RECURSIVE GLOBAL GAMES *

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Abstract. The present paper contributes to literature of dynamic games with strategic complementarities, in two interrelated ways. First, it identifies a class of fully dynamic games in which, under the assumption of complete information, contemporaneous and intertemporal complementarities and multiple equilibria can be fruitfully analyzed. Second, it extends the analysis to an incomplete information framework, where results from the literature on global games can be applied to identify a unique equilibrium.

1. Introduction

Over the last decade, it has been widely recognized that complementarities may be at the heart of a satisfactory explanation of why there are large shifts and fluctuations in economic activity. A variety of specific channels, such as search externalities, thick market externalities and increasing returns to scale in specific activities or sectors, yield a tendency to cluster and agglomerate. Models with multiple equilibria have been offered as descriptions of how such shifts may come about, arguing e.g. that when an economy slides into recession, it is nothing but a transition to a low activity equilibrium. While these types of explanations clearly have merit, they leave important issues unexplained. In particular, they are silent about the important question of how these equilibria are reached, and why there are shifts between them.[1] Partly as a response to this type of reservations about multiple equilibrium explanations, recent years have seen an increasing interest in equilibrium selection in games with strategic complementarities. Building on insights from Carlsson and van Damme (1993), a fast-growing literature has evolved in which several coordination games have been analyzed within the so-called global games framework.[2] The powerful results of the global games literature has allowed applied modelers to pin down equilibria in games that notoriously have very rich equilibrium sets, and extending the success to dynamic settings is now an active research field.

The present paper contributes to literature of dynamic games with strategic complementarities, in two interrelated ways. First, it identifies a class of fully dynamic games in

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[1] An exponent of this critique is Durlauf (1991) who states succinctly (but harshly) that 'Most of these models describe multiple steady states in economies rather than multiple non-degenerate time-series paths, and consequently cannot address issues of aggregate fluctuations. Further, this literature has not shown how economies can shift across equilibria, inducing periods of boom and depression.'

[2] See Morris and Shin (2002) for a survey of the literature and its relation to other strands of research.

which, under the assumption of complete information, contemporaneous and intertemporal complementarities and multiple equilibria can be fruitfully analyzed. Second, it extends the analysis to an incomplete information framework, where results from the literature on global games can be applied to identify a unique equilibrium. The analysis is carried out based on a model in which a continuum of players choose actions simultaneously over a finite number of periods. We show that under certain assumptions on payoff functions and the evolution of the stochastic variable, expected (current and future) payoffs are characterized by strategic complementarities. This approach allows us invoke the technology of the static global games literature to each stage of the dynamic game. It has the advantage that the relation to the static global games methodology become very transparent.

The class of games we analyze are characterized by two distinct types of complementarities, contemporaneous and intertemporal. Higher actions of other players make higher action for the remaining player more desirable. Furthermore, intertemporal complementarities are achieved by the introduction of state variables, one characterizing the players idiosyncratically and one characterizing a quantity common to all players. Assuming complementarities between these state variables and assuming that the transitions of these are monotone in players' actions and aggregate play, we show that these intertemporal links reinforce the contemporaneous complementarities. It is worth noting that while most models of dynamic complementarities circumvent the dynamic nature of the problem by assuming myopic behavior, we explicitly deal with the problem of how actions at both the individual and aggregate level influence the future evolution of aggregate and individual play. A notable feature of the model is that it can capture two polar cases of strategic complementarities (as well as hybrid cases in between). At one extreme is an economy in which complementarities are direct and contemporaneous, for e.g. through a product market, but where there are no intertemporal links. At the other extreme, firms do not interact directly within any period, and complementarities are purely intertemporal, working through lags of state variables. These two types of complementarities are interesting because they can produce multiple equilibria. Furthermore, contemporaneous complementarities can magnify shocks to the economy while intertemporal complementarities can propagate them. Specifically, within an infinite horizon version of the model, we show a momentum theorem which implies that at under certain conditions, growth in the economy can be self-sustaining, making future growth more likely. After an unexpectedly low realization of the economic shocks, this momentum is broken, and the economy may shift to a path with self-enforcing decline. Thus the model predicts paths of the economy where growth is followed by decline and so on, even when shocks to the underlying economy are small in magnitude.

As noted above, strategic complementarities can create multiple equilibria, even in static settings. Not surprisingly, dynamics do nothing but add to this multiplicity. If one assumes, as we do in our model, that a player's action in any given period influences this player's future prospects in a non-trivial way, multiplicity of equilibria may be increased because optimal play in a given period will depend on future play. Thus multiplicity in future stages of the game may create multiplicity now if there is not consensus about which equilibrium will be played in the future. To address this problem, we extend results from the literature on global games and find conditions under which there is a unique Markov perfect Bayesian equilibrium. We do this by employing a specific, but reasonable, information technology. In particular, we assume that common knowledge of fundamentals occurs only with a lag, and that in each period there is asymmetric information on the current realization.[3]

Our paper relates closely to two strands of literature. First, the basic model of complete information in which complementarities are analyzed shares features with an existing literature on supermodular games. Most closely related is work by Curtat (1996) which considers stochastic games with strategic complementarities. In his setup, there are a finite set of players who can influence the evolution of a state variable probabilistically. In contrast, we will work within a continuum player setup, and thus our result do not follow from his analysis. Moreover, our analysis has similarities with the work of Milgrom, Qian and Roberts (1991). They consider an up-stream down-stream model in which knowledge in the two industries is accumulated over time, and where actions in each period are complementary with the state of knowledge. They prove a momentum theorem which shows that when knowledge accumulation is increasing in actions, and actions are complementary with the state of knowledge, the system may enter a self-enforcing dynamic path where actions and knowledge increase over time.

Second, the analysis of the model of incomplete information is directly related to the literature of dynamic global games. Four somewhat different approaches to dynamic 'global' equilibrium selection have previously been developed. The first, exemplified by the work of Burdzy, Frankel and Pauzner (2001) is to consider players that are randomly matched over a sequence of periods to play a 2×2 coordination game, but where the payoff matrix is parametrized by a stochastically evolving economic fundamental. The second approach, exemplified by Frankel and Pauzner (2000), is to consider continuous time games in which the stochastic economic fundamental is governed by a Brownian motion and where players receive revision opportunities according to a Poisson process. The third approach is that of Levin (2001) and Oyama (2001). These authors consider players who only live for a single period, but whose payoffs may depend on actions of players living before or after them. The fourth approach is that of Toxvaerd (2002) who considers a timing game in which players decide when to move, and where players at each point in time trade off current payoffs with expected future payoffs. Although formally not a supermodular game, the recursive payoff structure allows for the static global games techniques to be employed at each stage of the game, and thus the full dynamic game can be analyzed as a sequence of (interrelated) static games. The present analysis is most closely related to this fourth approach.

