Cycles And Banking Crisis*

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Abstract

We extend Diamond and Dybvig's (1983)[11] model to a dynamic context where we study how the bank's financial stability is affected by successive withdrawal shocks during a crisis. We model a crisis as a series of these unanticipated events over a long period of time and not as isolated bank runs. We highlight the importance of banks' portfolio liquidity in surviving such crisis. The paper shows that external borrowing can smooth investment returns to guarantee that solvent but illiquid intermediaries can survive a crisis. In the presence of borrowing restrictions banks' liquidity exhibits an erratic behaviour.

*This paper derives from a chapter of my PhD thesis.

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1 Introduction

Bank runs are recurrent phenomena of considerable economic significance and historically they have been of major concern in the conduct of monetary policy. Runs can lead to bank failures, disruption of credit, and in most situations add a downward pressure to the real economy. Over the recent years, much work has been done, both theoretical and empirical, in examining the determinants of bank runs and institutional arrangements have been developed to avert the possibility of such events to occur. However, runs have been modelled in the literature as static phenomena and not as frequently recurring events. Indeed, some empirical research suggests that the banking sector has frequently experienced multiple run episodes over a given period of time. Caprio and Klingebiel (1996)[9] show that, from a sample of ninety episodes since 1970s, the average length of a bank crisis was 4.5 years. In this respect, the main contribution of this paper to the existing literature is the formulation of a model of banking crisis where successive withdrawal shocks take place over a long time interval. We use a three-period overlapping generations model to study the effects of a banking crisis on banks' liquidity, and as a consequence, on their solvency and survival. The paper shows that access to external borrowing plays an important role in the financial stability of the banking system. The cost of borrowing and the desired level of liquidity reserve holdings during the run episodes give a rise to different transition paths of equilibrium allocations during the course of the crisis.

The basic setup that we employ in this model is based Diamond and Dybvig's (1983)[11] framework which highlights the importance of asset transformation function of intermediaries in providing liquidity insurance to depositors, and explains their vulnerability to runs. In particular, in their seminal paper they focus on the liability side of the bank's balance sheet in a single-generation three-period model where a private preference shock is uncorrelated across agents, and the economy's productive technology has a long-term maturity. In this setting, the bank by pooling deposits can diversify risk so that the demand deposit contracts can provide liquidity insurance to depositors against their preference shock, and consequently improve on the competitive outcome. The one-sided asymmetry of information prevents the use of the standard Arrow-Debreu insurance contracts since the contracts can not be conditioned on depositors' privately observed preference shock. Therefore, when depositors' incentives are not distorted, the demand deposit contract can achieve the optimal risk-sharing equilibrium as a pure strategy Nash equilibrium. However, banks by providing liquidity become illiquid themselves. Another Nash-equilibrium may arise which is Pareto-inferior and is described as bank run. If depositors panic due to a commonly observed exogenous factor ("sunspot") and rush to withdraw their deposits, then the bank will not be able to honour all its liabilities and will finally fail. Bank runs are viewed as self-fulfilling prophecies where the public thinks that the bank is going to fail, and by rationally running to withdraw their funds it actually fails.

The literature pioneered by Diamond and Dybvig (1983)[11] has been primarily focused on the determinants of bank runs, and alternative ways to eliminate or reduce the possibility of the bank run equilibrium have been analysed. Similarly to Diamond and Dybvig (1983)[11] many authors have developed single-generation three-period models where they viewed bank runs as self-fulfilling prophecies (Smith (1984)[20], Engineer (1989)[13], Cooper and Ross (1998)[10] among others.). Others view bank runs as related to business cycle (Jacklin and Bhattacharya (1988)[18], Alonso (1996)[4], Alen and Gale (1998)[2] among others.). They argued that runs are triggered by depositors who receive information about an impending downturn in the cycle and withdraw their funds because they anticipate that the bank's portfolio of assets will not yield sufficient returns to meet their legitimate demands due to the low value of the fundamentals. An alternative approach has been introduced by Jacklin (1987)[17] and extended by Haubrich and King (1990)[16]. They show that the allocations that can be achieved through direct trading of firms' shares with a predetermined divided policy, or by financial institutions that issue tradeable securities (i.e. mutual funds), can provide the same liquidity insurance as depository intermediaries that issue standard debt contracts, and in addition, are free of bank runs.

Diamond and Dybvig's (1983)[11] model has also been extended to the realm of overlapping generations. This literature has focused on examining the respective roles of capital markets and financial intermediaries in providing liquidity insurance but little has been done in analysing bank runs. More specifically, in comparison to the single-generation three-period models, intergenerational transfers enable banks to invest a greater proportion of deposits in less liquid but more productive technology, and provide depositors with greater liquidity insurance. The intertemporal stock market allocation is dominated by the intermediated allocation, but these allocations become identical when an interbank market is introduced to the intermediated economy (Dutta and Karpur (1994)[12] and Fulghieri and Rovelli (1998)[15]). In this case, government intervention in the stock market can lead to second-best superior allocations (Bhattacharya and Padilla (1996)[6]). When uncertainty on the returns of the productive technology is introduced, an infinitely lived bank can achieve intertemporal smoothing of the returns of the risky technology whereas the competitive market can only achieve an intergenerational risk-sharing, and therefore is dominated (Allen and Gale (1997)[2]). The transition towards the Golden Rule steady state allocation can been modelled as a proposal game of generations between different financial configurations (Bhattacharya et al [7](1998)). Intermediation promotes economic growth by facilitating greater investment in the productive technology (Bencivenga and Smith (1991) [5]), but depositors' incentives may be distorted in a continuous time framework (von Thadden (2002)[21]). However, the limited amount of work that has been done in this literature on bank runs has modelled these shocks in a static way. First Bryant (1981)[8] argues that, due to the intergenerational transfer of resources, infinitely lived banks have a positive net worth and therefore are vulnerable to runs since agents of a given generation can gain from its collapse. Ennis and Keister (2003)[14] have developed an endogenous growth model where they show that run-proof contracts can promote economic growth. Qi's (1994)[19] findings suggest that apart from excessive withdrawals, bank runs in an intertemporal model can also occur due to a lack of new deposits.

In our attempt to model a banking crisis, similarly to Diamond and Dybvig (1983)[11], we view bank runs as sunspot phenomena which can not be anticipated by banks. Due to an extrinsic uncertainty that influences depositors' beliefs about banks' solvency but is unrelated to economics fundamentals, bank runs are triggered. In analysing bank runs in a dynamic context, we do not consider these phenomena as isolated events, but we rather analyse a banking crisis where successive bank runs take place over a time horizon characterised by the loss of depositors' confidence in the banking system, and therefore, by a turmoil in the banking sector. Starting from an intermediary with prior assets and liabilities, we allow for a sequence of runs to take place at regular time intervals, where the time interval between successive runs represents a "cycle", where cycles can be of different duration. The sequence of the steady state equilibria that we obtain can only be sustained if an intermediary relies periodically on the deposits made by the newborn generations of agents. The basic setup that we use to compute these steady state allocations within each cycle is a generalisation of the methodology that employed in Qi's (1994)[19] paper. However, in our model runs can not occur due to a lack of new deposits since newborn agents can not observe or anticipate the occurrence of a bank run. For this reason, we consider the following order of events in each time period, irrespectively of whether a run takes place; new deposits and withdrawals (standard or excess withdrawals due to a run) are made first, and then banks make their investment decisions with the remaining resources. In this process, we are able derive the dynamics of our model and describe the evolution of successive cycles' equilibrium allocations.

This myopic behaviour of newborn generations of agents implies that banks can survive a run when excess withdrawals can be financed by new deposits, and therefore, a withdrawal shock is not directly associated with a bank failure, as in the three-period models. We start our analysis with the condition that banks' financial stability is not severely damaged after experiencing the first run, so that intermediaries are still solvent but they become illiquid in order to remain competitive with respect to the new entrants in the market. In order to simplify the analysis of the transitional dynamics we consider that all the agents of a particular cycle are treated equally by the banks. In this way, a "short-term" steady-state within each cycle can be derived. Of course, when we consider the whole crisis, successive cycles overlap in the time period when a bank run is triggered. In accordance with the unanticipated nature of runs, agents that were born at this time period are treated as agents of the previous cycle. In the above setting, we analyse the transition of banks' liquidity over different cycles and we examine whether intermediaries can survive the crisis by converging to a feasible allocation.

Our results show that the system converges to a feasible "long-term" steady state when banks can smooth their portfolios' liquidity through external borrowing. The cost and the amount of borrowing determine the transition path towards these long-term allocations. We show that oscillations and monotonic transitions, or even constant periodic movements between successive equilibria, may arise. However, in the absence of such borrowing, banks' liquidity position becomes unpredictable.

The structure of this paper is as follows. The next section describes the model. In the third section we develop the planning problem of an intergenerational bank. In the fourth and fifth sections we analyse the dynamics of the model in the case where banks can obtain external borrowing from a Central Bank, and in the presence of borrowing restrictions, respectively. Finally, we conclude and discuss possible future extensions of our model in section six.

2 The Model

We consider an infinite-horizon version of Diamond and Dybvig (1983) [11] model. A new generation of agents, whose size is normalised to 1, is born in each period and live for three periods. Each newborn agent is endowed with 1 unit of the economy's single homogeneous good and receive no endowment in the remaining two periods. Agents of each generation are identical ex-ante at the time period they are born, and at the beginning of the middle period of their life they "privately" observe a preference shock which is assumed to be independently and identically distributed across agents. The shock is such as agents may become impatient consumers with probability π, or patient with probability $1-\pi$. On aggregate, however, from the law of large numbers the uncertainty is washed out and therefore the total number of impatient and patient depositors is ex ante known, and it is π and $1-\pi$, respectively. As in Diamond and Dybvig (1983) [11] we assume that impatient agents derive utility of consumption only in the middle period of their life, while patient depositors derive utility of consumption only in the last period of their life. The assumption of corner preferences implies that agents can consume only once in their lifetime, where $U(C_t)$ denotes the utility derived from the consumption of the commodity at period t ($t = 1, 2$). The utility function is assumed twice continuously differentiable, increasing and strictly concave and to satisfy the Inada conditions: $\lim_{C_t \to 0} U'(C_t) = \infty$ and $\lim_{C_t \to \infty} U'(C_t) = 0$.

There are two investment technologies available in this economy. There is a short-term technology which transforms one unit of the good invested at period t into one unit at period $t+1$ and is referred to as the storage technology. There is also a long-term investment technology which transforms one unit of the good invested at period t to $R > 1$ units at period $t+2$. If it is interrupted at $t+1$, it yields a positive return of $r \leq 1$.

The banking system consists of a large number of identical banks which are perfectly competing on the terms of their deposit contract and are subject to withdrawal shocks. Because banks and depositors are identical, all banks will choose the same contract. In addition, following Ennis and Keister (2003) [14], all depositors hold the same beliefs which are influenced by the same extrinsic factors that trigger the runs, and therefore either all banks will experience a run or none will. Hence, without loss of generality, we consider a representative intergenerational bank in this economy that has access to all the available technologies and maximises the expected welfare of depositors. At any time period t, the bank accepts newborn agents' endowment and offers in return a deposit contract that specifies a payment depending on the realisation of their consumption preferences at $t+1$; after they have received a preference shock at the beginning of the middle period of their life. The deposit contract has the form of $(D_1,\ D_2)$, where D_1 is the payment designed for depositors who turn out to be impatient and D_2 the payment for those who turn out to be patient. The bank is subject to a sequential service constraint[1] so that depositors are served on a first-come, first-served order. Depositors are assumed to be myopic in the sense that they can not observe how many depositors are in front or behind them in the line, and following a sequential service constraint, the bank can not distinguish the individual type of each depositor once they approach the counter to receive their payment.

Bank runs are exogenous and are modelled as unanticipated withdrawals where all depositors exercise their right to withdraw their funds at any time period, irrespectively of their consumption preference. A bank run occurs when due to a totally extraneous factor, unrelated to any fundamentals, each depositor believes that other depositors will rush do withdraw their funds. In addition, in order to analyse the dynamics in an environment where we allow the possibility of runs, we impose that the bank survives the first run.

In contrast with the vast majority of the literature which associates runs directly with banks' failure, we take a more general approach concerning banks' financial stability. In a dynamic framework, we consider that a bank that experiences an excess demand for withdrawals can find itself in one of the following three conditions[2]:

(i) Firstly, a bank is considered to be "bankrupt" when it fails to meet its promised payments even after utilising all its available assets.

