect to s, and using the inverse function theorem we obtain

$$\frac{dH}{ds} = \frac{K(s)\frac{d^2p}{ds^2} + (1 + K'(s))\frac{dp}{ds}}{\frac{\partial M}{\partial s}(K(s)p'(s) + p(s); 1-q)} \tag{B.2}$$

Equating this expression for dH/ds with (5.2) we can eliminate H to obtain a second order ordinary differential equation for $p(s)$,

$$\frac{d^2p}{ds^2} = \frac{p'}{K(s)}\left(\frac{\frac{\partial M}{\partial s}(K(s)p' + p; 1-q)}{M(s;q) - p}K(M^{-1}(K(s)p' + p; 1-q)) - 1 - K'(s)\right); \tag{B.3}$$

where $p' = dp/ds$. We can also use (5.1) to write two end-point conditions for $p(s)$ at $\bar{s}$ in terms of the shooting parameter P,

$$p(\bar{s}) = P; \qquad \left.\frac{dp}{ds}\right|_{s=\bar{s}} = \frac{M(\bar{s}; 1-q)}{K(\bar{s})} - \frac{1}{K(\bar{s})}P \tag{B.4}$$

In this form, we have reformulated our problem for $p(s), H(s)$, as a backward initial value problem for the second order equation (B.3), with initial conditions (B.4).

In Walter (1998), a comparison theorem for the initial value problem is proved,

$$\frac{d^2p}{ds^2} = F(s;p;u); \qquad \frac{dp}{ds} = u \tag{B.5}$$

$$p(0) = p_0; \qquad p'(0) = u_0 \tag{B.6}$$

If $F(s;p;u)$ is quasi-monotone increasing in p, that is if

$$\frac{\partial F(s;p;u)}{\partial p} > 0; \tag{B.7}$$

over the set of allowable solutions, then two solutions on $-a < s < 0$, $p_1(s); p_2(s)$ with the initial conditions,

$$p_1(0) < p_2(0); \qquad p_1'(0) \geq p_2'(0); \tag{B.8}$$

satisfy the inequalities for $-a < s < 0$,

$$p_1(s) < p_2(s); \qquad p_1'(s) \geq p_2'(s) \tag{B.9}$$

Therefore, if $F(s; p; u)$ for equation (B.3) satisfies condition (B.7), then for values of the parameter with $P_1 < P_2$, initial conditions (B.4) satisfy conditions (B.8, B.9) and we can conclude that $p_1(s) < p_2(s)$ for $0 < s < \bar{s}$. Graphically, the implications of this comparison theorem are that the trial solutions can not intersect each other in the interior of the domain, $0 < s < \bar{s}$ (see Figures 1, 2). This means that the solutions are ordered by the parameter P, and in particular the value of the solution at $s = 0$, $p(0)$, is a monotone decreasing function of P. Therefore, for some value of P, there is a unique solution of the boundary value problem satisfying $p(0) = p_0(q)$ at a given value of q.

B.1 A private values example

We now apply these comparison theorem results to the private values problem described by equations (6.2),

$$\begin{aligned} \frac{dp(s; q)}{ds} &= [v + H - (k - H)\rho(1 - q) - p]\frac{1}{s}; \\ \frac{dH(s; q)}{ds} &= \frac{[v + H - (k - H)\rho(1 - q) - p]}{[v + s - (k - s)\rho q - p]}\frac{H}{s} \end{aligned}$$

This example has

$$\frac{\partial M}{\partial s}(s; 1 - q) = 1 + \rho(1 - q) > 0 \tag{B.10}$$

which is independent of s and $K(s) = G(s)/g(s) = s$ is increasing in s. We observe that the comparison theorem from Appendix B applies to this example. For this problem, equation (B.1) takes the form

$$H(s) = \frac{sp'(s) + k\rho(1 - q) + p(s) - v}{1 + \rho(1 - q)}; \tag{B.11}$$

and the second-order differential equation for $p(s)$ is

$$\frac{d^2p}{ds} = \frac{p'}{s}\left(\frac{sp' + k\rho(1 + q) - 2(1 + \rho q)s - 3(v - p)}{v + s - (k - s)\rho q - p}\right): \tag{B.12}$$

The test of quasi-monotone behavior, (B.7), then reduces to

$$\frac{\partial F}{\partial p} = \frac{1}{s}\frac{dp}{ds}\frac{(sp' + k\rho(1-2q) + \rho q s)}{(v + s - (k-s)\rho q - p)^2} > 0 \tag{B.13}$$

and hence none of the potential solutions for $p(s)$ at fixed q can intersect each other, as shown in Figure 1. This result was used in our argument for the existence of the solution in Theorem 7.

