Dynamic optimal taxation with human capital

Leslie J. Reinhorn
Dept. of Econ. & Finance
University of Durham
23-26 Old Elvet
Durham DH1 3HY
United Kingdom

phone +44 191 334 6365
fax +44 191 334 6341
reinhorn@hotmail.com

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Abstract

This paper enhances the dynamic optimal taxation results of Jones, Manuelli, and Rossi (1993, 1997). They use a growth model with human capital and find that optimal taxes on both capital income and labor income converge to zero in steady state. For one of the models under consideration, I show that the representative household's problem does not have an interior solution. This raises concerns since these corners are inconsistent with aggregate data. Interiority is restored if preferences are modified so that human capital augments the marginal utility of leisure. With this change, the optimal tax problem is analyzed and, reassuringly, the Jones, Manuelli, and Rossi results are confirmed: neither capital income nor labor income should be taxed in steady state.

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1 Introduction

In seminal papers, Chamley (1986) and Judd (1985) have shown that capital income should not be taxed in steady state. Jones, Manuelli, and Rossi (1993, 1997) (hereafter JMR) extend this result to show that labor income should also be free from taxation in the limit. JMR add human capital to the model to derive their remarkable results. A key part of the analysis involves the manipulation of the household's first order conditions. However, it will be shown below that for a popular class of models nested within the JMR framework, the household's problem does not have an interior optimum: Given any interior solution to the first order conditions, there always exists a feasible variation that increases utility. This is of concern since an aggregative model with corners cannot fit the data. It also invites a closer look at the optimal tax results.

The class of models in which the difficulties arise is derived from Heckman (1976). However, in Heckman's original formulation utility depends on effective leisure — human capital multiplied by raw hours of leisure — while JMR's utility function depends only on raw leisure.[1] If utility is returned to the original Heckman form, the corners disappear and the standard first order characterization of equilibrium may be used. In this case, it is shown that the JMR extension of the Judd–Chamley optimal tax result continues to hold. That is, neither labor income nor capital income should be taxed in the limit.

Section 2 presents the JMR model and demonstrates that the household's optimum is not interior when technology has the Heckman specification. Section 3 restores interiority by taking the original Heckman utility function. With this change, the optimal tax policy is characterized in the limit. Section 4 is a brief conclusion.

2 Model

First the full JMR (1997) model is presented. Then the Heckman sub-class is considered. The representative household is a price taker with access to accumulation technologies for both physical capital and human capital. The optimization problem is

$$\begin{aligned}
\text{maximize} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{mt} - n_{ht}) \\
\text{subject to} \quad & \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t + x_{ht} + (1+\tau_t^m)x_{mt} + x_{kt} - (1-\tau_t^n)w_t z_t - (1-\tau_t^k)r_t k_t - (1+\tau_t^c)T_t] \le b_0 \\
& k_{t+1} \le (1-\delta_k)k_t + x_{kt} \\
& h_{t+1} \le (1-\delta_h)h_t + G(x_{ht}, h_t, n_{ht}) \\
& z_t \le M(x_{mt}, h_t, n_{mt})
\end{aligned}$$

[1] Ladrón-de-Guevara et al. (1999) consider similar issues in a related model without taxation. They find that non-convexities may arise when raw leisure enters the utility function.

with initial conditions b_0, h_0, and k_0 given. This appears as (P.1) on page 97 of JMR (1997). Utility depends on consumption c_t and hours of leisure. The time endowment is normalized to unity. Hours of work are divided between the market, n_{mt}, and human capital formation, n_{ht}. The purchased good is used for investment as well as consumption: x_{ht} is investment used in the production of human capital, x_{kt} is investment in physical capital, and x_{mt} is investment used in the production of effective labor. Thus z_t is effective labor. Physical capital is k_t; human capital is h_t; exogenous lump sum transfers are T_t; the τs are tax rates; b_0 is initial holdings of government debt. Non-negativity conditions apply; however, it is convenient to allow the household to choose $x_{kt} < 0$ (physical capital may be sold), though this will never occur in equilibrium. Current value prices r_t and w_t, and present value prices p_t, are determined in equilibrium.

The depreciation rates δ_k and δ_h are given positive parameters. The production functions G and M are smooth with positive and diminishing marginal products. Both G and M are homogeneous of degree one in (x, h).

Clearly the physical capital constraint will bind. This can be used to substitute for x_{kt} in the budget constraint. Then the problem becomes

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{mt} - n_{ht}) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t + x_{ht} + (1+\tau_t^m)x_{mt} + k_{t+1} - (1-\delta_k)k_t \\
& \qquad - (1-\tau_t^n)w_t z_t - (1-\tau_t^k)r_t k_t - (1+\tau_t^c)T_t] \leq b_0 \\
& h_{t+1} \leq (1-\delta_h)h_t + G(x_{ht}, h_t, n_{ht}) \\
& z_t \leq M(x_{mt}, h_t, n_{mt}).
\end{aligned}$$

For $t \geq 0$ the coefficient of k_{t+1} in the budget constraint is

$$p_t - p_{t+1}[1 - \delta_k + (1 - \tau_{t+1}^k)r_{t+1}]. \tag{1}$$

If there is an interior optimum the household must face prices and taxes such that (1) equals zero for all $t \geq 0$. See (1.d) in JMR (1997). If (1) were positive, $k_{t+1} = 0$ would be optimal. If (1) were negative, $k_{t+1} \uparrow \infty$ would be optimal. It is worth emphasizing this arbitrage condition for physical capital because its companion condition for human capital will play a key role in the analysis. Since (1) equals zero, and since the effective labor constraint clearly binds, the household's problem becomes