(ii) Secondly, a bank is said to "go out of business" when it serves all the

[1] The notion of a sequential service constraint has been introduced by Diamond and Dybvig (1983)[11] and analysed by Wallace (1988)[22].

[2] Allen and Gale (2000)[3] have used a similar characterisation of the banks' financial stability in order to explain the financial contagion in the banking system through an interbank market.

excess withdrawals but, by doing so, is left with less resources than the total endowment of a newborn generation.

(iii) Thirdly, a bank is said to "survive" the run when it serves the excess withdrawals and is left with more resources than the total endowment of the new generation.

In the first two scenarios, the newborn agents will prefer to form a new bank since they can attain superior allocations and a higher level of social welfare. In the last scenario, the surviving bank is "staying in business" since newborn agents can achieve a higher level of expected utility by depositing their endowments to the surviving bank than forming a new bank. Of course, in equilibrium, where the returns of the assets match the bank's standard liabilities, the bank is considered to be solvent.

In an infinitely repeated version of Diamond and Dybvig's (1983)[11] model with bank runs, let $m \in Z_+$ (where Z_+ is the set of nonnegative integers), represent the number of runs that have already taken place in this economy, where the time interval between any two successive runs represents a *cycle*. Within a cycle, time is indexed by $t \in \{0, ..., t_m\}$, where $t_m \geq 3$. Time is reset to zero when a run takes place and cycles are allowed to have different duration; where t_m can be different for every m. Our analysis shows that within each cycle, similarly to the discussion in Qi (1994)[19] about the stability of the intergenerational bank, if an intermediary survives the first run, the bank has to rely periodically on the new deposits made by the newborn agents of the current generation in order for the steady-state to be reached. However, in contrast to Qi (1994)[19], we assume that the myopic newborn agents can not anticipate a run. Therefore, we consider the following sequence of events in each time period; firstly, newborn agents of the current generation deposit their endowment to the bank[3] and all the withdrawal decisions are made by agents of the past generations who have realised

[3]Participation in the deposit contract is guaranteed since, similarly to the intragenerational bank, it offers liquidity insurance to depositors and therefore dominates the competitive outcome as it is shown in Fulghieri and Rovelli (1998)[15], among others.

their type (standard withdrawals or excess withdrawals due to a run), and then the bank makes its investment portfolio decisions with the resources available. In this respect, let $I_{t,m}$ be the the proportion of the good that the bank invests in the long-term technology and $S_{t,m}$ the proportion that it invests in the storage technology, for a given period t during cycle m.

Because $r < 1 < R$ holds, an intermediary will never choose to finance excess withdrawals by liquidating its investment in the productive technology when liquid assets are available. Hence, for any given m, denote x_m the amount of resources that remain available for investment at the initial period $t = 0$, and y_m the resources that are invested in the long-term technology during the previous cycle and comes to maturity at $t = 1$ after the run m has been occurred. We require $\{x_m, y_m\} \geq 0$ and as it arises from our analysis the values of x_m and y_m depend on the timing of runs within the crisis.

Within each cycle, we follow an analysis similar to Qi (1994) [19] to derive the steady-state equilibrium deposit contract payoff which we will refer to as "short-term" steady-sate equilibrium payoff. We assume that all relevant agents to the representative bank's planning problem are treated equally, regardless of their generation. Given that runs are unanticipated, it follows that all current and future depositors are offered a deposit contract that specifies a "short-term" steady-state payoff $(D_{1,m}, D_{2,m})$ which maximises their expected utility and is, therefore, ex-ante identical for a given m. We show that within any cycle, a feasible and stationary transition path of investment in both available technologies to a "short-term" steady-sate exists, along which the expected utility of all the members of all generations is constant.

However, in order to generalise the bank's planning problem for any cycle during the crisis, we need to address two important issues. The first concerns the treatment of the newborn agents at the time period when a run takes place, given the assumed sequence of events within each time period t and the assumption of equal treatment of all relevant agents. More specifically, for any given time period t within the cycle m, the deposit contract that the bank offers to newborn agents in exchange for their endowment is identical to the one offered to the newborn agents of the previous generation since the bank can not anticipate a sunspot run. If a run occurs at the same time period, then the current period of cycle m ($t = t_m$) is also the first period ($t = 0$) of the next cycle $m+1$ for the surviving bank. In other words, successive cycles overlap in the time period when a run takes place. Provided that runs are unanticipated, we treat these newborn agents as agents of the previous cycle so that the bank does not renege on the initial agreement and honour its promised payoffs[4].

The second point refers to the relationship between the payments of the two different types of depositors within a cycle. We assume that the coefficient of depositors' relative risk aversion is sufficiently high such that the incentive compatibility constraint that arises in our analysis in order to prevent depositors' misrepresentation of their type, binds for a newly formed bank. This is in accordance with Bencivenga and Smith (1991)[5] who provide a necessary condition that financial intermediation can be justified for a high level of depositors' risk aversion, and with von Thadden (2002)[21] where a non-arbitrage incentive compatibility constraint is more likely to bind at all times. In our model, this assumption provides a necessary condition which ensures that the relationship between the payments, offered to both types of depositors within the same cycle, is the same for any surviving bank during the crisis. It also determines whether the resulting difference equation of successive "short-term" equilibrium payoffs takes a linear or nonlinear form.

3 Bank's Planning Problem

We commence the analysis of financial intermediation by considering a representative bank's planning problem at time $t = 0$ for a given cycle m. At any particular point of time, the bank in question accepts deposits and makes

[4]The case where agents born in the time period when a run takes place are treated as agents of the next cycle, so that the bank renege on the initial contract, has been analysed in my PhD thesis where i have shown that similar results can be obtained.

investment in the available technologies. As any other bank within the competitive market, it is modelled as an infinitely lived financial intermediary since bank runs are assumed to be "sunspot" phenomena that are unanticipated by the bank. Provided that all the agents participate in the bank's deposit contract and deposit their endowment, the relevant agents to this planning problem are agents who are born at and after time $t = 0$. Since the bank does not renege on the initial contract, following the assumed sequence of events within each period, the agents who deposit their funds at $t = 0$ and have been offered a contract similar to that offered to the agents born in the previous cycle, they receive the full proceeds of their contract, as they are specified by the contract's terms. For the bank to be able to honour the promised payoffs to depositors, the following sequential budget constraints must be satisfied which specify that in each period bank's liabilities (LHS) should be equal to its assets (RHS) due to the assumed intense competition in the banking system.

$$
\begin{aligned}
0 &= x_m - (S_{0,m} + I_{0,m}) && \text{for } t = 0 \\
\pi D_{1,0,m-1} &= 1 + y_m R + S_{0,m} - (S_{1,m} + I_{1,m}) && \text{for } t = 1 \\
\pi D_{1,1,m} + (1-\pi) D_{2,0,m-1} &= 1 + R I_{0,m} + S_{1,m} - (S_{2,m} + I_{2,m}) && \text{for } t = 2 \\
\pi D_{1,t-1,m} + (1-\pi) D_{2,t-2,m} &= 1 + R I_{t-2,m} + S_{t-1,m} - (S_{t,m} + I_{t,m}) && \text{for } t \geq 3
\end{aligned}
\tag{3.1}
$$

For any cycle m, let $\{S_{t,m}\}$ and $\{I_{t,m}\}$ denote sequences $\{S_{0,m}, S_{1,m}, S_{2,m}, ...\}$ and $\{I_{0,m}, I_{1,m}, I_{2,m}, ...\}$, respectively. In the same way, let $D_{1,t,m}$ and $D_{2,t,m}$ denote the payments offered to newborn agents who deposit their endowments at period t, if they withdraw their deposits at $t+1$ or $t+2$, respectively. The highly competitive environment in the banking system compels the representative bank to choose its investment portfolio such as to maximise the expected utility of the newborn agents within each time period in order to attract new deposits, subject to the above sequential budget constraints, which therefore should hold with equality. Bank's liabilities incorporate the payments to depositors who are eligible to withdraw (i.e. without regard to runs) and become stationary at $t \geq 3$ after all the impatient and patient depositors born at $t = 0$ have been served at $t = 1$ and $t = 2$, respectively. On the other hand, for any time period within a cycle, the bank's assets consist of the deposits made by the current newborn generation, and the returns of the investment made in the productive and storage technology (two-periods and one-period ago, respectively) and comes to maturity at the current period. After all legitimate liabilities have been met, the rest of the available resources are allocated in the two available investment technologies. In order to generalise our analysis, we have denoted as x_m the resources available for investment at $t = 0$ which are allowed to be different from new deposits, and y_m the resources that are invested in the long-term technology during the previous cycle, where $\{x_m, y_m\} \geq 0$.

From the above formation of the bank's planning problem we can observe that one solution would be for the bank to offer a high payoff to newborn agents at $t = 1$ and without regard to future generations, subject to the above sequential budget constraints and the incentive compatibility constraint that arises in our model, in order to drive its competitors out of the market and enjoy monopoly profits in future periods. However, due to major conceptual and technical difficulties in forming the set of all possible strategies, we assume that all relevant agents, regardless of their generation, are treated equally. Hence, our analysis is limited to steady-state payoffs that offer all current and future depositors an identical ex ante payoff under incentive compatibility. Thus, let $(D_{1,m}, D_{2,m})$ denote the ex ante steady-state payoff. Of course, the problem of determining the optimal payoffs is the same in any time period and therefore we consider that the bank solves the expected utility maximisation problem at $t = 0$. In addition, the resulting allocation in this economy is different from that of the social planner. The social planner does not face any competitive pressures and its only objective is to maximise the expected utility of the depositors across the whole time horizon of a cycle according to some aggregate welfare function (as in Allan and Gale (1997)[1]).

Since the deposit contract terms are time independent, the sequential budget constraints can be written as:

$$
\begin{aligned}
0 &= x_m - (S_{0,m} + I_{0,m}) && \text{for } t = 0 \\
\pi D_{1,m-1} &= 1 + y_m R + S_{0,m} - (S_{1,m} + I_{1,m}) && \text{for } t = 1 \\
\pi D_{1,m} + (1-\pi) D_{2,m-1} &= 1 + R I_{0,m} + S_{1,m} - (S_{2,m} + I_{2,m}) && \text{for } t = 2 \\
\pi D_{1,m} + (1-\pi) D_{2,m} &= 1 + R I_{t-2,m} + S_{t-1,m} - (S_{t,m} + I_{t,m}) && \text{for } t \geq 3
\end{aligned}
\tag{3.2}
$$

Provided that all relevant agents receive the same payoff independently of their generation, a budget-feasible steady-state payoff $(D_{1,m}, D_{2,m})$ is defined.

Definition 3.1 *A steady-state payoff $(D_{1,m}, D_{2,m})$ is budget feasible if there exist some nonnegative $\{I_{t,m}\}$ and $\{S_{t,m}\}$ such that the conditions described by 3.2 hold.*

To simplify the notation, denote $a_m = \pi D_{1,m} + (1-\pi) D_{2,m}$ the bank's standard liabilities for each time period $t \geq 3$. Given the above definition of the steady state, we derive the steady-state feasibility conditions and the conditions on $\{I_{t,m},\, S_{t,m}\}$ which satisfy the feasibility condition and determine the transition path of the investment technologies towards this steady-state allocation.

Proposition 3.2 *A necessary condition for a steady-state payoff $(D_{1,m}, D_{2,m})$ to be budget feasible is that $D_{1,m}$ and $D_{2,m}$ satisfy*

$$
B_m(D_{1,m},\, D_{1,m-1},\, R) : \frac{2\,(1-a_m)}{R-1} + 2 - \pi D_{1,m} + R\,(x_m + y_m) - a_{m-1} = 0 \tag{3.3}
$$

In particular, the sequential budget constraints are satisfied by the following nonnegative $\{I_{t,m},\, S_{t,m}\}$ when $y_m R - \pi D_{1,m-1} > 0$:[5]

[5] The alternative set of conditions on $\{I_{t,m},\, S_{t,m}\}$, when $y_m R - \pi D_{1,m-1} < 0$ that

$$S_{t,m} = \begin{cases} 0 & \textit{for } t = 0 \textit{ and } t = 2\lambda, \lambda \in Z_{++} \\ y_m R - \pi D_{1,m-1} & \textit{for } t = 1 \\ R - a_m & \textit{for } t = 2\lambda + 1, \lambda \in Z_{++} \end{cases}$$

$$I_{t,m} = \begin{cases} x_m & \textit{for } t = 0 \\ \frac{2(a_m - 1)}{R - 1} - 1 & \textit{for } t = 2\lambda, \lambda \in Z_{++} \\ 1 & \textit{for } t = 2\lambda + 1, \lambda \in Z_{+} \end{cases}$$

where Z_+ *and* Z_{++} *denote the set of nonnegative and positive integers, respectively.*

(Proof: see appendix)

Despite the fact that there is no aggregate uncertainty about the withdrawal demand in our model, so that the bank's deposit contract can offer the optimal payoffs to both types of depositors, when depositors are served sequentially on a first-come first-served order, the bank cannot distinguish their individual type. Due to this informational disadvantage of the bank, patient depositors in the middle period of their life, instead of waiting to withdraw $D_{2,m}$ that is designed for their type may find optimal to misrepresent themselves initially as impatient by withdrawing $D_{1,m}$, and then emulate the newborn agents of the current generation by redepositing their funds. In this way, they will receive a payment of $D_{1,m}^2$ at the last period of their life in which they derive utility from consumption. An incentive compatibility constraint should be introduced to ensure that the utility from consumption they derive in the final period of their life by misrepresenting themselves as impatient and redepositing their funds is at least less than the utility from consumption that they could derive by waiting to withdraw and consume the payment designed for their type[6]. Simplifying for the utility function,

satisfy the sequential budget constraints with equality violate the nonnegativity condition and, therefore, is ignored.