Throughout the paper, we have based the analysis on first principles, in order to present the results in a self-contained and accessible way. For this reason, Section 2 provides all the definitions and relevant results from lattice programming which we will be using. In most cases, these can be generalized to sets other than $\mathbb{R}^n$ and we refer the interested reader to Topkis (1998) for a thorough exposition. Section 3 provides the setup of the class of models that can be studied with our approach and analysis, under the assumption of complete information. Section 4 analyzes the incomplete information case. Finally, Section 5 offers some concluding remarks.

[3]While this assumption effectively sidesteps issues of learning from the observation of past behavior, learning is not within the scope of this paper.

2. Mathematical Preliminaries

In this first section, we present definitions and known results that will be used throughout the rest of the paper. For more general definitions and results see Topkis (1998).

Throughout the paper, we endow $\mathbb{R}^n$ with the element-wise order, i.e. for $x^1 = \left(x_1^1, ..., x_n^1\right)$ and $x^2 = \left(x_1^2, ..., x_n^2\right)$ we write $x^1 \leq x^2$ if $x_i^1 \leq x_i^2$ for all $i = 1, ..., n$. Let

$$\begin{aligned} x^1 \wedge x^2 &= (\min(x_1^1, x_1^2), ..., \min(x_n^1, x_n^2)) \\ x^1 \vee x^2 &= (\max(x_1^1, x_1^2), ..., \max(x_n^1, x_n^2)) \end{aligned}$$

Definition 1. *A set $X \subseteq \mathbb{R}^n$ is called a **sublattice of** $\mathbb{R}^n$, if $x^1 \wedge x^2 \in X$ and $x^1 \vee x^2 \in X$ for all $x^1, x^2 \in X$.*

Consider a vector $x = (x_1, ..., x_n) \in X$. Let $\widehat{x} = (x_{m_1}, ..., x_{m_r})$, where $\{m_1, ..., m_r\} \subseteq \{1, ..., n\}$ and $x^c = (x_{n_1}, ..., x_{n_q})$, where $\{n_1, ..., n_q\} = \{1, ..., n\} \backslash \{m_1, ..., m_r\}$. I.e., the notation $\widehat{x}$ is used to denote a tuple of x constructed by picking out specific coordinates of interest from x. With this notation and by treating the rest of the coordinates of x as constant (fixed), we can write $x = (\widehat{x}; x^c)$. The following gives the definition of supermodularity of a function in all its arguments, or specific tuples of its arguments.

Definition 2. *Let $X \subseteq \mathbb{R}^n$ be a sublattice of $\mathbb{R}^n$.*

(a) *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **supermodular on** X, if for all $x^1, x^2 \in X$*

$$f(x^1) + f(x^2) \leq f(x^1 \wedge x^2) + f(x^1 \vee x^2)$$

(b) *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **supermodular in** $\widehat{x}$, if for all $x^1 = (\widehat{x}^1; x^c)$ and $x^2 = (\widehat{x}^2; x^c)$*

$$f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) \leq f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c)$$

for fixed x^c.

(c) *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **strictly supermodular on** X, if for all unordered $x^1, x^2 \in X$*

$$f(x^1) + f(x^2) < f(x^1 \wedge x^2) + f(x^1 \vee x^2)$$

(d) *A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **strictly supermodular in** $\widehat{x}$, if for all $x^1 = (\widehat{x}^1; x^c)$ and $x^2 = (\widehat{x}^2; x^c)$, such that $\widehat{x}^1$and $\widehat{x}^2$ are unordered*

$$f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) < f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c)$$

for fixed x^c.

Given these definitions, it follows trivially that a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is (strictly) supermodular on X, if it is (strictly) supermodular in $x \in X$. Also trivially, if f is strictly supermodular then it is supermodular as well. Furthermore, if f and g are (strictly) supermodular and $\delta > 0$, then $f + \delta g$ is (strictly) supermodular. Moreover, if f is (strictly) supermodular, then taking the max of f over one of its arguments yields a (strictly) supermodular function. Next, the limit of a sequence of supermodular functions (if it exists) is also supermodular.[4] Finally, the following holds:

Lemma 3. *Let $X \subseteq \mathbb{R}^n$ be a sublattice of $\mathbb{R}^n$. If $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly supermodular in $\widehat{x}$ and $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is supermodular in $\widehat{x}$, then the sum $f + g$ is strictly supermodular in $\widehat{x}$.*

Proof. See Appendix A. ■

Next we give the definition of increasing differences and a result that relates it to supermodularity.

Definition 4. *Let $X \subseteq \mathbb{R}^n$ be a sublattice of $\mathbb{R}^n$.*

(a) *Consider a vector $x = (x_1, ..., x_n)$. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has **increasing differences in** (x_i, x_j) if for all $x_i' \leq x_i''$ and $x_j' \leq x_j''$*

$$f((x_1, ..., x_i'', ..., x_j'', ..., x_n)) - f((x_1, ..., x_i'', ..., x_j', ..., x_n) \geq$$
$$f((x_1, ..., x_i', ..., x_j'', ..., x_n)) - f((x_1, ..., x_i', ..., x_j', ..., x_n))$$

(b) *Consider a vector $x = (x_1, ..., x_n)$. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has **increasing differences on** X, if it has increasing differences in (x_i, x_j) for all $i \neq j$, $i, j = 1, ..., n$.*

(c) *Consider a vector $x = (x_1, ..., x_n)$. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has **strictly increasing differences in** (x_i, x_j) if for all $x_i' < x_i''$ and $x_j' < x_j''$*

$$f((x_1, ..., x_i'', ..., x_j'', ..., x_n)) - f((x_1, ..., x_i'', ..., x_j', ..., x_n) >$$
$$f((x_1, ..., x_i', ..., x_j'', ..., x_n)) - f((x_1, ..., x_i', ..., x_j', ..., x_n))$$

(d) *Consider a vector $x = (x_1, ..., x_n)$. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has **strictly increasing differences on** X, if it has strictly increasing differences in (x_i, x_j) for all $i \neq j$, $i, j = 1, ..., n$.*

Proposition 5. *Let $X \subseteq \mathbb{R}^n$ be a sublattice of $\mathbb{R}^n$. If a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is (strictly) supermodular in $\widehat{x} = (x_{m_1}, ..., x_{m_r})$ then it has (strictly) increasing differences in (x_{m_i}, x_{m_j}), for all $m_i \neq m_j$, $m_i, m_j \in \{m_1, ..., m_r\} \subseteq \{1, ..., n\}$.*

Proof. See Appendix B. ■

Finally, let $\{F(x; \theta),\ \theta \in \mathbb{R}\}$ be a family of distribution functions on $\mathbb{R}^n$, parametrized by θ and let $\int_S dF(x; \theta)$ be the probability measure of $S \subseteq \mathbb{R}^n$ with respect to the distribution $F(x; \theta)$. The following introduces the concept of stochastically increasing distribution and is followed by a necessary and sufficient condition for it to hold.

