Bayesian *versus* robust control approach towards parameter uncertainty in monetary policymaking: how close are the outcomes? Some illustrating evidence from the EMU economies

Juha Kilponen* Marc-Alexandre Senegas† Jouko Vilmunen‡

February 28, 2006
Draft - incomplete and preliminary version

*Juha Kilponen, Economics Department, Bank of Finland, P.O. Box 160, FIN-00101 Helsinki, Finland. email: juha.kilponen@bof.fi.

†Corresponding author: Marc-Alexandre Sénégas, GRAPE, Université Montesquieu-Bordeaux IV, Avenue Léon Duguit, 33608 Pessac Cedex, France. email: senegas@u-bordeaux4.fr.

‡Jouko Vilmunen, Research Department, Bank of Finland, P.O. Box 160, FIN-00101 Helsinki, Finland. email: jouko.vilmunen@bof.fi.

Abstract

This paper tries to assess the proximity of the macroeconomic outcomes which could arise from a monetary policymaking process based upon either a robust control or a Bayesian (à la Brainard) approach towards parameter uncertainty. We use a small, structural, backward-looking, aggregate model of the EMU economies as the basis for this empirical exercise. After deriving the optimal feedback rules which correspond to the two approaches that we consider in this study, we assess their relative performances with respect to the behavior of the output gap and the inflation rate volatilities and compare with the no-uncertainty benchmark case. We are particularly interested in the output-inflation variability trade-off which is usually associated with the implementation of the optimal monetary policy rule in the literature and in the distortions that the presence of parameter uncertainty and its taking into account via the robust control approach or the Bayesian method may induce to this trade-off.

The results show that the performances of the rules are not too divergent but they appear to be highly contingent upon the preference parameters in the model, ie the relative weight that the monetary authorities attach to output variability (w.r.t. inflation variability) in the loss function and the robustness aversion of the policymaker which is associated to the robust control approach. In particular, non-standard shapes of the output-inflation variability trade-off obtain in the robust control case what may be due to the way the misspecifications associated with the worst case scenario feedback into the structural equations of the model. When the rules are considered within the nominal model, the volatility outcomes appear to be closer to each other.

1 Introduction

How shall policymakers account for the degree in uncertainty which surrounds the practice of monetary policy? How does this behaviour affect the stance and the performances of the policy which is then implemented, c.p.? Since one decade at least, a lot of monetary policy studies (either theoretically or empirically oriented) have addressed those two issues. There is however no consensus about the answer, what is in fact not surprising. Depending on the features of the economy, depending on the kind of uncertainty they are faced with, the monetary authorities may deliver quite different monetary policy reactions which will in turn affect the economy in diverse ways. It appears then that the answer is both contingent upon the representation of the economy the policymaker adopts and upon the way she chooses to model the degree of uncertainty which can surround the latter.

Leaving aside the question of the model contingency (and beyond, the contingency upon the monetary policy regime (inflation targeting,...)), this paper focuses on the approaches which may be retained in the monetary policymaking process to account for uncertainty. Two strands of the literature have emerged concerning this issue and may be set apart from each other according to the statistical underpinnings they give to the uncertainty question.

The Bayesian approach tackles this problem by giving a probabilistic content to the magnitude of uncertainty. Following Brainard seminal article (1967), this approach has been magnified in the treatment of parameter uncertainty where the policymaker considers a prior (probability) distribution for the parameters of the model she uses as a way to account for her imperfect knowledge of the true representation of the economy functioning and designs the monetary policy rule on the basis of this randomizing exercise[1].

To the contrary, and as raised Marcellino and Salmon (2002), "*robust decision theory has recently emphasized a deterministic approach to modelling the unstructured shocks hitting the decision problem of the policymaker*" when uncertainty prevails. (...) "*The development of the H^∞ theory* [has allowed for formalizing] *the lack of knowledge regarding the environment facing the decision maker, enabling robust rules to be used in the face of norm-bounded deviations from a nominal model*". Accordingly, as in the Bayesian approach, the nominal model estimated by the policymaker is considered to be an approximate representation to the true structure of the economy. However, the two approaches differ in the way they proceed to account for the distance between the true and the approximate model. Instead of capturing it in probabilistic terms, the robust control approach refers to the uncertainty aversion of the policymaker which delimits a given range for the misspecifications which can be allowed. Given this range, it is assumed that the policymaker wants to act optimally so as to minimise the loss undergone when the worst case in terms of the structural parameter settings prevails. Thus, "*rather than viewing the set of possible mis-specifications as simply random, the policy maker assumes*

[1] The distribution usually relies upon the statistical outcome of the estimation of the model (i.e. the empirical variance-covariance matrix of the estimated parameters)

"nature" is an evil agent who will choose the mis-specification that makes the policy maker look as bad as possible" (Walsh, 2001).

Given this rather drastic difference in the approach towards uncertainty, this paper tries to assess the proximity of the macroeconomic outcomes which could arise from a monetary policymaking process based upon either a robust control or a Bayesian approach of parameter uncertainty. In particular, we attempt to single out the factors which may explain the main differences observed in this respect. We use a small structural model of the Euroland economy (at the start of EMU) as the basis for this empirical exercise.

This study may be related to different papers which have tried to bridge the gaps between the two approaches such as, *inter alii*, those by Onatski (1999) and Dupor and Liu (2004). The quest for shaping the comparison between the robust control and the Bayesian approach within a single framework may be argued upon two further reasons.

First, to some extent, model uncertainty to which the robust control approach is generally related, may encompass parameter uncertainty as the former concept usually includes broader forms of uncertainty than the latter. However, parameter uncertainty is typically thought in multiplicative terms while the robust control approach to model uncertainty brings about additive disturbances. It has been shown however that both approaches may be addressed within a linear stochastic control framework (see Hansen and Sargent (2001) or Kendrick (2002) for extensive illustrations). In this respect and as we mentioned *supra*, it has to be noted that, while the recognition for parametric uncertainty rests upon one specific distribution for the parameters (and thus a specific form of the variance-covariance matrix in the related stochastic optimal control problem), there is no prior made about the distribution of the (additive) error term when the robust control method is considered. Therefore, each method implicitly entails a specific treatment of uncertainty by the policymaker which may or may not interfere with her standard preferences over traditional macroeconomic objectives. This is one element to take into account when comparing the two approaches towards uncertainty.

Second, there is the difference emphasized in the literature regarding the stance of monetary policy related to the two aforementioned approaches. In particular, several studies have assessed the link between multiplicative (parameter) uncertainty and the cautious nature of monetary policy actions (especially in a dynamic setting where we have to distinguish between the amplitude of the response in itself (compared to the no-uncertainty case) and the pattern of the responses over time which may be smoother than the benchmark - no uncertainty - case). In the robust control literature, concern for model uncertainty is imputed into the actual decision-making problem of the agents by allowing additional cost that arises from making large mistakes due to the uncertainty in the underlying model. Typically, such robust rules deliver more aggressive policy responses. It would thus be interesting to confront these opposite aspects of monetary policy stance.