[6] According to Fulghieri and Rovelli (1998)[15] this informational asymmetry is removed if the bank could introduce age-dependant restrictions on the deposit contract. Under these circumstances, from the bank's maximisation problem it follows that the

the incentive compatibility constraint can be expressed in terms of payments.

The problem that the representative intergenerational bank has to solve within any cycle in order to determine the optimal deposit contract's payoffs is;

Problem 3.3 *The intergenerational bank maximises depositors' expected utility*

$$\max_{D_{1,m}\, D_{2,m}} \{\pi\, U\,(D_{1,m}) + (1 - \pi)\, U\,(D_{2,m})\} \tag{3.4}$$

subject to:

the feasibility constraint as described in Proposition 3.2

$$B_m(D_{1,m},\, D_{1,m-1},\, R) : \frac{2\,(1 - a_m)}{R - 1} + 2 - \pi D_{1,m} + R\,(x_m + y_m) - a_{m-1} = 0$$

and an incentive compatibility constraint

$$D_{2,m} \geq D^2_{1,m} \tag{3.5}$$

where, the bank's investment decisions between the two alternative investment technologies are chosen according to Proposition 3.2. To simplify notation, denote $w_m(D_{1,m-1},\, R) = R\,(x_m + y_m) - a_{m-1}$ and $v_m(D_{1,m},\, R) = B_m - w_m$, where w_m captures the net resources available for investment after all the bank's commitments with respect to the previous cycle have been met so that it depends on the previous cycle's equilibrium payoff, and v_m represents the assets and liabilities of the current cycle so that the budget constraint holds with equality.

In our analysis about the transition towards the "long-term" steady-state equilibrium, we must ensure that the steady-state allocation the surviving bank can attain in any cycle is superior to the steady-state allocation that

"Golden Rule" levels of investment and consumption smoothing could be achieved. However, the anonymity of the deposit contract implies that this information asymmetry always exists.

newborn agents of the next generation can obtain if they form a new bank. This requirement imposes a lower boundary on the resources available for investment after a run has taken place. In other words, the intermediary will "stay in business" if after any given run, it has been left with more resources than the aggregate endowment of the next generation of agents.

Hence, when newborn agents of a particular generation decide to form a new bank (we use the subscript N) at any period, then the current available resources of the newly formed intermediary are equal to the endowments of the newborn agents, and no prior investment comes to maturity in the following period.

$$\begin{aligned} x_N &= 1 \\ y_N &= 0 \end{aligned} \tag{3.6}$$

However, in this setting there is no previous cycle, and therefore, no outstanding liabilities that depend on the previous cycle's equilibrium payoff. As a consequence, the sequential budget constraints of the bank's planning problem can not be described by (3.2)[7]. In order to check whether a potential "long-term" steady-state equilibrium payoff is sufficiently high so that new generations do not have incentives to form a new bank, we need to derive a condition such as the budget constraint 3.3 becomes identical to the budget constraint of a newly formed bank. This is when $R\,(x_N + y_N) - a_{N-1} = 0$, or alternatively, $w_N = 0$. Consequently, when a run takes place, the intergenerational bank survives the run when $w_m \geq 0$.

In order to simplify the transitional dynamics towards a "long-term" steady-state equilibrium payoff when we allow for successive runs to occur, we have assumed that the incentive compatibility constraint binds for a newly formed bank, which is a necessary assumption to ensure that the constraint also binds for any surviving bank, and is also consistent with our model. This

[7] The sequential budget constraints, as well as the general budget constraint, are discussed in Proposition 5 by Qi (1994)[19], p. 401.

is more likely to be the case when the agents' degree of relative risk aversion is sufficiently high [8]. The necessity of this assumption derives from the bank's maximisation problem where we can observe that if the constraint binds for a newly formed bank, which has the lowest available resources for investment and therefore offers the lower equilibrium payoff, it also binds for any other allocation that a surviving bank can offer. In terms of consistency, we start with a bank that prior to any run could achieve the "Golden Rule" level of consumption smoothing but due to the asymmetry of information the about depositors' age, the incentive compatibility constraint binds. Therefore, is consistent in our analysis to consider that the relationship between the payments of both types of depositors for any cycle is described by this constraint so that it binds at all times. Hence, welfare comparison between "short-term" steady-states can be simplified to a simple comparison of a payment designed for either type of depositor or to a comparison of the net resources that remain available for investment (provided that $\{I_{t,m}, S_{t,m}\}$ are nonnegative), which in turn determines the relationship between bank's liabilities across different "short-term" steady-states. Hence, when $w_m = w_N$ the incentive compatibility constraint 3.5 binds. It also indicates that payments designed for the two types of depositors are related in a nonlinear manner, so that the relationship between successive equilibria is also nonlinear. This gives rise to a nonlinear system of first-order difference equations.

Commencing the analysis on the effects of a sequence of bank runs to the bank's financial stability, suppose that, initially ($m = 0$) an infinitely-lived bank is in place. Following the standard approach in the literature in determining the planning problem of an infinitely lived intermediary at $t = 0$, the relevant agents, apart from the agents who are born at and after

[8] More specifically, assuming a utility of consumption function of the form $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$, where γ the coefficient of relative risk aversion, the first order condition of the maximisation problem 3.3 will be $D_{2,m} = \left(\frac{R+1}{2}\right)^{\frac{1}{\gamma}} D_{1,m}$. In order for the incentive compatibility constraint to bind for a newly formed bank it follows that $\left(\frac{R+1}{2}\right)^{\frac{1}{\gamma}} \leq D_{1,N}$ should hold. Substituting for $D_{1,N}$, which can be obtained by substituting $w_m = 0$ into 3.3, and solving for γ we derive that the coefficient of relative risk aversion should be $\gamma \geq \frac{\ln\left(\frac{R+1}{2}\right)}{\ln D_{1,N}}$ in order for the incentive compatibility constraint to bind.

$t = 0$, are also the impatient depositors of the previous generation ($t = -1$) who have deposited all their endowment. Analysing the sequential budget constraints as they described by (3.2), we observe that, at period $t = 0$, the resources available for investment are equal to the endowment of the newborn agents, deposited in the current period. In addition, at period $t = 1$, the investment made in the long-term technology at period $t = -1$, (i.e. $I_{-1,0} = 1$), comes to maturity and the bank has also to serve the impatient agents of generation $t = -1$ who are in the last period of their life and withdraw on aggregate $(1-\pi)D_{2,0}$. For computational ease and in order to keep our analysis consistent, we incorporate this extra liability into y_0 which is discounted by R. Hence, for $m = 0$;

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 1 - \frac{(1-\pi)D_{2,0}}{R} \end{aligned} \tag{3.7}$$

Note also that the previous cycle is identical to the current cycle, and therefore $a_{m-1} = a_m$. Substituting $\{x_0, y_0\}$ into the feasibility constraint 3.3, the latter it becomes:

$$a_0 = R \tag{3.8}$$

where $a_0 = \pi D_{1,0} + (1-\pi)D_{2,0}$.

Hence, the steady-state payoff $(D_{1,0}, D_{2,0})$ is such as the bank's standard liabilities are equal to the returns of the long-term investment. Solving the bank's maximisation problem we can derive that the incentive compatibility constraint binds, so that $1 < D_{1,0} < R < D^2_{1,0}$. Obviously, new generations of agents do not have incentive to form a new bank since $w_0 > 0$, and therefore can achieve a higher level of expected utility (i.e. $D_{1,N} < D_{1,0}$). Furthermore, by substituting $\{x_0, y_0\}$ and equation 3.8, where $a_0 = a_{m-1} = a_m$, into the conditions for $\{I_{t,m}\}$ and $\{S_{t,m}\}$ in Proposition 3.2, we observe that $\{I_{t,0}\} = 1$ and $\{S_{t,0}\} = 0$ so that for any given period during $m = 0$, the bank invests all the available resources on the productive technology.

In examining how bank's financial stability evolves during the crisis we first need to determine the initial allocation of the discrete dynamical system that we will derive. Of course, this allocation should be feasible and therefore we require that that the bank survives the first run, or $w_1 > 0$. In particular, provided that bank runs are unanticipated shocks, let the first run occur at any time period t where any excess withdrawals are financed out of new deposits of the current generation.[9] This implies that the available resources for investment at $t = 0$ are what is left out of new deposits after all the impatient depositors of the previous generation who withdraw early have been served. The investment made in the previous period in the long-term technology remains unaffected, where $I_{t,0} = 1$. Therefore, for $m = 1$;

$$\begin{aligned} x_1 &= 1 - (1-\pi) D_{1,0} \\ y_1 &= 1 \end{aligned} \tag{3.9}$$

where we require that $w_1 > 0$ for the bank to survive at least the first run. The "short-term" steady-state is such as $D_{1,1} < D_{1,0} < R$[10] so that $a_1 < a_0 = R$. Checking its feasibility we derive that the conditions on $\{I_{t,0},\, S_{t,0}\}$, as they described in Proposition 3.2, are satisfied and therefore the steady-state equilibrium can be sustained.[11]

[9] If the excess withdrawals just exhaust the new deposits from the newborn agents, then $x_1 = 0$ and $y_1 = 1$ so that $w_1 = 0$. Similarly, in the alternative case where the bank liquidates the long-term investment made in the previous period in order to meet any excess withdrawals which just exhaust the resources obtained, then $x_1 = 1$ and $y_1 = 0$ so that again $w_1 = 0$ and the bank "goes out of business". However, since $r < 1$ resources are wasted and therefore liquidation is never an option.

[10] It is sufficient to show that the difference between w_0 and w_1 is negative. From our analysis, $w_0 - w_1 = \pi D_{1,0} - (R(x_1 + y_1) - a_0)$ and from $a_0 = R$ the difference can be simplified to $(1-\pi) D_{1,0} (R - D_{1,0}) > 0$ since $D_{1,0} < R$.

[11] For the investment in productive and storage technology, we observe that $I_{2\lambda,1} > 0$, $S_{1,1} = R - \pi D_{1,0} > 0$ and $S_{2\lambda+1,1} = R - a_1 > 0$ since $a_1 < a_0 = R$, respectively. If instead the excess withdrawals where financed by liquidating the long-term investment made in the previous period, then the resources available for investment at the first two periods of the first cycle would be $x_1 = 1$ and $y_1 = 1 - \frac{(1-\pi)D_{1,0}}{r}$ where $r \in (0, (1-\pi)D_{1,0}]$ for the

After the first run has taken place, Proposition 3.2 indicates an important property of the transition path of the investment in the two available technologies towards feasible steady state equilibria.

Property 3.4 *For any $m \geq 1$, the bank's investment in $\{I_{t,m}, S_{t,m}\}$, and therefore its liquidity position, becomes periodic with a two-period periodicity in order for a steady-state equilibrium to be sustained. During even periods the bank holds an illiquid portfolio of assets, and during odd periods it holds a liquid portfolio of assets.*

This property of the "short-term" equilibria arises in our model from the equal treatment assumption and the nature of the investment technologies. When the first run occurs, the smooth pattern of the investment in the two technologies ($I_{t,0} = 1$ and $S_{t,0} = 0$) is disrupted. The patient agents who trigger the run receive the payment that is designed for impatient depositors instead of the higher payment that is designed for their type ($D_{1,0} < D_{2,0}$). Hence, the *surviving* bank finances relatively inexpensive its rather expensive liabilities out of the new deposits so that the total resources for investment at $t = 0$ are less than the endowment of a new generation, but greater at $t = 1$ since the investment in the productive technology that was planned to meet its standard liabilities comes to maturity. Provided that the bank does not renege on the initial contract, the sequential budget constraints become stationary for $t \geq 3$; after all depositors born at $t = 0$ have been served. This process is repeated for any other cycle. From the equal treatment assumption, the two-period periodicity of the bank's portfolio is perpetuated for the whole duration of each cycle as the bank offers the same allocation to all agents within the current cycle.