[4] For detailed statements, proofs and discussion of these results, see Topkis (1998), sections 2.6.2 and 2.7.2.

Definition 6. *$F(x;\theta)$ is* ***stochastically increasing*** *in θ if $\int_S dF(x;\theta)$ is an increasing function of θ for each increasing set $S \subseteq \mathbb{R}^n$.*

Proposition 7. *$F(x;\theta)$ is stochastically increasing in θ if and only if the expectation*

$$\int h(x)dF(x;\theta)$$

is an increasing function of θ, for all increasing real valued functions $h(x)$ on $\mathbb{R}^n$.

Proof. For a proof of this last proposition, see Topkis (1998), section 3.9.1. ■

3. Complete Information

3.1. Framework. Assume that time is discrete and that there is a finite horizon, so $t = 1, ..., T$. In each period, a continuum of players of measure one simultaneously choose an action from a binary action set. Denote by i an arbitrary player, and by I the set of players. To make the dynamics interesting, we assume that there are intertemporal links such that play at each point in time will be influenced by past play and will influence the game to be played subsequently. Specifically, we introduce three such links as follows. First, we assume that a randomly evolving economic fundamental θ influences players' payoffs directly. This variable is exogenous. Second, we assume that there exists a state variable x which depends on past aggregate play, but not on the actions of any individual player. Last, we assume that each player is characterized by a personal state variable τ which is only influenced by the specific player's past play, and thus independent of aggregate play. Let z denote that measure of players choosing high action in a given period. Let x denote the aggregate state. Last, we shall refer to a player's personal state τ as the player's type. A *strategy* for a player is a mapping from the state space into the action space (i.e. the strategy is a mapping that yields, for every triple τ, x, θ, an action a), while a *strategy profile* is a collection of strategies, one for each player. A *policy* for a player is a sequence of strategies, one for each period, while a *policy profile* is a collection of such policies, again one for each player. In what follows, we shall restrict attention to Markov policies. This restriction is crucial, since strategic complementarities are difficult to obtain in dynamic games when fully history dependent strategies are allowed. But recall that equilibria in Markov strategies remain equilibria when a larger space of non-Markov strategies is considered.

In what follows, we shall begin by analyzing the decision problem of a representative player under the assumption of a fixed profile of policies of all other players, and common knowledge of both the current realization of the economic fundamental θ, the current value of the state x and the distribution of player types τ. Given such a policy profile, the remaining player essentially solves a Markov decision process, in which other players' actions, the distribution of player types and the endogenous common state can all be treated as elements of an exogenous state vector. Specifically, fix a policy profile. With this knowledge, a player can perfectly forecast the future evolution of aggregate actions, the endogenous state variable and the distribution of player types. Thus the player may view all these as state variables.[5]

[5]For a discussion of the relationship between stochastic games and Markov decision problems, see e.g. Heyman and Sobel (1984).

To summarize the above, let $\chi = (z, \tau, x, \theta)'$ and a denote the state and action respectively in any given period, where in any period $z \in [0, 1]$ is the measure of players choosing action 1, $\tau \in \{1, 2, ..., k\}$ is the personal state variable for the player (his type), $x \in \mathbb{R}$ is the common state variable and $\theta \in \mathbb{R}$ is the exogenous economic fundamental. The collection of all possible states in any period is $\mathcal{X} = [0, 1] \times \{1, 2, ..., k\} \times \mathbb{R} \times \mathbb{R} \subseteq \mathbb{R}^4$, and $a \in A = \{0, 1\}$. For later use, note that the set $\mathcal{X} \times A$ is a sublattice of $\mathbb{R}^5$.

We assume that the evolution of types and common states is determined by the following relations:

$$\tau' = \phi(\tau, a, \theta), \ \ x' = \psi(x, z, \theta)$$

where primes denote next period's variables. These formulations mean that the type of a player depends on his type and action in the previous period, and on the last period's exogenous shock θ, and that the common state variable depends on the common state and aggregate play in the previous period, and on the last period's exogenous shock θ. Moreover, θ evolves according to a First Order Markov process. Finally we assume that the only source of uncertainty is θ (i.e. θ is the only stochastic variable).

Denote by $r_t(a, z, \tau, x, \theta)$ the one-stage return for an arbitrary player choosing action $a \in A = \{0, 1\}$ in period t given a vector of state variables $\chi = (z, \tau, x, \theta)$. Let $0 < \delta < 1$ denote the common discount factor and assume that $r_t(a, z, \tau, x, \theta)$ is bounded in all arguments. Next, define the following recursion:

$$\begin{aligned} W_t(a, z, \tau, x, \theta) &= r_t(a, z, \tau, x, \theta) + \delta \int V_{t+1}(z', \tau', x', \theta') dF_t(\theta'|\theta) \\ V_t(z, \tau, x, \theta) &= \max_a W_t(a, z, \tau, x, \theta) \\ W_T(a, z, \tau, x, \theta) &= r_T(a, z, \tau, x, \theta) \end{aligned}$$

where the expectation is taken with respect to the distribution function $F_t(\theta'|\theta)$ of next period's θ' conditional on θ. The interpretation of these expressions is as follows. The function W_t denotes the expected discounted future payoffs from playing action a in period t and then playing optimally thereafter. The function V_t simply expresses the maximum possible expected discounted future payoffs from periods t through T. By definition, last period's W_t is just the one period return, as there is no further play thereafter.

Next we summarize the assumptions required for the proceeding results. Let Ω denote all pairs of variables that can be formed by combining elements of (a, z, τ, x, θ), i.e.

$$\Omega = \{(a, z), (a, \tau), (a, x), (a, \theta), (z, \tau), (z, x), (z, \theta), (\tau, x), (\tau, \theta), (x, \theta)\}$$

(A1) The functions ϕ and ψ are increasing in all their arguments.