While not directly addressing the former issues, the paper may allow for an indirect insight on them when looking at the performances of the aforementioned rules in the different modelling frameworks we consider. To this aim, we proceed as follows

At the onset, the derivation of three monetary policy rules is contemplated. We suppose that the rules are fully contingent upon the state variables (optimal feedback rules) in the model: thus, we do not consider restricted (instrument) rules like Taylor rules. We first derive the optimal (called *benchmark*) rule which does not take uncertainty into account ; second, we derive the optimal (called *Brainard*) rule which fully acknowledges the existence of parameter uncertainty and, finally, the *robust* rule which performs under the robust control approach. The first two rules are optimal in the sense that they minimise the expected value of the loss function of the policymaker given a particular distribution of shocks and, in the case of parametric uncertainty, given a particular distribution for the (random) parameters. On the contrary, the robust rule does not exploit the specific information regarding parameter's estimated variance-covariance matrix. Rather the robust rule is "optimised" with respect to the worst-case model only, which arises naturally as a result of the solution to the robust control problem. In this sense, the robust rule is designed with the specific aim to deal with the occurrence of the worst setting of structural parameters.

After the rules have been derived and compared in terms of their feedback parameters, we compute the pairs of output gap and inflation (unconditional) variances which result from the implementation of the rules within the related models. We are particularly interested in the pattern of these volatilities which usually obtains when the preferences of the monetary authorities (over inflation and output) vary, i.e. the well-known Efficient Policy Frontier (first derived by Taylor in 1979). We try to assess how the trade-off which is usually associated with the implementation of the optimal monetary policy rule in the no-uncertainty environment may be distorted by the presence of uncertainty and its taking into account through either the Bayesian or the robust control approach.

Rest of the paper is organised as follows. We first present the model used as the basis for our simulations (section 2). The way the three monetary policy rules are derived from this modelling framework is then presented in section 3 while emphasizing how the approaches towards parameter uncertainty frame this computation. Finally, we analyse in section 4 the pattern of the output and inflation unconditional variances which follow from the implementation of the rules in a given model of the economy (nominal or worst case model).

2 The baseline model: a parsimonious view on "the monetary transmission mechanism" at the start of EMU

TO BE COMPLETED

2.1 A backward-looking model for policy analysis

The design of monetary policy is usually presented as an optimisation problem solved upon the knowledge of the dynamics of the state variables (which depend in a way or another on the policy control variable) and the stochastic disturbance process driving the economy. A class of models, which are widely used for the purpose of policy analysis, is based on a backward-looking specification of the economy. Often, in these types of models, the model specification is reduced to a dynamic aggregate supply equation, and an aggregate demand equation as provided by Rudebusch and Svensson (1999). A generic representation of the model consists of the two following equations:

$$y_t = a_1^y \cdot y_{t-1} + a_2^y \cdot y_{t-2} + c_1^y \cdot [i - \pi]_{t-1} + \varepsilon_{yt} \tag{2.1}$$

$$\pi_t = a_1^\pi \cdot y_{t-1} + a_2^\pi \cdot y_{t-2} + b_1^\pi \cdot \pi_{t-1} + b_2^\pi \cdot \pi_{t-2} + \varepsilon_{\pi t} \tag{2.2}$$

The model captures the main problems facing the policy-maker in practice. Both output and inflation are subject to unexpected shocks. Monetary policy that is conducted by controlling the short-term nominal interest rate influences the economy with lags. According to the previous formulation, it takes a period for policy to affect output, and a period for output to affect inflation, therefore it takes for policy two periods to affect inflation. Thus this structure captures the stylised fact that the monetary policy affects output more quickly than it affects inflation.

This representation departs from recent models including (forward looking) expectations[2]. While this element could be considered as a drawback for this study, it has to be noted that our main objective is to compare the two different ways of treating uncertainty without intervening choice of expectations formation in the context of uncertainty[3]. We leave the assessment of such concerns for future research and consider the study as a first step in this respect. Considering the case for parameter uncertainty, we will also abstract from learning and assume that the degree of uncertainty remains constant for the forecastable future. This is an unrealistic assumption but once one allows for a changing uncertainty regarding model parameters, one must take a stand on exactly what information related to the model parameters that the policymaker does and does not possess, what proves to be a complex and very diversified task. We prefer in a first stage to adopt such a simplifying assumption all the more as our objective is to compare two approaches for uncertainty in terms of their

[2] See, for example, McCallum and Nelson (1999), and Rotemberg and Woodford (1999) in a framework with forward-looking expectations. For an application of robust control methods to model-uncertainty in this framework, see Leitemo and Söderstrom (2004). While these models have strong theoretical foundations, they may fail to fit key facts like the inertia of inflation that appears in the data. For instance, Fuhrer (1997) found that the backward-looking version reproduces much closer empirically observed inflation (inertial) dynamics. See Galí (2001) for a much qualified view

[3] Kilponen (2004) has studied the expectations formation in the context of forward looking model and robust control.

2.2 Union-wide perspective on EMU variables

Inasmuch as our modelling aims at describing (in a reasonable way) the functioning of the EMU economy at its starting point, we opt for the approach of building a Union-wide model by first aggregating the relevant macroeconomic time series across the now EMU member economies, and then estimate a model for the euro area as a whole. The main alternative would have been to consider separate, country-specific models, and then linking them together in a multi-country model in order to design common monetary policy rules. This second approach would have amounted to enable the common central bank to make an intensive use of the national data in order to achieve its different targets while we adopt here a minimalist view on the information management by the monetary authorities[4].

Allowing for the presence of uncertainty in and about the union-wide model may alleviate the drawbacks of this shortcut however, especially once we take into account the break introduced by the switch to the monetary union. The main weakness of both approaches relates indeed to the Lucas critique as it is very difficult to predict (on the basis of the pre-EMU economic relationships) how the economies will work under the new EMU-regime. Not only is the monetary regime changing but at the same time many other structural changes are taking place, which are likely to impact the national transmission mechanisms (as well as their linkages), and the structure of stochastic disturbances in the euro area countries[5].

As a consequence, and even if the results need to be treated very cautiously in any case, we may expect that the Lucas critique could have been of more damage if we have adopted a multi-country setting rather than one Union-wide aggregate modelling all the more as in the latter case we allow for the presence of model and/or parameter uncertainty. One possible interpretation of the analysis we will perform in this respect would be to consider that it all happens as if the central banker would be aware of potential misspecification errors in the model she uses, because of the switch to EMU, and tries to tackle them by allowing for a specific treatment of the ensuing uncertainty regarding either the parameters or the structural relationships of this model. This is why we furthermore restrict the estimation sample prior to the actual implementation of EMU.

2.3 Empirical evidence for the pre-Euroland economy

[TO BE COMPLETED]

Our empirical work is based on the quarterly data taken from the EU-

[4] It seens indeed plausible that the European System of Central Banks as a decentralised structure makes use of various types of multi-country models in the decision process.

[5] Furthermore and even if we looked at euro-aggregates, the model postulates that the aggregate demand relationship is affected by the real interest rate defined as the difference between the short-term interest rate that will be common once the economies will be in EMU and the average of country specific inflation rates.

ROSTAT database. The estimations were performed on the period (1987.03-1998.04).