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{mt} - n_{ht}) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t + x_{ht} + (1+\tau_t^m)x_{mt} - (1-\tau_t^n)w_t M(x_{mt}, h_t, n_{mt}) - (1+\tau_t^c)T_t] \\
& \qquad \leq b_0 + p_0 k_0[1 - \delta_k + (1-\tau_0^k)r_0] \\
& h_{t+1} \leq (1-\delta_h)h_t + G(x_{ht}, h_t, n_{ht}).
\end{aligned}$$

2.1 Heckman household technology — no interior optimum

At this point the household's production functions G and M are specialized as follows: $G(x_{ht}, h_t, n_{ht}) = \hat{G}(x_{ht}, n_{ht}h_t)$ with $\hat{G}$ homogeneous of degree one in its two arguments, and $M(x_{mt}, h_t, n_{mt}) = n_{mt}h_t$. See Heckman (1976, p. S13) and also JMR (1993, § III). Since x_{mt} is no longer relevant, the household's problem now becomes

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{mt} - n_{ht}) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t + x_{ht} - (1-\tau_t^n)w_t n_{mt} h_t - (1+\tau_t^c)T_t] \\
& \qquad \leq b_0 + p_0 k_0[1 - \delta_k + (1-\tau_0^k)r_0] \\
& h_{t+1} \leq (1-\delta_h)h_t + \hat{G}(x_{ht}, n_{ht}h_t).
\end{aligned}$$

The purpose of this section is to show that this problem does not have an interior solution. The method will be to assume an interior solution and derive a contradiction. The Lagrangian for the problem is

$$\begin{aligned}
\mathcal{L} \quad = \quad & \sum_{t=0}^{\infty} \Big\{ \beta^t u(c_t, 1 - n_{mt} - n_{ht}) - \lambda p_t[(1+\tau_t^c)c_t + x_{ht} - (1-\tau_t^n)w_t n_{mt} h_t] \\
& \qquad + \mu_t[(1-\delta_h)h_t + \hat{G}(x_{ht}, n_{ht}h_t) - h_{t+1}] \Big\}.
\end{aligned}$$

If the solution were interior the first order conditions for c_t, n_{mt}, n_{ht}, x_{ht}, and h_{t+1} respectively would be

$$\beta^t u_1(c_t, 1 - n_{mt} - n_{ht}) = \lambda p_t(1+\tau_t^c) \tag{2}$$

$$\beta^t u_2(c_t, 1 - n_{mt} - n_{ht}) = \lambda p_t(1-\tau_t^n)w_t h_t \tag{3}$$

$$\beta^t u_2(c_t, 1 - n_{mt} - n_{ht}) = \mu_t h_t \hat{G}_2(x_{ht}, n_{ht}h_t) \tag{4}$$

$$\lambda p_t = \mu_t \hat{G}_1(x_{ht}, n_{ht}h_t) \tag{5}$$

$$\lambda p_{t+1}(1-\tau_{t+1}^n)w_{t+1}n_{mt+1} = \mu_t - \mu_{t+1}[1 - \delta_h + n_{ht+1}\hat{G}_2(x_{ht+1}, n_{ht+1}h_{t+1})]. \tag{6}$$

Again, the goal is to use the first order conditions, and more generally use the assumption of interiority, to generate a contradiction. From (3), (4), and (5),

$$\hat{G}_2(x_{ht}, n_{ht}h_t)/\hat{G}_1(x_{ht}, n_{ht}h_t) = (1-\tau_t^n)w_t, \quad t \geq 0. \tag{7}$$

This looks very much like the first order condition for a cost minimization problem. Indeed, this is the case. Let $\ell_t = 1 - n_{mt} - n_{ht}$ denote hours of leisure. Then the household's problem can be written as

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t[(1+\tau_t^c)c_t + x_{ht} - (1-\tau_t^n)w_t h_t(1-\ell_t-n_{ht}) - (1+\tau_t^c)T_t] \\
& \qquad \leq b_0 + p_0 k_0[1-\delta_k + (1-\tau_0^k)r_0] \\
& h_{t+1} \leq (1-\delta_h)h_t + \hat{G}(x_{ht}, n_{ht}h_t) \\
& 1 - \ell_t - n_{ht} \geq 0.
\end{aligned}$$

Since h, x_h, and n_h do not appear in the utility function, they will be chosen to maximize income, i.e., to minimize $\sum_{t=0}^{\infty} p_t[x_{ht} - (1-\tau_t^n)w_t h_t(1-\ell_t-n_{ht})]$ subject to the human capital accumulation constraint and the non-negativity of market hours. In particular, for any $\{\ell_t, h_t\}_{t=0}^{\infty}$, the household will choose x_{ht} and n_{ht} to minimize $x_{ht} + (1-\tau_t^n)w_t h_t n_{ht}$ subject to these constraints. This is a standard cost minimization problem with an upper bound on one of the inputs. Under the maintained assumption that the solution is interior, it will satisfy (7). Since $\hat{G}$ is homogeneous of degree one, the minimized cost is proportional to "output:" $x_{ht} + (1-\tau_t^n)w_t h_t n_{ht} = [h_{t+1} - (1-\delta_h)h_t]\xi(1, (1-\tau_t^n)w_t)$ where ξ is the unit cost function for $\hat{G}(x_h, z_h)$.