(A2) For a fixed policy profile, and for all $t = 1, ..., T$, $r_t(a, z, \tau, x, \theta)$ is strictly supermodular in all the elements of Ω.

(A3) $F_t(\theta'|\theta)$ is stochastically increasing in θ.

3.2. Complementarities. With these assumptions in place, we can state the first main result of the paper. Roughly, the result is that under the maintained assumptions, complemenarities between players' actions, between actions and the economic fundamental and between the endogenous state variable and players' types, are all reinforced from one period to the next.

Theorem 8. *Under (A1) - (A3),* $W_t(a, z, \tau, x, \theta)$ *is strictly supermodular in all the elements of* Ω *for each* $t = 1, ..., T$.

Proof. The proof is by induction. At the last period $t = T$, $W_T(a, z, \tau, x, \theta) = r_T(a, z, \tau, x, \theta)$ thus the result holds by (A2). Next, assume that at period $t = k + 1$ the result holds, i.e. that $W_{k+1}(a', z', \tau', x', \theta')$ is strictly supermodular in all elements of

$$\Omega' = \left\{(a', z'), (a', \tau'), (a', x'), (a', \theta'), (z', \tau'), (z', x'), (z', \theta'), (\tau', x'), (\tau', \theta'), (x', \theta')\right\}$$

Under this assumption consider the function in period $t = k$

$$W_k(a, z, \tau, x, \theta) = r_k(a, z, \tau, x, \theta) + \delta \int V_{k+1}(z', \tau', x', \theta') dF_k(\theta'|\theta)$$

First note that $r_k(a, z, \tau, x, \theta)$ is strictly supermodular in all elements of Ω by (A2).

Next, let

$$h_k(a, z, \tau, x, \theta) \equiv \int V_{k+1}(z', \tau', x', \theta') dF_k(\theta'|\theta) = \int V_{k+1}(z', \phi(\tau, a, \theta), \psi(x, z, \theta), \theta') dF_k(\theta'|\theta)$$

Recall that

$$V_{k+1}(z', \tau', x', \theta') = \max_{a'} W_{k+1}(a', z', \tau', x', \theta')$$

Since $W_{k+1}(a', z', \tau', x', \theta')$ is strictly supermodular in all elements of Ω', it follows that it is also supermodular, and therefore $V_{k+1}(z', \tau', x', \theta')$ is supermodular in (z', τ'), (τ', θ'), (τ', x'), (τ', θ'), (x', θ'). Using this, it is possible to show that $h_k(a, z, \tau, x, \theta)$ is supermodular in all elements of Ω. For a detailed proof, see Appendix C. Thus, since r_k is strictly supermodular and h_k is supermodular in all elements of Ω, and $\delta > 0$, from Lemma 3 it follows that $W_k(a, z, \tau, x, \theta)$ is strictly supermodular in all elements of Ω. This concludes the induction. ■

An immediate consequence of Theorem 8 is the following:

Corollary 9. *Under (A1) - (A3),* $W_t(a, z, \tau, x, \theta)$ *has strictly increasing differences in* (a, z) *and* (a, θ).

Proof. Follows immediately from Theorem 8 and Proposition 5. ■

Note that by relaxing (A2) so that r_t is supermodular in all elements of Ω, we can obtain an analogous result to Theorem 8 stating that W_t is supermodular in all elements of Ω, and thus restate Corollary 9 so that W_t has increasing differences in (a, z) and (a, θ). Nevertheless, we state the theorem in terms of strict supermodularity so that we can apply results by Frankel, Morris and Pauzner (2002), henceforth FMP, in the incomplete information case.

Next, consider the role of the horizon T. The above analysis was carried out for arbitrary, but finite horizon. As one may be interested in extending the analysis to infinite horizon, the following result is useful. Let W_t^T denote the value of W at period t, when the length of the horizon is T.

Proposition 10. *Under (A1) - (A3), if the limit* $\bar{W}_t = \lim_{T\to\infty} W_t^T(a, z, \tau, x, \theta)$ *exists for all time periods* t*, then it is strictly supermodular in all the elements of* Ω*.*

Proof. Consider the constituent parts of the function $W_t^T(a, z, \tau, x, \theta)$. The first part, namely $r_t(a, z, \tau, x, \theta)$ is independent of the horizon length and thus remains strictly supermodular when the limit is taken, i.e. as $T \to \infty$. The second part, $\delta \int V_{t+1}(z', \tau', x', \theta')dF_t(\theta'|\theta)$, is supermodular in all elements of Ω for all t, for each horizon length T (as shown in Theorem 8). But since supermodularity is preserved when taking limits, $W_t^T(a, z, \tau, x, \theta)$ converges to the sum of a strictly supermodular function and a supermodular function, and this limit is thus itself strictly supermodular by Lemma 3. ■

Next, assume that the horizon is infinite, that the limit $\bar{W}_t = \lim_{T\to\infty} W_t^T(a, z, \tau, x, \theta)$ exists for all time periods t and suppose that the players use stationary Markov policies. Then, the limiting $\bar{W}$ is time invariant. The following result is a momentum theorem, for the infinite horizon case.

Theorem 11. ***Momentum***. *Suppose that*

$$(\theta_t, x_t, \tau_t) \geq (\theta_{t-1}, x_{t-1}, \tau_{t-1})$$

for some period t*. Furthermore, assume that the representative player conjectures that* $z_t \geq z_{t-1}$*. Then*

(i) $a_t \geq a_{t-1}$.

(ii) *the conjecture that* $z_t \geq z_{t-1}$ *is confirmed.*

(iii) $\tau_{t+1} \geq \tau_t$ *and* $x_{t+1} \geq x_t$

Proof. Part (i) follows from Theorem 8 and Topkis' monotonicity theorem. To see this, let $S(\chi)$ be the set of maximizers of $\bar{W}$ with respect to a, i.e.

$$S(\chi) = \arg\max_a [\bar{W}(a, \chi), a \in \{0, 1\}]$$

Then, by Theorem 8, a is complementary with each component of $\chi = (\tau, z, x, \theta)$. Therefore, from Theorems 2.8.1 and 2.6.1, and Corollary 2.6.1 of Topkis (1998), the set of maximizers $S(\chi)$ is increasing in χ. Thus, since $(\theta_t, x_t, \tau_t) \geq (\theta_{t-1}, x_{t-1}, \tau_{t-1})$ and the player believes that $z_t \geq z_{t-1}$, it follows that

$$S(\chi_t) \geq S(\chi_{t-1})$$

If the set of maximizers consists of a unique element, then $a = S(\chi)$ and it follows that $a_t \geq a_{t-1}$. If there is more than one maximizer, we can assume that players use a consistent way of choosing among those maximizers in order to take their action, e.g. $a_t = \min S(\chi_t)$, in which case again the result holds.