After several representations have been estimated and compared, we retain the following parsimonious, but obviously in many ways incomplete estimated relationships regarding output and price dynamics for the Euroland economy (standard errors are into brackets):

$$y_t = \underset{(0.075)}{0.905} y_{t-1} - \underset{(0.064)}{0.125} [i - \pi]_{t-2} + \widehat{\varepsilon}_t \tag{2.3}$$

$$\pi_t = \underset{(0.088)}{0.209} y_{t-1} - \underset{(0.09)}{0.116} y_{t-2} + \underset{(0.156)}{0.820} \pi_{t-1} - \underset{(0.152)}{0.154} \pi_{t-2} + \widehat{u}_t \tag{2.4}$$

In (2.3)-(2.4) y_t represents the output gap variable; π_t the quarterly year to year inflation rate (taken as a deviation from the trend) and $[i - \pi]_t$ the (ex-post) real interest rate. $\widehat{\varepsilon}_t$ stands for the estimated (structural) demand shock whereas $\widehat{u}_t$ refers to the (estimated)

It is likely that the specific lag structure we obtain for the impact of the monetary policy instrument on output and inflation has to be related to the way we choose to model the aggregate behavior of the euroland economy. It is indeed widely acknowledged that the euro-aggregated equations will generally differ in the polynomial order (possibly infinite) from the national ones. In this respect, if we have assumed that the relationships (2.1) and (2.2) were correctly specified for the national economies, imposing that the euro-wide aggregate equations should have the same polynomial order as the national ones, or to be finite, would have generally resulted in an incorrect model specification.

By the way, since we are estimating a system of two equations separately, there might exist some cross-correlation between the error terms of the equations that can be exploited to obtain more efficient estimators with a system estimator such as seemingly unrelated regressions (SUR). To check whether the separate estimation of each of the two equations is efficient relative to system estimation, we tested the contemporaneous correlation of the error terms of the two-equation model. We were not able to reject the null hypothesis of zero contemporaneous correlation at a 10 percent level

2.4 State-space form representation

For the purpose of further discussion, it is useful to represent the (small) structural model (2.3)-(2.4) in its reduced form, arising from the restrictions put on the first two equations of a trivariate vector autoregression model containing the output gap, the inflation rate and the short-term interest rate as dependent variables[6]. Let Y_t denote the related vector, $Y_t \equiv \begin{pmatrix} y_t & \pi_t & i_t \end{pmatrix}^\top$. Then we can write that:

[6] We borrow the notations and approach from Söderström (1999).

$$Y_t = \begin{bmatrix} A_1^y & B_1^y & C_1^y \\ A_1^\pi & B_1^\pi & C_1^\pi \\ A_1^i & B_1^i & C_1^i \end{bmatrix} Y_{t-1} + \begin{bmatrix} A_2^y & B_2^y & C_2^y \\ A_2^\pi & B_2^\pi & C_2^\pi \\ A_2^i & B_2^i & C_2^i \end{bmatrix} Y_{t-2} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{\pi t} \\ \varepsilon_{it} \end{bmatrix}$$

The restrictions imposed to obtain the structural model (2.3) (2.4) can be written as:

$$\begin{aligned} B_i^y &= -C_i^y \quad \text{for } i = 1,2 \\ C_i^\pi &= 0 \quad \text{for } i = 1,2 \\ B_1^y &= C_1^y = 0 \\ A_2^y &= 0 \end{aligned}$$

The first two restrictions refer to the way the interest rate enters into the dynamics of the economy (the impact of the policy instrument on the output equation goes only through the *real* interest rate while there is no feedback from the interest rate to the inflation rate). The last two restrictions refer to the lag structure in the IS equation. It is assumed that the real interest rate impacts on the output gap with a two-period delay while we do not retain more than one autoregressive lag in the output gap equation.

Accordingly, we adopt the following state-space representation for the structural model (2.3) (2.4):

$$X_t = \mathbf{A} \cdot X_{t-1} + B \cdot i_{t-1} + \varepsilon_t \tag{2.5}$$

where the state vector and the matrix of parameters are defined as follows: $X_t \equiv \begin{pmatrix} y_t \\ y_{t-1} \\ \pi_t \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix}$ $A \equiv \begin{bmatrix} A_1^y & 0 & 0 & -B_2^y & B_2^y \\ 1 & 0 & 0 & 0 & 0 \\ A_1^\pi & A_2^\pi & B_1^\pi & B_2^\pi & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $B \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is the column vector describing how the instrument (i_t) affects the *current* level of the state variable and finally $\varepsilon_t \equiv \begin{pmatrix} \varepsilon_{yt} \\ 0 \\ \varepsilon_{\pi t} \\ 0 \\ 0 \end{pmatrix}$ is the vector of disturbances whose unconditional variance-covariance matrix is denoted by $\mathbf{\Sigma}_\varepsilon$.

3 Optimal monetary policy rules under different approaches towards uncertainty

3.1 The standard approach: designing benchmark and Brainard rules

Following Rudebusch and Svensson (1999), we next derive the optimal (linear) monetary policy rules on the basis of the previous modelling framework. The exercise is however implemented in two different contexts, depending on whether we assume that some uncertainty surrounds the parameters in (2.5) or not[7]. In other terms, we model the design of the optimal rule in the following way: the policymaker is supposed to behave either as if she knew the true values of the parameters in the model (the no-uncertainty case) or, alternatively, as if she could only rest upon the expectations of those true values which are unknown to her (parameter uncertainty case). As an empirical counterpart of this issue, we assume in the following that the estimated equations (2.3) and (2.4) provide either the true values for the parameters in (2.5) in the no-uncertainty case or, alternatively, the expected values of the latter when parameter uncertainty prevails[8].

To make this distinction operational, we assume that, in the no uncertainty case, $\mathbf{A}$ and B contain non-random elements in the state-space representation. In the parameter uncertainty case, the latter are random such that

$$
\begin{aligned}
\widetilde{\mathbf{A}} &= \mathbf{A} + \boldsymbol{\eta}_A \\
\widetilde{B} &= B + \eta_B
\end{aligned}
$$

and where $\boldsymbol{\eta}_A$ (resp. η_B) refers to a matrix (resp. vector) of disturbances with specific variance-covariance matrices, $\boldsymbol{\Omega}_A$ and $\boldsymbol{\Omega}_B$ respectively[9]. The variance-covariance matrices reflect the degree of uncertainty surrounding the estimated mean values of $\mathbf{A}$ and B for the policymaker. We further assume that the random parameters are independent both of each other across the matrices[10] and of the economic disturbances contained in ε_t.

[7] We define parameter uncertainty in a broad sense as it *a priori* concerns all the parameters involved in the model (and not only those pertaining to the transmission of monetary policy *per se*).

[8] This does not mean that the sole expected values of the parameters are required to design the optimal rule in the context of parameter uncertainty. As it will become apparent in what follows, the knowledge of the multivariate distribution of the parameters (at least of the first two moments) is used to derive the rule.

[9] In the specific case in which we place ourselves, B is a non random vector. We use the general formula however.

[10] If B was effectively randon, this assumption would imply that we do not take some parameters covariance terms (those between the a_{ij} and b_i) into account when looking for the optimal rule under parameter uncertainty. This could be restrictive as Brainard originally demonstrates that covariance between parameters may reverse the cautious nature of monetary policy responses under parameter uncertainty (see Mercado and Kendrick (1999) for an extension). In an empirical study devoted to the euro area, Sahuc (2004) finds however that the "caution argument" remains valid for the optimal monetary policy rule even if

Accordingly, we refer to the representation which is used by the policymaker under the certainty case, as $\Pi(\mathbf{A},\ B)$ - this corresponds to equation (2.5) - whereas, under parameter uncertainty, we denote it by $\Pi\left(\widetilde{\mathbf{A}},\ \widetilde{B}\right)$. This corresponds to equation (2.5) in which where $\widetilde{\mathbf{A}}$ and $\widetilde{B}$ replace $\mathbf{A}$ and B.