Substitute the minimized cost back into the budget constraint. The household's problem becomes

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} p_t \Big\{ (1+\tau_t^c)c_t + [h_{t+1} - (1-\delta_h)h_t]\xi(1, (1-\tau_t^n)w_t) - (1-\tau_t^n)w_t h_t(1-\ell_t) - (1+\tau_t^c)T_t \Big\} \\
& \qquad \leq b_0 + p_0 k_0[1-\delta_k + (1-\tau_0^k)r_0].
\end{aligned}$$

Collect together the terms with h_t and re-write the budget constraint as follows:

$$\begin{aligned}
& p_0(1+\tau_0^c)(c_0 - T_0) + \sum_{t=1}^{\infty} \Big\{ p_t(1+\tau_t^c)(c_t - T_t) \\
& \qquad + h_t[p_{t-1}\xi(1, (1-\tau_{t-1}^n)w_{t-1}) - p_t(1-\delta_h)\xi(1, (1-\tau_t^n)w_t) - p_t(1-\tau_t^n)w_t(1-\ell_t)] \Big\} \\
& \qquad \leq b_0 + p_0 k_0[1-\delta_k + (1-\tau_0^k)r_0] + p_0 h_0[(1-\delta_h)\xi(1, (1-\tau_0^n)w_0) + (1-\tau_0^n)w_0(1-\ell_0)].
\end{aligned} \tag{8}$$

At an interior optimum, the coefficient of h_t in (8) must equal zero:[2]

$$p_{t-1}\xi(1,(1-\tau^n_{t-1})w_{t-1}) - p_t(1-\delta_h)\xi(1,(1-\tau^n_t)w_t) - p_t(1-\tau^n_t)w_t(1-\ell_t) = 0, \quad t \geq 1. \tag{9}$$

This states that the present value cost of a marginal unit of human capital ($p_{t-1}\xi_{t-1}$) equals its stock value next period ($p_t(1-\delta_h)\xi_t$) plus its flow return ($p_t(1-\tau^n_t)w_t(n_{mt}+n_{ht})$). The flow rate of return is equalized across market hours and training hours due to the static optimality conditions. But note that the marginal unit of human capital does not augment the value of leisure hours since human capital does not enter the utility function (cf. Ladrón-de-Guevara et al., 1999).

If the expression in (9) were strictly negative (positive), any increase (decrease) in h_t would provide more income for consumption, so optimality and interiority would be incompatible. This is much like the reasoning associated with (1) above. However, there is a key difference here. There is a choice variable in (9): ℓ_t. So this is not a standard arbitrage condition on prices.

Let the superscript zero denote the hypothesized interior optimum, e.g., ℓ^0_t. Consider a variation $\tilde{c}_s$, $\tilde{\ell}_s$, $\tilde{h}_s$ at a given $s \geq 1$.[3] Since the interior optimum satisfies (9), and since the variation must also satisfy the budget constraint (8), it must be that

$$p_s(1+\tau^c_s)c^0_s = p_s(1+\tau^c_s)\tilde{c}_s + \tilde{h}_s\Big[p_{s-1}\xi(1,(1-\tau^n_{s-1})w_{s-1}) - p_s(1-\delta_h)\xi(1,(1-\tau^n_s)w_s) - p_s(1-\tau^n_s)w_s(1-\tilde{\ell}_s)\Big].$$

In particular, if $\tilde{\ell}_s = \ell^0_s + \epsilon_\ell$, then

$$p_s(1+\tau^c_s)c^0_s = p_s(1+\tau^c_s)\tilde{c}_s + \tilde{h}_s p_s(1-\tau^n_s)w_s\epsilon_\ell \tag{10}$$

since (9) vanishes at the interior optimum. If also $\tilde{h}_s = h^0_s + \epsilon_h$ then (10) yields

$$\tilde{c}_s = c^0_s - (h^0_s+\epsilon_h)(1-\tau^n_s)w_s\epsilon_\ell/(1+\tau^c_s).$$

Hence $u(\tilde{c}_s,\tilde{\ell}_s) = u(c^0_s - (h^0_s+\epsilon_h)(1-\tau^n_s)w_s\epsilon_\ell/(1+\tau^c_s), \ell^0_s+\epsilon_\ell)$ and

$$\left.\frac{\partial u(\tilde{c}_s,\tilde{\ell}_s)}{\partial\epsilon_\ell}\right|_{\epsilon_\ell=0} = -(h^0_s+\epsilon_h)(1-\tau^n_s)w_s u_1(c^0_s,\ell^0_s)/(1+\tau^c_s) + u_2(c^0_s,\ell^0_s) = -\epsilon_h(1-\tau^n_s)w_s u_1(c^0_s,\ell^0_s)/(1+\tau^c_s) \tag{11}$$

where the last equality follows from the static interior optimum condition $u^0_2/u^0_1 = (1-\tau^n_s)w_s h^0_s/(1+\tau^c_s)$, implied by (2) and (3). From (11), for any $\epsilon_h < 0$ and for sufficiently small $\epsilon_\ell > 0$, the variation raises welfare. This is inconsistent with the optimality of the interior solution.[4] Thus, the assumption of an interior optimum leads to a contradiction.

[2]Lemma A.1 in the appendix confirms that the first order conditions (3)–(6) imply (9).