To show part (ii), since $a_t \geq a_{t-1}$ for the representative player, it follows trivially that $z_t \geq z_{t-1}$. Last, to show for part (iii), we use parts (i) and (ii) and A1:

$$\begin{aligned} \tau_{t+1} &= \phi(\tau_t, a_t, \theta_t) \geq \phi(\tau_{t-1}, a_{t-1}, \theta_{t-1}) = \tau_t \\ x_{t+1} &= \psi(x_t, z_t, \theta_t) \geq \psi(x_{t-1}, z_{t-1}, \theta_{t-1}) = x_t \end{aligned}$$

■

3.3. Discussion. The results derived so far merit some further comments. Under the maintained assumptions, an increase in aggregate activity z induces any agent to increase his own action a, for two separate reasons. First, there is a direct contemporaneous effect, captured by the notion of complementarity in the pair (a, z). Second, there is an intertemporal effect through the state variables. An increase in aggregate activity today increases the level of tomorrow's common state variable x'. But since there is complementarity in the pair (τ, x), a player may want to increase the level of his type the next period, which in turn is achieved by increasing the current action. Similar reasoning can be applied to the pair (a, θ). An increase in the economic fundamental induces an increase in the current action through the contemporaneous complementarity. But because an increase in θ causes and upward shift in the distribution of tomorrow's realization, and there are complementarities in the pair (τ, θ), the agent has a further incentive to increase the current action to benefit from a higher future realization of θ by having a higher type.

As for the momentum theorem, it shows the possibility of multiple self-confirming equilibria, in that an initial conjecture about high activity may indeed bring about high activity as an equilibrium outcome. This momentum theorem comes with more qualifications than do those of Milgrom, Qian and Roberts (1991) and Topkis (1998), and for good reasons. First, their results are derived within deterministic frameworks, allowing them to predict perpetual growth in the state variables. For example, Milgrom, Qian and Roberts (1991) state that '[...] once the system begins along a path of growth of the core variables, it will continue forever along that path or, more realistically, until unmodeled forces disturb the system'. Apart from our conditions on the distribution of types and the common state variable, which closely mirrors those in Milgrom, Qian and Roberts (1991) and Topkis (1998), our result hinges on the fact that the economic fundamental does not decrease from one period to the next. In a sense, this is an instance of the unmodeled forces mentioned in the quote. A second, and more crucial difference is that we, in contrast to their result, derive a momentum theorem with a game and not in a decision theoretic framework. Thus, in our setup a player cannot determine (or even influence) some of the key variables of the model. Thus we are forced to state our result under the conjecture that all players believe that aggregate activity will be high, a conjecture which is indeed confirmed in equilibrium.

Before turning to the incomplete case, consider the following special case of the model. Assume that $\tau' = \phi(\tau, a, \theta) = \tau$ for all a. That is, a model in which players' types are exogenous. Under this assumption, the statement of Corollary 9 holds under assumptions (A1)-(A2) alone, i.e. also for distributions $F_t(\theta'|\theta)$ that are not stochastically increasing. In this case, since the players cannot influence their own future prospects by changing current actions, the future plays no role in current decisions. A model with such features is that of Morris and Shin (1999) in which speculators can attack a currency in each period, but where neither the speculators' ability to attack, nor the monetary authority's ability to defend the peg is influenced by past play.

4. Incomplete Information Framework

We now extend the model to one of incomplete information in order to invoke the global games methodology. The basic idea is as follows. In FMP, a very general but static game is considered in which there are strategic complementarities, each player is characterized by a specific type and where higher levels of the economic fundamental raises the relative desirability of higher actions. What we will do is to reinterpret their results so that instead of receiving a level of utility once and for all, the players receive expected discounted payoffs. FMP's results cannot be applied directly though, for the following reason. In FMP, the payoffs are uniquely determined, given a strategy profile and a realization of the economic fundamental. But recall that in the present paper, all the analysis has been carried out given a fixed profile of policies. Thus, even if a player knows what other players will do in the current period, there may still be multiple equilibria in any future period. In turn, this implies that the expected future payoffs (i.e. the value functions) are not well defined. But if players do not agree on the continuation values, equilibrium in the current period cannot be unique. The uniqueness proof below shows by an inductive argument that FMP's results yield well defined value functions, and so their results may be invoked at each stage of the game.

Assume that all assumptions and definitions of the complete information framework are maintained, but consider the following change in the informational structure. In each period, rather than observing the economic fundamental directly, a player obtains a noisy signal s where $s = \theta + \sigma\eta$ with $\sigma > 0$ and where η is distributed according to G with density g (signals may vary across types and across individuals within the same type, and the realizations of the noise terms η will be independent across players). We also assume that the noise η is independent of the economic fundamental θ. Last period's realization is common knowledge, and provides a prior $F_t(\theta'|\theta)$ over the current period's realization. After a signal is received the player forms a posterior of θ. Define

$$\Delta_t(a_1, a_2, z, \tau, x, \theta) \equiv W_t(a_1, z, \tau, x, \theta) - W_t(a_2, z, \tau, x, \theta)$$

where W_t is defined as in the previous section. In addition to (A1) - (A3), we make the following assumptions.

(A4) For all τ, x, z, there exist realizations $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ with $\underline{\theta} < \overline{\theta}$ such that $\Delta(a_1, a_2, z, \tau, x, \theta) > 0$ for $\theta > \overline{\theta}$ and $\Delta(a_1, a_2, z, \tau, x, \theta) < 0$ for $\theta < \overline{\theta}$.

(A5) The return function $r_t(a, z, \tau, x, \theta)$ is continuous in z, x and θ.

4.1. Uniqueness of Equilibrium. With assumptions (A1) - (A5), the present setup satisfies the conditions in FMP in each period. We can thus state the following result:

Theorem 12. *Under (A1) - (A5), as $\sigma \to 0$ the game in any period $t \leq T$ has an essentially unique strategy profile surviving iterative strict dominance.*

Proof. We prove the statement by induction, using Theorem 5 of FMP. First note that, to apply this theorem, we need to ensure that the required assumptions of FMP are satisfied.[6] FMP1 (strategic complementarities) is satisfied by Corollary 9, under (A1) - (A3). FMP2 (dominance regions), is assumption (A4). FMP3 (state monotonicity) is ensured by Corollary 9, under (A1) - (A3). FMP4 is assumption (A5). Finally, we do not need to check FMP5, because the action space is finite.

