A Generalized Theory of Monetary and Macroeconomics

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Abstract

In a closed form, we investigate why an expansionary policy rises beyond the equilibrium, and alters its positive impact into a negative one upon output growth, employment, and price inflation. Thus, without control, real business cycles arise. First, we find that such a generalized prevailing relationship leads to a minmax solution and implies a symmetric demand and supply curves. Monetarists and Keynesians find only asymmetric positive or negative curves. Second, our main contribution is to detect the sign of the first-order and the second-order coefficients, and predict the optimal policy and the non-zero equilibrium with a minimum shrinking variance for all horizons. Third, using dynamic quadratic regression without iterations, this paper proves that with a super-convergence probability close to one, an incentive aggressive policy can steer stochastic output growth. The heteroschedasticity of shocks is reduced into Brownian motions. The first-order solution can be misleading. As sensitivity tests, the optimum in real interest rate policy is found invariant and consistent across time and spaces. When the observations converge towards the equilibrium and policies, their volatility decreases. The optimal policy is discretionary and evolutionary, because the disturbances are not zero. The data of Taiwan, United States, and Argentina over different time periods are estimated. The optimal policies are verified through sensitivity tests. (JEL: C43; D6; D9; H2, G1).

Keywords: Stochastic growth theory; convergence probability, limit cycles; minmax; optimum equilibrium; heteroschedasticity, extreme policy; switching models.

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1. Introduction

1.1 Motivations

This paper addresses the generalized macroeconomic theory. In New Keynesian models, the equilibrium is underdetermined when policy is passive. Linear rational expectations models yield multiple equilibria which are also indeterminate (Lubik, et al. 2004). Thus, without control, there occur real business cycles. With a closed form, however, we introduce an innovative convergence probability to reflect the government's ability to enact an output stabilization policy in a general equilibrium framework in continuous time. The big innovation is to detect the generalized, positive and negative, macro relationship and stochastic growth through dynamic quadratic regression. The non-zero equilibrium is the turning point of discretionary aggressive policy. With the minmax policy and non-zero equilibrium, we solve conflicting theories in the infinite dimension.

1.2 The Existing Literature

A consistent policy can widely promote output and the firm's value at the infinite horizons, while the sign of the first-order solution can be misleading. Monetarists and Keynesians, however, are incorrect to assume only a positive relation between output and prices(Almonacid, 2003). Under a competitive equilibrium, Kamihigashi (2001, p. 1007) suggests that the equilibrium is zero at the origin without growth. Lucas (1996) and Friedman (1977) indicate that the zero inflation and a zero interest rate maximize the labor effort. In Taylor's comparative static linear rule, the interest rate is a dependent (endogenous) variable and tends to decrease as the inflation and output decline to zero and fall into liquidity traps. Estrella et al. (2002) detect the inconsistent direction of interest rate policy when the disturbances are unobserved. Government spending also has negative and positive impacts on output (Baxter et al. 1999; Blanchard, et al. 2002; Granelli, 2003;Graham 1994). In the existing literature (Caner et al. 2001; Amato et al. 2004), two questions arise: What is the best scope for stabilization policies with a consistent sign and confidence interval? How can a high yet stable equilibrium of output growth be ascertained when consumption and policy have habit persistence? Few previous works (Douglas, et al. 2004) have estimated the optimum non-zero equilibrium as a basis for integrating these mathematic and statistical models in overlapping generation models.

1.3 Contributions

The contribution of this study is to provide the minmax solution of quasi-stationary output growth and aggressive incentive policy, reduction of heteroschedastcity, and the probability p test of convergence. Our surprising findings are that in imperfect markets, a low convergence probability indecates the inequality constraint and the underutilization of resources. Even if the first-order coefficient is negative, the minmax solution of equilibrium and optimal policy may still be a positive value in the sense that the consumers' welfare is maximized and the disturbances are minimized. Thus, the law of motion used is the second-order polynomial where both demand and supply shocks may coexist. Thus, policy is non-neutral; and persistent output fluctuations arise from crowding-out effects of government spending. Deviations from the optimum equilibrium and inflation policy can be enacted at the cost of recession, bankruptcy, unemployment, and currency depreciation. Thus, in our comparative dynamic and feedback closed form, we estimate the non-zero equilibrium of the state variable, while the policy is an explanatory variable. The volatility of policy can increase the test power, insulating disturbances, bounding and steering the output to converge to the non-zero equilibrium with the minimum and shrinking variance.

Thus we test the null hypothesis on the best parameter estimate θ for the non-zero equilibrium x*:

$$H_0 : \theta = 0 \notin \theta(\text{x*}) \text{ versus } H_1 : \theta \in \theta(\text{x*}) \neq 0$$

We use a superior technique for hypotheses testing which is dynamic quadratic regression, where a continuous model is approximated by discrete data as the time unit becomes scaled infinitely small or as time goes to infinity. The convergence probability is the dependent variable and denotes the p value of a dynamic controllability. We prove that the equilbrium meets Opial's (1967) property of minimum and shrinking variance and satisfies nearly all constraints. A super-convergence probability implies that within a finite period, the control policy steers the output and coefficients toward the equilibrium and satisfies Berge's (1959) maximum theorem, which requires that the constraint sets change continuously. The deviations of the observations from the equilibrium converge to zero.

With this stochastic growth model, the non-zero equilibrium of growth exists as a fixed-point. The simultaneous system is treated as a set of non-constant entire functions between the state and control variables. It is solved as the reduced, closed, and feedback form of the law of motion, which yields the fixed point in the nonexpansive compact convex domain. The equilibrium is the minimum or maximum or maxmin solutions, where the expected errors tend to be zero and is independent of the quadratic incentives of policy. Beyond the equilibrium, the government raises policies to alter the real interest rate; thus a positive relationship between the output (or input supply) and factor prices may become a negative one. An incompatible-incentive policy is used to bound and limit business cycles, when the equilibrium is not equal to optimal extreme policy, and business cycles occur. Thus, the scope for policy intervention is limited to the boundary domain. Such a solution is consistent with the data, because the sign of coefficients reflects the increasing or decreasing tendency of output as the policy steers the output up or down. Therefore, such policy switchings cause structural changes, where we find business cycles, output bifurcation, as well as policy option pricing.

We present a new econometric process. Our assumptions are that in equilibrium, the unobservable random variables are reduced into Brownian motions. During depression, below the equilibrium, an expansionary policy may turn from a negative demand-sided shock into a positive supply-sided stochastic shock. Thus, the observations of random variables are serially correlated, nonlinear, and nonstationary data. The main advantages of such analysis are that dynamic quadratic incentives (and regression) yields a superconvergent probability within a finite time; and such convergence does not require intensive iterative computation. The Student t^2 statistics converge to a standardized central distribution as the observations converge to the equilibrium with a consistent and shrinking confidence interval. The coefficient of determination $\Re^2$ can denote the convergence probability with no required specific distribution of observations. Other advantages are that our solution is exact in mechanical sciences and an approximation in other sciences; also the estimate of optimum equilibrium is uniformly consistent and unbiased. We use the optimal extreme policy to minimize the variance of the high, yet stable, output growth. The solution is dynamic consistency of the general equilibrium in a continuous sense; this equilibrium is unique, stable, and robust to time-varying parameters, disturbances, and simulations.

The prediction is the equilibrium of outcomes. The nonlinear simultaneous equations are solved recursively as a reduced closed form for each variable. Our estimators of equilibrium and coefficients are the maximum likelihood estimators and are thus invariant to the equation delected, because in equilibrium, the covariance matrix of disturbances is singular and independent (Ravikumar, 2000). Our equilibrium serves as the center of a probability distribution of the Student t_n test. Unlike a χ^2 statistics, our quadratic terms is additive, deductive, interactive, independent, and invariant to the ordering of optimizors or regressors. In contrast, the methods of linear regression, vector autoregression, nonparameteric regression, and the bootstrap statistic all have weak convergent rates, are biased, and cannot predict *ex ante* structural change (Fan et al. 2000). For some computations, our regression outperforms the Cramer rule which requires constant coefficients of linear models and is better than the polynomial time algorithms which require intensive iterative computations and cannot indicate the convergence probability when the initial value or the order of variables changes.

Our contribution, first, is to estimate the non-zero equilibrium under the incentive-compatible policy, whose outcome outperforms Nash(1950)'s noncooperative equilibrium. Also, unlike the Heckscher-Ohlin approach(Melvin, 1989), our equilibrium is independent of the initial endowments. In contrast to Black and Scholes(1973), our policy options do not require normality distributions of observations; in imperfect markets, our convergence probability indicates the convergence towards the equilibrium which follows the standardized central distribution of statistics. And, unlike Solow (1956), the government enters into the economy as a production factor. When resource constraints are potential, policy can alter the negative relationship between input demands and factor prices into a positive coefficient; thus, around the equilibrium, production technologies and the time preferences for consumption are not constant. In a perfectly competitive economy, there is no scope for welfare improvement; because preferences and technology are such that a competitive equilibrium achieves the first best.

In an imperfect economic environment, however, the market structure is imperfectly competitive, nonlinear and nondifferentiable. Moreover, monopolistic firms innovate and increase output, and reduce the cost of welfare loss while the policy converges toward the optimal equilibrium.