[3]I.e., for all $t \neq s$, consumption, leisure, and human capital remain c^0_t, ℓ^0_t, and h^0_t. Also, note that the variation $\tilde{h}_s$ induces variations $\tilde{x}_{hs-1}$, $\tilde{n}_{hs-1}$, $\tilde{x}_{hs}$, $\tilde{n}_{hs}$ through the cost minimization problem associated with (7). Similarly, there are induced variations in $\tilde{n}_{ms-1}$ and $\tilde{n}_{ms}$.

[4]Another possibility that could prevent the variation from raising utility is $w_s = 0$. But then the household would have no incentive to do any market work at s: $n^0_{ms} = 0$. Again, this would violate the interiority assumption.

3 Optimal taxation with Heckman utility

This section revisits the optimal taxation results in JMR (1997). JMR show that both capital and labor income should not be taxed in the limit. This result is derived from the assumption of an interior solution to the household's problem. But as shown in section 2.1 above, the household's optimum is not interior when household technology takes the Heckman (1976) form. Interiority may be restored, however, if the household not only has Heckman technology but also Heckman (1976) utility:

$$u(c_t, (1 - n_{mt} - n_{ht})h_t).$$

That is, utility is generated from effective leisure and not merely hours of leisure. With this change, it will be shown that the JMR result continues to hold: the steady state tax rate is zero for both capital income and labor income.

3.1 Household's problem

With the modified utility function, the first order conditions for an interior optimum become

$$\begin{aligned}
\beta^t u_1(c_t, (1 - n_{mt} - n_{ht})h_t) &= \lambda p_t(1 + \tau_t^c) && (2')\\
\beta^t u_2(c_t, (1 - n_{mt} - n_{ht})h_t) &= \lambda p_t(1 - \tau_t^n)w_t && (3')\\
\beta^t u_2(c_t, (1 - n_{mt} - n_{ht})h_t) &= \mu_t \hat{G}_2(x_{ht}, n_{ht}h_t) && (4')\\
\lambda p_t &= \mu_t \hat{G}_1(x_{ht}, n_{ht}h_t) && (5')\\
\lambda p_{t+1}(1 - \tau_{t+1}^n)w_{t+1}n_{mt+1} &= \mu_t - \mu_{t+1}[1 - \delta_h + n_{ht+1}\hat{G}_2(x_{ht+1}, n_{ht+1}h_{t+1})]\\
&\quad - \beta^{t+1}(1 - n_{mt+1} - n_{ht+1})u_2(c_{t+1}, (1 - n_{mt+1} - n_{ht+1})h_{t+1}). && (6')
\end{aligned}$$

The analysis of the household's problem proceeds as in section 2.1. The cost minimization problem associated with (7) still applies, and hence so does the budget constraint (8). With hours of leisure given by $\ell_t = 1 - n_{mt} - n_{ht}$, the household solves

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t h_t)\\
\text{s.t.} \quad & p_0(1 + \tau_0^c)(c_0 - T_0) + \sum_{t=1}^{\infty}\Big\{p_t(1 + \tau_t^c)(c_t - T_t)\\
& \quad + h_t[p_{t-1}\xi(1, (1 - \tau_{t-1}^n)w_{t-1}) - p_t(1 - \delta_h)\xi(1, (1 - \tau_t^n)w_t) - p_t(1 - \tau_t^n)w_t(1 - \ell_t)]\Big\}\\
& \quad \le b_0 + p_0 k_0[1 - \delta_k + (1 - \tau_0^k)r_0] + p_0 h_0[(1 - \delta_h)\xi(1, (1 - \tau_0^n)w_0) + (1 - \tau_0^n)w_0(1 - \ell_0)].
\end{aligned}$$

Unlike the analysis of (8) and (9) above, here the first order conditions do not cause the coefficient of h_t in this budget constraint to vanish. To see this, re-write the budget constraint but separate out the term with effective leisure:

$$\begin{aligned} p_0(1+\tau_0^c)(c_0-T_0) + \sum_{t=1}^{\infty}\Big\{ & p_t(1+\tau_t^c)(c_t-T_t) + p_t(1-\tau_t^n)w_t\ell_t h_t \\ & + h_t[p_{t-1}\xi(1,(1-\tau_{t-1}^n)w_{t-1}) - p_t(1-\delta_h)\xi(1,(1-\tau_t^n)w_t) - p_t(1-\tau_t^n)w_t]\Big\} \\ & \leq b_0 + p_0k_0[1-\delta_k+(1-\tau_0^k)r_0] + p_0h_0[(1-\delta_h)\xi(1,(1-\tau_0^n)w_0) + (1-\tau_0^n)w_0(1-\ell_0)]. \quad (12) \end{aligned}$$

At an interior optimum, the coefficient of h_t in the second line of (12) must equal zero:[5]

$$p_{t-1}\xi(1,(1-\tau_{t-1}^n)w_{t-1}) - p_t(1-\delta_h)\xi(1,(1-\tau_t^n)w_t) - p_t(1-\tau_t^n)w_t = 0, \quad t \geq 1. \tag{13}$$

This is based on a standard arbitrage argument. E.g., if the expression in (13) were negative, any increase in h_t would provide more income for consumption. And this increase in h_t could be matched with a reduction in ℓ_t to leave effective leisure unchanged — both in the first line of (12) and also in the utility function. The key distinction between (13) and the more problematic (9) is the presence of the choice variable ℓ_t in the latter. By contrast, in (13) the flow return to a marginal unit of human capital, $p_t(1-\tau_t^n)w_t$, applies to all hours, even leisure, due to the Heckman specification for preferences.