Similarly, we can examine the common values example given by (6.14), though for simplicity, we will neglect the affiliation in the distributions. As in the private values example, we have

$$\frac{\partial M}{\partial s_i}(s;0;1-q) = 1 + \rho(1-q) \tag{B.14}$$

which is independent of s and $K(s) = G(s)/g(s) = s$ is increasing in s. For this problem, equation (B.1) takes the form

$$H(s) = \frac{sp'(s) + k\rho(1-q) + p(s) - v - \theta s}{1 + \rho(1-q)} \tag{B.15}$$

Substituting this expression for $H(s)$ into equation (6.18) yields the second order differential equation for $p(s)$. For $\theta = 0$, this equation has many more terms than the corresponding equation for the private values case (B.12) and we will not present the details. It is also much more analytically challenging to show that Walter's comparison principle applies, so we will defer to the numerical solutions of the specific examples shown earlier to illustrate the form of the equilibrium.

B.2 Conditions for $p(s;q)$ to be decreasing in q

Another consequence of Walter's comparison theorem is the ability to characterize the q-dependence of the solution $p(s;q)$ using differential inequalities. Walter shows that if $\bar{p}(s)$ satisfies

$$\frac{d^2\bar{p}}{ds^2} \geq F(s;\bar{p};\bar{u}); \qquad \frac{d\bar{p}}{ds} = \bar{u} \tag{B.16}$$

$$\bar{p}(0) \geq p_0; \qquad \bar{p}'(0) < u_0 \tag{B.17}$$

then the following inequalities hold in comparison to the solution of (B.5, B.6), on $-a < s < 0$,

$$\bar{p}(s) \geq p(s); \qquad \bar{p}'(s) < p'(s): \tag{B.18}$$

We now restore the explicit dependence of the solution on q, let $p = p(s;q)$ and let $\tilde{q}$ be a value near, but less than q, $\tilde{q} < q$. Let $\tilde{p} = p(s;\tilde{q})$ be a solution of (B.3, B.4) with $F = F(s;p;u;\tilde{q})$. In the limit that $\tilde{q} \to q$, we can expand F as a Taylor series in $(\tilde{q} - q)$,

$$\frac{d^2\tilde{p}}{ds^2} = F(s;\tilde{p};\tilde{u};\tilde{q}) \approx F(s;\tilde{p};\tilde{u};q) - (q - \tilde{q})\frac{\partial F}{\partial q}(s;\tilde{p};\tilde{u};q). \tag{B.19}$$

Equation (B.19) satisfies the hypothesis of the comparison theorem, (B.16), if F is a decreasing function of q,

$$\frac{\partial F(s;p;u;q)}{\partial q} < 0; \tag{B.20}$$

then (B.18) states that $p(s;q)$ is a monotone decreasing function of q,

$$p(s;q) \geq p(s;\tilde{q}) \qquad \text{for} \qquad q < \tilde{q}. \tag{B.21}$$

We note that it is not necessary for condition (B.20) to hold for all possible trial shooting solutions, in general, it will not. However, through careful estimates, including the dependence of the end-point conditions on q, $H_0(q)$, if it can be shown to hold on the solution of the boundary value problem for $p(s;q)$, for $F = F(s;p(s;q);\partial_s p(s;q);q)$, then the comparison theorem proves that $p(s;q)$ is a valid equilibrium. Equation (5.51) is a necessary condition on $H_0(q)$ for $p(s;q)$ to be decreasing in q locally at $s = 0$. From the class of solutions that satisfy that constraint, a smaller subset will also satisfy condition (B.20) that yields the global conditions for an equilibrium. Showing this is non-trivial, even for the simplest private values problem.

C Details of the local structure of solutions at $s = 0$

Further details of the local analysis are different for problems involving a common component or pure private values and hence we separate these cases.

C.1 Common component problems

Assuming condition (5.42) holds, we can apply L'Hopital's rule to obtain an equation for the value of dp/ds at $s = 0$,

$$\left.\frac{dp}{ds}\right|_{s=0} = \frac{1}{1 + J(0|H_0)}\left(\frac{\partial M}{\partial s_i}(H_0;0;1-q)\left.\frac{dH}{ds}\right|_{s=0} + \frac{\partial M}{\partial s_j}(H_0;0;1-q)\right) \tag{C.1}$$

This derivative is given in terms of the derivative dH/ds at $s = 0$, which can be obtained similarly from

$$\frac{dH}{ds} = \frac{[M(H,s,1-q) - p]}{[M(s,H,q) - p]}\frac{HJ(H|s)}{sJ(s|H)} \tag{C.2}$$

using L'Hopital's rule,

$$\left.\frac{dH}{ds}\right|_{s=0} = \frac{\left(\frac{\partial M}{\partial s_j}(H_0,0,1-q) - \left.\frac{dp}{ds}\right|_{s=0}\right)\frac{G(H_0|0)}{g(H_0|0)}}{J(0|H_0)[M(0,H_0(q),q) - p_0(q)]\frac{g(H_0|0)}{G(H_0|0)} - \frac{\partial M}{\partial s_i}(H_0,0,1-q)}. \tag{C.3}$$