Second, dynamic quadratic regression combines the symmetry of both demand and supply curves and shocks. Our convergence solution reduces the heteroschedasticity of errors. Stochastic demand and supply shocks may coexist, and policy has delay impacts and cause limit cycles. The welfare problem over an infinite horizon is continuously approximated over a finite life by the summation of a dynamic quadratic incentives (or regression) with a non-zero equilibrium, when the measurement time unit of discrete data becomes infinitely small. Therefore, the high yet stable equilibrium of output growth is estimated as the valued policy option and is stabilized by the optimum policy in the absence of commitment problems. In contrast, the Keynesian investment-saving curve is asymmetric and shows a negative relation between consumption and interest rates. The money demand and supply curve shows a positive relation between the equilibrium output and the interest rate.

Third, the convergence probability detects the non-profitable deviations from the maximum equilibrium, which is multiple under rational expectations but is dynamically consistently determinate under the optimization function. When investors have rational, yet inaccurate expectations, investors tend to lose and withdraw from capital formation in the stock market. Thus the policy has a negative and a positive impact, and steers the state variable to converge to the expected value, which equals the equilibrium. Variability and aggressiveness in policy increases the power of tests. Deviations from the equilibrium prices of assets, capital stocks, and foreign exchange rates may overshoot or undershoot and discourage production. Assume that the output level is shifting and output growth is quasi-stationary, and that the economy grows, endowments change, and some capital stock is renewable and reproducible through positive incentive policies. Optimal policy stimulates output growth from the interactive equilibrium between the supply and demand; the mutual causality between the marginal policy cost and the marginal product is reflected in the first-order and second-order sign changes in coefficients or elasticity.

Fourth, the equilibrium is a turning point. The perturbations turn the policy to become discretionary. Deviations from optimal policy, such as inflation, interest rates, and tax rates, present an opportunity for both a shadow cost (or price) and an equilibrium incentive. Therefore, in our assumptions, the Lagrangian multiplier, $\lambda = \theta$ (x*), can be transformed into coefficients and variables. Thus the policy of interest rate and prices may converge towards the equilibrium shadow price of resources, θ (x(t)-x*)=0 as x(t)$\rightarrow$x*; the equilibrium need not always be unified or constant, x*=1 or x*=0, because the shadow price and the parameters are time-varying and determined by positive and negative demand and supply forces.

Fifth, we find that the optimal path of policy, such as the inflation and the interest rates, is non-negative. The dynamic quadratic regression we apply is in the closed form of a bifurcation model, which can be decomposed into demand and supply equations for both consumption and capital goods. Below the optimum, the sign of coefficients and policy responses changes; the low inflation, interest rates, and tax rates have a positive incentive for output growth, consumption, and employment. Beyond the optimum, the high inflation, interest rates, and tax rates have a negative impact upon output growth. The equilibrium is the minmax solution. The optimal policy promotes high capital formation and high effort, enhances wages, and reduces unemployment rates. The optimum equilibrium is an attractor which attracts all unstable solutions and business cycles. In this sense the optimal fiscal and monetary policy reduces the unemployment rate and offsets the nominal rigidity of high wages in the public sector. The central bank exercises the option to maintain the optimal real interest rate, while the value of this policy option is sustainable consumption growth. This mixed policy taget is evolutionary and solves the problems between outcomes and incentives. As a sensitivity test, we use different data sets to verify our result of the optimal real interest rate across countries and time.

In the following, Section 2 constructs the model. Section 3 analyzes the model and explains the new contribution and its impacts. Section 4 supplies empirical examples for the equilibrium, policy, the convergence p value as well as sensitivity tests and simulations. The link between the model and examples is the reduced, closed, recursive form of the law of motion. Section 5 concludes with remarks. In Appendices 1 and 2, the preliminary estimates confirm the existence of our non-zero equilibrium.

2. The Model

2.1 Assumptions

Assumption 1: The unobservable random variables v are stochastic shocks in demand and supply, such as ineffective demand and technological changes. In equilibrium, v follows the random walk in discrete data, and Brownian motions in a continuous data.

Assumption 2: The observations of random variables (Y, x, u) are serially correlated, nonlinear and nonstationary. The wages are relatively rigid in the imperfect public sector. (Y, x, u) are output, output growth, and control variables at time t.

2.2 The Methodology and the New Contributions

Our main contributions are the estimation of the non-zero equilibrium growth and optimal policy, the reduction of heteroschedasticity, and the test of convergence probability. The stochastic growth model is consistent with data.

The households maximize the welfare H:

x*=argmaxF(x)

The problem is solved as the ordinary differential equation is defined as,

$$\lim_{t\to\infty} dx/dt = \lim_{t\to\infty} f(x^*) = \lim_{t\to\infty} \theta_2 \ (x(t) - x^*)^2 = 0$$

Since response parameters are unobservable, we control the equilibrium and estimate the value of feedback control policy such that the equilibrium x* is nonlinearly stable. To solve the stochastic differential equations, we allow the coefficient θ_2(t) to be time-varying. The equilibrium is the turning point after we transform the nonhomogenous equation dlnY/dt=f(Y) into a homogenous equation, $dx/dt=\theta_2 (x(t)-x^*)^2$, where θ_2 is a parameter. We control the quasi-stationary equilibrium of output growth, x(t)=x*>0, which is positive rather than at the origin x(t)=0. We also transform the incentive of interest rates and prices into output along the balance law. Due to the near symmetry of demand and supply curves and feedback policies, our non-zero equilibrium is the attractor in Newton's gravitation equation and Einstein's general relativity theory. In contrast, the previous studies often assume that the equilibrium is zero and at the origin. In heat equations (Ortega et al.,2001; Sakaguchi, 2001), $dx/dt=\theta x(t)$, the origin x(t)=0 is the stationary critical point if the initial data satisfy the balance law. To stabilize cycles, Magnitskii, et al.(2000) suggest the control of parameters and require that all eigenvalues are negative real parts.

Our equilibrium path of output is unique and Pareto-improving:

$$Y(t) = Y^* = Y(0)\exp(x^*t)$$

where x*>0 is a quasi-stationary equilibrium of output growth.

In contrast, Ito's problem (Chern et al. 2000, p.573) is dY=Ydx. Ito's stochastic differential equation reads as follows:

$$Y(t) = Y(0)\exp(x(t)-x(0)) = Y(0)\exp\int_0^t dx$$

The Maxwell-Boltzman distribution (Cherm, et al., 2000, p. 109) is:

$$Y(x, v, t) \to C\exp(-\beta v^2)$$

where C is a constant. $\frac{\partial Y}{\partial t} = \sum v\frac{\partial Y}{\partial x} + \sum x\frac{\partial Y}{\partial v}$.

The Ornstein-Uhlenbeck process (Bishwai and Bose, 2001) is

$$dx(t) = \theta x(t)dt + dv(t)$$

where {v(t), t} is a standard Wiener process, and $\theta \in (-\infty, 0)$. The regression coefficient is $\theta(T) = \int_0^T x(t)dx(t) / \int_0^T x^2(t)dt$, where E(x(0))=0; and $E(x^2(0)) = 1/(2\theta)$. These existing works assume that the equilibrium is zero, and lack the efficient algorithm for computing the non-zero equilibrium in the nonhomogeneous system.

2.3 Main results on Nonlinear systems and attractor

The nonlinear system is solved recursively as a reduced form for each equation. The solution is improved through the equilibrium growth rate and optimal policy, such as (x*, u*, q*, r**, ((G-T)/Y)**).

We use the U.S. annual data over various periods, as reported in *International Financial Statistics* by the International Monetary Fund (IMF). The Appendix 2 shows our preliminary estimate of the positive optimal inflation. Equation (4.3) is the consumption function. To stabilize consumption growth, the optimal real interest rate is r**=2.5%. It is consistent with the average interest rate in Europe over 1880-1980 and in the United States over 1950-1990. When the interest rate declines, the consumption increases. Savings, investment, and output decline. Conversely, when the interest rate rises, the consumption and investment fall, and ineffective demand leads to declines in output growth.

As Tables 1 through 3, the optimal inflation rate is about 3%. The optimal equilibrium of the real interest rate is about r*=1%. The share of the optimal budget deficit in income (GDP) is around (G** -T**)/Y*=3%. The optimal share of the government spending in income (GDP) is G**/Y*=25%. The optimal income tax rate is T**/Y*=23%. The optimum equilibrium of the unemployment rate is around u=u*=2%. The real exchange rate q*=1 is a competitive equilibrium.