As a result, the liquidity of the bank's investment portfolio becomes periodic with a two-period periodicity so that periods of high liquidity are followed by periods of low liquidity. From Proposition 3.2 we observe that

bank to survive the first run. Note, however, that $w_0 - w_1 = \pi D_{1,0} - Ry_1 = -S_{1,1}$ and therefore the "short-term" equilibrium is not feasible.

bank's total resources during even periods ($t = 2\lambda$) in any cycle, net of new deposits, are equal to the return of the long-term term and short-term investment made two and one periods ago, respectively; $RI_{2\lambda-2,\,m} + S_{2\lambda-1,\,m} = R\left(\frac{2(a_m-1)}{R-1} - 1\right) + R - a_m$. Subtracting the standard liabilities a_m we obtain that the total available resources for investment, net of new deposits, during even periods are $\frac{2(a_m-R)}{R_1} < 0$, since $a_m < R$ is required in order for $S_{2\lambda+1,\,m} > 0$ and the allocation to be feasible. Hence, at even periods the bank relies on new deposits in order to meet its standard liabilities, and therefore we argue that during even periods it holds an illiquid portfolio of assets. In a similar manner, the resources net of new deposits during odd periods ($t = 2\lambda + 1$) in any cycle are $RI_{2\lambda-1,\,m} + S_{2\lambda,\,m} = R$, and its total resources $R - a_m > 0$ for a feasible allocation. Hence, the returns from past investment are greater than its standard liabilities and therefore we argue that during odd periods it holds a very liquid portfolio of assets. Overall, the assumptions of our model impose a transition path of bank's investment portfolio towards these equilibria. The liquidity of bank's portfolio, therefore, depends only on the model's parameter values and on the timing of runs; during periods of low or high liquidity. The periodicity of the investment of the two available technologies implies that the liquidity of bank's portfolio becomes also periodic, which in turn determines the bank's ability to meet any excess withdrawals in case of a run for any $m \geq 1$.

From the above property, x_m and y_m can be determined and the system of difference equations can be derived. In this way, consider the period t_{m-1} which is the last period of cycle $m-1$ during which the m run takes place and therefore constitutes the initial period of the m cycle.

If t_{m-1} is a period of low liquidity (even period), the total resources available for investment, incorporating new deposits, are $\frac{2(a_{m-1}-1)}{R-1} - 1$. Therefore, x_m is equal to the total resources minus the excess withdrawals, and y_m is equal to the investment in the productive technology made in the odd period of the previous cycle (i.e. $I_{2\lambda+1,\,m-1}$).

$$
\begin{aligned}
x_m &= \frac{2(a_{m-1}-1)}{R-1} - 1 - (1-\pi)D_{1,\,m-1} \\
y_m &= 1
\end{aligned} \tag{3.10}
$$

On the other hand, if t_{m-1} is a period of high liquidity (odd period), the total resources available for investment, incorporating new deposits are $1 + R - a_{m-1}$. Therefore, x_m and y_m are:

$$
\begin{aligned}
x_m &= 1 + R - a_{m-1} - (1-\pi)D_{1,\,m-1} \\
y_m &= \frac{2(a_{m-1}-1)}{R-1} - 1
\end{aligned} \tag{3.11}
$$

Hence, depending on t_{m-1}, by substituting x_m and y_m into the budget constraint, incorporating the incentive compatibility constraint, we obtain a function $B_m(D_{1,\,m},\, D_{1,\,m},\, R) = 0$ which is an implicit function of the difference equation between the two successive "short-term" equilibrium payoffs. Let the "long-term" steady-state be $(D_{1,\,M},\, D_{2,\,M})$. Setting $D_{1,\,m} = D_{1,\,m-1}$, we obtain a function $B_m(D_{1,\,m},\, R)$, the solution of which (fixed points) may constitute potential "long-term" equilibria if they satisfy our conditions on $\{S_{t,\,M},\, I_{t,\,M}\}$ (i.e. $B_M(D_{1,\,M},\, R) = 0$). The "long-term" steady state is budget feasible and therefore can be sustained only when $w_M > 0$, $S_{1,\,M} > 0$, and $D_{1,\,M} \leq D_{1,\,0}$ in order for $S_{2\lambda+1,\,M}$ to be positive.[12]

As the reader can notice, the evolution of bank's liquidity position during the whole crisis is described by a set of difference equations. However, provided the randomness of a bank run event within each cycle when we consider the whole crisis, the system becomes unpredictable since through the process of iteration, fixed points and orbits will depend on the timing of these events. Hence, the system becomes very unstable and may exhibit chaotic behaviour.

[12]From equation 3.8 we can observe that for any $D_{1,\,M} \leq D_{1,\,0}$ it follows $a_M \leq a_0$, and therefore $S_{2\lambda+1,\,M} > 0$.

4 Bank's Financial Stability with Reserve Requirement and Borrowing

The erratic behaviour that arises in our model results from the periodicity of the liquidity of bank's portfolio, which in turn depends only on the timing of runs and the model's parameter values, since the investment in the two available technologies is determined by Proposition 3.2. However, the introduction of the possibility of external borrowing can resolve the unpredictability of banks' liquidity provision. In this respect, we consider that a Central Bank responsible for the conduct of monetary policy and the soundness of the banking system is in place. On its role as a Lender of Last Resort can improve the financial stability of the banking system through injections of capital to illiquid intermediaries by providing access to a discount window. In addition, as a part of its responsibility in the formulation and implementation of monetary policy, a minimum reserve requirement is introduced as a way to control banks' liquidity.

These policy measures can smooth the liquidity of banks' portfolio over periods of financial distress and ensure the soundness of the banking system over the whole period of crisis. Following the standard literature, we view a Central Bank as being largely motivated by the negative consequences that bank failures, originating in liquidity problems, have on the stability of the financial system. Banks are identical and, according to property 3.4 are modelled as illiquid but solvent institutions since we require that they survive the first withdrawal shock. In addition, the Central Bank has full information about their liquidity position, and therefore, moral hazard problems involved in direct lending do not arise in our model.

In this setting, suppose that a Central Bank is in place which has access to unlimited resources and lends funds through a discount window to illiquid but solvent banks. Let $\tau \in [0, 1]$ be the one-period interest rate for one unit of resources borrowed, and $\kappa_m \geq 0$ the amount of resources that the central bank lends during a period of low liquidity. Therefore, a bank with an illiquid portfolio of assets that borrows an amount of resources κ_m at

$t = 0$ has to repay an amount $\kappa_m(1+\tau)$ at period $t = 1$, during which it has high liquidity. In this way, the Central Bank can impose τ and κ_m through a minimum reserve requirement, to smooth the liquidity of banks' portfolio over the periods characterised by an abnormal increase in withdrawals.

We follow a simple formulation of the problem where τ and κ_m are choosen such as banks' liquidity is perfectly smoothed during the event of a run. In this way, the timing of runs becomes irrelevant in our model since the evolution of liquidity over different cycles can be described by a single difference equation which we can analyse and examine whether or not is possible for a representative bank to survive a crisis.

Let the amount κ_m borrowed by an illiquid bank that experiences a withdrawal shock be sufficiently high such that its available resources for investment at the current period are equal to the resources that it holds during the periods of high liquidity, or alternatively $x_{2m-1} = \kappa_m + x_{2m-2}$, for every $m \geq 2$. This amount is repaid after one period where the bank has sufficient liquidity after the investment in the long-term technology comes to maturity. The interest rate charged by the central bank is set such as to ensure that y_m remain the same independently of the time periods of runs, or alternatively $y_{2m-1} = y_{2m-2} - \frac{\kappa_m(1+\tau)}{R}$, for every $m \geq 2$. Clearly, the cost of borrowing an amount κ for one period at an interest rate τ, has to be discounted over two periods by R in order to be subtracted from the investment in the long-term technology. Doing the relevant substitutions from equations 3.10 and 3.11, we derive;

$$\begin{aligned} \tau &= \frac{R-1}{R+1} \\ \kappa_m &= \frac{R-a_{m-1}}{\tau} \end{aligned} \tag{4.1}$$

What we consider here is a variation of the minimum reserve requirement regulatory policy which we will refer to as *"adjustable" reserve requirement* as it provides more flexibility to banks' investment portfolio. Such policy is implemented by imposing that at the time period where resources are below the level that a solvent bank has during periods of low liquidity, they should increase to the level at which they are at periods of high liquidity. When the run m is triggered and resources fall below $\frac{2(a_{m-1}-1)}{R-1} - 1$, the bank has to borrow from the central bank a sufficient amount of resources k_m to restore its liquidity position bank to its high level during the odd periods. Provided that the unit cost of borrowing for one period is determined by 4.1, the timing of runs during the crisis becomes irrelevant as x_m and y_m are always given by equation 3.11. This "adjustable" policy of reserve holdings, apart form eliminating the unpredictability of banks' liquidity position during runs, it also encourages an optimal allocation of resources, in contrast with a flat minimum reserve requirement which reduce investment in the productive technology and therefore social welfare. This is simply because the threshold level of reserves at which the policy is enforced is lower than the actual level of liquid reserves that the bank is required to hold when the policy is implemented, and reserves are adjusted only when runs are triggered. This facilitates higher investment in the productive technology, and therefore, an optimal allocation of resources for the whole duration of a cycle, in contrast with a constant minimum required level of reserves for the whole duration of the cycle.

Consequently, irrespectively of the time period of runs, the bank's resources available for investment and distribution at the first two periods of each cycle are given by equation 3.11 for any $m \geq 2$. By substituting w_m into the budget constraint and setting $D_{1,m} = D_{1,m-1}$, the later becomes $B_m(D_{1,m}, R) = R^2 - (1-\pi)(R-1)D_{1,m}^2 - RD_{1,m}$ the roots of which determine the potential "long-term" equilibria. The discriminant of the quadratic function is always positive, and therefore, bifurcations do not occur for our model's parameters. The two fixed points are;

$$D_{1,M} = \frac{\pm R\left(\sqrt{1+4(1-\pi)(R-1)} - 1\right)}{2(1-\pi)(R-1)} \tag{4.2}$$

from which we accept the positive root by the nonnegativity condition on equilibrium payments. This payoff can constitute a potential “long-term” equilibrium only if it is feasible.

Lemma 4.1 *The positive root satisfies the conditions on $\{I_{t,M},\, S_{t,M}\}$ and is monotonically increasing in R. It constitutes a potential “long-term” steady-state equilibrium when the return of the long-term investment is sufficiently high such as $w_M > 0$.*

(Proof: see appendix)

The above lemma guarantees that the “long-term” steady state equilibrium payoff we have derived is feasible. However, whether or not the system converges towards this payoff depends on its nature which is characterised by the derivative of any particular “short-term” equilibrium payoff with respect to the previous equilibrium payoff, when is evaluated at this fixed point. By substituting x_m and y_m as described by equation 3.11 into the budget constraint B_m in equation 3.3 and evaluating the derivative at $D_{1,M}$, we derive;

$$\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = 1 - \left(\frac{R + 2(1-\pi)(R-1)D_{1,M}}{\pi + \frac{2\left(\pi + 2(1-\pi)D_{1,M}\right)}{R-1}} \right) \tag{4.3}$$

The value and sign of the above derivative characterises the transition of successive equilibrium payoffs. If the absolute value of the derivative is less than unity, then the “long-term” equilibrium is referred to as “attractive” and the successive equilibrium payments will converge towards that equilibrium. Otherwise, the “long-term” equilibrium is referred to as “repelling” since successive equilibrium payoffs diverge from that payoff, and consequently it can not be achieved. Moreover, the time path towards or away from the “long-term” equilibrium steady-state is determined by the sign of this derivative; if positive, the system converges (or diverges) monotonically, and if negative, the system oscillates towards to (or away from) the equilibrium.

However, an important property of 4.3 enables us to characterise the “long-term” steady-state equilibrium.

Property 4.2 *The derivative described in 4.3 is always less than unity and monotonically decreasing in R.*

(Proof: see appendix).