At $t = T$, uniqueness follows trivially from Theorem 5 of FMP. Now assume that for some period $k + 1 < T$ there is a unique equilibrium, and consider the problem in period k.

[6] In what follows, we denote assumptions A1 - A5 in FMP, by FMP1 - FMP5 correspondingly, in order to avoid confusion with the assumptions (A1) - (A5) we make here.

Since the continuation values are well defined as a function of the current period's data by the inductive hypothesis that there is a unique equilibrium in period $k+1$, uniqueness in period k follows. ■

4.2. Discussion. As the analysis above has shown, it is possible to identify a unique equilibrium for any finite horizon version of the model. It is interesting to know if uniqueness carries over to the case of an infinite horizon. As shown in the section on the complete information model, strict supermodularity at each stage continues to hold when the horizon recedes. Whether uniqueness also obtains for the infinite horizon is not a straightforward question, and we are still working on that. Our strategy is to employ the techniques of Fudenberg and Levine (1983, 1986) and Harris (1985). This literature studies the behavior of the equilibrium set of games when taking limits of simpler truncated versions of the games.

5. Conclusion

In this paper we have set out a fully dynamic model in which a continuum of players simultaneously choose actions in an uncertain environment. By assuming complementarity between players' actions and components of a state vector, we have shown that complementarities at each point in time spill over across periods. Using these results, methodology from the static global games literature has been shown to yield a unique equilibrium.

While still at an early stage, this approach to dynamic global games seems promising, in particular from an applied perspective. There are also a number of interesting features of the model that remain to be studied. For example, empirical studies have shown convincing evidence of dynamic complementarities, whereby higher activity in a given period raises productivity in subsequent periods. Such evidence may be rationalized through models like the one presented here. Last, there are a plethora of intertemporal links in the industrial organization literature that merit further study within a dynamic framework of strategic complementarities. For example, time to build, learning by doing, irreversibility of investment decisions, consumer switching costs, capital accumulation and contractual obligations spanning several periods are but some of the many features that may link a firm's current decisions and future prospects, and influence the strategic interaction between industry participants.

Appendices

A. Proof of Lemma 3

Consider $x^1 = (\widehat{x}^1; x^c) \in X$ and $x^2 = (\widehat{x}^2; x^c) \in X$, such that $\widehat{x}^1$and $\widehat{x}^2$ are unordered and let $h(x) = f(x) + g(x)$. Then

$$\begin{aligned} h(\widehat{x}^1; x^c) + h(\widehat{x}^2; x^c) &= f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) + g(\widehat{x}^1; x^c) + g(\widehat{x}^2; x^c) \\ &\leq f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) + g(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + g(\widehat{x}^1 \vee \widehat{x}^2; x^c) \\ &< f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c) + g(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + g(\widehat{x}^1 \vee \widehat{x}^2; x^c) \\ &= h(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + h(\widehat{x}^1 \vee \widehat{x}^2; x^c) \end{aligned}$$

where the first inequality follows from supermodularity of f in $\widehat{x}$ and the second inequality follows from strict supermodularity of g in $\widehat{x}$. Thus, h is strictly supermodular in $\widehat{x}$. ■

B. Proof of Proposition 5

We first show that supermodularity implies increasing differences. Consider two arbitrary $m_i \neq m_j$ and suppose $x'_{m_i} \leq x''_{m_i}$ and $x'_{m_j} \leq x''_{m_j}$. Let

$$\begin{aligned}\widehat{x}^1 &= (x_{m_1}, ..., x''_{m_i}, ..., x'_{m_j}, ..., x_{m_r})\\ \widehat{x}^2 &= (x_{m_1}, ..., x'_{m_i}, ..., x''_{m_j}, ..., x_{m_r})\end{aligned}$$

Then

$$\begin{aligned}\widehat{x}^1 \vee \widehat{x}^2 &= (x_{m_1}, ..., x''_{m_i}, ..., x''_{m_j}, ..., x_{m_r})\\ \widehat{x}^1 \wedge \widehat{x}^2 &= (x_{m_1}, ..., x'_{m_i}, ..., x'_{m_j}, ..., x_{m_r})\end{aligned}$$

Since f is supermodular in $\widehat{x}$, it follows from the definition of supermodularity that

$$\begin{gathered}f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) \leq f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c) \Longrightarrow\\ f((x_{m_1}, ..., x''_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c) + f(x_{m_1}, ..., x'_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c) \leq\\ f((x_{m_1}, ..., x''_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c)) + f(x_{m_1}, ..., x'_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c)\end{gathered}$$

Therefore

$$\begin{gathered}f((x_{m_1}, ..., x''_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c)) - f((x_{m_1}, ..., x''_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c) \geq\\ f(x_{m_1}, ..., x'_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c) - f(x_{m_1}, ..., x'_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c)\end{gathered}$$

which means that f has increasing differences in (x_{m_i}, x_{m_j}), for all $m_i \neq m_j$.

To show that strict supermodularity implies strict increasing differences, consider two arbitrary $m_i \neq m_j$ and suppose now that $x'_{m_i} < x''_{m_i}$ and $x'_{m_j} < x''_{m_j}$ and notice that the two vectors

$$\begin{aligned}\widehat{x}^1 &= (x_{m_1}, ..., x''_{m_i}, ..., x'_{m_j}, ..., x_{m_r})\\ \widehat{x}^2 &= (x_{m_1}, ..., x'_{m_i}, ..., x''_{m_j}, ..., x_{m_r})\end{aligned}$$

are unordered. By the definition of strict supermodularity it follows that

$$f(\widehat{x}^1; x^c) + f(\widehat{x}^2; x^c) < f(\widehat{x}^1 \wedge \widehat{x}^2; x^c) + f(\widehat{x}^1 \vee \widehat{x}^2; x^c)$$

thus

$$\begin{gathered}f((x_{m_1}, ..., x''_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c)) - f((x_{m_1}, ..., x''_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c) >\\ f(x_{m_1}, ..., x'_{m_i}, ..., x''_{m_j}, ..., x_{m_r}; x^c) - f(x_{m_1}, ..., x'_{m_i}, ..., x'_{m_j}, ..., x_{m_r}; x^c)\end{gathered}$$

i.e. f has strictly increasing differences in (x_{m_i}, x_{m_j}), for all $m_i \neq m_j$. ■

C. Remaining Part of the Proof for Theorem 8

We want to show that under (A1) - (A3), if $V_{k+1}(z',\tau',x',\theta')$ is supermodular in (z',τ'), (τ',θ'), (τ',x'), (τ',θ'), (x',θ) then $h_k(a,z,\tau,x,\theta)$ is supermodular in all elements of Ω.