Under both cases, the policymaker defines its strategy according to the (scaled) intertemporal loss function defined as:

$$\Lambda = (1-\beta)\, E_t \left[\sum_{\tau=0}^{+\infty} \beta^{\tau} L_{t+\tau}\right]$$

and where the period-by-period loss function of the policymaker may be written as:

$$L_t \equiv X_t^{\top}.\mathbf{Q}(\lambda).X_t \tag{3.1}$$

with diagonal weighting matrix $\mathbf{Q}(\lambda) \equiv diag([\lambda\ 0\ (1-\lambda)\ 0\ 0])$.

λ represents the relative weight put by the authorities towards output gap variability (with respect to inflation volatility[11]). Following Rudebusch and Svensson (1999) we place ourselves in the limiting case where the discount factor (β) is equal to one. Under this assumption, it can be proved that the loss function Λ corresponds, at the limit, to the unconditional mean of the period-by-period loss function (denoted as $E[L_t]$), which, in turn, equals the weighted sum of the unconditional variances of the goal variables (y_t and π_t). We may thus write:

$$E[L_t] = \lambda \cdot var[y_t] + (1-\lambda) \cdot var[\pi_t] \tag{3.2}$$

where $E[.]$ and $var[.]$ design unconditional expectation and variance.

Under both contexts, the problem is written in a form convenient to apply the standard stochastic linear regulator method (see Rudebusch and Svensson [1999]). Considering the class of linear feedback rules and minimizing (3.2), subject to $\Pi(\mathbf{A},\ B)$ or $\Pi\left(\widetilde{\mathbf{A}},\ \widetilde{B}\right)$ and the current state of the economy X_t results in the optimal rule for the instrument, i_t. Let f (resp. f_u) denote the (1×5) vector of optimal parameters for the full-contingent rule under certainty (resp. under uncertainty)[12]. As each of the two vectors is derived for a given (λ), we make explicit reference of this contingency

covariance between parameters is acknowledged.

Moreover, it must be noted that other factors than those pertaining to the interdependence between the parameters may play against the cautious nature of monetary policy rules under uncertainty, in particular the features of the dynamic setting in which monetary policy impinges on the economy (see, on this specific issue, Craine (1979), Mercado and Kendrick (2000) and Söderström (1999).

[11] We also assume that the targeted rate of inflation is zero and that the objective in terms of output consists in stabilising the output gap around 0.

[12] See Söderström (1999) for an excellent presentation of the dynamic optimal control problems associated to the certainty and the parameter uncertainty case.

We then have two different expressions of the optimal linear feedback rule depending on whether we consider that the latter accounts for parameter uncertainty or not:

$$\begin{aligned} i_t &= f(\lambda) \cdot X_t & (3.3) \\ i_t &= f_u(\lambda) \cdot X_t & (3.4) \end{aligned}$$

The **benchmark rule** corresponds to the certainty case where the f vector fulfills:

$$f = -\left(B^\top \cdot \mathbf{V} \cdot B\right)^{-1} \cdot \left(B^\top \cdot \mathbf{V} \cdot \mathbf{A}\right) \tag{3.5}$$

and where $\mathbf{V}$ is determined by the Ricatti equation:

$$\mathbf{V} = \mathbf{Q}(\lambda) + (\mathbf{A} + B \cdot f)^\top \cdot \mathbf{V} \cdot (\mathbf{A} + B \cdot f) \tag{3.6}$$

The **Brainardian rule** corresponds to the parameter uncertainty case where the vector of policy responses f_u fulfills:

$$\begin{aligned} f_u &= -\left[B^\top \cdot \left(\mathbf{V}_u + \mathbf{V}_u^\top\right) \cdot B + 2\nu_u^{11} \cdot \Sigma_B^{11} + 2\nu_u^{55} \cdot \Sigma_B^{55}\right]^{-1} \\ &\quad \cdot \left[B^\top \cdot \left(\mathbf{V}_u + \mathbf{V}_u^T\right) \cdot B + 2\nu_u^{11} \cdot \left(\Sigma_B^{11}\right)^T + 2\nu_u^{55} \cdot \Sigma_B^{55}\right] \end{aligned}$$

and where $\mathbf{V}_u$ satisfies the following equation:

$$\begin{aligned} \mathbf{V}_u &= \mathbf{Q}(\boldsymbol{\lambda}) + (\mathbf{A} + B \cdot f_u)^\top \cdot \mathbf{V}_u \cdot (\mathbf{A} + B \cdot f_u) \\ &\quad + \nu_u^{55} \cdot \left(\Sigma_A^{55} + f_u^\top \cdot \left(\Sigma_B^{55}\right)^\top \cdot f_u\right) \\ &\quad + \nu_u^{11} \cdot \left(\Sigma_A^{11} + 2 \cdot \Sigma_{AB}^{11} \cdot f_u + f_u^\top \cdot \left(\Sigma_B^{11}\right)^\top \cdot f_u\right) \end{aligned}$$

ν_u^{ij} is the (i,j)th element of $\mathbf{V}_u$, $\boldsymbol{\Sigma}_{AB}^{ij}$ is the covariance matrix of the ith row of $\mathbf{A}$ with the jth row of B, $\boldsymbol{\Sigma}_A^{ij}$ is the covariance matrix of the ith row of $\mathbf{A}$ with the jth row of $\mathbf{A}$ and finally $\boldsymbol{\Sigma}_B^{ij}$ the covariance term between the elements located on the ith and jth rows of B.

3.2 Robust control approach

3.2.1 The robust rule

The rapidly developing literature on robust policy rules considers decision problems in circumstances where the true model is not exactly known, but where the applied decisions rules should perform reasonably well even under the worst-case. In this robustness literature, concern for model uncertainty is imputed into the actual decision-making problem of the agents, in the sense that model uncertainty directly distorts the decision maker´s preferences to a particular direction. This distortion in preferences is effectively achieved by combining the minimisation problem with the maximisation problem and by introducing a Lagrange multilier of the relevant constraint directly into the loss function of the decision maker.

It is typically assumed that the policymakers have a reference model or the nominal model that reflects their best knowledge of the believed laws of motion of the economy, yet they acknowledge that their information about the complete dynamics of the economy is limited. The agents hedge against the uncertainty by making mental constructs of the model sets. In order to construct a robust decision rules, the decision-maker computes a Markov Perfect Equilibrium of a particular zero-sum game, where each player chooses sequentially and/or simultaneously in each period, taking the other player's decision rule as given (see Hansen and Sargent 2004).

In our backward looking case, the policymaker attempts to minimise his distorted loss function, while the nature, the second player, chooses the distortions to the nominal model's dynamics. The policymaker thus achieves robustness with the help of evil nature's distorted laws of motion.

Following the standard setup in the robust control literature, we assume that the policymaker considers the model uncertain in the sense that there is an additive (vector) term w

$$X_t = \mathbf{A} \cdot X_{t-1} + B \cdot i_{t-1} + \varepsilon_t + w_t \tag{3.7}$$

which is allowed to feedback in a general but restrained way on the state variables of the structural representation of the economy. In other words, we let

$$w_t = g_t(X_{t-1}) \tag{3.8}$$

The constraint arises from the fact that w_t is assumed to be a vector process, where the size of the model approximation errors is constrained such that

$$E_0 \sum_{t=0}^{\infty} \beta^t . w_{t+1}^{\top} . w_{t+1} \leq \eta_0 \tag{3.9}$$

and where E_0 denotes a mathematical expectation conditioned on the initial values of system variables and where β is the respective discount factor. η_0 defines a set of models and provides, in a backward looking setting, a constraint under which the maximising agent can distort the model's dynamics. The set of models, that are "possible" around the approximating (or nominal) model are therefore constraint by η_0. Given that the uncertainty surrounding the model is presented here in an unstructured way, it can be thought of as capturing a wide range of misspecified dynamics (associated with different combinations of the model parameters). As will be seen in due course, w_t appears as a feedback control sequence which maximises the assigned loss function of the policy maker.