The above property suggests that, for a given π, cycles (orbits) nearby the "long-term" steady-state will converge towards the equilibrium for low values of R, but diverge as the returns of the productive technology increase. The monotonicity of $\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}}$ in terms of R enables us to define two threshold values of R which determine the nature of the "long-term" steady-state equilibrium. Let $\underline{R}$ the value of R for which $\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = 0$ and $\overline{R}$ the value of R for which $\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = -1$, for a given π, as it is indicated by equation 4.3. In the analysis that follows, the the transition towards "long-term" steady states in each case is graphically displayed through graphical iteration using web diagrams.

Initially, for low values of R such as $R < \underline{R}$ it follows that $0 < \left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} < 1$ so that B_m has a positive slope. The fixed point is an attracting equilibrium, and the system of the difference equation which describes successive cycles' equilibrium payoffs converge monotonically to this "long-term" equilibrium. This case is illustrated in figure 1 where on the horizontal and vertical axis we measure the successive equilibrium payoffs. The area where a "long-term" equilibrium is feasible is delimitated by the area between $D_{1,N}$ and $D_{1,0}$. The payoff $D_{1,N}$ is the lower boundary value for $D_{1,M}$ because for any equilibrium payment below the payment that newborn agents can reach by forming a new bank it would imply that $w_m < 0$, so one of our conditions would be violated, in a given cycle m. Also, $D_{1,0}$ is the upper boundary value for $D_{1,M}$ because for any equilibrium payments above $D_{1,0}$ the nonnegativity condition on $S_{2\lambda+1,m}$ would be violated. The "long-term" steady-state payoff is determined by the intersection of $B_m(D_{1m}, D_{1,m-1}, R) = 0$ function, which is the graphical illustration of the general budget constraint and represents the relationship between payoffs of successive cycles, and the 45 degrees line through the origin. Successive equilibrium payments converge monotonically towards the long-term equilibrium.

Despite the fact that the bank holds a very liquid portfolio of assets, for very low values of R, the "long-term" equilibrium may violate the lower boundary so that $w_M < 0$. This case is represented by the $B_{m,1} = 0$ line and the fixed point M_1 which does not constitute a long-term equilibrium since, after a particular run has taken place, newborn agents will prefer to form a new bank. For example, for $\pi = 0.4$ and $R = 1.2$, $D_{1,M} = 1.0827 < D_{1,N} = 1.0943 < D_{1,0} = 1.1196$ and $\left.\frac{dD_{1,m}}{dD_{1,m-1}}\right|_{D_{1,M}} = 0.916$. However, for higher values of R such that $w_M > 0$, the fixed point such as M_2 constitutes a "long-term equilibrium" and is determined by the intersection of $B_{m,2} = 0$ and the 45 degrees line.

In terms of figure 1, we analyse the path of successive cycle's equilibrium payoffs by the process of graphical iteration, where the arrows on the graph indicate the direction of iteration. From an initial value, we use $B_{m,2} = 0$ line to map successive payoffs on the vertical axis, and the 45 degrees line to transplot them to the horizontal axis. Starting from $D_{1,1}$ on the 45 degrees line which is the first point of the orbit, we move down towards the $B_{m,2} = 0$ line which maps $D_{1,1}$ into $D_{1,2}$ and we read its height on the vertical axis as the value of $D_{1,2}$. Next, in order to map $D_{1,2}$ into $D_{1,3}$, firstly we transplot $D_{1,2}$ on the horizontal axis (the second point of the orbit), and then down towards the $B_{m,2} = 0$ line that maps $D_{1,2}$ into $D_{1,3}$ as we can read on the vertical axis. By repeating this process we can trace out all the subsequent values of $D_{1,m}$ until we reach the "long-term" steady state equilibrium at point M_2. Hence, for low values of R, but sufficiently high to ensure that the surviving bank "stays in business" in the long-term for a given π (i.e. $w_M > 0$), the system converge monotonically to the "long-term" equilibrium such as point M_2 in figure 1. For example, for $\pi = 0.75$ and $R = 2$, $D_{1,N} = 1.3642 < D_{1,M} = 1.6568 < D_{1,0} = 1.7015$ and $\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = 0.276035$.

The monotonic transition towards the "long-term" steady state is explained by the positive relationship between successive "short-term" equilibrium payoffs, and the effect of R on $D_{1,0}$ which, in turn, affects the first allocation of the system $D_{1,1}$. Indeed, from 3.8 we can observe that $D_{1,0}$ is increasing in R and therefore, for very low values of R, the cost of financing excess withdrawals out of new deposits is low. The remaining resources available for distribution and investment during the first cycle are sufficiently high such as $D_{1,1}$ is relatively high in comparison to the initial allocation. Of course, this effect is perpetuated between successive equilibrium payoffs through the iteration process, and diminishes over time as is indicated by the slope of B_m which is less than unity, until we reach the "long-term" steady-state.

Moreover, for higher values of R such as $\underline{R} < R < \overline{R}$, it follows that $0 < \left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} < -1$ so that the "long-term" equilibrium is attracting and the system of difference equation converges to this equilibrium in an oscillatory way. The relationship between successive "short-term" equilibrium payoffs is graphically represented by $B_{m,3} = 0$ line which has a negative slope less than unity, and the long-term equilibrium by point M_3 where the $B_{m,3} = 0$ line crosses the 45 degrees line. Similarly to the previous case, $D_{1,M}$ is bounded to the area between $D_{1,N}$ and $D_{1,0}$ in order for our conditions to be satisfied. As before, we use the $B_{m,3} = 0$ line to map successive equilibrium payoffs on the vertical axis, and then use the 45 degrees line to transplot them to the horizontal axis. Starting from $D_{1,1}$ on the 45 degrees line, we map $D_{1,1}$ into $D_{1,2}$ by moving upwards towards the $B_{m,3} = 0$ line. In order to map $D_{1,2}$ into $D_{1,3}$, firstly we transplot $D_{1,2}$ on the horizontal axis by moving towards the 45 degrees line, and then downwards towards the $B_{m,3} = 0$ line. By continuing this process of graphical iteration, where the direction of iteration is indicated by the arrows, successive equilibrium payoffs oscillate and converge to the "long-term" steady-state equilibrium point M_3. For example, for $\pi = 0.75$ and $R = 2.5$, $D_{1,N} = 1.4848 < D_{1,M} = 1.9371 < D_{1,0} = 2$ and $\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = -0.29967$.

For the limit case where $R = \overline{R}$, it follows that $\left.\frac{dD_{1,m}}{dD_{1,m-1}}\right|_{D_{1,M}} = -1$, and therefore, the system of difference equations of the "short-term" equilibrium payoffs rotates on a 2-period constant orbit, as it is illustrated in figure 3. In this case, R is sufficiently high so that, as the system oscillates, $D_{1,2}$ becomes identical to $D_{1,0}$. In general, $D_{1,2m-2} = D_{1,0}$ and $D_{1,2m-1} = D_{1,1}$ for any $m \geq 2$. Despite the fact that non of the conditions on $\{I_{t,M}, S_{t,M}\}$ is violated, the "long-term" equilibrium can not be reached as successive equilibrium payoffs rotate on a constant orbit. For example, for $\pi = 0.75$ and $R = 3$, $D_{1,N} = 1.5825 < D_{1,M} = 2.1961 < D_{1,0} = 2.2749$ and $\left.\frac{dD_{1,m}}{dD_{1,m-1}}\right|_{D_{1,M}} = -1$.

Consequently, for even higher values of R such $\overline{R} < R$, successive equilibrium payoffs also oscillate but the system of difference equations naturally "explodes" away from the fixed point so that the upper boundary condition is violated for $m = 2$, and therefore $D_{1,2}$ will not feasible. In our example, for any $R > 3$ when $\pi = 0.75$, $S_{2\lambda+1,2}$ becomes negative. This, however, does not mean that another "short-term" payoff is not feasible. In fact, the intermediary has sufficient liquidity to offer even a higher allocation than the one that it could offer prior to any withdrawal shock. For such a high payoff though, the binding sequential feasibility constraints are not satisfied as $S_{2\lambda+1,2}$ becomes negative. The highest feasible payoff that could be offered in this case is $D_{1,0}$ and the system returns to its initial allocation. In this process, some of the extra resources are wasted and the system rotates on a 2-period constant orbit as is presented in figure 3.

For high values of R, oscillations arise due to the effect of $D_{1,0}$ on the first allocation $D_{1,1}$ and the negative relationship between successive cycles' payoffs. High values of R imply that the cost of financing excess withdrawals out of new deposits at $m = 1$ is high, which implies a low $D_{1,1}$, but this cost becomes low at $m = 2$, which implies a high $D_{1,2}$. Again, through iteration this effect is perpetuated between successive payoffs and diminishes when the absolute value of the slope of B_m is less than unity so that we converge to the "long-term" steady state, or increase when the absolute value of the slope of B_m is greater than unity, so that we diverge from the fixed point.

From the above analysis we generalise our results in the following proposition.

Proposition 4.3 *An infinitely-lived intergenerational bank with an illiquid portfolio of assets can survive a crisis when external borrowing is available. The cost of borrowing and the desired level of liquidity determine the transition path towards a feasible "long-term" steady state allocation.*

In our analysis of this section we have consider only some particular values of κ_m and τ. Of course there are many other values of these the policy parameters that the Central Bank can consider in order to ensure the financial stability of the banking system during a crisis. However, for any other set of values, the evolution of banks' liquidity during the crisis will be described by a different set of difference equations which will complicate the problem. In addition, depending on the parameters of our model, this will result different fixed points and transition paths of "short-term" equilibrium payoffs which is impossible to predict and analyse.

5 Banks' Financial Stability without a Central Bank

In this section, we examine the evolution of banks' liquidity over the whole period of the crisis in the presence of borrowing restrictions. In this case, banks' inability to borrow implies that the liquidity of their portfolio of assets will necessarily be periodic during bank run episodes, new fixed points will emerge, and on the whole, banks' financial stability becomes unpredictable during a crisis. We derive some general properties of these fixed points and we identify the factors that affect the transition path of successive "short-term" equilibria .

In the absence of a Central Bank that can perfectly smooth banks' liquidity during the course of a crisis as we have analysed in the previous section, new fixed points can arise in our model as potential "long-term" steady state allocations. The general property 3.4 of our model implies that in this case, the evolution of banks' liquidity depends on the timing of bank run episodes, which determine the available resources in each cycle. By substituting the corresponding values of x_m and y_m into the budget constraint, incorporating the incentive compatibility constraint, we derive a set of implicit functions of difference equations (denoted as $B_m(D_{1,m}, D_{1,m-1}, R) = 0$) that, depending on whether a bank run occurs at a period of high or low liquidity, describe the relationship between successive allocations. In this respect, new possible fixed points may exist. In particular, if the last run before the system settles takes place at a period of high liquidity then, following our analysis in the previous section, the "long-term" steady state is given by equation 4.2. On the other hand, if the last run before the system settles happens at a period in low liquidity, by substituting x_m and y_m as given by equation 3.10 into the budget constraint and setting $D_{1,m} = D_{1,m-1}$, we obtain the quadratic function $B(D_{1,m}, R) = -(1-\pi)(R - D_{1,m})D_{1,m}$, the solutions of which (fixed points) may constitute potential "long-term" equilibria if they satisfy our conditions. We derive that this function has two fixed points; $D_{1,M} = R$ which violates the nonnegativity condition on $S_{2\lambda+1,M}$, since $D_{1,M} > D_{1,0}$ and may result to a misallocation of resources, and $D_{1,M} = 0$ which violates the lower boundary since $D_{1,N} > 0$. Despite the fact that these new fixed points violate the feasibility conditions, it is interesting to examine their properties in order to characterise the behaviour of nearby orbits. By differentiating the budget constraint with respect to successive equilibrium payoffs and evaluating the derivative at $D_{1,M} = R$ we obtain;

$$\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}=R} = 1 + \frac{R(1-\pi)(R-1)}{\pi(R-1) + 2\left(\pi + 2(1-\pi)R\right)} \tag{5.1}$$

which is clearly greater than unity and therefore this is a repelling fixed point.

However, when we evaluate the derivative at $D_{1,M} = 0$, we obtain;

$$\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}=0} = 1 - \frac{R(1-\pi)(R-1)}{\pi(R+1)} \tag{5.2}$$

From the above derivative we can observe that $D_{1,M} = 0$ is an attracting fixed point since the absolute value of its derivative is less than one, provided that $w_1 > 0$ holds.[13] Hence, $D_{1,M} = 0$ is another fixed point to which the "short-term" steady states will converge if bank runs occur at periods of low liquidity. Of course, it is not a feasible "long-term" steady state equilibrium since in the process of convergence, during a particular run incident, new generations will prefer to form a new bank and therefore the existing bank will go "out of business".