Following some algebraic manipulations, explicit nontrivial values for the local derivatives can be obtained if; (i)

$$\frac{\partial M}{\partial s_j}(H_0(q),0,1-q) \neq 0, \tag{C.4}$$

hence the private values problem must be treated separately, and (ii)

$$(1 + J(0|H_0))[M(0,H_0,q) - M(H_0,0,1-q)] > \frac{\partial M}{\partial s_i}(H_0(q),0,1-q)\frac{G(H_0|0)}{g(H_0|0)}. \tag{C.5}$$

where we remind the reader that $J(0|H) = \lim_{s\to 0} G(s|H)/sg(s|H)$. The latter condition is necessary to ensure that $dp/ds > 0$ at $s = 0$. It yields an upper bound on the allowable values for $H_0(q)$. This analysis can be extended to obtain all of the higher order derivatives of $p(s,q)$ and $H(s,q)$ at $s = 0$ to construct the Taylor series expansions for these solutions in terms of any allowable value of the end-point condition $H(0,q) = H_0(q)$. This establishes the local existence and regularity of the nontrivial, smooth solutions.

C.2 Pure private values problems

We now consider problems for pure private values, described by $\partial M(s_i,s_j,q)/\partial s_j \equiv 0$, but we still allow for affiliation. In the literature, this is termed correlated private values. This case violates the second condition in (C.5), and the solutions exhibit a more complicated structure. To examine this, we use the change of variables,

$$z = s^{\alpha}, \qquad \alpha > 0, \tag{C.6}$$

then equations (5.1, 5.2) take the form

$$\frac{dp}{dz} = \frac{[M(H,1-q) - p]}{\alpha z J(z^{1/\alpha}|H)}, \tag{C.7}$$

$$\frac{dH}{dz} = \frac{[M(H,1-q) - p]}{[M(z^{1/\alpha},q) - p]}\frac{HJ(H|z^{1/\alpha})}{\alpha z J(z^{1/\alpha}|H)}. \tag{C.8}$$

Applying L'Hopital's rule to this system at $s = 0$, we obtain the coupled equations

$$\left.\frac{dp}{dz}\right|_{z=0} = \frac{1}{\alpha J(0|H_0)}\left(\frac{\partial M}{\partial s}(H_0(q);1-q)\left.\frac{dH}{dz}\right|_{z=0} - \left.\frac{dp}{dz}\right|_{z=0}\right); \tag{C.9}$$

$$\left.\frac{dH}{dz}\right|_{z=0} = \frac{\left(\frac{\partial M}{\partial s}(H_0(q);1-q)\left.\frac{dH}{dz}\right|_{z=0} - \left.\frac{dp}{dz}\right|_{z=0}\right)\frac{G(H_0|0)}{g(H_0|0)}}{\alpha J(0|H_0)[M(0;q) - M(H_0;1-q)]}. \tag{C.10}$$

This linear homogeneous system of two equations has a nontrivial solution for $p'(0); H'(0) > 0$ only if it is a singular system. This condition determines the exponent, called the indicial exponent (see Ince (1956)), in the change of variables (C.6) to be

$$\alpha = -\frac{\frac{\partial M}{\partial s}(H_0;1-q) + [M(H_0;1-q) - M(0;q)]\frac{g(H_0|0)}{G(H_0|0)}}{J(0|H_0)[M(H_0;1-q) - M(0;q)]\frac{g(H_0|0)}{G(H_0|0)}} \tag{C.11}$$

and yields the solution

$$\left.\frac{dp}{dz}\right|_{z=0} = H_1[M(H_0(q);1-q) - M(0;q)]\frac{g(H_0|0)}{G(H_0|0)} > 0; \qquad \left.\frac{dH}{dz}\right|_{z=0} = H_1 > 0 \tag{C.12}$$

where $H_1 > 0$ is a positive free-parameter that characterizes the set of local solutions according to their slope.

The allowable range of values for α is $0 < \alpha < \infty$. Since α is given in terms of H_0, if (C.11) is inverted, then it determines the allowable range of H_0 values. The limit $\alpha \to \infty$ determines an upper bound for H_0,

$$M(\overline{H}_0;1-q) = M(0;q); \tag{C.13}$$

this is a special case of the result given by the p-nullcline, (5.21, 5.22). A new lower bound on H_0 is given by the limit $\alpha = 0$,

$$\frac{\partial M}{\partial s}(\underline{H}_0;1-q) + [M(\underline{H}_0;1-q) - M(0;q)]\frac{g(\underline{H}_0|0)}{G(\underline{H}_0|0)} = 0. \tag{C.14}$$

If the left side of (C.14) is a monotone function of H_0, then can obtain a unique positive lower bound $H(0;q) > \underline{H}_0(q) > 0$. We note that this equation is the denominator of equation (5.50) that determines the derivative bound.

If $\alpha = 1;2;3;\ldots$ any positive integer, then $p(s);H(s)$ have Taylor series expansions; this occurs at special values of $H_0(q)$ determined by (C.11). If α is not an integer, then $p(s);H(s)$ have Frobenius series expansions containing rational powers of s, with the form s^{α} times a Taylor series in powers of s.

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