We first introduce some convenient notation and state some auxiliary results. For any two $\omega_1,\omega_2\in\mathbb{R}^k$ let $\bar{\omega}=\max(\omega_1,\omega_2)$ and $\underline{\omega}=\min(\omega_1,\omega_2)$. It is trivial to show the following two results:

Lemma 13.

(i) *For any* $\omega_1,\omega_2\in\mathbb{R}^n$, $\max(\omega_i,\bar{\omega})=\bar{\omega}$ *and* $\min(\omega_i,\bar{\omega})=\omega_i$, $i=1,2$.

(ii) *If* $\xi:\mathbb{R}^n\rightarrow\mathbb{R}$ *is increasing in all its arguments, then*

$$\min_i \xi(\hat{\omega}^i;\omega^c)=\xi(\underline{\omega};\omega^c)\quad \textit{and}\quad \max_i \xi(\hat{\omega}^i;\omega^c)=\xi(\bar{\omega};\omega^c)$$

(a) Supermodularity in (a,z). Consider (a_1,z_1) and (a_2,z_2). Then

$$\begin{aligned}
& h_k(a_1,z_1,\tau,x,\theta)+h_k(a_2,z_2,\tau,x,\theta)\\
= & \int\left[V_{k+1}(z',\phi(\tau,a_1,\theta),\psi(x,z_1,\theta),\theta')+V_{k+1}(z',\phi(\tau,a_2,\theta),\psi(x,z_2,\theta),\theta')\right]dF_k(\theta'|\theta)\\
\leq & \int\left[V_{k+1}(z',\phi(\tau,\bar{a},\theta),\psi(x,\bar{z},\theta),\theta')+V_{k+1}(z',\phi(\tau,\underline{a},\theta),\psi(x,\underline{z},\theta),\theta')\right]dF_k(\theta'|\theta)\\
= & h_k(\bar{a},\bar{z},\tau,x,\theta)+h_k(\underline{a},\underline{z},\tau,x,\theta)
\end{aligned}$$

where the inequality follows from (i) the fact that $V_{k+1}(z',\tau',x',\theta')$ is supermodular in (τ',x'), (ii) (A1) and (iii) Lemma 13.

(**b**) Supermodularity in (a,τ). Consider (a_1,τ_1) and (a_2,τ_2). Then

$$\begin{aligned}
& h_k(a_1,z,\tau_1,x,\theta)+h_k(a_2,z,\tau_2,x,\theta)\\
= & \int\left[V_{k+1}(z',\phi(\tau_1,a_1,\theta),\psi(x,z,\theta),\theta')+V_{k+1}(z',\phi(\tau_2,a_2,\theta),\psi(x,z,\theta),\theta')\right]dF_k(\theta'|\theta)\\
\leq & \int\left[V_{k+1}(z',\phi(\bar{\tau},\bar{a},\theta),\psi(x,z),\theta')+V_{k+1}(z',\phi(\underline{\tau},\underline{a},\theta),\psi(x,z,\theta),\theta')\right]dF_k(\theta'|\theta)\\
= & h_k(\bar{a},z,\bar{\tau},x,\theta)+h_k(\underline{a},z,\underline{\tau},x,\theta)
\end{aligned}$$

where the inequality follows from (i) the fact that $V_{k+1}(z',\tau',x',\theta')$ is trivially supermodular in τ' (in fact this holds with equality), (ii) (A1) and (iii) Lemma 13.

(c) Supermodularity in (a,x), (z,τ), (τ,x) follows repeating similar steps as in (a).

(d) Supermodularity in (x,z) follows repeating similar steps as in (b).

(e) Supermodularity in (a,θ). Consider (a_1,θ_1) and (a_2,θ_2), and assume without loss of generality that $\theta_1 \leq \theta_2$ i.e. that $\bar{\theta} = \theta_2$ and $\underline{\theta} = \theta_1$. Then

$$\begin{aligned}
& h_k(a_1, z, \tau, x, \theta_1) - h_k(\underline{a}, z, \tau, x, \underline{\theta}) \\
= & \int \left[V_{k+1}(z', \phi(\tau, a_1, \theta_1), \psi(x, z, \theta_1), \theta') - V_{k+1}(z', \phi(\tau, \underline{a}, \underline{\theta}), \psi(x, z, \underline{\theta}), \theta')\right] dF_k(\theta'|\theta_1) \\
\leq & \int \left[V_{k+1}(z', \phi(\tau, \bar{a}, \bar{\theta}), \psi(x, z, \bar{\theta}), \theta') - V_{k+1}(z', \phi(\tau, a_2, \theta_2), \psi(x, z, \theta_2), \theta')\right] dF_k(\theta'|\theta_1) \\
\leq & \int \left[V_{k+1}(z', \phi(\tau, \bar{a}, \bar{\theta}), \psi(x, z, \bar{\theta}), \theta') - V_{k+1}(z', \phi(\tau, a_2, \theta_2), \psi(x, z, \theta_2), \theta')\right] dF_k(\theta'|\theta_2) \\
= & \int V_{k+1}(z', \phi(\tau, \bar{a}), \psi(x, z), \theta') dF_k(\theta'|\bar{\theta}) - \int V_{k+1}(z', \phi(\tau, a_2), \psi(x, z), \theta') dF_k(\theta'|\theta_2) \\
= & h_k(\bar{a}, z, \tau, x, \bar{\theta}) - h_k(a_2, z, \tau, x, \theta_2) \Longrightarrow
\end{aligned}$$

$$h_k(a_1, z, \tau, x, \theta_1) + h_k(a_2, z, \tau, x, \theta_2) \leq h_k(\bar{a}, z, \tau, x, \bar{\theta}) + h_k(\underline{a}, z_2, \tau, x, \underline{\theta})$$