Having identified a second fixed point that successive allocations may converge to, we can characterise the general behaviour of the system around these fixed points. Differentiating the implicit functions of the difference equations with respect to successive "short-term" steady-state equilibrium payoffs, we derive;

$$\frac{\partial D_{1,m}}{\partial D_{1,m-1}} = -\frac{\partial B_m/\partial D_{1,m-1}}{\partial B_m/dD_{1,m}} = -\frac{\partial w_m/\partial D_{1,m-1}}{\partial v_m/\partial D_{1,m-1}} \tag{5.3}$$

since only the w_m term of the budget constraint depends on $D_{1,m-1}$, and correspondingly, only the v_m term depends on $D_{1,m}$. However, when we evaluate the above derivative at the fixed point $D_{1,M} = D_{1,m} = D_{1,m-1}$, from the balance budget constraint it follows that $\partial w_m/\partial D_{1,m-1}|_{D_{1,M}} = \partial B_M/\partial D_{1,M} - \partial v_m/\partial D_{1,m}|_{D_{1,M}}$, and therefore it can be simplified to;

$$\left.\frac{\partial D_{1,m}}{\partial D_{1,m-1}}\right|_{D_{1,M}} = 1 - \frac{\partial B_M/\partial D_{1,M}}{\partial v_m/\partial D_{1,m-1}|_{D_{1,M}}} \tag{5.4}$$

Provided that the budget constraint, which is the difference between assets and liabilities, balances at all times we can infer that at the "long-term" steady state the constraint decreases monotonically in $D_{1,M}$ since higher equilibrium payoffs correspond to higher liabilities, and therefore more resources are required for the constraint to remain balanced (i.e. $\partial B_M/\partial D_{1,M} < 0$). In a similar manner, the part of the budget constraint that captures the current resources (endowments of the new generation) and liabilities of a particu-

[13]It is clear that the derivative in 5.2 is less than 1. Thus, it is sufficient to show that is always greater than -1. By developing the inequality and substituting R for a_0, we derive that $0 < \pi \left(D_{1,0}\left(1 + (1-\pi)D_{1,0}\right) + 2\right) + w_1 D_{1,0}\left(\pi^2 + (1-\pi)\left(D_{1,0} + (1-\pi)D_{1,0}^2\right)\right)$ which always holds for $w_1 > 0$.

lar cycle (v_m) decreases in $D_{1,\,m}$ as higher equilibrium payoffs correspond to higher liabilities (i.e. $\left.\frac{\partial v_m}{\partial D_{1,\,m}}\right|_{D_{1,\,M}} < 0$).[14] Therefore, the change in the budget constraint relative the change in the current cycle's resources and liabilities, due to a change in the equilibrium payoff is positive when evaluated at the "long-term" steady-state (i.e. $\frac{\partial B_M/\partial D_{1,\,M}}{\partial v_m/\partial D_{1,\,m-1}|_{D_{1,\,M}}} > 0$).[15] From the above expression we derive an important property about the transition of successive equilibrium payoffs around the fixed points we have identified.

Property 5.1 *The system of difference equations does not "explode" monotonically away from a "long-term" equilibrium since* $\left.\frac{\partial D_{1,\,m}}{\partial D_{1,\,m-1}}\right|_{D_{1,\,M}} < 1$, *independently of the timing of runs (i.e.* $\{x_m,\, y_m\}$*).*

In order to identify the factors that affect the nature of the fixed points that arise in our model and provide some intuition about the behaviour of the system away from these fixed points, we need to identify the factors that affect the relationship between successive "short-term" steady state payoffs. For this reason we use the expression for w_m in terms of x_m and y_m which captures the resources that remain available for investment from the previous cycle in order to highlight the importance of the liquidity of the bank's portfolio at the time period when runs take place. We have shown that the denominator in equation 5.3 is negative whereas the numerator can take any sign and can be written as:

$$\frac{\partial w_m}{\partial D_{m-1}} = R\left(\frac{\partial x_m}{\partial D_{1,\,m-1}} + \frac{\partial y_m}{\partial D_{1,\,m-1}}\right) - \frac{\partial a_{m-1}}{\partial D_{1,\,m-1}} \tag{5.5}$$

Hence, the sign of the derivative described in equation 5.3 depends on the effect that a change in the equilibrium payoff has on the resources that become available for investment from the previous cycle's investment decisions (w_m). In a similar manner, the magnitude of the derivative depends on the magnitude of this effect relative to the magnitude of the effect due

[14] $\left.\frac{\partial v_m}{\partial D_{1,\,m}}\right|_{D_{1,\,M}} = -\left(\frac{2\,\frac{\partial a_m}{\partial D_{1,\,m}}\,{}_{D_{1,\,M}}}{R-1} + \pi\right) < 0$, where $\frac{\partial a_M}{\partial D_{1,\,M}} > 0$.

[15] This can be shown by equations 4.3 and 5.2, when the last bank run before the system settles occurs at a period of high and low liquidity, respectively.

to a change in the equilibrium payoff on the assets and liabilities of the current cycle (v_m). From Property 3.4 it is clear that the values of x_m and y_m depend on the timing of runs for the whole duration of the crisis. However, despite the fact that $D_{1,\,m-1}$ can not be explicitly defined when runs occur at random time periods during a crisis, we can make inferences about the sign of the above derivative during periods of high and low liquidity of the bank's portfolio of assets, since the sequential budget constraints and the investment in $\{S_{t,\,m},\, I_{t,\,m}\}$ become stationary at $t \geq 3$ for any cycle. In this way, we can make inferences not only about the orbits around the "long-term" equilibria, but also about the transition paths towards these equilibria.

Consider the last period of the cycle $m-1$ during which the m run takes place and therefore constitutes the initial period of the m cycle. As we have seen in the previous section, during periods of high liquidity (odd periods), the total resources available for investment, x_m and y_m, are given by equation 3.11 from which we compute that $\frac{\partial x_m}{\partial D_{1,\,m-1}} < 0$ and $\frac{\partial y_m}{\partial D_{1,\,m-1}} > 0$.[16] In this case, the liquid assets which are used to finance the excess withdrawals, and consequently the resources available for investment at $t = 0$, are negatively related to the previous cycle's equilibrium payoff. However, the investment in the long-term technology that comes to maturity at $t = 1$ is positively related to the previous cycle's equilibrium payoff as more resources are required to be invested in the productive technology in order to maintain higher payments.

Substituting the above derivatives into equation 5.5 we derive that for any payoffs away from the fixed points, this will become;

$$\frac{\partial w_m}{\partial D_{1,\,m-1}} = \frac{\partial a_{m-1}}{\partial D_{1,\,m-1}} \left(R \left(\frac{2}{R-1} - 1 \right) - 1 \right) \frac{R+1}{R-1} - R(1-\pi)$$

Of course, we do not know the explicit value of $D_{1,\,m-1}$ since this depends on the timing of withdrawal shocks that have happened in the past, but we can make general inferences about the sign of the above derivative: since

[16]When a run happens at odd periods $\frac{\partial x_m}{\partial D_{1,\,m-1}} = -\frac{\partial a_{m-1}}{\partial D_{1,\,m-1}} - (1-\pi) < 0$ and $\frac{\partial y_m}{\partial D_{1,\,m-1}} = 2\left(\frac{\partial a_{m-1}}{\partial D_{1,\,m-1}}\right)/(R-1) > 0$.

$\frac{\partial a_{m-1}}{\partial D_{1,\,m-1}} > 0$ for any surviving intermediary, its sign depends on the parameter values. For example, for any R not exceeding 2 the above derivative will always be positive, whereas for any R no lower than 3 it will always be negative.[17] Hence, independently of the value of $D_{1,\,m-1}$, in the case when runs take place in odd periods, the transition path of successive "short-term" equilibrium payoffs of the surviving bank does not "explode" monotonically away from a fixed point. In the case when the system settles at the fixed point described in equation 4.2 so that runs around this equilibrium take place at odd periods, nearby orbits may be attracted or repelled depending on the responsiveness of y_m due to a change in $D_{1,\,m-1}$ as opposed to that of x_m and a_{m-1}, and therefore, on the values of our parameters. This is consistent with our analysis in the previous section where Central Bank intervention guarantees that, independently of the timing of withdrawal shocks within each cycle, bank's liquidity remains always at the level that it has during odd periods. As we have seen from property 4.2 and in our numerical examples, for a given π, when $R = 2$ the system monotonically converges to M_2, and for any other higher value less than 3 it oscillates and converges towards M_3, as we have illustrated in figures 1 and 2.

On the other hand, if the run m takes place during a period of low liquidity (even periods), the total resources available for investment, x_m and y_m, are given by equation 3.10 where y_m is independent of $D_{1,\,m-1}$, and $\frac{\partial x_m}{\partial D_{1,\,m-1}} > 0$ for a surviving bank since w_1 is required to be positive[18]. Clearly, despite the fact that the bank holds an illiquid portfolio of assets during the time period that a run takes place and all excess withdrawals are financed by its liquid assets, the resources that are available for investment at $t = 0$ are pos-

[17]Indeed, for $R = 2$ we derive that $\frac{\partial w_m}{\partial D_{1,\,m-1}} = 3\,(\pi + 2(1-\pi)D_{1,\,m-1}) - 2(1-\pi)$ which is always positive for a surviving bank since $D_{1,\,m-1} > D_{1,\,N} > 1$. However, for $R = 3$ the derivative becomes $\frac{\partial w_m}{\partial D_{1,\,m-1}} = -2\,(\pi + 2(1-\pi)D_{1,\,m-1}) - 3(1-\pi)$ which is always negative.

[18]When a run happens during periods of low liquidity $\frac{\partial x_m}{\partial D_{1,\,m-1}} = 2\left(\frac{\partial a_{m-1}}{\partial D_{1,\,m-1}}\right)/(R - 1) - (1-\pi)$. Substituting $R = a_0$, the expression will become $4(1-\pi)D_{1,\,m-1} + w_1\pi + 1 - (1-\pi)^2 D_{1,\,0}^2$, which can be simplified to $4(1-\pi)D_{1,\,m-1} + w_1\,(1 + \pi + (1-\pi)D_{1,\,0})$ which is positive from the requirement that the bank survives the first run.

itively related to the equilibrium payoff of the previous cycle. This is simply due to the effect the change in equilibrium payoff has on the investment in the productive technology. In other words, the higher the previous cycle's equilibrium payoff, the greater the investment in the long-term technology that is required for this payoff to be sustained, and therefore, the greater the returns for distribution following a run. Hence, for any cycle m for which the m run has been triggered in an even period, we derive;

$$\frac{\partial w_m}{\partial D_{1,\,m-1}} = \frac{\partial a_{m-1}}{\partial D_{1,\,m-1}}\left(\frac{R+1}{R-1}\right) - (1-\pi)R$$

The sign of the above derivative depends on whether the positive effect on the resources that are available for investment at $t = 0$ is greater than the negative effect of the standard liabilities of the previous cycle, due to a change in the previous cycle's equilibrium payoff. However, we can not make positive conclusions about the sign of this derivative since these effects depend on the parameters of our model and on the value of $D_{1,\,m-1}$ which in turn depends on the timing of the withdrawal shocks that have happened in the past. Hence, we can not make any inferences about the transition of successive allocations when a runs take place at periods of low liquidity away from a fixed point. Property 5.1 describes only the behaviour of the system before successive allocations settle at $D_{1,\,M} = 0$ so that successive bank runs occur at periods of low liquidity. As we have mentioned earlier, this fixed point is not feasible and is associated with bank failure. In the extreme example when we allow runs to take place only at periods of low liquidity, successive payoffs may oscillate or move monotonically towards $D_{1,\,M} = 0$ until, for a given cycle m, x_m becomes negative and the newborn generation forms a new bank. For example, when $\pi = 0.22$ and $R = 1.5$, $\left.\frac{\partial D_{1,\,m}}{\partial D_{1,\,m-1}}\right|_{D_{1,\,M}=0} = -0.6363$ so that successive payoffs oscillate towards $D_{1,\,M=0}$ but the bank "goes out of business" at $m = 2$ since x_2 becomes negative. In addition, when $\pi = 0.75$ and $R = 2$, $\left.\frac{\partial D_{1,\,m}}{\partial D_{1,\,m-1}}\right|_{D_{1,\,M}=0} = 0.7777$ so that successive payoffs monotonically converge towards $D_{1,\,M=0}$ but the bank "goes out of business" at $m = 7$ as x_7 becomes negative.