On the basis of this representation (we design (3.7) as $\Pi\left(\mathbf{A},\ B,\ w\right)$), the robust control exercise is implemented as a min max strategy which is defined as:

$$\begin{array}{ll} \min\limits_{\{i_t\}_0^{+\infty}} \max\limits_{\{w_t\}_1^{+\infty}} E_0 \left[\sum\limits_{t=0}^{+\infty} \beta^t \left(X_t^\top.\mathbf{Q}(\lambda).X_t \right) \right] & \\ s.t. \quad \begin{array}{c} \Pi(\mathbf{A},\ B,\ w) \\ E_0 \sum_{t=0}^{\infty} \beta^t.w_{t+1}^\top.w'_{t+1} \leq \eta_0 \end{array} & \end{array} \tag{3.10}$$

Introducing the bound constraint prevailing upon the range of the misspecifications which are allowed for by the policymaker into the objective function, the problem may be re-written now as follows:

$$\begin{array}{l} \min\limits_{\{i_t\}_0^{+\infty}} \max\limits_{\{w_t\}_1^{+\infty}} E \left[\sum\limits_{t=0}^{+\infty} \beta^t \left(X_t^\top.\mathbf{Q}(\lambda).X_t - \theta.w_{t+1}^\top.w'_{t+1} \right) \right] \\ s.t.\ \Pi(\mathbf{A},\ B,\ w) \end{array}$$

θ (the Lagrangian multiplier) may be interpreted as the parameter which reflects the robustness (or uncertainty) aversion of the policymaker. The higher its value, the lesser the policymakers allows for taking uncertainty into account. The latter bounds the range in the misspecifications which are allowed by the monetary authorities[13].

The solution to the problem can be expressed as a joint product of two rules: the first corresponds to the monetary policy rule implemented by the policymaker for a given set of preferences $(\lambda,\ \theta)$ and a given path of X_t. This rule is optimal with respect to the worst case model. The other rule pertains to the evolution of the disturbance term reflecting the misspecifications which are allowed for by the policymaker. As a consequence let f_r denote the (5×1) vector of optimal parameters for the full-contingent rule performed under the robust control approach. As this vector is derived for a given pair of preferences (λ, θ), we make explicit reference of this contingency. We also consider f_w the vector of the optimal parameters governing the law of motion of w at equilibrium. We thus have that:

$$i_t = f_r(\lambda, \theta) \cdot X_t \tag{3.11}$$

$$w_t = f_w(\lambda, \theta) \cdot X_t \tag{3.12}$$

We refer to (3.11) as the **robust** rule and to (3.12) as the associated model (worst case) disturbance[14]**.**

[13]It may be shown that when $\eta_0 = 0, \theta$ goes to infinity: we are back to the benchmark (no-uncertainty) case. However, for the min-max problem to have a well-behaved solution, the robustness aversion parameter has to lie above a specified lower bound which depends, among other things, on λ (see *supra*). For an assessment of the impact of the changes in θ on the features of the robust rule, see Gonzalez and Rodriguez (2005).

[14]Regarding the simulations, we use and adapt GAUSS code programs which have been kindly provided to us by Ulf Söderström for the benchmark and Brainard cases. The GAUSS codes provided by Söderlind were used to compute the robust rule and implement the detection probability method. A parallel checking was implemented using procedures in MATLAB.

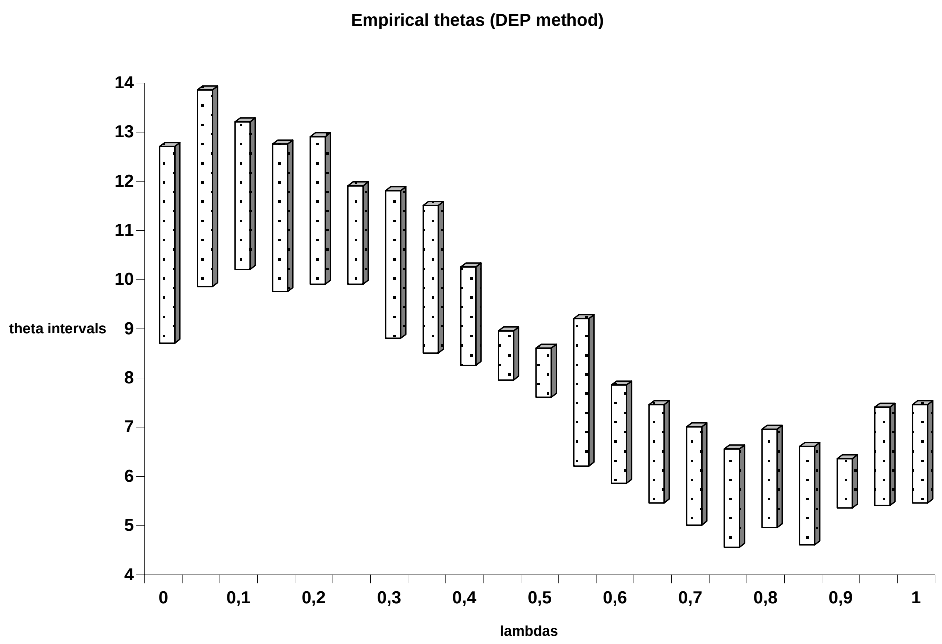
3.2.2 Choice of θ

θ appears to be the free parameter in the robust control approach. Two caveats have to be noted in considering this parameter.

First, the choice for the range for θ cannot be determined in a fully independent way from the preferences of the policymaker in terms of output and inflation variability (i.e. λ). There is a threshold value for θ over which, the maximizing part of the program is known to be well behaved (the threshold value depends on λ).

Second, within the allowed range, it remains to be seen how the value may be chosen in order to perform the simulations. A first solution would be to posit a sufficiently high arbitrary value for this parameter. On the other hand, empirically relevant values for θ may be found through the detection probability method (DEP in what follows) (see Hansen and Sargent, 2003). The idea behind this process is the following: the models in the set allowed for by the policymaker should not be easy to distinguish with the available data. We may choose the value of θ so that the probability to make the wrong choice with respect to the true generating process of the data should not be high.

In this process, the choice of θ still depends on λ as this parameter enters into the determination of the set of models which would be considered as a worst case for the policymaker (the worst case being evaluated through the loss function). As a result, we implement the error detection probability method in a consistent way with respect to the threshold values for the robustness aversion parameter. For each λ, then, we choose to select a range of values for θ which provide an error detection probability between 10% and 20%. Results are shown on figure 1.



3.3 Uncertainty and monetary policy reactions

[TO BE COMPLETED]

The coefficients in the policy rule vector allow to quantify the extent to which monetary policy reacts to the macroeconomic environment (which is captured by the state vector). How the presence of uncertainty may impinge on this feedback can be identified throughout the comparison between the optimal rule coefficients across the different settings we have considered (no uncertainty, Bayesian. Three values of λ are considered (0, 0.5, 1) while, regarding the robust rule, the value of θ has been chosen as the mean value for the interval associated with DEP method (for a given λ). Figures are in the Appendix

We observe that the coefficient values are close to each other across the kinds of rules considered and whatever the value of λ. Very big values are observed however what could be partly due to the fact that we look at fully contingent and not restricted rules. Shifts logically occur when the preference parameter varies from 0 (no concern for inflation variability in the loss function) to 1 (inflation nutter). The Brainard case seems to be consistently associated with a dampening in the monetary policy reaction compared to the two other strategies, suggesting an moderate attenuation effect.