2.4 The detailed model

Assuming that the supply and demand shocks exist, this study finds that the dynamic continuous model can be approximated by the discrete data as the time unit becomes infinitely small or as time goes to infinity. Suppose the production is a function of capital stock K and labor input L. The households intend to maximize the sustainable consumption flow series {C(t)} over a finite life. C is consumption; $0<t\leq n$ denotes a time horizon. n is a scalar and denotes the finite sample period. With stochastic death, the welfare over an infinite horizon is approximated by the overlapping generation utility as

(2.1) $$U\{C(t), \rho(t)\}=\lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} \left(\frac{1}{1+\rho(t)}\right)^{t}(C(t))$$

subject to the identity:

(2.2) $$Y(t)=C(t)+dK/dt+G(t)+qX(t)$$
$$= C(t)+dK/dt+G(t)+(r(t)-r^*(t))B(t)$$

and the budget constraint :

(2.3) $$\Delta(B(t)/Y(t))= \{B(t+1)/Y(t+1)\}-\{B(t)/Y(t)\}$$
$$=(r(t)-\Delta Y(t)/Y(t))(B(t)/Y(t))+P(t)(G(t)-T(t))/Y(t+1)- (M(t)-M(t-1))/Y(t+1)$$

the production technology,

(2.4) $$Y(t)=F(K, L)$$

the income distribution

(2.5) $$Y(t)=rK+wL$$

and the real interest rate parity

(2.6) $$r(t)=r^* +\theta\, d \log q(t)/dt+v(t)$$

where the initial endowment values are given. In Equation (2.1), the utility U(C, ρ) is a compact, concave and intertemporal function. ρ is the consumers' rate of time preference which discounts the future utility of consumption in the present value. Equation (2.2) is the identity. Y is the gross domestic product (GDP) and equals consumption C, investment dK/dt, government spending G, and net exports qX. The net exports are equal to the accumulation of foreign debts B>0 or foreign assets B<0. r is the domestic real interest rate. r* is the foreign real interest rate. K is capital stock and denotes the total assets, which equal the equity capital and debts. In the steady state, $\lim_{t\to\infty}$ dK/dt=0, X=0, and Y=C, output is equal to the consumption, including private and government consumption expenditure. In Equation (2.3), T is tax revenue; B is public debts; M is money supply. When the interest rate r does not exceed the economic growth ΔY(t)/Y(t), the share of public debt does not accumulate. In Equation (2.4), the production is a function of capital K and labor L. Labor includes the government services G. The income tax rates, G/Y $\approx$ T/Y, redistribute income and reduce income inequality and ineffective demand. In Equation (2.5), income equals the returns to capital, rK, plus the wages of labor, wL. r is the real interest rate; it denotes the user's cost of capital and equals the nominal interest rate i minus the inflation rate dlog P/dt. w is the wage rate. In (2.6), for an open economy, the interest rate differential is equal to changes in real exchange rate plus the risk premium. q is the real exchange rate. θ is time-varying response parameters. The noise v(t) denotes risk premium. In equilibrium, q*=1, d log q/dt=0, and the expected noise is zero.

3. Solution and Analysis of the Model

*Definition 1: The optimal equilibrium growth x** between supply and demand is quasi-stationary as follows:

$x(t+1) \le x(t) = x^* = x(t-1) \ge x(t-2)$ for all t>0.

$0 < \partial F(K,L)/\partial (K) = r \le x$ if $x \le x^*$; and $0 < x \le r$ if $x^* \le x$

$\partial F(K, L)/\partial K > 0$ and $\partial^2 F(K,L)/\partial (K)^2 \le 0$

This equilibrium, x*=dlogF/dt, is supported by optimal policy, when the marginal product equals the marginal cost. For example, the government spending is as productive as an increase in capital and as incentive-compatible as a promotion in labor effort. Dynamic quadratic regression yields the equilibrium solution, which satisfies the necessary and sufficient conditions of optimization.

Definition 2:The heteroschedasticity of shocks and errors is reduced into homoschedasticity when observations of the state variable converge towards the equilibrium which is the center of the distribution. The expected error is zero with the unit variance; residual errors are independent of the explanatory variables. Thus, output fluctuations are decomposed into dynamics in equilibrium, policy oscillations, and the Brownian motion of residual errors after the shocks have been controlled by policy.

Definition 3: The strong convergence probability p(x) is close to one as $t\to\infty$:

$$p(x)=\exp(dx/dt)=\exp\{(\partial^2 F(K,L)/\partial K^2) +(\partial^2 F(K,L)/\partial L^2)+ (\partial^2 F(K,L)/\partial t^2)\}$$
$$=1- o(1/n^2)\to 1$$

As the sample size or time goes to infinity, the observations converge to the equilibrium, and their variances tend to decrease towards zero. *The superconvergence probability* implies $p(x)=1- o(1/p(x)n^2) \to 1$, such that the control policy steers the output to the maximum equilibrium within all finite times $t<\infty$.

Definition 4: The model is a non-constant entire function F(.), where the equilibrium is not a constant. Each variable can be solved as the endogeneous or exogenous variable, or the state or policy variable. The given optimal policy steers the observations of the state variable to converge towards the equilibrium, $\max_{K,L} F(K,L)=Y^*$. The recursive, reduced, and closed form is a link between the model and the solution of each variable.

State variables and control policies exhibit interactions and serial correlation. Nevertheless, under the perfectly competitive economy, the competitive equilibrium achieves the first best solution (see Appendix 1). There is zero economic growth. From (2.1) through (2.6), at equilibrium, the Hamiltonian function H(.) is simplified as

$$(3.1) \qquad H=\left(\frac{1}{1+\rho(t)}\right)^t(C(t))+ \lambda (Y(t) - C(t)-G(t))+ \lambda (F(K, L) - rK - wL)$$

where λ is the Lagrangian multiplier. The optimal real interest rate equals the marginal product of capital and nearly equals the rate of time preference which is unobservable and time-varying due to stochastic death.

Assumption 3: The non-zero equilibrium exists as the benchmark attractor and is used to detect the deviations of abnormal data. When the economy grows and the output is shifting, the output growth is a quasi-stationary equilibrium. In a complete economic environment, the market is imperfectly competitive and wages are sticky. This composite hypothesis is that as the demand for resources increases, a sustainable ouput growth, the inflation rate, and the real interest rate are all positive and unique.

(3.2) $i(t) = r^{**}(t) + d \ln P/dt + \theta \, d\log Y/dt + v(t)$,

$\theta = 0$ if $r^{**} = \partial F(K,L)/\partial K$ and $d\log Y/dt = d\log Y^*/dt$,

$\theta > 0$ if $r(t) < r^{**}$, and $d\log Y/dt < d\log Y^*/dt$,

$\theta < 0$ if $r(t) > r^{**}$, and $d\log Y/dt < d\log Y^*/dt$

where an asterisk denotes the equilibrium state variable. Two asterisks denote the optimal (extreme) policy which stabilizes the equilibrium. Suppose Y* is the maximum equilibrium. r** is the optimal policy. The return for savings is the real interest rate r(t). In (3.2), the feedback rules of interest rates satisfy the first and the second-order conditions for minimizing the variance of output. The nominal interest rate i(t) is equal to the optimal real interest rate r**(t) plus the inflation rate π (=dln P/dt), when the output growth equals the equilibrium growth. P is the domestic price level. The stabilization parameter θ varies over time. v is noise. Unlike Fisher's concept of expected inflation, the inflation rate here is the backward-looking average of, say, the previous four quarters' GDP price inflation.

Suppose the Hamiltonian function H(.) in (3.1) is homogeneous of degree one:

(3.3) $H = U(C, \rho) + \lambda(Y - C) = U(C, \rho) + (\lambda Y - \lambda C)$

(3.4) $H = U(C, \rho) + (PY - PC)$ if $\lambda = P$

where the target PY is the nominal gross domestic product(GDP). The shadow price λ denotes the Lagrangian multiplier and can be approximated by either the real interest rate or the price level. To maximize the welfare U(C, ρ), the central bank adjusts the interest rate to control the output and the price level. The first derivatives of H (.) are the Euler equations:

(3.5) $dC/dt = dH/dC = (\partial H/\partial C)(\partial C/\partial t) + (\partial H/\partial r)(\partial r/\partial t) = f(C, r)$ for all $t \geq 0$

where f(.) is assumed to be a compact, strictly concave, and thrice continuously differentiable function. The marginal utility, dH/dC=exp(dC/dt)=p(C), is equal to the price p(C) of policy option or convergence probability.

Property 1 on the existence of a solution. *In a non-expansive convex compact domain, the solution of rational expectations exists as a fixed point, and is consistent with the optimization function.* Suppose output is growing. In the non-expansive closed domain, the growth rates and ratios of the variables converge to the equilibrium. Let dlog C/dt $\approx$ dlogY/dt=x. The convergence probability implies that the p value is close to one, $\lim_{t\to\infty}$ p(x) = $\lim_{t\to\infty}$ exp(∂ H/ ∂ t) = $\lim_{t\to\infty}$ exp(dx/dt) = 1. As t $\to\infty$, x(t) $\to$ x*=x(t-1); $\lim_{t\to\infty}$ dx/dt=x(t) – x(t-1)=f(x*, r**)=0, and H(.) $\to$ H*(.) for H(.) $\le$ H*(.).

The general solution of (3.5) is approximated by various envelope conditions,

(3.6) ∂ H/ ∂ t= f(x*, (r**))

$=(\partial H/\partial x)(\partial x/\partial t)+(\partial H/\partial r)(\partial r/\partial t)$

$=(\partial H/\partial x)^2(\sigma^2)^{-1}$

$=\theta_2(\partial^2 H/\partial x^2)+\theta_4(\partial^2 H/\partial r^2)=0$

where x is output growth. $\sigma^2\approx(\partial^2 H/\partial r^2)$ is the volatility of the control policy which stabilizes and steers the state variable towards the equilibrium.