Given (13), the household's problem is (after re-arranging)

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty}\beta^t u(c_t, \ell_t h_t) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty}\Big[p_t(1+\tau_t^c)(c_t-T_t) + p_t(1-\tau_t^n)w_t\ell_t h_t\Big] \\ & \qquad \leq b_0 + p_0k_0[1-\delta_k+(1-\tau_0^k)r_0] + p_0h_0[(1-\delta_h)\xi(1,(1-\tau_0^n)w_0) + (1-\tau_0^n)w_0]. \end{aligned}$$

This is a completely standard dynamic utility maximization problem with choice variables c_t and $z_{\ell t} := \ell_t h_t$. The first order conditions are (2′) and (3′).

In summary, for $t \geq 0$ the conditions for a household optimum are as follows:

$$p_t/p_{t+1} \;=\; 1-\delta_k+(1-\tau_{t+1}^k)r_{t+1} \tag{1′}$$

$$\hat{G}_2(x_{ht}, n_{ht}h_t)/\hat{G}_1(x_{ht}, n_{ht}h_t) \;=\; (1-\tau_t^n)w_t \tag{7}$$

$$p_t\xi(1,(1-\tau_t^n)w_t) - p_{t+1}(1-\delta_h)\xi(1,(1-\tau_{t+1}^n)w_{t+1}) - p_{t+1}(1-\tau_{t+1}^n)w_{t+1} \;=\; 0 \tag{13}$$

[5]Lemma A.2 in the appendix confirms that the first order conditions (3′)–(6′) imply (13).

$$\beta^t u_1(c_t, \ell_t h_t) = \lambda p_t (1+\tau_t^c) \tag{2'}$$

$$\beta^t u_2(c_t, \ell_t h_t) = \lambda p_t (1-\tau_t^n) w_t \tag{3'}$$

$$k_{t+1} = (1-\delta_k)k_t + x_{kt} \tag{14}$$

$$h_{t+1} = (1-\delta_h)h_t + \hat{G}(x_{ht}, n_{ht}h_t) \tag{15}$$

$$1 = \ell_t + n_{mt} + n_{ht} \tag{16}$$

$$\sum_{t=0}^{\infty} \Big[p_t(1+\tau_t^c)(c_t - T_t) + p_t(1-\tau_t^n) w_t \ell_t h_t \Big]$$
$$= b_0 + p_0 k_0 [1 - \delta_k + (1-\tau_0^k) r_0] + p_0 h_0 [(1-\delta_h)\xi(1, (1-\tau_0^n)w_0) + (1-\tau_0^n)w_0]. \tag{17}$$

3.2 Equilibrium

The optimal tax problem is to select the equilibrium that gives the greatest utility to the household. In addition to the household's optimality conditions, equilibrium is characterized by the firms' optimality conditions and by goods market clearing (details below).

It is convenient to express the equilibrium entirely in terms of primal variables — quantities rather than prices. Thus some of the household's optimality conditions may be regarded as definitions of after tax prices, for a given allocation. Specifically, (7) defines the after tax wage, then (13) defines the price ratio p_t/p_{t+1}:

$$\frac{p_t}{p_{t+1}} = (1-\delta_h) \frac{\xi(1, (1-\tau_{t+1}^n) w_{t+1})}{\xi(1, (1-\tau_t^n) w_t)} + \frac{\hat{G}_2(x_{ht+1}, n_{ht+1}h_{t+1}) / \hat{G}_1(x_{ht+1}, n_{ht+1}h_{t+1})}{\xi(1, (1-\tau_t^n) w_t)}. \tag{18}$$

Since ξ is the unit cost function for $\hat{G}$, the cost minimizers satisfy

$$\begin{aligned} \hat{G}(x_{ht}, n_{ht}h_t)\xi(1, (1-\tau_t^n)w_t) &= x_{ht} + n_{ht}h_t(1-\tau_t^n)w_t \\ &= x_{ht} + n_{ht}h_t \frac{\hat{G}_2(x_{ht}, n_{ht}h_t)}{\hat{G}_1(x_{ht}, n_{ht}h_t)} \\ &= \frac{\hat{G}(x_{ht}, n_{ht}h_t)}{\hat{G}_1(x_{ht}, n_{ht}h_t)} \end{aligned}$$

where the last line follows from homogeneity. Therefore $\xi(1, (1-\tau_t^n)w_t) = 1/\hat{G}_1(x_{ht}, n_{ht}h_t)$ and (18) yields

$$\frac{p_t}{p_{t+1}} = \frac{\hat{G}_1(x_{ht}, n_{ht}h_t)}{\hat{G}_1(x_{ht+1}, n_{ht+1}h_{t+1})} \Big(1 - \delta_h + \hat{G}_2(x_{ht+1}, n_{ht+1}h_{t+1}) \Big), \quad t \geq 0. \tag{19}$$

With this result, (1′) then defines the after tax interest rate for $t \geq 1$. Note that p_0 and $(1-\tau_0^k)r_0$ are not restricted. Also, (2′) at $t \geq 1$ divided by (2′) at time 0 yields τ_t^c, but τ_0^c is unrestricted at this point.