4 Inflation-output variability trade-offs and the approach towards uncertainty

Given the dynamic setting usually held in the analysis of monetary policy actions, and in order to assess the relative performances of different types of optimal rules on the economy, it has been usual to look at the long-run behavior of the volatility in inflation and output, a so-called "second-order Phillips curve" as Taylor (1979, p. 1280) defined it. It is generally found that a trade-off between those volatilities prevails when the optimal rule is adopted by the monetary authorities. When the policymaker's relative preferences parameter varies, "*a minimum variability efficiency locus between output and inflation* [appears]. *This efficiency locus is the trade-off curve*" (1979, p. 1280). We refer to this concept as the efficient policy frontier (EPF in the following)[15]. With respect to Taylor's original presentation of the latter concept, several points are worth being emphasized:

- The Efficient Policy Frontier is only relevant for optimal rules. In other terms, for each class of rules there will be only one EPF. In the following we only look at fully state-contingent rules class

[15]Another one is generally used in the litterature on the monetary policy performances accross industrialised countries, namely the sacrifice ratio (see Mankiw (1994) and Fuhrer (1994)).

- The EPF locus and its shape in the output/inflation variances mapping are contingent upon the structural parameters of the model (see Fuhrer and Moore (1994) for a thorough analysis of this point) and on the specification of the rule.

It ensues that the EPF shape and location both depend, among others, on the degree of uncertainty when the latter is to be taken into account in the design of the rule (in the case of the robust rule, the degree of uncertainty may be tackled by θ). This explains why this analytical tool can be used both to rank different optimal rules in a no uncertainty framework but also to assess how the recognition of uncertainty may affect the performances of a given class of rules.

4.1 Computing the inflation output variability frontiers

To plot the EPF loci, we have to compute the unconditional variance-covariance matrix of the state vector obtained when the monetary authorities implement the optimal rule while using the specific representation of the economy which this rule refers to.

4.1.1 Efficient policy frontiers in the "standard" case

Let us start from the case where uncertainty regarding the model's parameters has not been taken into account.

Substituting equation (3.3) into (2.5), we may obtain closed-loop representation of the economy, now re-written as:

$$\mathbf{X}_t = \mathbf{G} \cdot \mathbf{X}_{t-1} + \boldsymbol{\varepsilon} \tag{4.1}$$

and where $\mathbf{G} \equiv \mathbf{A} + B \cdot f$. The unconditional variance-covariance matrix of the state vector can be directly obtained from (4.1) by applying the unconditional expectation operator. By denoting Σ_X this matrix, we then find that:

$$\boldsymbol{\Sigma}_X = \mathbf{G} \cdot \boldsymbol{\Sigma}_X \cdot \mathbf{G}^\top + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} \tag{4.2}$$

Using the **vec** operator the last expression may be rewritten as:

$$\mathbf{vec}\left(\boldsymbol{\Sigma}_X\right) = \left[\mathbf{I} - \mathbf{G} \otimes \mathbf{G}\right]^{-1} \mathbf{vec}\left(\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right) \tag{4.3}$$

where $\mathbf{I}$ an identity matrix of size appropriate dimensions (25×25).

When the uncertainty is explicitly taken into account, we can apply the same procedure, but realizing the matrices are now random. Thus, we start writing the closed loop representation of the economy as

$$X_t = \widetilde{\mathbf{G}} \cdot X_{t-1} + \boldsymbol{\varepsilon} \tag{4.4}$$

with $\widetilde{\mathbf{G}} \equiv \widetilde{\mathbf{A}} + \widetilde{B} \cdot f_u$. The computation of the unconditional variance-covariance matrix of the state vector (denoted as $\boldsymbol{\Sigma}_X^u$) is tedious as now $\widetilde{\mathbf{G}}$ is random.

Assuming that ε and $\boldsymbol{\eta}$ (the matrix of disturbances pertaining to $\widetilde{\mathbf{G}}$) are uncorrelated, we obtain the unconditional expectation on the basis of (4.4), leading to:

$$E\left[X_t X_t^\top\right] = E\left[\widetilde{\mathbf{G}} \cdot X_{t-1} X_{t-1}^\top \cdot \widetilde{\mathbf{G}}^\top\right] + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} \tag{4.5}$$

Using the **vec** operator, we then find that

$$\mathbf{vec}\left(\boldsymbol{\Sigma}_X^u\right) = \left\{\mathbf{I} - E\left[\widetilde{\mathbf{G}} \otimes \widetilde{\mathbf{G}}\right]\right\}^{-1} \mathbf{vec}\left(\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right) \tag{4.6}$$

In order to highlight an obvious difference to the certainty case, let us recall that we may write $\widetilde{\mathbf{G}} = \mathbf{G} + \boldsymbol{\eta}$ where $\boldsymbol{\eta} \equiv B \cdot (f_u - f) + \eta_B \cdot f_u + \boldsymbol{\eta}_A$. Then equation (4.5) may be written as:

$$\boldsymbol{\Sigma}_X^u = \mathbf{G} \cdot \boldsymbol{\Sigma}_X^u \cdot \mathbf{G}^\top + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} + E\left[\boldsymbol{\eta} \cdot X_{t-1} X_{t-1}^\top \cdot \boldsymbol{\eta}^\top\right] \tag{4.7}$$

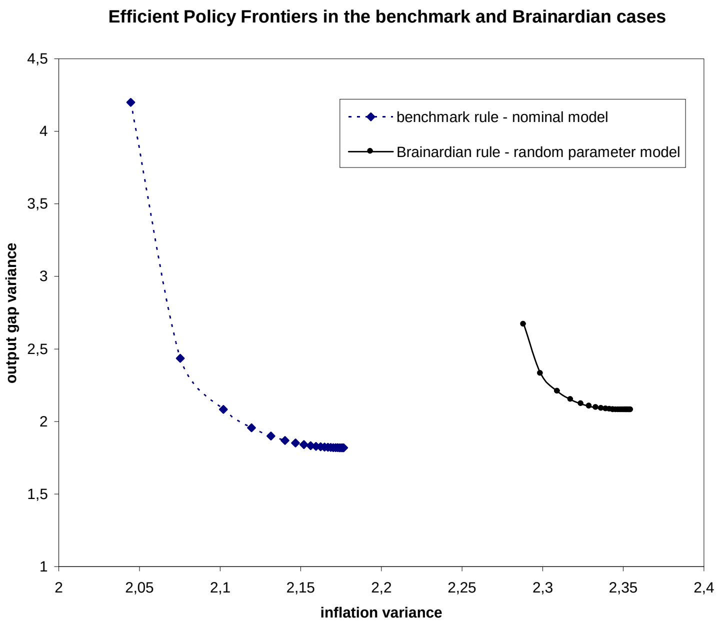
The former has common features with (4.2) except for the last term which reflects the random nature of the monetary transmission parameters. The main difference between the uncertainty case and no-uncertainty cases regarding the efficiency frontier is reflected by the matrix $\boldsymbol{\eta}$ in (4.7).