The econometric process is as follows:

Step 1: Estimate the equilibrium and optimal policy

The Euler equation (3.6) is approximated by the dynamic quadratic regression as:

Min $\sum_{t=1}^{n} v_3^2(t)$

(3.7) Δ C(t)= θ_0 - θ_1 C(t-1)+ θ_2 C^2(t-1)- θ_3 r(t-1)+ θ_4 r^2(t-1) +v_3(t)

where dC/dt $\approx \Delta$ C(t)= C(t) – C(t-1). The equilibrium is Y* $\approx$ C*= $-\theta_1/2\theta_2$; and the optimal policy is r**= $-\theta_3/2\theta_4$. Suppose v=(v_1, v_2,…) is a vector of i.i.d. random disturbances with a mean of zero and a constant variance; in disequilibrium, the expected noise is not zero, E(v) $\neq$ 0 (*ad hoc*).

Step 2: Reduce the heteroschedasticity:

(3.8) $\Delta C(t)=\theta_2(C(t-1)-C^*)^2+\theta_4(r(t-1)-r^{**})^2$

$+\theta_5(C(t-1)-C^*)(r(t-1)-r^{**})^2$

$+\theta_6\,\Delta C(t-1)+\theta_7\,\Delta C(t-2)+v_4(t)$

where in (3.8), the optimal path C*(t) is integrable. In our regression the integration is close to the summation when the sample size increases. Suppose the first derivative is positive, dY/dr= $\theta_1\ge 0$ and the second-order derivative is negative, $d^2Y/dr^2=\theta_2<0$, the solution (C*, r**) satisfies the necessary and the sufficient condition of maximization.

(3.8) is a nonexpansive closed domain.

Step 3: Test the convergence probability, p(C)=exp(dC/dt)=F(C*, r**)+v, through Student t_n statistic, and Durbin-Watson statistic.

Property 2 on the stability. The stability of the non-zero equilibrium output Y*(t) requires an active policy with the negative parameter but a passive policy with a positive response; for dC/dt=θ_5(C(t-1)-C*) and C*>0, θ_5<0 if C(t-1)>C* and θ_5>0 if C(t-1)<C*.

Property 3 on strong and super-convergence: In the steady state, $\lim_{t\to\infty}\Delta$ Y(t)=0; $\lim_{t\to\infty}$dK/dt=0 with a strong convergence $\lim_{t\to\infty}$p(C)=exp(dC/dt)=1. The super-convergence probability is p=exp(dC/dt) $\to 1$ within a finite time n$<\infty$, as the interest rate steers the output and consumption towards the equilibrium, that is, as r$\to$r**, C$\to$C*$\approx$Y*.

Definition 5: The maximum equilibrium is the maximum sustainable output Y* such that Y(t+1)$\le$Y*=Y(t)$\ge$Y(t-1). The equilibrium implies dY/dt=0. Its necessary and sufficient conditions are dY/dt=$\theta_1\ge 0$ for Y(t) $\le$Y*, and dY/dt=θ_2<0 for Y(t)>Y*; conversely, θ_1<0 and θ_2>0 denotes the minimum equilibrium. If the maximum equilibrium equals the minimum equilibrium, the solution is unique for θ_1>0 and $\theta_2\ge 0$.

Definition 6 on bifurcation points: The stability implies that the expected value is the equilibrium. The errors follow a Brownian motion with mean zero, $\lim_{t\to\infty}$ dY/dt=$\lim_{t\to\infty}$dC/dt=$\lim_{t\to\infty}$f(C*, r**)=0. The turning point (Y*, C*, r**) is a bifurcation point.

In the investment-saving (IS) equation, output and the interest rate are negatively related.

(3.9) Y=θ_3r+v_1 for dY/dr=θ_3<0 if Y(t)>Y*(t)

The interest rate rule is that when output increases, the interest rate tends to rises:

(3.10) r=θ_4Y+v_2 for dr/dY=θ_4>0 if Y(t)<Y*(t)

where as output decreases, Y<Y*, the interest rate needs to fall, r<r**, and vice versa. *Definition 7 on limit cycles*: *If the equilibrium is not equal to the optimal policy, x*$\ne$x**, limit cycles occur due to delay impacts.* The optimal policy r*$\ne$r** is not sustainable with an unbalanced budget. In equation (3.8), Δ C= θ_7 Δ C((t-t*)^2) implies that t*=-θ_6/2θ_7 is the cyclical period.

Theorem 1: Dynamic consistency implies that the equilibrium tends to equal the expected value, x*=E(x)=x(t)=x(t-1) and C*=E(C), and is consistent with the support of policy r=r** in the optimization function. The super-convergence probability is close to one,

p(x) =exp(dx/dt)=exp(0) $\to 1$ as r=r**, x $\to$ x* for t $\in$ (0, n)

Proof: Suppose the likelihood ratio is 2log $\lambda \equiv$ 2log p(x*)/p(x) $\equiv \log(\frac{p(x^*)}{p(x)})^2 > 0$

where p(x*) is the maximum likelihood estimator. x* is the equilibrium growth. Suppose

(3.11) p (log $\lambda > \theta_2\, \sigma_x^2$)= p((log p(x*)- log p(x))> $\theta_2\, \sigma_x^2$)=p($\sigma > \theta_2\, \sigma_x^2$)< α

or equivalently, p(x)=p((dlog C/dt $\geq \theta_2\, \sigma^2$) $\leq \alpha$ for dlogC(t)/dt=x(t)

where p(x*)>p(x) as x*>x. The critical probability is α, as x*-x= $\sigma_x > \frac{t_n^2}{2} = \theta_2\, \sigma_x^2$.

Let $\theta_2\, \sigma_x^2 = \frac{1}{2}\frac{1}{n}\sum_{t=1}^{n}$ (x(θ)-x*(θ))^2 $= \frac{t_n^2}{2}$. As the observations converge to the equilibrium, the equilibrium equals the expected value and is the center with a central distribution of the Student t_n statistic. The convergence probability is ensured and bounded by the variance of policy.

If p(x)= p(reject $H_0: \theta$ (x*)=0) $\leq$ p(dlog C/dt> $\theta_2\, \sigma_x^2$) $\equiv \alpha \to 0$,

as r $\to$ r**, C $\to$ C* and $\lim_{t\to\infty}$ dlogC/dt=f(C*, r**)=0.

$\lim_{i\to\infty}$ p(accept $H_1: \theta$ (x*) $\neq$ 0) $\geq$ p(dlog C/dt $\leq \theta_2\, \sigma_x^2$) $\equiv$ 1- α $\to 1$ for t< ∞, dlogC/dt=0.

(3.12) p(C)=p(θ_2 (C(t)- C*(t))$^2 \leq \theta_4$ (r(t) – r**)^2) $\equiv$ 1- α $\to 1$ for t< ∞, dlogC/dt=0.

where the interest rates have mutual causality and interaction with consumption expenditure and steer the consumption to converge. Thus the equilibrium is forward and backward consistent. However, the optimal equilibrium (C*, x*) is a backward and forward consistent solution:

C(t)=C(0)exp(tx(t)) $\leq$ C(0)exp(tx*(t)) and C(0)=C(t)exp(-x*t) for all t>0, x $\leq$ x*

The monotonicity implies that the consumption monotonely increases, C(0)<C(1)<…<C(n) and logC*(t)=logC(t-1)+x*(t) for x*>0. As t $\to \infty$ and p(x) $\to 1$, thus the equilibrium x* is the uniformly most powerful unbiased (UMPU) estimator which coincides with the maximum likelihood estimator (MLE). The general test covers the Wald test, the likelihood ratio(LR), and the Lagrangian multiplier(LM) tests:

p(x)=exp(dx/dt)=exp(log{ θ_2 (x(t-1)- x*(t))^2/ θ_4 (r(t-1) – r**)^2}) $\to 1$

as $r \to r^{**}$, $x(t) \to x^*(t)$

which holds for finite sample sizes, $0< t< \infty$. If r=r**, x=x*. The noises v are independently, identically and normally distributed with mean zero and follow Brownian motion. The super-convergence implies that the equilibrium x(t-1)=x* is attained within a finite time or sample size. Like the χ^2 statistic in the Hilbert space, the quadratic regressors are independent, additive, deductive, and interactive. Q.E.D.

Theorem 2: In comparative dynamics, the output x declines, if $dx/dt=\theta_4 \ (r(t-1)-r^{**})^2<0$. $\theta_4<0$ implies that if beyond the optimum r>r**, a further increase in real interest rate has negative impacts upon the output x.

$p(x)=\exp(dx/dt)\approx\exp(x(t)-x(t-1))=\exp(\theta_2 (x(t-1)-x^*)^2+\theta_4 \ (r(t-1)-r^{**})^2)>0$

for $\theta_2<0$ and $\theta_4<0$, implying that x(t) is decreasing and the equilibrium x* is maximal.