Our analysis so far has only been concerned with explaining the transition between successive payoffs away from or close to the fixed points that we have identified, provided that a series of run episodes (at least 2) occur during periods of high or low liquidity between successive cycles. If, however, these episodes occur at periods characterised by different levels of liquidity, then it becomes impossible to predict the relationship of these equilibrium allocations, and more importantly, the system may not converge to a fixed point. In particular, the relationship of allocations away from the fixed points will depend on our model's parameters and on the timing of past run events. This determines the value of the previous cycle's equilibrium allocation, which in turn through the process of iteration, affects the whole transition path. Furthermore, the system can settle to a fixed point only if a series of successive bank runs occur at periods characterised by the same level of liquidity. If this is not the case then successive cycles' equilibrium payoffs will rotate on constant orbits of n-cycles when the timing of the withdrawal shocks and the model's parameters are such that $D_{1,m} = D_{1,m+n}$, for any n that belongs to the set of nonnegative integers. Hence, it becomes evident that due to the unpredictability that arises in the model, it is impossible to establish a complete description of the effects that a crisis has on banks' financial stability in the presence of borrowing restrictions.

Overall, we have shown that this erratic behaviour of banks' liquidity during a crisis results from the periodicity of the investment in the two available technologies that is required in order for the "short-term" steady-state payoff to be offered to all the generations of agents within the same cycle. Our results suggest that when borrowing restrictions are in place, intermediaries may survive the crisis if they converge to a feasible "long-term" steady state (or rotate on constant orbits of feasible allocations), or fail. This chaotic behaviour that characterises the liquidity of banks' portfolios of assets can threaten the soundness of a banking system that consists of solvent but illiquid banks. However, when external borrowing is available, illiquid intermediaries can smooth the returns of their investment portfolio during periods of financial distress and may survive a crisis. We have presented a particular case where the amount of resources borrowed and the cost of borrowing are predetermined, and we analysed the transition paths that arise towards a feasible "long-term" steady state payoff for different values of the model's parameters.

6 Discussion

In this paper we have developed a model that highlights the importance of the liquidity of banks' portfolio, which determines their ability to survive a crisis. External borrowing by a Central Bank acting as a Lender of Last Resort by providing a discount window that enables transfer of resources across time, and a minimum reserve requirement can provide the necessary stability in the banking system during a period characterised by a series of withdrawal shocks. In this respect, several assumptions of our model merit some comments.

In our analysis, we have restricted our attention to steady state payoffs within a cycle by treating each generation of agents the same. As we have mentioned in the formulation of a representative bank's planning problem, starting with a large number of identical banks, competition in the market compels banks to perfectly compete on the terms of their deposit contract in each period in order to attract new deposits. This does not only mean that sequential budget constraints should hold with equality, as with the equal treatment assumption in place, but also banks may have incentives to offer even higher payoffs to new generations under the premise that such high payments will drive their competitors out of the market and enable them in the future periods to make monopoly profits. However, there are major conceptual and technical difficulties associated with this approach in determining the whole strategy set in the formulation of banks' planning problem.[19] This

[19] In this respect, Bencivenga and Smith (1991)[5] in developing an endogenous growth model using an overlapping generations framework they viewed banks as coalitions of agents of the same generation without prior liabilities or additional deposits by other

interesting issue is a subject for future research.

Another important issue in our analysis is that bank runs are modelled as massive withdrawals where all the depositors who hold an outstanding liability, exercise their right to withdraw. The requirement of our model that the bank survives the first run where all the excess withdrawals are satisfied by liquid assets, therefore, imposes restrictions on our model parameters in order to be satisfied. In other words, w_1 is required to be positive where at the same time is decreasing with respect to the returns of the long term technology. Therefore, for a given π, there is a small range of values that R can take so that the bank will "remain in business" after the first run has taken place so that excess withdrawals are relatively inexpensive to be financed by new deposits. This requirement affects the behaviour of the system since it determines the initial orbit of the system and consequently, through iteration, the final outcome. However, these restrictions on the parameters can be relaxed if we allow instead only a small fraction of impatient depositors observe this extinsic factor that affects their beliefs about banks' solvency, and consequently trigger the run (as in as in Allen and Gale (2000)[3]). Under these circumstances, it is more likely that these excess withdrawals can be met out of new deposits and therefore, our results can be extended for higher values of R.

Moreover, an interesting issue that arises in our model concerns the oscillations of successive equilibrium payoffs and withdrawal behaviour of impatient agents. As we have seen, in case of oscillations, a run triggered by one generation can lead to an improvement of welfare for future generations. However, this behaviour is incompatible with the assumption of our model that the bank does not renege on the initial contract. In other words, patient depositors do not have incentives to trigger a run and misrepresent themselves as newborn agents of the first generation of the new cycle simply because the payment that will receive in the last period of their life is identical to the payment that impatient depositors of the same generation

generations of agents.

had received. Provided that the incentive compatibility constraint binds, "strategic" withdrawals do not arise in our model, irrespectively of whether generations of the new cycle will receive a higher equilibrium payoff.

The final point concerns our approach to banks' financial stability. We view bank runs unanticipated withdrawal shocks where any external event that triggers depositors to belief that other depositors will withdraw their deposits results in a run. An interesting extension of our paper would be to model explicitly the mechanism that causes agents' beliefs to change as in Ennis and Keister (2003)[14], but this is left for future research.

Figure 1

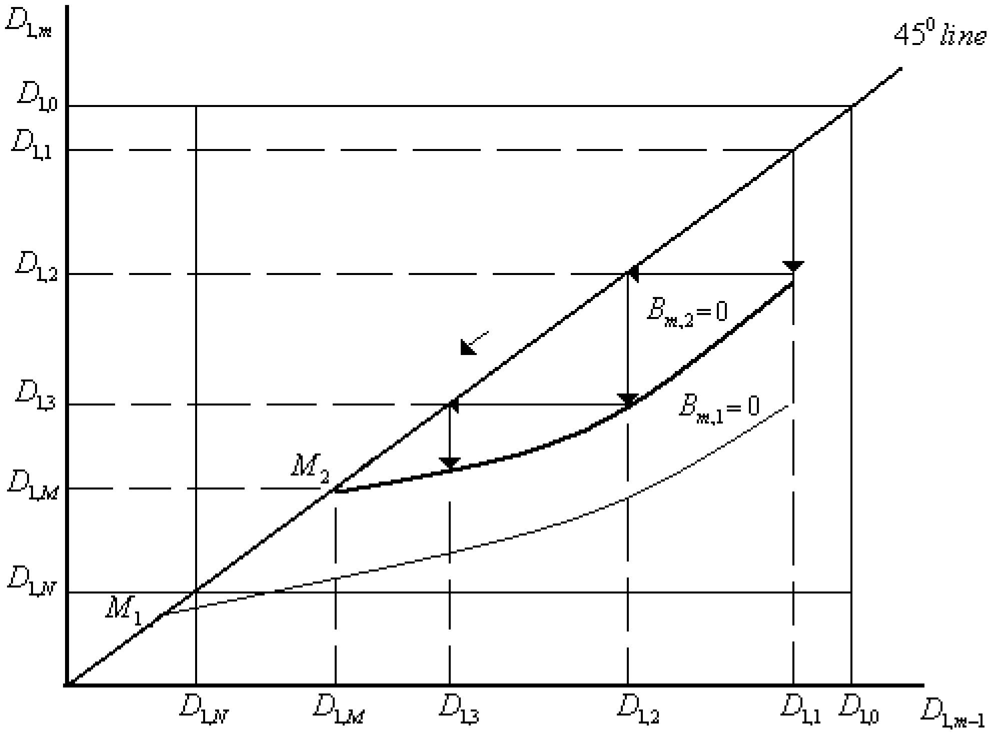


Figure 2

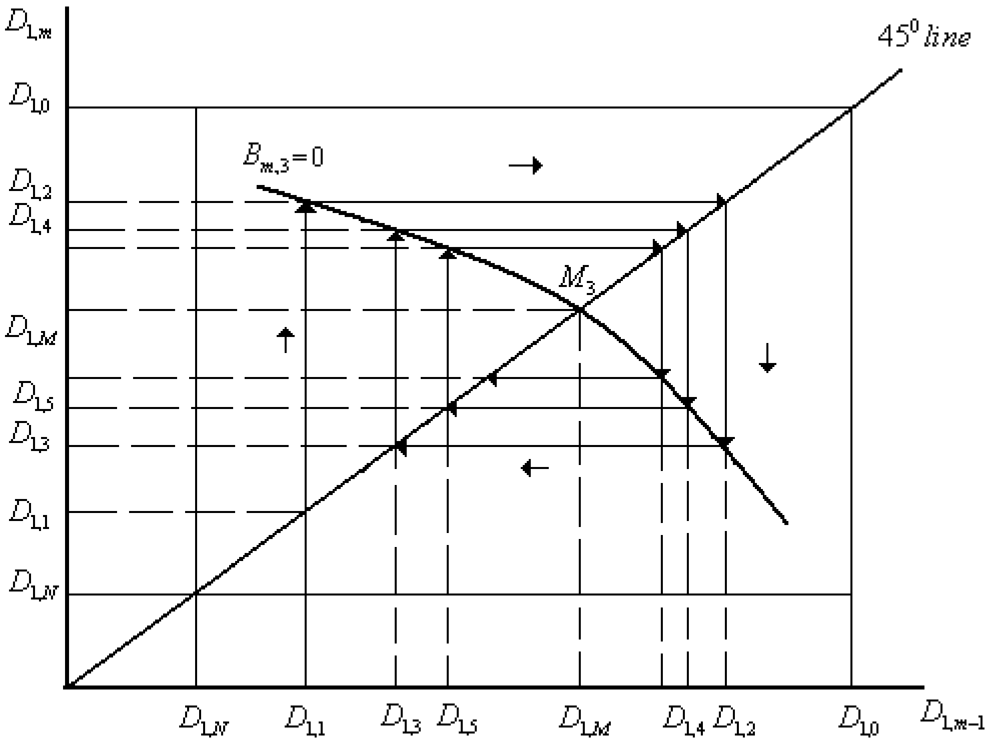
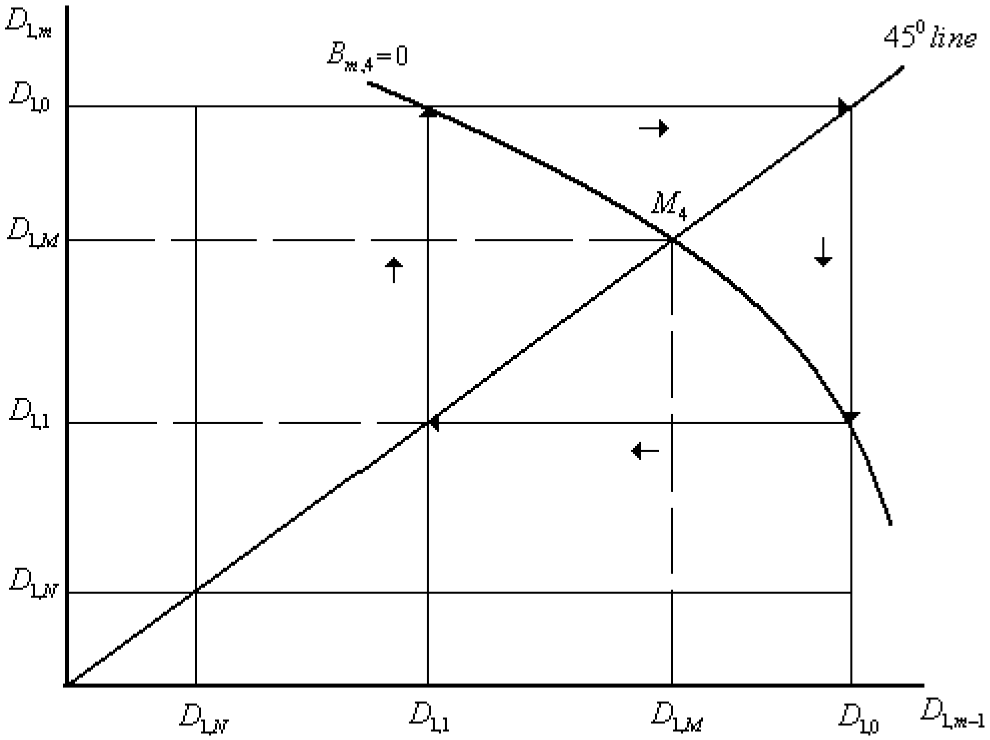


Figure 3



A Appendix

Proof of Proposition 3.2
This proof is a generalisation of Qi's (1994)[19] earlier work.