(Footnote 6 will identify τ_0^c.) Equation (3′) presents a real restriction since the after tax wage and the present value price p_t have already been determined. From (3′) and (7),

$$\begin{aligned}\frac{u_2(c_t,\ell_t h_t)}{\beta u_2(c_{t+1},\ell_{t+1}h_{t+1})} &= \frac{\hat{G}_2(x_{ht},n_{ht}h_t)/\hat{G}_1(x_{ht},n_{ht}h_t)}{\hat{G}_2(x_{ht+1},n_{ht+1}h_{t+1})/\hat{G}_1(x_{ht+1},n_{ht+1}h_{t+1})}\cdot\frac{p_t}{p_{t+1}}\\ &= \frac{\hat{G}_2(x_{ht},n_{ht}h_t)}{\hat{G}_2(x_{ht+1},n_{ht+1}h_{t+1})}\Big(1-\delta_h+\hat{G}_2(x_{ht+1},n_{ht+1}h_{t+1})\Big),\quad t\geq 0 \qquad (20)\end{aligned}$$

where the second line follows from (19).[6] This is analogous to the restriction $\phi(v_{t-1},v_t)=0$ in problem (P.2) on page 100 of JMR (1997).

In order to express the budget constraint (17) in primal form, multiply through by the Lagrange multiplier λ and use (2′) and (3′) to get

$$\begin{aligned}&\sum_{t=0}^{\infty}\beta^t\Big[(c_t-T_t)u_1(c_t,\ell_t h_t)+\ell_t h_t u_2(c_t,\ell_t h_t)\Big]\\ &= \lambda b_0+\lambda p_0 k_0[1-\delta_k+(1-\tau_0^k)r_0]+\lambda p_0 h_0[(1-\delta_h)\xi(1,(1-\tau_0^n)w_0)+(1-\tau_0^n)w_0]\\ &= \frac{u_2(0)\hat{G}_1(0)}{\hat{G}_2(0)}\Big\{b_0/p_0+k_0[1-\delta_k+(1-\tau_0^k)r_0]+h_0[(1-\delta_h)+\hat{G}_2(0)]/\hat{G}_1(0)\Big\} \qquad (21)\end{aligned}$$

where the last line uses (3′) and (7) at $t=0$ to substitute for λ, and also uses previous results to substitute for terms with $(1-\tau_0^n)w_0$.

The other equilibrium conditions are that firms are price taking profit maximizers and that the goods market clears. The production function F for the purchased good is homogeneous of degree one in physical capital and effective labor. Thus profits will be zero. For $t\geq 0$,

$$F_1(k_t,n_{mt}h_t) = r_t \qquad (22)$$

$$F_2(k_t,n_{mt}h_t) = w_t \qquad (23)$$

$$F(k_t,n_{mt}h_t) = c_t+k_{t+1}-(1-\delta_k)k_t+x_{ht}+g_t. \qquad (24)$$

In (24), government purchases, g_t, are exogenously given. Also, (24) subsumes the physical capital accumulation equation (14). Equations (22) and (23) do not impose constraints on the primal form of the optimal tax problem. They can be used to define the before tax interest rate and wage.

Thus, in terms of primal variables, equilibrium is characterized by (15), (16), (20), (21), and (24). In the last line of (21), p_0 and τ_0^k are unrestricted while r_0 is defined by (22). By Walras' law, when these conditions are satisfied the government's infinite horizon present value budget constraint is redundant.

[6] It appears (3′) also presents a restriction at time 0. From (7), (2′), and (3′) at $t=0$, $u_2(c_0,\ell_0h_0)/u_1(c_0,\ell_0h_0)=(1+\tau_0^c)^{-1}\hat{G}_2(x_{h0},n_{h0}h_0)/\hat{G}_1(x_{h0},n_{h0}h_0)$. However, this is not a restriction. Rather, it can be used to define τ_0^c since (2′) left τ_0^c available.

3.3 Optimal tax problem

The tax problem is not particularly interesting if the government can confiscate initial wealth. Thus it is customary to assign reasonable exogenous values to p_0 and τ_0^k in (21). Let $z_{mt} := n_{mt}h_t$ and $z_{ht} := n_{ht}h_t$ denote effective labor for the market and for human capital accumulation. The tax problem is as follows:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, h_t - z_{mt} - z_{ht}) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} \beta^t \Big[(c_t - T_t)u_1(c_t, h_t - z_{mt} - z_{ht}) + (h_t - z_{mt} - z_{ht})u_2(c_t, h_t - z_{mt} - z_{ht})\Big] = A_0 \\
& h_{t+1} = (1-\delta_h)h_t + \hat{G}(x_{ht}, z_{ht}) \\
& F(k_t, z_{mt}) = c_t + k_{t+1} - (1-\delta_k)k_t + x_{ht} + g_t \\
& \frac{u_2(c_t, h_t - z_{mt} - z_{ht})}{\beta u_2(c_{t+1}, h_{t+1} - z_{mt+1} - z_{ht+1})} = \frac{\hat{G}_2(x_{ht}, z_{ht})}{\hat{G}_2(x_{ht+1}, z_{ht+1})}\Big(1 - \delta_h + \hat{G}_2(x_{ht+1}, z_{ht+1})\Big)
\end{aligned}$$

where A_0 in the first constraint denotes the last line of (21). Let θ be the Lagrange multiplier for the first constraint. As in JMR (1997), introduce the auxiliary function

$$\begin{aligned}
& W(c_t, h_t - z_{mt} - z_{ht}, T_t, \theta) := \\
& \quad u(c_t, h_t - z_{mt} - z_{ht}) + \theta\Big[(c_t - T_t)u_1(c_t, h_t - z_{mt} - z_{ht}) + (h_t - z_{mt} - z_{ht})u_2(c_t, h_t - z_{mt} - z_{ht})\Big].
\end{aligned}$$