Proof: Let Δ be the backward difference operator. Suppose an equation is

$$\Delta \ x(t)\approx x(t)-x(t-1)=\theta_2 (x(t-1)-x^*)^2=\theta_2 \ \chi^2 (t-1)>0,$$

where $x(t)-x(t-1)=\theta_2>0$ when the value x(t) is increasing. The parameter estimator $\hat{\theta}_2$ is

$$\hat{\theta}_2 =(\sum_{t=1}^{n} \Delta x \ \chi^2(t-1))(\sum_{t=1}^{n} \chi^2 \ (t-1)\chi^2(t-1))^{-1} \qquad \text{for } \chi^2(t-1)>0$$

$$\hat{\theta}_2 =\theta_2+ (\sum_{t=1}^{n} v(t) \ \chi^2(t-1))(\sum_{t=1}^{n} \chi^2 \ (t-1)\chi^2(t-1))^{-1} \qquad \text{for } \chi^2(t-1)>0$$

where the estimate $\hat{\theta}_2$ is the unbiased and consistent estimator of θ_2 at equilibrium, when x(t-1)=x*, $\chi^2(t-1)=0$, and E(v)=E(dx/dt)=0. Beyond the maximum x*, $\Delta x(t)=\theta_4 \ (r(t-1)-r^{**})^2<0$. As $t\to\infty$, the estimators, $E(\hat{\theta}_2)=\theta_2$ and E(x)=x*, are unbiased and consistent with the minimum shrinking variance $\frac{1}{n}\sum_{t=1}^{n} (x(t-1)-x^*)$:

$$dx/dt\approx x(t)-x(t-1)=\theta_2 (x(t-1)-x^*)^2<\theta_2 (x(t-1)-\bar{x})^2 \quad \text{for all } x^*\neq \bar{x} \text{ and } t>0. \text{ Q.E.D.}$$

4. Empirical Evidence of New International Macroeconomics

The link of the model and the following examples is the recursive, reduced, closed form of the law of motion. The equilibrium solution satisfies the integration and differentiation of the non-constant entire function of the model. The equilibrium of consumption is stabilized by the optimal real interest rate and the optimal inflation rate.

Example 4.0 Economic Growth

Suppose economic growth is a function of fiscal and monetary policy or the interest rate and the money supply. In Taiwan, the equilibrium output growth is 9% and the optimal real interest rate is 2.5%. The quarterly data are more able to reflect the dynamic cycles. The data used are available for the period 1983:1 - 2001:4, as published by the government in Taiwan. Let x =dlog Y/dt is output growth; r is the real interest rate. M is the money supply (M1). The econometric process of equations (3.7) and (3.8) is as follows:

Step 1: Compute the dynamic quadratic regression

(4.1) $$\Delta x(t)=-3.48+0.09x(t-1)+0.05x^2(t-1)+0.73r(t-1)-0.13r^2(t-1)$$

(-2.74) (0.46) (3.70) (1.30) (–1.93)

$$+0.07(dlogM/dt)-0.005(dlogM/dt)^2$$

(1.12) (-3.70)

$$+ 0.50\Delta x(t-1)-0.20\Delta x(t-2)$$

(4.42) (-2.31)

R^2 =0.78 $\bar{R}$ =0.75 n=76 D.W.=1.61 1st order autocorrelation =0.18

where the values in the parentheses are Student t_n statistic. Δx(t)=x(t)-x(t-4); and x=dlogY/dt is the annualized growth rate of national income Y.

Step 2: Compute the equilibrium growth. From (4.1), x*=θ_1=0.09=9%>0. For θ_1, θ_2 >0, x* is the unique solution. The maximum real interest rate is r**=-0.73/(2)(-0.13)$\approx$2.5%; the maximum growth rate of money supply M1 is dlogM*/dt=-0.07/2(0.005)=7%.

Step 3: Reduce the heteroschedasticity and estimate the convergence probability:

(4.2) $$dlogx/dt\approx \frac{x(t)-x(t-4)}{x(t-4)}=13.88-0.49(x(t-1)- 9\%)^2+0.51(x(t-1)- 9\%)(r(t-1)- 2.5\%)^2$$

(1.83) (-2.10) (2.75)

$$+0.86\frac{x(t-1)-x(t-5)}{x(t-5)} -0.37\frac{x(t-2)-x(t-6)}{x(t-6)}$$

(7.93) (-3.32)

R^2 =0.63 $\bar{R}$ =0.61 n=76 D.W.=2.09 1st order autocorrelation =-0.04

where the convergence probability is p(x)=exp(-exp(x))=$\frac{1}{1+\exp(v^2)}$ = R^2 =0.63. v is the percentage errors. If dx/dt=v=0 and x=x*, then p(x)=0.5. The error estimates are close to a lognormal distribution. Since output growth sometimes is negative, we do not take the logarithm on x and let dlogx/dt$\approx$(x(t)-x(t-1))/x(t-1). In equation (4.1), if dx/dt=0 and r=r**, θ_1 x+ θ_2 x^2=0.09x+0.05x^2>0; in (4.2), dlogx/dt $\approx$ d^2logY/dt^2 $\approx$ (x(t)-x(t-1)/x(t-1) $\approx$-0.49(x(t-1)- x*)$^2 \leq 0$. Therefore, $0 \leq x^* \leq 9\%$. The upper bound of the equilibrium growth is x*=9% .

Table 1. Analyses of Equilibrium and Optimal policy in Taiwan (1983-2001)

	Mean	Standard Deviation	Minimum	Maximum
Output growth x	8.2%	5.89	-4%	24%
Money growth	13%	12.8	-6.8%	51%
Real Interest Rate	1.9%	7.83	-41%	9.6%
Optimum real interest rate	2.5%	1.3		
Equilibrium output growth	9%	3.0		

Note:The standard deviation of the optimal interest rate and equilibrium growth are obtained from the Student t statistic in the regression. The real interest rate is defined as the central bank's discount interest rate minus the consumers' price inflation rate.

The following preliminary estimates are based on the annual U.S. data from International Financial Statistics (IFS) for 1953-1997.

Example 4.1 Consumption Function

In the United States, the equilibrium consumption growth is 5% and the optimal real interest rate 2.5%. Let C be the aggregate consumption; dC/C$\approx\Delta$C/C is the growth rate of consumption. r is the real interest rate, which is the Treasury bill rate minus the inflation rate of the consumers' price index. We estimate the maximum equilibrium, dln C*/dt$\approx\Delta$C*/C*=5.5%, stabilized by the real interest rate, r**=0.025 over a finite sample or horizon to be

(4.3) $d(\Delta C/C)/dt = -0.008 + 0.64\,(\Delta C/C) - 7.23(\Delta C/C)^2 + 1.25r - 15.09r^2$
(0.58) (1.86) (-4.07) (4.82) (-2.37)

That is

(4.4) $d(\Delta C/C)/dt = -72.3(\Delta C/C - 0.055)^2 - 15.09(r - 0.025)^2$
(-4.07) (-2.37)

$$-18.19(\Delta C/C-0.055)(r-0.025)+v_1(t)$$
(-2.42)

$\bar{R}^2$=0.83 D.W.=1.89 1st order autocor.=0.23; Dep. Mean =0.00008

where all the values in the parentheses are Student t_n statistics in the following regression. Beyond the equilibrium, as the interest rate increases, consumption decreases.

Example 4.2 The turning point of the real interest rate is around 1.2% for the whole model.

Equations (4.5) and (4.6) show the demand and supply curves as follows:

Table 2

Consumption Growth and Interest Rate Parity

	(4.5)		(4.6)	
	Demand Curves If r>1.2%		Supply Curves If r<1.1%	
Dependent variables	Δ C/C (4.5a)	Δq (4.6a)	Δ C/C (4.5b)	Δq (4.6b)
constant	0.02 (2.80)	-0.07 (-1.74)	0.03 (4.80)	-0.4 (-2.1)
real interest rate r	-0.51 (-3.36)	4.22 (2.26)	1.08 (1.63)	1.21 (5.2)
$\bar{R}^2$	0.38	0.28	0.21	0.3

Note: In (4.5a) and (4.5b), the dependent variable ΔC/C is the consumption growth. The explanatory variable r is the real interest rate, i.e., the treasury bill rate minus the inflation rate. In (4.6a) and (4.6b), the dependent variable q is the real exchange rate. If r>1.2, the demand curve implies that as the interest rate increases, consumption growth tends to decline; and vice versa. The turning point is r=1.2%. When interest rates are too high or the inflation rate is too low, the further increase in interest rate tends to reduce output growth and consumption and cause the depreciation of domestic currency; and vice versa. When the interest rate is either too high or too low, the output growth falls and consumption declines. Any deviation from the optimum interest rate leads to depreciation. The values in parentheses are Student t_n statistics. The system is estimated as two sets of simultaneous equations. (4.6a) and (4.6b) imply the real interest rate parity, $\Delta q/q=\theta(t)(r(t)-r^*(t))$, where r*(t) is the foreign (optimal) real interest rate and the real exchange rate is q*=1.