First, and for a given rule, we have to consider that, in the uncertainty case, the variance of the relevant state variables depend not only on the variance of the disturbances but also on the variance of the parameters (this is captured by the $(\eta_B \cdot f_u + \boldsymbol{\eta}_A)$ element in the expression of $\boldsymbol{\eta}$). The latter element is obviously absent in the certainty case.

Second, the variances of the state variables depends, in both contexts, on the features of the rule and thus on the extent to which the latter takes the presence of parameter uncertainty into account (the $(B \cdot (f_u - f))$ element in the definition of $\boldsymbol{\eta}$).

The EPF can be obtained by making the policymaker's preference parameter λ vary from 0 to 1 and computing the pairs $(\Sigma_X^{11}(\lambda),\ \Sigma_X^{33}(\lambda))$ and $(\Sigma_X^{u\ 11}(\lambda),\ \Sigma_X^{u\ 33}(\lambda))$, $\Sigma_X^{ij}(\lambda)$ $\left(\text{resp. } \Sigma_X^{u\ ij}(\lambda)\right)$ denoting the element of $\boldsymbol{\Sigma}_X$ (resp $\boldsymbol{\Sigma}_X^u$) on the ith row and jth column. $\boldsymbol{\Sigma}_X$ (resp. $\boldsymbol{\Sigma}_X^u$) depends on λ through f (resp. f_u).

We plot the two frontiers we obtain on figure 2. We observe that the recognition of uncertainty leads both to a shift in the frontier and to a reduction in the volatility range for the output gap.



[TO BE COMPLETED]

4.1.2 The robust case

In the robust control approach, the relevant representation to consider to plot the EPF is *the worst case model* as the policymaker tries to solve for the rule which minimises the loss function in such a context. By substituting equations (3.11) and (3.12) into (3.7), the dynamics of the state vector in the worst case model is given by

$$\mathbf{X}_t = \mathbf{G}_r \cdot \mathbf{X}_{t-1} + \boldsymbol{\varepsilon} \tag{4.8}$$

and where $\mathbf{G}_r \equiv \mathbf{A} + B \cdot f_r + f_v$. The unconditional variance-covariance matrix of the state vector is directly obtained from (4.8) by applying the unconditional expectation operator. By denoting this matrix with Σ_X^r, we then find, using again the **vec** operator that

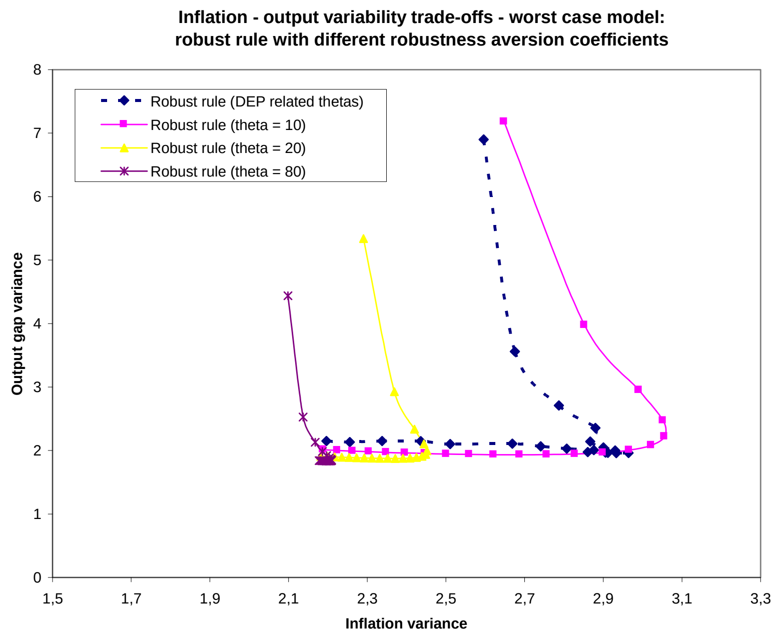
$$\mathbf{vec}\left(\boldsymbol{\Sigma}_X^r\right) = \left[\mathbf{I} - \mathbf{G}_r \otimes \mathbf{G}_r\right]^{-1} \mathbf{vec}\left(\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right) \tag{4.9}$$

with $\mathbf{I}$ an identity matrix of appropriate dimension (25×25).

Additional complication of calculating the EPF in the robust control context is that there is an additional parameter θ which, not only distorts the policymakers preferences to a particular direction but also defines the worst-case model against which the EPF is to be evaluated. Given that this parameter captures the extent to which the policymaker feels his model is uncertain, the choice of θ becomes of crucial importance.

Accordingly we obtain a particular EPF **for** any **given** θ $(\theta = \theta_0)$ by making λ vary from 0 to 1 and, for each value of the latter parameter, by computing the pairs $\left(\Sigma_X^r{}^{11}(\lambda,\theta_0),\ \Sigma_X^r{}^{33}(\lambda,\theta_0)\right)$ [16].

We plot different frontiers corresponding to different values of the robustness aversion coefficient (including those arising from the DEP method). As we observe, those frontiers appear to deliver non-convexities. We also note that the higher the value of θ, the closer we are to the frontier related to the no-uncertainty case. Non-convexities thus seem to be associated with the magnitude in the misspecifications which are allowed for by the policymaker.



[TO BE COMPLETED]

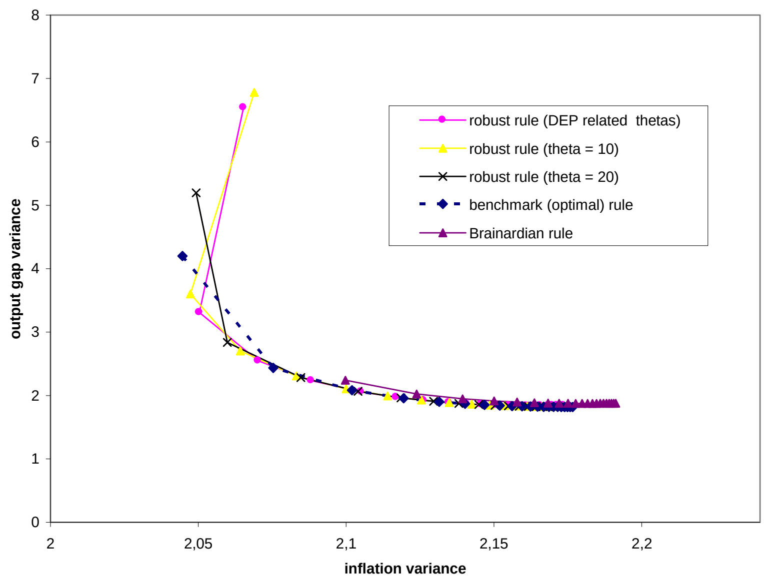
4.1.3 Simple Policy frontiers

We may also consider some policy frontiers arising from the combination of a given monetary policy rule and a given representation with respect to which this rule is not necessarily optimal. (those frontiers are thus not efficient).

For example, plotting the frontiers with respect to the nominal model for different kinds of rulesand allows to assess how the use of a suboptimal policy rule (as the Brainard and robust rules are with respect to this model) may bring about excessive variability when the policymaker accounts in the latter rule for what it perceives as uncertainty. The way this perceived uncertainty may affect economic variability (through the monetary policy reaction) can also be measured through the distance between the efficient policy frontier (which hinges on the combination of the optimal rule and the related model) and the policy frontier which is computed with respect to the benchmark representation (nominal model).