The first inequality follows from A1 and from the fact that V_{k+1} is supermodular in (τ', x'). The second inequality follows from the fact that the function

$$p(\theta') \equiv V_{k+1}(z', \phi(\tau, \bar{a}, \bar{\theta}), \psi(x, z, \bar{\theta}), \theta') - V_{k+1}(z', \phi(\tau, a_2, \theta_2), \psi(x, z, \theta_2), \theta')$$

is increasing in θ' and Proposition 7. To see why p is increasing, consider $\theta_1' \leq \theta_2'$. Also, let

$$\begin{aligned}
\tau_1' &= \phi(\tau, \bar{a}, \bar{\theta}),\ \tau_2' = \phi(\tau, a_2, \theta_2) \\
x_1' &= \psi(x, z, \bar{\theta}),\ \ x_2' = \psi(x, z, \theta_2)
\end{aligned}$$

Then, by A1 and lemma 13 we have that

$$\begin{aligned}
\bar{\tau} &= \max(\tau_1', \tau_2') = \tau_1',\ \ \underline{\tau} = \min(\tau_1', \tau_2') = \tau_2' \\
\bar{x} &= \max(x_1', x_2') = x_1',\ \ \underline{x} = \min(x_1', x_2') = x_2' \\
\bar{\theta}' &= \max(\theta_1', \theta_2') = \theta_2',\ \ \underline{\theta} = \min(\theta_1', \theta_2') = \theta_1'
\end{aligned}$$

Next, we use the above and the supermodularity of V_{k+1} in (τ', θ') and in (x', θ') to get

$$\begin{aligned}
& V_{k+1}\left(z', \tau_1', x_1', \theta_1'\right) + V_{k+1}\left(z', \tau_2', x_1', \theta_2'\right) \\
\leq & V_{k+1}\left(z', \tau_1', x_1', \theta_2'\right) + V_{k+1}\left(z', \tau_2', x_1', \theta_1'\right) \\
\leq & V_{k+1}\left(z', \tau_1', x_1', \theta_2'\right) + V_{k+1}\left(z', \tau_2', x_1', \theta_2'\right) + V_{k+1}\left(z', \tau_2', x_2', \theta_1'\right) - V_{k+1}\left(z', \tau_2', x_2', \theta_2'\right)
\end{aligned}$$

Therefore

$$V_{k+1}\left(z', \tau_1', x_1', \theta_1'\right) - V_{k+1}\left(z', \tau_2', x_2', \theta_1'\right) \leq V_{k+1}\left(z', \tau_1', x_1', \theta_2'\right) - V_{k+1}\left(z', \tau_2', x_2', \theta_2'\right)$$

Thus

$$\begin{aligned}
p(\theta_1') &\equiv V_{k+1}(z', \phi(\tau, \bar{a}, \bar{\theta}), \psi(x, z, \bar{\theta}), \theta_1') - V_{k+1}(z', \phi(\tau, a_2, \theta_2), \psi(x, z, \theta_2), \theta_1') \\
&= V_{k+1}\left(z', \tau_1', x_1', \theta_1'\right) - V_{k+1}\left(z', \tau_2', x_2', \theta_1'\right) \\
&\leq V_{k+1}\left(z', \tau_1', x_1', \theta_2'\right) - V_{k+1}\left(z', \tau_2', x_2', \theta_2'\right) \\
&\leq V_{k+1}(z', \phi(\tau, \bar{a}), \psi(x, z), \theta_2') - V_{k+1}(z', \phi(\tau, a_2), \psi(x, z), \theta_2') \\
&= p(\theta_2')
\end{aligned}$$

(f) Supermodularity in (τ, θ), (z, θ), (x, θ) follows by repeating similar steps as in (e). This concludes the proof. ■

REFERENCES

[1] Burdzy, K., D. Frankel and A. Pauzner, 2001. Fast Equilibrium Selection by Rational Players Living in a Changing World, *Econometrica*, 68, 163-190.

[2] Carlsson, H. and E. van Damme, 1993. Global Games and Equilibrium Selection, *Econometrica*, 61, 989-1018.

[3] Curtat, L. O., 1996. Markov Equilibria of Stochastic Games with Complementarities, *Games and Economic Behavior*, 17, 177-199.

[4] Frankel, D., 2001. Payoff Shocks and Equilibrium Selection with Endogenous Flexibility, *mimeo.*

[5] Durlauf, S. N., 1991. Multiple Equilibria and Persistence in Aggregate Fluctuations, *American Economic Review*, 81(2), 70-74.

[6] Frankel, D. M. and A. Pauzner, 2000. Resolving Indeterminacy in Dynamic Settings: The Role of Shocks, *Quarterly Journal of Economics*, 115, 283-304.

[7] Frankel, D. M. , S. Morris and A. Pauzner, 2002. Equilibrium Selection in Global Games with Strategic Complementarities, *forthcoming in Journal of Economic Theory.*

[8] Fudenberg, D. and D. Levine, 1983. Subgame-Perfect Equilibria of Finite- and Infinite Horizon Games, *Journal of Economic Theory*, 31, 251-268.

[9] Fudenberg, D. and D. Levine, 1986. Limit Games and Limit Equilibria, *Journal of Economic Theory*, 38(2), 261-279.

[10] Harris, C., 1985. A Characterisation of Perfect Equilibria of Infinite Horizon Games, *Journal of Economic Theory*, 37, 99-125.

[11] Heyman, D. P. and M. J. Sobel, 1984. Stochastic Models in Operations Research, Volume II: Stochastic Optimization, McGraw-Hill Book Company.

[12] Levin, J., 2001. A Note on Global Equilibrium Selection in Overlapping Generations Games, *mimeo.*

[13] Milgrom, P., Y. Qian and J. Roberts, 1991. Complementarities, Momentum, and the Evolution of Modern Manufacturing, *AEA Papers and Proceedings*, 81(2), 84-88.

[14] Morris, S. and H. S. Shin, 2002. Global Games: Theory and Applications, *forthcoming in Advances in Economic Theory and Econometrics: Proceedings of the Eighth World Congress of the Econometric Society* (M. Dewatripont, L. Hansen and S. Turnovsky, eds.), Cambridge University Press.

[15] Morris, S. and H. Shin, 1999. A Theory of the Onset of Currency Attacks, *in The Asian Financial Crisis: Causes, Contagion and Consequences* (Agenor, Vines and Weber, eds.), Cambridge University Press.

[16] Oyama, D., 2001. Booms and Slumps in a Game of Sequential Investment with Changing Fundamentals, *mimeo.*

[17] Topkis, D. M., 1998. Supermodularity and Complementarity, Princeton University Press.

[18] Toxvaerd, F., 2002. Strategic Merger Waves: A Theory of Musical Chairs, *mimeo.*