Example 4.3 The real exchange rate under the floating exchange rate regime

The equilibrium real exchange rate is q*=1 and is consistent with the purchasing power parity. Let q be the real exchange rate on the pound/dollar, which is adjusted by the value-added price deflators. The quarterly data of the real exchange rate over 1978.1 to 1996.4 are used, as reported by *International Financial Statistics*. The pound/dollar exchange rate is predicted by the following preliminary estimate:

Suppose that t denotes time or quarters. Our unique equilibrium of the real exchange rate, q*=1, is derived as follows:

(4.7) $\Delta q = q(t) - q(t-1) = \theta_0 + (\theta_1 - 1)q(t-1) + \theta_2 q^2(t-1)$

$$= -0.17 + 0.46q(t-1) - 0.23q^2(t-1) + v_3(t)$$

(0.8) (3.38) (-1.2)

$$= -0.23(q(t-1) - (0.46)/2(0.23))^2 + v_3(t)$$

$\bar{R}^2 = 0.8$; D.W.=1.71

where Student t_n statistics are reported in the underlying parentheses, but are biased due to endogeneity and autocorrelation. It is noteworthy that the Dickey-Fuller test is applicable to the linear model. By treating $(q(t-1) - 1)^2$ as one explanatory variable, we can reestimate the nonlinear model and reduce nonlinearity and nonstationarity.

Example 4.4 The optimal fiscal and monetary policy

The equilibrium real interest rate is around r=0.8%. The optimal share of budget deficit to gross national product (GDP) is 3.5%.

(4.8) $dr(t)/dt = 0.032 + 0.77r(t-1) - 0.48\, r^2(t-1) + 11.78(T(t)-G(t))/Y(t))$

(1.05) (1.92) (-5.65) (3.13)

$$+166.73(T(t)-G(t))/Y(t))^2$$

(2.80)

$\bar{R}^2 = 0.60$ D.W.=1.93

where the equilibrium in real interest rate is determined by the budget deficit.

(4.9) $\Delta r = -0.48(r(t) - 0.8\%)^2 + 166.73\,((G(t) - T(t))/Y(t) - 3.5\%)^2$

where (G-T)/Y is the share of budget deficits in GDP. The three percent of optimal deficit in the United States satisfies the requirement of member countries for the European Monetary Union. θ_4 =166.73>0 implies that the real interest rate tends to increase if the budget deficit deviates from the minimum 3.5%.

Example 4.5 The balanced budget rule, no crowding-out effect, and economic growth

The maximum equilibrium of output growth is 5% , the maximum government spending is 25%, and the maximum flat income tax rate is 23%. When the government spending exceeds the optimum share (G/Y)**>25%, and the tax revenues exceed (T/Y)>23%, the expansionary fiscal policy crowds out investment and consumption, and reduces output growth.

(4.10) $dx/dt=-0.08 + 1.27\, x(t) - 0.57x^2(t) + 0.63(G(t)/Y(t)) - 1.36(G/Y)^2$
(-4.05) (6.01) (-4.90) (2.45)
$+0.68(T(t)/Y(t)) - 1.77(T(t)/Y(t))^2 + v_4(t)$
(1.89) (-1.41)

$\bar{R}^2$=0.77; D.W.=1.87

(4.11) $\Delta x = -0.57\, (x(t) - 0.05)^2 - 1.36\, ((G(t)/Y(t) - 0.25)^2$
$-1.77(T(t)/Y(t) - 0.23)^2 + v_4(t)$

where x=dln Y/dt is output growth. Y is output (GDP); G is government spending; T is tax revenues. θ_2 =-0.57<0, θ_4 =-1.36<0 and θ_6 =-1.77<0 imply that output growth is at the maximum 5% and starts to decrease if the government spending (G/Y) and the income tax rate (T/Y) deviate from the optimum, respectively. From (4.8) and (4.9), the stabilization policy r*=0.8% is actually achievable by the maximum efficient share (G/Y)** =25% of government spending in GDP and the maximally efficient income tax rates, (T/Y)** =23%.

Example 4.6 No blow-out of the inflation rate

The response of the optimal money supply(M1) to inflation rates is

(4.12) $\Delta M/M = 0.02 - 2.04\, dln\, P/dt + 17.7(dln\, P/dt)^2$
(1.39) (-2.66) (2.40)
$= 17.7(dln\, P/dt - 5.7\%)^2$

$\bar{R}^2$=0.46 D.W.=1.80

where, when the inflation rate exceeds the maximum of 5.7% , the Federal Reserve Bank tends to reduce the money growth in order to ensure no blow-up of inflation. Otherwise, people desire the real money balances (M/P).

Example 4.7 The unemployment rate is stabilized by the optimal interest rate.

The equilibrium unemployment rate is around 2% and the optimal interest rate is around 1.2%.

In Table 3, Equations (4.13) and (4.14) show the optimal feedback rule. When the inflation rate is high, there is no trade-off between inflation and unemployment. An increase in the interest rate tends to reduce both inflation and unemployment rate. Equation (4.13) is the Phillips curve and implies a trade-off. Beyond r=1.2%, when the interest rate decreases, the inflation rate and nominal wages rise, and the unemployment rate falls. Rational expectations provide no policy choice.

Table 3

Causality analysis of the unemployment rate

	(4.13)	(4.14)
Dependent variable	u(t) for all r(t)	u(t) for r>1.2%
Lagged unemployment rate u(t-1)	0.82 (11.7)	0.75 (8.70)
Real interest rate r(t-1)	- 42.3 (-2.49)	57.0 (1.66)

Note: The system is estimated as a part of simultaneous equations. The Student t_n statistics are in parentheses. u is the unemployment rate; r is the real interest rate.

Similarly, in Singapore, the unemployment rate is stabilized by the interest rate and real wage growth as follows:

$$(4.15)\quad \Delta u(t)=\underset{(1.18)}{0.01}\ \underset{(-2.21)}{-0.46u(t-1)}\ \underset{(-1.14)}{-0.12r(t-1)}\ \underset{(3.21)}{+7.05r^2(t-1)}\ \underset{(-0.15)}{-0.004\dot{w}(t-1)}\ \underset{(-0.29)}{-0.07\dot{w}^2(t-1)}$$

$\bar{R}^2$=0.61; D.W.=2.4

$$(4.16)\quad \Delta u(t)=-\ 0.46\ (u(t)-2\%)+7.05(r(t)-0.015)^2-0.07(\dot{w}(t)-0.03)^2+v_5(t)$$

where u is the unemployment rate; r is the real interest rate; $\dot{w}$ is the real wage growth. As $\Delta u(t)=0$, $u^*=0.01/0.46\approx 2\%$ is the minimum equilibrium of the unemployment rate.

Example 4.8 Investment function is a roundabout process of production:

The minimum equilibrium investment rate is 9% when the wage growth is as high as 10%.

(4.17) $d(I/Y)= -0.007 -0.20(I/Y) +1.50(I/Y)^2 +0.33(T/Y) -1.44(T/Y)^2$
(-1.68) (-1.58) (2.31) (3.17) (-2.99)

$+0.15(d(PY)/PY) +0.05(d(PY)/PY)^2+0.13(wg) -0.07(wg)^2$
(5.96) (2.22) (3.67) (-1.38)

That is

(4.18) $d(I/Y)=1.57((I/Y) - 9\%)^2-0.85(T(t)/Y(t) -11\%))^2$

$-0.05((d(PY)/(PY)-12\%)^2 -0.07(wg -10\%)+v_2(t)^2$

R^2=0.76 D.W.=2.00 1st order autocor.=-0.02

where I is investment; Y is the gross domestic product(GDP); P is the GDP price deflator. dPY/PY is a proxy of growth rate of the nominal GDP, or sales, or the cash flow. wg is the growth rate of nominal wages. T/Y is the income tax rate. Investment is determined by income tax rates, (nominal) cash flows (or Tobin's Q), and nominal wage rates.

Remark 1 on conditional investment: Beyond the optimum, the income tax rates have a negative impact upon output growth and investment. In the United States, θ_2 =1.57>0 implies that the investment I/Y=9% is the minimum investment rate, as the wage growth, inflation, and nominal GDP increase. Hsieh(2003c) finds that in Taiwan, the minimum investment rate is around 17%, when the flat income tax rate increases beyond 17%, and when the nominal GDP growth is as low as 3%.

Example 4.9 The limited cycles exist among output growth, interest rate and wage growth:

(4.19) $(1/13)\ x=0.01 +(1/13)0.62\dot{w} - \dot{w}^2 + x^2$
(3.0) (2.28) (-1.6) (9.4)

or $a^2=(x -0.04)^2 -(w -0.02)^2$

$\bar{R}^2$=0.8 D.W.=2.2

Furthermore,

(4.20) $a^2=13.3(r -0.02)^2 - 27.5(\dot{w} -0.02)^2$
(3.35) (2.77)

$\bar{R}^2$=0.38 D.W.=1.82

where x is output growth; r is the real interest rate; $\dot{w}$ is real wage growth.

Remark 2 on numerical methodology. Suppose the central bank's target is the nominal output, H(PY)=PY, which is the gross domestic product (GDP). The saddle proximal method (Shpirko, 2000) has the derivative form dY/dt=-bP, dP/dt=bY. b is a positive constant. Thus, YdP+PdY=0. The solution is a ball, $Y^2+P^2=r^2$. For time t ≥ 0, the numerically iterative trajectory (Y(t), P(t)) may not converge to the unique saddle solution (Y*(t), P*(t))=(0, 0). Thus the prediction may not be correct. Our alternative is dynamic quadratic regression. The convergence to the equilibrium is controlled by the feedback policy.