[16] $\Sigma_X^r{}^{ij}(\lambda,\theta_0)$ denotes the element of $\Sigma_X^r(\lambda,\theta_0)$ on the ith row and jth column. Notice that $\mathbf{\Sigma}_X^u$ depends on λ through f_r for a given θ.

Inflation - output variability frontiers with different robust rules (nominal model)



[TO BE COMPLETED]

5 Conclusion

[TO BE COMPLETED]

6 References

[TO BE COMPLETED]

References

Altavilla C. (2001), "Assessing monetary policy rules performances across EMU countries", *CES Discussion Paper*, Katholieke Universiteit Leuven, June.

Batini N., Martin B. and C. Salmon (1999), "Monetary policy and uncertainty", Bank of England Quarterly Bulletin, May, p. 183-189.

Bean C. (1998), "The new UK monetary arrangements: a view from the literature", The Economic Journal, vol. 108, p. 1795-1809, November, p. 1795-1809

Blinder A. (1998), *Central banking in theory and practice*, MIT Press, Cambridge (US).

Brainard W. (1967), "Uncertainty and the effectiveness of policy", *American Economic Review*, vol. **57** (2), May, p. 411-425.

Cecchetti S. (1996), "Practical issues in monetary policy targeting", *Economic Review*, Federal Reserve Bank of Cleveland, vol. 32 (1), p. 2-15.

Cecchetti S. (1998), "Policy rules and targets: framing the central banker problem", Economic Policy Review, Federal Reserve Bank of New York, vol. ?, June, p. 1-14.

Cecchetti S., McConnell M. and G. Perez Quiros, (1999), "Policymakers' revealed preferences and the output-inflation variability trade-off: implications for the European System of Central Banks", Federal Reserve Bank of New York, mimeo, February.

Chow G. (1975), *Analysis and control of dynamic economic systems*, Wiley, New York.

Craine R. (1979), "Optimal monetary policy with uncertainty", Journal of Economic Dynamics and Control, vol. 1, p. 59-83.

Fuhrer J. (1997), "Inflation/Output variance trade-offs and optimal monetary policy", Journal of Money, Credit and Banking, vol. 29 (2), May, p. 214-234.

Gerlach S. and F. Schnable (1999), "The Taylor rule and interest rates in the EMU area", CEPR Working Paper, No 2271, October.

Gerlach S. et F. Smets (1999), "Output gaps and monetary policy in the EMU area", European Economic Review, vol. 43, p. 801-812.

Levin A., Wieland V. and J. Williams (1998), "Robustness of simple monetary policy rules under model uncertainty", Finance and Discussion Papers Series, 1998-45, Board of Governors of the Federal Reserve System, November.

Rudebusch G. and L. Svensson (1998), "Policy rules for inflation targeting", NBER Working Paper, No 6512, April.

Rudebusch G. and L. Svensson (2000), "Eurosystem monetary targeting: lessons from US data", mimeo, Federal Reserve Bank of San Francisco, May.

Söderström U. (1999),"Monetary Policy under Uncertainty."*Ph.D. Dissertation, Stockholm School of Economics.*

Srour G. (1999), "Inflation targeting under uncertainty", *Technical report*, Bank of Canada, No 85, April.

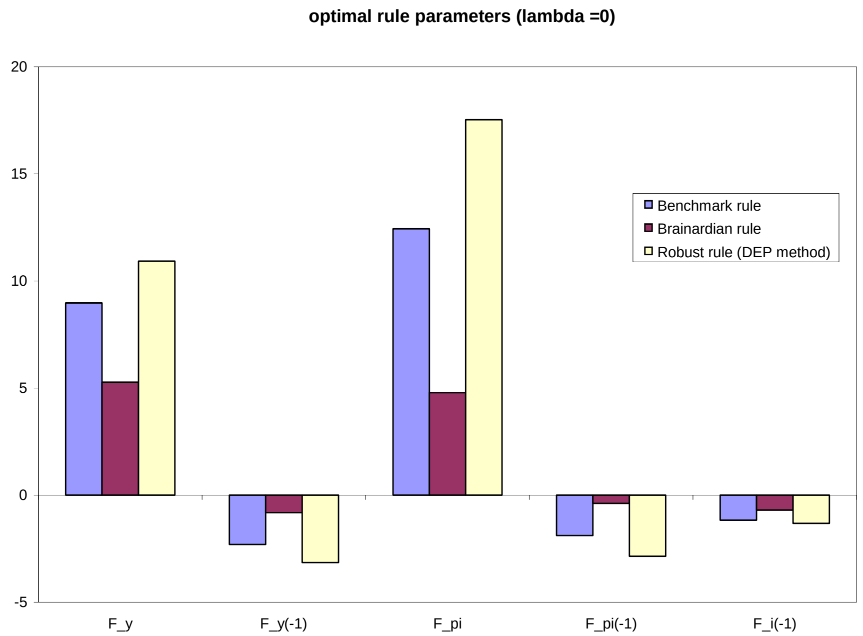
Taylor J. (1979), "Estimation and control of a macroeconomic model with rational expectations", Econometrica, Vol. 47 (5), September, p. 1267-1286.

Taylor J. (1994), "The Inflation/Output variability trade-off revisited" in Goals, Guidelines and Constraints Facing Monetary Policymakers, J. Fuhrer (eds), Federal Reserve bank of Boston, p. 21-38.

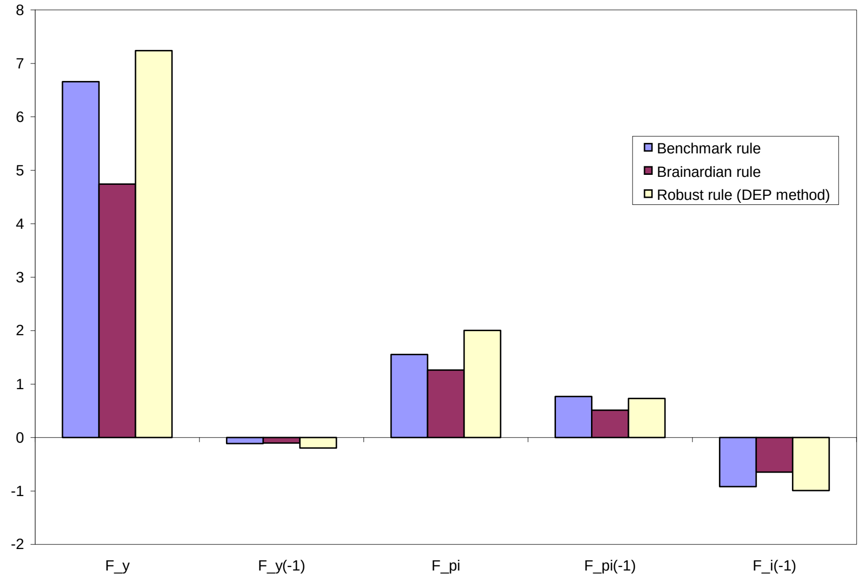
Taylor J. (1999), "The robustness and efficiency of monetary policy rules as guidelines for interest rate setting by the European Central Bank", mimeo, Stanford University, February

Tetlow R. and P. von zur Muehlen (2001), "Robust monetary policy with misspecified models: does model uncertainty always call for attenuated policy", *Journal of Economic Dynamics and Control*, vol. 25, p. 911-949.

7 Appendix



optimal rule parameters (lambda =0.5)



optimal rule parameters (lambda =1)