The sequential budget constraints can be written as:

$$
\begin{aligned}
\text{At } t=0 \quad &: \quad I_{0,m} \leq x_m - S_{0,m} && \text{(A.1)}\\
\text{At } t=1 \quad &: \quad I_{1,m} \leq 1 - \pi D_{1,m-1} + y_m\,R + S_{0,m} - S_{1,m} && \text{(A.2)}\\
\text{At } t=2 \quad &: \quad I_{2,m} \leq 1 - \pi D_{1,m} - (1-\pi)D_{2,m-1} + I_{0,m}R + S_{1,m} - S_{2,m} && \text{(A.3)}\\
&: \quad I_{2,m} \leq 1 - \pi D_{1,m} - (1-\pi)D_{2,m-1} + R\,(x_m - S_{0,m}) + S_{1,m} - S_{2,m} &&\\
\text{At } t=3 \quad &: \quad I_{3,m} \leq 1 - a_m + I_{1,m}\,R + S_{2,m} - S_{3,m} && \text{(A.4)}\\
&: \quad I_{3,m} \leq 1 - a_m + R\,(1 - \pi D_{1,m-1} + y_m R) + R\,(S_{0,m} - S_1) + S_{2,m} - S_{3,m} &&\\
\text{At } t=4 \quad &: \quad I_{4,m} \leq 1 - a_m + I_{2,m}R + S_{3,m} - S_{4,m} && \text{(A.5)}\\
&: \quad I_{4,m} \leq 1 - a_m + R\,(1 - \pi D_{1,m} - (1-\pi)D_{2,m-1}) &&\\
&+ \quad R^2\,(x_m - S_{0,m}) + R\,(S_{1,m} - S_{2,m}) + S_{3,m} - S_{4,m} &&\\
\text{At } t=5 \quad &: \quad I_{5,m} \leq 1 - a_m + I_{3,m}R + S_{4,m} - S_{5,m} && \text{(A.6)}\\
&: \quad I_{5,m} \leq 1 - a_m + R\,(1 - a_m) &&\\
&+ \quad R^2\,(1 - \pi\,D_{1,m-1} + y_m\,R) + R^2\,(S_{0,m} - S_{1,m}) + R\,(S_{2,m} - S_{3,m}) + S_{4,m} - S_{5,m} &&
\end{aligned}
$$

The above constraints can be described in two sequences for λ the set of positive integers for $t \geq 3$.

$$
\begin{aligned}
I_{2\lambda+1,m} \quad &\leq \quad 1 - a_m + I_{2\lambda-1,m}\,R + S_{2\lambda,m} - S_{2\lambda+1,m} \text{ or} && \text{(A.7)}\\
I_{2\lambda+1,m} \quad &\leq \quad \sum_{i=1}^{\lambda} R^{\lambda-i}(1-a_m) + R^{\lambda}(1 + y_m R - \pi D_{1,m-1}) + \sum_{i=1}^{\lambda+1} R^{\lambda+1-i}\,(S_{2i-2,m} - S_{2i-1,m}) &&
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2\lambda+2,\,m} &\leq 1 - a_m + I_{2\lambda,\,m}\,R + S_{2\lambda+1,\,m} - S_{2\lambda+2,\,m} \text{ or} \qquad \text{(A.8)}\\
I_{2\lambda+2,\,m} &\leq \sum_{i=1}^{\lambda} R^{\lambda-i}(1-a_m) + R^{\lambda}(1-\pi D_{1,\,m} - (1-\pi)D_{2,\,m-1})\\
&+ R^{\lambda+1}\,(x_m - S_{0,\,m}) + \sum_{i=1}^{\lambda+1} R^{\lambda+1-i}\,(S_{2i-1,\,m} - S_{2i,\,m})
\end{aligned}
$$

Suppose that the payoff $(D_{1,\,m}, D_{2,\,m})$ is budget feasible; there exist some nonnegative $\{I_{t,\,m}\}$ and $\{S_{t,\,m}\}$ such as the above sequential budget constraints hold.

Take $S_{t,\,m} \geq 0$. It follows that:

$$
\begin{aligned}
I_{2\lambda+1,\,m} + I_{2\lambda+2,\,m} &\leq 2\sum_{i=1}^{\lambda} R^{\lambda-i}(1-a_m) + R^{\lambda}\,(2 - \pi D_{1,\,m} - a_{m-1} + R\,(y_m + x_m)) - R^{\lambda+1}S_{0,\,m}\\
&+ \sum_{i=1}^{\lambda+1} R^{\lambda+1-i}\,(S_{2i-2,\,m} - S_{2i,\,m}) \qquad \text{(A.9)}
\end{aligned}
$$

Simplifying the above expression, we obtain:

$$
\begin{aligned}
I_{2\lambda+1,\,m} + I_{2\lambda+2,\,m} &\leq 2\sum_{i=1}^{\lambda} R^{\lambda-i}(1-a_m) + R^{\lambda}\,(2 - \pi D_{1,\,m} - a_{m-1} + R\,(y_m + x_m))\\
&- (R-1)\sum_{i=1}^{\lambda} R^{\lambda+1-i}S_{2i-1,\,m} - S_{2\lambda,\,m} \qquad \text{(A.10)}
\end{aligned}
$$

or alternatively,

$$
I_{2\lambda+1,\,m} + I_{2\lambda+2,\,m} \leq 2\sum_{i=1}^{\lambda} R^{\lambda-i}(1-a_m) + R^{\lambda}\,(2 - \pi D_{1,\,m} - a_{m-1} + R\,(x_m + y_m))
$$

where the last inequality obtains because $\{S_t\} \geq 0$ is taken as given.

Taking the limit as λ approaches infinity, it follows:

$$\lim_{\lambda\to\infty}(I_{2\lambda+1,\,m}+I_{2\lambda+2,\,m}) \leq \lim_{\lambda\to\infty} R^{\lambda}\left(\frac{2(1-a_m)}{R-1}+2-\pi D_{1,\,m}-a_{m-1}+R(x_m+y_m)\right) \tag{A.11}$$

By contradiction, if 3.3 does not hold then $\lim_{\lambda\to\infty}(I_{2\lambda+1,\,m}+I_{2\lambda+2,\,m}) = -\infty$, implying that either $I_{2\lambda+1,\,m}$ or $I_{2\lambda+2,\,m}$ (or both) is negative for a large λ. That is, there is no nonnegative $I_{2\lambda+1,\,m}$ and $I_{2\lambda+2,\,m}$ that makes the sequential budget constraints true. This is contradictory to the assumption of budget feasibility. Therefore, a budget-feasible steady-state payoff $(D_{1,\,m},\, D_{2,\,m})$ must satisfy the condition 3.3.

When $y_mR-\pi D_{1,\,m-1} < 0$, the other set of conditions on $\{I_t,\, S_t\}$, is described by:

$$S_{t,\,m} = \begin{cases} 0 & \text{for } t=0 \text{ and } t=2\lambda, \lambda\in Z_{++} \\ 1+y_mR-\pi D_{1,\,m-1} & \text{for } t=1 \\ 1-a_m & \text{for } t=2\lambda+1, \lambda\in Z_{++} \end{cases}$$

$$I_{t,\,m} = \begin{cases} x_m & \text{for } t=0 \\ \frac{2(a_m-1)}{R-1}-1 & \text{for } t=2\lambda, \lambda\in Z_{++} \\ 0 & \text{for } t=2\lambda+1, \lambda\in Z_{+} \end{cases}$$

We can observe that this set of conditions on $\{I_t,\, S_t\}$ also satisfies the sequential budget constraints with equality but violates the nonnegativity condition since when $S_{2\lambda+1,\,m} > 0$ it follows that $I_{2\lambda,\,m} < 0$ and vice versa. For this reason is ignored

Proof of Lemma 4.1

From the definition of this case $S_{1,\,M} = y_mR-\pi D_{1,\,M} > 0$. In addition, $\{x_M, y_M\} > 0$ so that $I_{0,M} = x_M > 0$ and $I_{2\lambda,\,M} > 0$. The last condition on the investment technologies for the "long-term" steady-state equilibrium to be feasible is that $S_{2\lambda+1,\,M} = R - a_M$ is positive. Hence, is sufficient to show that $D_{1,\,0} > D_{1,\,M}$ which implies that $a_0 > a_M$, where $a_0 = R$.

Evaluating the general budget constraint for the "long-term" steady state at $D_{1,\,0}$, we obtain:

$$B_M(D_{1,\,0},\, R) = R^2 - (1-\pi)(R-1)D_{1,\,0}^2 - R\,(\pi + (1-\pi))\,D_{1,\,0}$$

where $B_M(D_{1,\,M},\, R) = 0$ is strictly decreasing and concave in $D_{1,\,M} > 0$. Hence, if $D_{1,\,0} > D_{1,\,M}$ then we need to show that $B_M(D_{1,\,0},\, R) < 0$. Indeed by substituting for $a_0 = \pi D_{1,\,0} + (1-\pi)D_{1,\,0}^2$ into the above expression, this can be simplified to:

$$B_M(D_{1,\,0},\, R) = R^2 - R\,a_0 + (1-\pi)D_{1,\,0}^2 - R(1-\pi)D_{1,\,0}$$

and from $a_0 = R$, it follows:

$$B_M(D_{1,\,0},\, R) = (1-\pi)D_{1,\,0}\,(D_{1,\,0} - R) < 0$$

since $D_{1,\,0} < R$. Hence, from the monotonicity of $B_M(D_{1,\,M},\, R) = 0$ in $D_{1,\,M}$ it follows that $D_{1,\,0} > D_{1,\,M}$, or equivalently $a_0 > a_M$, which implies that $S_{2\lambda+1,\,M} = R - a_M > 0$.

Differentiating the general budget constraint the "long-term" steady state $D_{1,\,M}$ with respect to R we obtain:

$$\frac{dB_M(D_{1,\,M},\, R)}{dR} = \frac{\partial B_M}{\partial R} + \left(\frac{\partial B_M}{\partial D_{1,\,M}}\right)\frac{dD_{1,\,M}}{dR} = 0$$

or alternatively,

$$\frac{dD_{1,\,M}}{dR} = -\frac{\partial B_M/\partial R}{\partial B_M/\partial D_{1,\,M}} = -\left(\frac{(1-\pi)D_{1,\,M}^2 + D_{1,\,M} - 2R}{2(1-\pi)(R-1)D_{1,\,M} + R}\right)$$

where $\partial B_M/\partial R$ can be written as $a_M - R + (1-\pi)D_{1,\,M} - R < 0$ since $a_M < R$.

Therefore, $\frac{dD_{1,M}}{dR} > 0$ indicating that the "long-term" steady-state equilibrium payoff is increasing in R.

Proof of Property 4.2

In order to determine the sign of the the derivative of equation 4.3 with respect to R, we simply focus on the derivatives numerator and denominator with respect to R, respectively. If the the later has a different sign from the former we can conclude that the sign of the whole derivative. Note that $D_{1,M}$ depends also on R.

Differentiating the nominator with respect to R we obtain:

$$1 + 2(1-\pi)\left(D_{1,M} + (R-1)\frac{dD_{1,M}}{dR}\right) > 0$$

Differentiating the denominator with respect to R we obtain:

$$\frac{4(1-\pi)\left((R-1)\frac{dD_{1,M}}{dR} - D_{1,M}\right) - 2\pi}{(R-1)^2}$$

Focusing on the terms inside the brackets, where $D_{1,M} = \frac{R\ \sqrt{1+4(1-\pi)(R-1)}-1}{2(1-\pi)(R-1)}$ and $\frac{dD_{1,M}}{dR} = \frac{R}{(R-1)\sqrt{1+4(1-\pi)(R-1)}} - \left(\frac{\sqrt{1+4(1-\pi)(R-1)}-1}{2(1-\pi)(R-1)^2}\right)$. Hence:

$$(R-1)\frac{dD_{1,M}}{dR} - D_{1,M} = -\frac{D_{1,M}}{R} - \frac{R\left(1 + 2(1-\pi)(R-1) - \sqrt{1+4(1-\pi)(R-1)}\right)}{2(1-\pi)(R-1)\sqrt{1+4(1-\pi)(R-1)}} < 0$$

where $1 + 2(1-\pi)(R-1) > \sqrt{1+4(1-\pi)(R-1)}$.

Therefore, the derivative of the denominator with respect to R is negative, and on the whole $\frac{d(dD_{1,m}/dD_{1,m-1})}{dR} < 0$.

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