Remark 3 on optimum equilibrium: The equilibrium is time-invariant and has no bubbles. The optimal policy r** is gotten from an optimization model with market-clearing equilibrium Y*. This extreme policy is efficient since it minimizes the variance of output Y(t). The solution Y* is feasible since the feedback rule ensures its convergence $\lim_{t\to\infty}$ dY/dt=0, as r $\to$ r**, Y(t) $\to$ Y(t-1)=Y*. The estimate is variance-stationary and risk-bounded with a probability close to one, p=$\lim_{t\to\infty}$ exp(dY/dt)=1.

Example 4.10 **Sensitivity and Rubustness Tests on Interest Rates and Inflations**

Remark 4: The optimal interest rate corresponds to the optimal inflation.

The Fisherian real interest rate is neither unique nor stable, as different rationally expected inflations are yielded by different models. It will not be optimal nor unique unless it satisfies the stability condition. Banks with rational expectations tend to raise the nominal interest rate in proportion to the excess of the domestic inflation rate over the optimal foreign inflation. If the controllable response exists and is stable, we can estimate the comparative-static control rule. When the nominal interest rate rises faster than the inflation rate, the real interest rate rises while inflation rates fall.

Using the U.S. monthly data over 1954.1 through 1988.6, as reported in the IMF's *International Financial Statistics,* we ascertain an optimal real interest rate. The interest rate used is the three-month treasury bill rate. Inflation denotes consumers' price-index inflation. If the nominal interest rate, i, rises faster than inflation,

π, the real interest rate will rise. We estimate the nominal rate as the switching feedback rule:

$$i = a_0 + b\pi + v \qquad a_0 > 0,\ b > 0$$

where π is the inflation rate. v is the residual error. Beta, b, denotes the correlation and covariances between the expected inflation and the interest rate (or the rate of stock return). When rational expectations cause the nominal interest rate to rise as fast as the expected inflation, b=1. a_0 equals a constant real interest rate. The natural rate exists around a_0= r*=2%, implying the Fisher effect as follows:

(4.21a) $i(t)=1.84+1.1\pi(t)$ as $\pi(t) <2\%$ $\bar{R}^2 =0.38$
(12.11) (9.00)

(4.21b) $i(t)=3.05 + 0.66\pi(t)$ as $\pi(t)>3\%$ $\bar{R}^2 =0.41$
(10.26) (14.12)

(4.21c) $i(t)=3.37+0.62\pi(t)$ as $\pi(t)>4\%$ $\bar{R}^2 =0.30$
(6.07) (8.77)

From (4.21a)-(4.21c), Fisher's real interest rate is around r*=2%:

(4.22) $i=2.0+1.0\pi$ for $\pi(t)=3\%$

where the values in parentheses are the Student t_n statistics. The optimal stable inflation is 3%. In (4.21a), if b>1, the actual inflation falls below π <2%. This implies that the nominal interest rate is rising faster than the inflation rate. The real interest rate, r, tends to rise, while inflation will fall. As in equations (4.21) through (4.22), high interest rates, r(t)>r*, will generate cumulative deflation; the residuals of Eq.(4.21a) are, as expected, mostly positive. Conversely, in Eq.(4.21b), low interest rates, r<r*, generate cumulative inflation; its residuals of Eq.(4.21b) are mostly negative before residuals are adjusted by the methods such as co-integration. Since various models of rational expectations may not yield a stable expected inflation, they could yield multiple rates of Fisherian real interest. From (4.21) through (4.22), it is clear that the regression rejects Kinal and Lahiri’s (1988) hypothesis of the random real interest rate since Fisher’s effect is not random but varies systematically. During depression, the real interest rate rises procyclically while the inflation falls; and vice versa.

Figure 4.1. Output growth and Inflation in Saudi Arabia

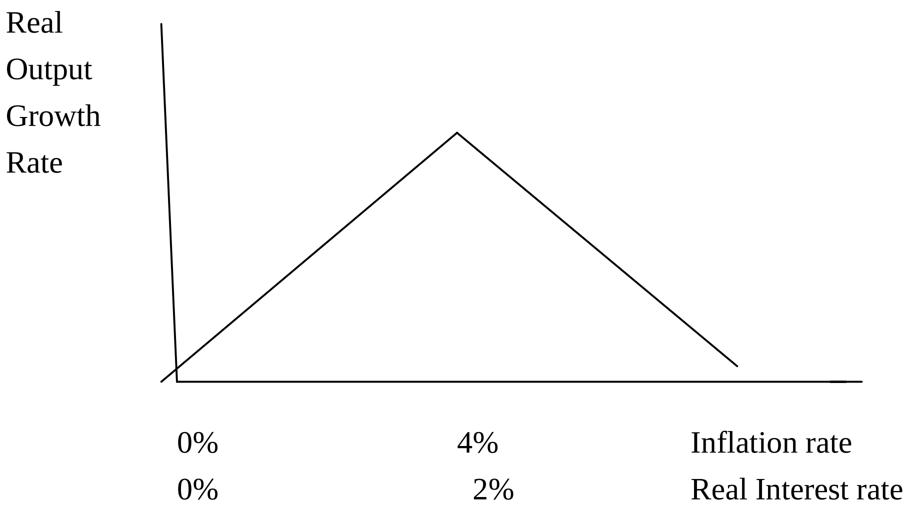
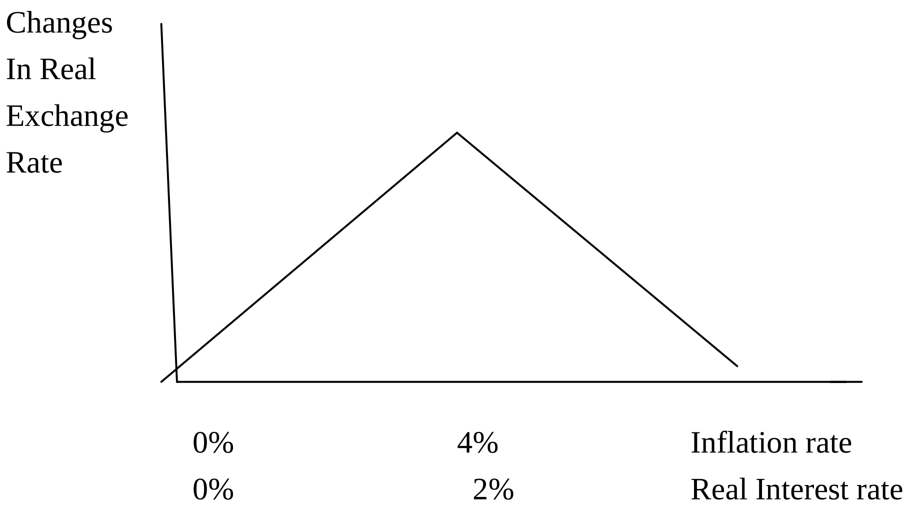


Figure 4.2. Exchange Rates and Inflation in Saudi Arabia



Example 4.11 **Sensitivity tests on internal and external balances**:

Before 1979 and after 1987, the Federal Reserve in the United States adopted the nominal interest rate target. As the nominal interest rate rises, the real interest rate also increases, letting the inflation rate respond in time lags. We can verify the turning point using the three month Treasury bill rate as a proxy for the Federal funds rate. During deflation, when the interest rate rises, output growth falls. The relationship between output and interest rates turns from a positive into a negative one. In simulations, a model of business cycles is as follows:

$$r(t+1)=b_0 + b_1r(t) - b_2r^2(t)- b_3x(t) + v_7(t+1)$$

where for i=0,1,2,3, b_i's are positive. v_7, the residual error, denotes shocks, such as money supply, velocity, and productivity shocks. r is the real interest rate; x is real output growth. Over business cycles, a different initial value r tends to yield a different solution x(t). In equilibrium, $\lim_{t\to\infty} dr(t)/dt=0$, the optimum interest rate is detected through a feedback polynomial,

$$x(t)=d\log Y/dt=d\log F(K(r), L(r))/dt= f(r(t)))= a_0+a_1r(t) + a_2r^2(t)+a_3r^3(t)+ v_8(t).$$

where a_i's are parameters. x (t)=f(K(r), L(r)) denotes the vertical axis; r is measured by the horizontal axis. We estimate the annual data over 1950-1994 for the United States and Taiwan, and 1960-1994 for Singapore, Brazil, Saudi Arabia and Mexico. When the variance of noise v increases, the results tend to deviate from the benchmark projections. The following figures show the representative trends across countries and over time. Fig.1.1 shows that the confidence region of the optimum real interest rate is around $0.0<r^*<4\%$. Its deviation tends to reduce the growth rates of output (real GDP) and other macro variables. In contrast, the horizontal axis for the conventional IS-LM curve denotes the output level instead of the output growth. In Fig. 1.2a, beyond the purchasing power parity, a further rise in real interest rates tends to either induce capital inflows and a real appreciation of the domestic currency, as in Brazil, or, after overshooting, cause net imports and a real depreciation of the domestic currency due to domestic ineffective demand, capital flight, and the exodus of the U.S. exporting industries. In Figs. 1.2b and 1.2c, when the real interest rate falls or the inflation rate rises beyond the optimum, there is a tendency to reduce the ratio of saving to GDP and the ratio of net exports to GDP. We find that a similar result occurred in Saudi Arabia, Taiwan, Japan, and other countries. The optimal inflation rate corresponds to the optimal real interest rate where the output growth is at a maximum. When the inflation rate is low, savings exceed the investment and net exports occur.