Then the Lagrangian for the problem is

$$\begin{aligned}
\mathcal{L} \;=\; -\theta A_0 + \sum_{t=0}^{\infty} \beta^t \Bigg\{ & W(c_t, h_t - z_{mt} - z_{ht}, T_t, \theta) + \nu_{1t}[(1-\delta_h)h_t + \hat{G}(x_{ht}, z_{ht}) - h_{t+1}] \\
& + \nu_{2t}[F(k_t, z_{mt}) - c_t - k_{t+1} + (1-\delta_k)k_t - x_{ht}] \\
& + \nu_{3t}\left[\frac{u_2(c_t, h_t - z_{mt} - z_{ht})}{u_2(c_{t+1}, h_{t+1} - z_{mt+1} - z_{ht+1})} - \frac{\beta(1-\delta_h)\hat{G}_2(x_{ht}, z_{ht})}{\hat{G}_2(x_{ht+1}, z_{ht+1})} - \beta\hat{G}_2(x_{ht}, z_{ht})\right]\Bigg\}.
\end{aligned}$$

Suppose the optimal tax equilibrium converges to an interior steady state in which all variables have finite limits, including Lagrange multipliers. Let asterisks denote steady state values. Then in the limit the first order conditions for c_t, h_t, z_{mt}, z_{ht}, x_{ht}, and k_t yield

$$W_1^* - \nu_2^* + (1-\beta^{-1})\nu_3^* u_{21}^*/u_2^* = 0 \tag{25}$$

$$W_2^* + (1-\delta_h)\nu_1^* - \beta^{-1}\nu_1^* + (1-\beta^{-1})\nu_3^* u_{22}^*/u_2^* = 0 \tag{26}$$

$$-W_2^* + \nu_2^* F_2^* - (1-\beta^{-1})\nu_3^* u_{22}^*/u_2^* = 0 \tag{27}$$

$$-W_2^* + \nu_1^*\hat{G}_2^* - (1-\beta^{-1})\nu_3^*\big(u_{22}^*/u_2^* + \beta(1-\delta_h)\hat{G}_{22}^*/\hat{G}_2^*\big) - \beta\nu_3^*\hat{G}_{22}^* = 0 \tag{28}$$

$$\nu_1^*\hat{G}_1^* - \nu_2^* - (1-\beta^{-1})\nu_3^*\beta(1-\delta_h)\hat{G}_{21}^*/\hat{G}_2^* - \beta\nu_3^*\hat{G}_{21}^* = 0 \quad (29)$$

$$\nu_2^*(F_1^* + 1 - \delta_k) - \beta^{-1}\nu_2^* = 0. \quad (30)$$

The main result may now be stated.

Theorem *If $\nu_2^* \neq 0$ then the steady state tax rate is zero for both capital income and labor income.*

Proof The Chamley–Judd result that $\tau^{k*} = 0$ is straightforward. If $\nu_2^* \neq 0$, (30) yields $F_1^* + 1 - \delta_k = \beta^{-1}$. From (19) and (20), p_t/p_{t+1} converges to β^{-1}. Hence (1′) yields $1 - \delta_k + (1-\tau^{k*})r^* = \beta^{-1}$. Since $r = F_1$ from (22), the result follows.

Next it will be shown that $\nu_3^* = 0$. This will then be used to prove $\tau^{n*} = 0$. Add (26) and (28) to get

$$\nu_1^*(1 - \delta_h + \hat{G}_2^* - \beta^{-1}) + \beta\nu_3^*\hat{G}_{22}^*[(\beta^{-1}-1)(1-\delta_h)/\hat{G}_2^* - 1] = 0.$$

In steady state, (20) yields $\beta^{-1} = 1 - \delta_h + \hat{G}_2^*$. Hence $\beta\nu_3^*\hat{G}_{22}^*[(\beta^{-1}-1)(1-\delta_h)/\hat{G}_2^* - 1] = 0$. Since $\delta_h \neq 0$, it follows that $\nu_3^* = 0$.

Finally it will be shown that $\tau^{n*} = 0$. With $\nu_3^* = 0$ and $\nu_2^* \neq 0$, (29) implies $\nu_1^* \neq 0$. Thus (28) and (29) yield $\hat{G}_2^*/\hat{G}_1^* = W_2^*/\nu_2^*$, while (27) yields $W_2^*/\nu_2^* = F_2^*$. Hence $\hat{G}_2^*/\hat{G}_1^* = F_2^*$. And from (7) and (23), this is precisely the condition under which $\tau^{n*} = 0$. ■

4 Conclusion

For the model in section 2.1, the household's problem fails to have an interior optimum. Thus, this model is not well suited for the analysis of optimal taxation. With the modification to utility in section 3, interiority is restored and the optimal tax problem yields the JMR (1997) result: neither labor income nor capital income should be taxed in steady state.

The intuition for the Judd–Chamley zero capital tax result is based on the idea that a tax on capital income is a tax on future consumption. This creates a distortionary wedge between the intertemporal marginal rate of substitution for consumption and the corresponding marginal rate of transformation. This wedge grows exponentially through time — just like compound interest (Judd, 2002). Hence even a small capital income tax can generate large distortions if it is left in place a long time.