When the inflation rate is high, investment exceeds the domestic savings, and imports exceed exports. The high, yet stable output growth is a fixed point where the domestic saving is equal to investment. Furthermore, as the inflation rate increases to the optimum, the exchange rate tends to appreciate to the purchasing power parity. When the inflation and interest rates deviate from the optimum, the domestic ineffective demand leads to capital outflow. Output growth declines and the domestic currency depreciates. High inflation rates, however, could be associated with high or low interest rates since there are time-lagged impacts of interest rates upon inflation.

Example 4.12 **Evidence of Interest Rates and Unemployment Rates in Mexico**

As in Figures 2 and 3, in 1992 in Mexico, the real interest rate was around 2%, while the employment index attained a peak. The optimal real interest rate corresponded to the maximal output growth and employment, cheap credit, and balanced trade. Imports grew gradually for several years before 1994. The Mexican government liberalized, signed, and started operating a free trade area with North America (NAFTA) in January 1994. In December 1994 when the real interest rate rose beyond 6%, unemployment rates soared. The central bank announced its intention to devaluate the over-appreciated peso by 15 percent. Its net imports and capital flight led to a negative output growth in 1995. During the period of peso devaluation and depression, the domestic real interest rate was too high. After capital inflows, the Mexican government sold foreign currency, while reducing the domestic interest rate.

5. Concluding Remarks

This paper contributes a new generalized macroeconomic theory, stochastic growth theory and an exact technique to compute the optimal equilibrium and aggressive incentive policy along with convergence probability at p value. The generalized relationship between the state and the control variable is evolutionary and found to be negative and positive around the equilibrium and policy, and is widely applicable to the attractor with symmetry of demand and supply curves and shocks. The dynamic quadratic regression reduces the heteroschedastic errors into Brownian motion. Empirical evidence is consistent with the nonzero equilibrium growth and optimal policy. Sticky prices can shift with the demand and supply incentives.

Suppose the equilibrium value is the strike price of the policy option. Deviations from the equilibrium justify the policy option for the central bank. The price policy varies over time and stabilizes the output (sales). For example, the commitment to inflation or pollution taxes could be over-taxation. After the inflation or pollution has been abated, the reoptimization process will alter the earmarking rule of tax revenues. Here, the Lagrangian multiplier is represented by a proxy of the price and the interest rate. This real interest rate is an incentive for a balanced growth of consumption and investment. When the price inflation rate is higher or lower than the optimum positive inflation, either the cost of capital is too high, or the marginal product or the output growth decreases below the interest rate. Deviations from the optimum lead to declines in output growth and employment rates. This solves many conflicting theories about the coexistence equilibrium and no crowding-out policy.

Note: The behaviour of the sample mean is less stable than the equilibrium. Suppose the sample mean is $\frac{1}{n}\sum_{t=1}^{n}$ v(t). Assume that $2^m \leq n \leq 2^{m+1}$, and $\frac{1}{n}\sum_{t=1}^{n} v^2(t) \leq 2^{(m+1)\theta}$**.** With the covariance nonstationarity and with an additional parameter θ, the optimum equilibrium, θ(x(t)-x*)=v(t), converges to zero faster than the sample mean:

$$\limsup_{n\to\infty} \frac{1}{n^{\theta}} \left| \sum_{t=1}^{n} v(t) \right| \leq \limsup_{m\to\infty} \frac{1}{2^{\theta m}} \max_{0\leq k\leq 2^{m+1}} \left| \sum_{t=1}^{k} v(t) \right| = 0$$

with probability of one (see Ninness, 2000).

Appendix 1: The First-Best Policy Rules

Suppose consumers and firms maximize the utility and the capital value λK:

$$H=\sum_{t} \{\exp(-\rho(t)t)(C(t)) + \lambda(t)\frac{dK}{dt} + K(t)\frac{d\lambda}{dt}\}$$

As a first-order solution, the factor price is equal to the marginal product:

$\frac{\partial H}{\partial C(0)} = \frac{\partial U(C(0))}{\partial C(0)}$ - λ (0)=1- λ (0)=0 for U(C(t), ρ (t)) = $\sum_t$ C(t) exp(- ρ t) and t=0

$\frac{\partial H}{\partial C(t)} = \frac{\exp(-\rho t)\partial C(t)}{\partial C(t)}$ - λ (t)=exp(- ρ t) - λ (t)=0

$\frac{\partial H}{\partial K} = \lambda$ ($\frac{\partial F(K,L)}{\partial K}$ - r)=0 or $\frac{\partial F(K,L)}{\partial K}$ =r

$\frac{\partial H}{\partial K} = \lambda$ ($\frac{\partial F(K,L)}{\partial L}$ - w)=0 or $\frac{\partial F(K,L)}{\partial L}$ =w

$\frac{\partial H}{\partial K} = \frac{d\lambda}{dt}$ =- ρ exp(- ρ t)

The transversality condition is

$\lim_{t\to\infty}$ λ (t)(K(t) - K*) exp(- ρ t)=0 as $\lim_{t\to\infty}$ dK/dt=0

where K* is the desirable capital stock. The shadow price λ will compensate for the rent-free opportunity cost of the total productivity of all factors, including land (or technology progress). The price level of non- renewable and renewable resources tends to grow at a rate higher (or lower) than the nominal interest rate.

Appendix 2: Monetary Policy

Corollary 1 on the optimal inflation and feedback rule: The stability of the treatment effect of discretionary policy is ensured if the policy is time-consistent and stabilizes high, yet stable output growth.

Suppose Ω is a convex compact domain of the historical data. The maximum feasible domain is on the boundary $\partial\Omega$. By importance sampling, we estimate the first-order derivative as the optimal feedback rule to be:

(A2.1) ∂ H(C(r))/ ∂ r=f(x, r) $\approx$ a_{11} x(t)+ a_{12} r(t)+ v_9(t) on x, r $\in$ $\partial\Omega$

or

(A2.2) dlogM/dt=f(x, dlogP(t)) on x, r, M $\in \partial\Omega$

where (A2.1) is an optimal feedback rule of the interest rate. (A2.2) is the optimal money growth. It is assumed in both equations (A2.1) and (A2.2) that high output growth are correlated with and caused by the optimal interest rate, inflation rate, and money growth.

Our optimum equilibrium is detected through an approximate sampling of the first-order derivative of optimization. We select the observations of the optimal-experience subsample in which the annual output growth x(t) exceeds the average growth and in which the annual inflation rates are lower than the average inflation rate. If the observations x(t) with such good experiences are in the optimal subsample space, we give the dummy weight 1 on regression coefficients; otherwise, we give those bad observations the dummy weight zero.

Our purpose is to compare the different impacts of the average and the optimal response rules upon the performances of output and inflation. We estimate the U.S. data for the period 1954-1991, as published in *International Financial Statistics* by the International Monetary Fund.

Table A.1:
Payoffs of the U.S. Optimal Money Growth

	Actual (1954-1981)	Optimal (1954-1981)	Actual (1981-1982)	Optimal (1981-1982)
Inflation	4.58%	3.19%	6%	5%
Monetary Growth	4.6%	6.8%	5.5%	11%
Unemployment Rate	5.56%	5.22%	9.5%	5.2%
Output Growth	2.93%	5.49%	-1.08%	5.49%

Note 1: As predicted over 1983-87, the *ex post* actual average growth rate of the money supply(M1) rose to 10.6% while the inflation rate was lower than before. The *ex ante* prediction is consistent with the *ex post* actual value. The optimal feedback rule is estimated through the SAS and SPSS regression and based on the condition SELECT IF x(t)> $\frac{1}{n}\sum_{t=1}^{n}$ x(t) and dlnP(t)/dt< $\frac{1}{n}\sum_{t=1}^{n}$ dlnP/dt.

Note 2: In the United States, in 1903-1994, the income elasticity of the nominal interest rate i remained in the range of $-0.1 \le$ dlog (Y)/dlog i ≤ 1, varying from a negative value of -0.1 to a positive value of 1. The income elasticity of money is always positive, $0.8 \le$ d log Y/d log (M/P) ≤ 1.05. We cannot disentangle the effects of interest rates from the effect of output. As in Graham (1995), the income multiplier of the government spending is around –0.5<dY/dG=(a/b)<2.0, if $\Delta C=-a\Delta G+b\Delta Y+v(t)$ for various periods. At equilibrium, $\Delta C=C(t)-C(t-1)=0$.

During the period of deflation, increases in money supply may finance production, raise real output and reduce price inflation. Table A.1 shows that inflation would fall while the output growth would rise if the government followed the optimal money supply. The optimal real interest rates are in the range of from 1% to 2% . Under importance sampling, such as in Table A.1, the optimal money growth is a discretion rather than a rule because the money supply shows a time-varying delay of impacts upon output.

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