When human capital is present, similar reasoning applies. The accumulation technology for human capital creates an additional channel through which taxation can potentially cause explosive intertemporal distortions. Hence, at an optimal steady state, the intertemporal marginal rate of substitution for effective leisure will equal the corresponding marginal rate of transformation. In terms of the model, this implies that constraint (20) will automatically be satisfied at an optimal steady state. Since this constraint does not bind, in the limit the economy behaves as if there were no restrictions on the tax instruments.[7] Thus

[7] Equation (20) is an instrument restriction. It imposes the same tax rates on both the consumer and the human capital producer.

the Diamond–Mirrlees (1971) production efficiency theorem applies: Marginal rates of transformation must be equal in the two production sectors, human capital and physical capital/consumption. Inspection of (7) and (23) reveals that this efficiency condition yields a zero tax rate on labor income. In the limit, this tax must vanish; otherwise the two production sectors would face different relative prices.

Finally, given the problems with non-interiority that were considered here, one is led to wonder if other human capital models may lead to similar difficulties.[8] It seems that caution is warranted in these situations.

[8] Ortigueira and Santos (2002) use a continuous time model with human capital and find that equilibrium is characterized by corners for some parameter values and tax rates.

Appendix

A.1 Lemma *If* (3)–(6) *are satisfied for all* $t \geq 0$ *with* $\lambda \neq 0$ *then* (9) *must hold for all* $t \geq 1$.

Proof Use (3) and (4) to substitute for $\mu_{t+1}\hat{G}_2(t+1)$ in (6):

$$\lambda p_{t+1}(1-\tau^n_{t+1})w_{t+1}n_{mt+1} = \mu_t - \mu_{t+1}(1-\delta_h) - \lambda p_{t+1}(1-\tau^n_{t+1})w_{t+1}n_{ht+1}, \quad t \geq 0.$$

Therefore, since $\ell_{t+1} + n_{mt+1} + n_{ht+1} = 1$,

$$0 = \mu_t - \mu_{t+1}(1-\delta_h) - \lambda p_{t+1}(1-\tau^n_{t+1})w_{t+1}(1-\ell_{t+1}), \quad t \geq 0$$

or

$$0 = \mu_{t-1} - \mu_t(1-\delta_h) - \lambda p_t(1-\tau^n_t)w_t(1-\ell_t), \quad t \geq 1.$$

A comparison with (9) shows that the lemma will be proved if $\mu_t/\lambda = p_t\xi(1,(1-\tau^n_t)w_t)$ for all $t \geq 0$. Recall that ξ was defined to be the unit cost function for the production function $\hat{G}$. Therefore,

$$\xi(1,(1-\tau^n_t)w_t)\hat{G}(x_{ht}, n_{ht}h_t) = x_{ht} + (1-\tau^n_t)w_t n_{ht}h_t, \quad t \geq 0 \tag{31}$$

at an interior optimum. Multiply both sides of (4) by n_{ht}, and both sides of (5) by x_{ht}, then add:

$$n_{ht}\lambda p_t(1-\tau^n_t)w_t h_t + x_{ht}\lambda p_t = \mu_t \hat{G}(x_{ht}, n_{ht}h_t), \quad t \geq 0$$

by (3) and homogeneity of $\hat{G}$. Compare this with (31) to get $\lambda p_t \xi(1,(1-\tau^n_t)w_t) = \mu_t$ as required. ■

A.2 Lemma *If* (3′)–(6′) *are satisfied for all* $t \geq 0$ *with* $\lambda \neq 0$ *then* (13) *must hold for all* $t \geq 1$.

Proof Use (4′) to substitute for $\mu_{t+1}\hat{G}_2(t+1)$ in (6′), then simplify to get

$$\begin{aligned}\lambda p_{t+1}(1-\tau^n_{t+1})w_{t+1}n_{mt+1} &= \mu_t - \mu_{t+1}(1-\delta_h) \\ &\quad - \beta^{t+1}(1-n_{mt+1})u_2(c_{t+1},(1-n_{mt+1}-n_{ht+1})h_{t+1}), \quad t \geq 0.\end{aligned}$$

Next, use (3′) to substitute for $\beta^{t+1}u_2(t+1)$, then simplify to get

$$0 = \mu_t - \mu_{t+1}(1-\delta_h) - \lambda p_{t+1}(1-\tau^n_{t+1})w_{t+1}, \quad t \geq 0$$

or

$$0 = \mu_{t-1} - \mu_t(1-\delta_h) - \lambda p_t(1-\tau^n_t)w_t, \quad t \geq 1.$$

A comparison with (13) shows that the lemma will be proved if $\mu_t/\lambda = p_t\xi(1,(1-\tau^n_t)w_t)$ for all $t \geq 0$. Recall that ξ was defined to be the unit cost function for the production function $\hat{G}$. Therefore,

$$\xi(1,(1-\tau^n_t)w_t)\hat{G}(x_{ht}, n_{ht}h_t) = x_{ht} + (1-\tau^n_t)w_t n_{ht}h_t, \quad t \geq 0 \tag{32}$$

at an interior optimum. Multiply both sides of (4′) by $n_{ht}h_t$, and both sides of (5′) by x_{ht}, then add:

$$n_{ht}h_t\lambda p_t(1-\tau^n_t)w_t + x_{ht}\lambda p_t = \mu_t \hat{G}(x_{ht}, n_{ht}h_t), \quad t \geq 0$$

by (3′) and homogeneity of $\hat{G}$. Compare this with (32) to get $\lambda p_t \xi(1,(1-\tau^n_t)w_t) = \mu_t$ as required. ■

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