# Inflation prediction from the term structure: the Fisher equation in a multivariate SDF framework

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## Abstract

This paper proposes a new way of extracting inflation information from the term structure. We rehabilitate the Fisher equation, by setting it in the context of the stochastic discount factor (SDF) asset pricing theory. We develop a multivariate estimation framework which models the term structure of interest rates in a manner consistent with the SDF theory while at the same time generating and including an often omitted time varying risk component in the Fisher equation. The joint distribution of excess holding period bond returns of different maturity and fundamental macroeconomic factors is modelled on the basis of the consumption CAPM, using multivariate GARCH with conditional covariances in the mean to capture the term premia. We apply this methodology to the U.S. economy, re-examining the Fisher equation at horizons of up to one year. We find it offers substantial evidence in support of the Fisher equation and greatly improves its goodness of fit, at all horizons.

Keywords: Inflation, Fisher equation, Term structure, the stochastic discount factor model, term premia, GARCH

JEL Classification: G1, E3, E4, C5

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# 1 Inflation Prediction and the Term Structure of Interest Rates

Macroeconomists and financial analysts have rarely found that they have a lot to discuss. All too often it has seemed as if their perspectives on the same economic phenomena were vastly different. The increasingly widespread adoption of inflation targeting is causing this to change. They now have a shared interest in the term structure of interest rates. Macroeconomists examine the level and slope of the yield curve for its information content on future inflation and output as indicators of the current stance of monetary policy and of central bank credibility. Financial analysts try to assess the impact of current and future inflation and output on the shape of the yield curve and on term premia. In this paper we propose a new way to extract inflation forecasts from the term structure based on taking account of term premia. Previous methods of trying to recover information about the future path of inflation from the term structure have ignored the information contained in term premia. We show that using this additional financial information considerably improves inflation forecasts.

Although our interest in this paper is the information contained in the term structure about future inflation, there is an entirely different literature on forecasting inflation which is based on the use of standard time series forecasting methods. This literature aims to forecast inflation largely from its past, but also from the past values of other variables, see for example Cecchetti, Chu and Steindel (2001). An extensive comparison of the forecasting performance of some leading models of inflation for the cross-section of G-7 countries has been undertaken by Canova (2002). He shows that simple, univariate, autoregressive models can often outperform the bivariate and trivariate models suggested by economic theory. Although there is no necessary reason for a good forecasting model to have theoretical underpinnings, theory may still be able to help in the choice of model to use. It is partly with this observation in mind that we take the approach of using the information contained in the term structure.

The main advantage of using the term structure is that it instantly and efficiently incorporates new market information about inflation. As is well known, the yield to maturity on an $n$-period bond is the expected value of the average of future one-period rates plus the expected value of the average term premium on that bond for the rest of its life. Further, through the Fisher equation, future one-period rates depend on expected future inflation. This provides a connection between the current yield to maturity and the market's expectation of future inflation. As a result, the term structure of yields in the current period provides information about inflation over each of the next $n$ periods.

In addition to giving forecasts of future inflation, this also provides a useful check on the credibility of monetary policy. It enables a comparison to be made between the market's expectations of inflation and the inflation pronouncements of the monetary authority. This information is now commonly taken into account when setting monetary policy as it enables the monetary authority to respond to the market's expectations or to adopt forward looking Taylor rules which rely on forecasts of inflation for implementation (see Batini and Haldane (1998)).

The problem that remains is how best to extract inflation expectations from the term structure. There is a large literature on this that began in the early 1980's, see for example Fama and Gibbons (1982). Notable more recent contributions include those by Schich (1999), Stock and Watson (2003), Estrella, Rodrigues and Schich (2003) and Hardouvelis and Malliaropulos (2004).

The basis of forecasting inflation from the term structure is to form an $n$-period version of the Fisher equation by combining the term structure with the one-period Fisher equation. This relates the yield to maturity on an $n$-period bond to the expected value of average future inflation and the underlying real interest rate over the next $n$-periods plus a risk premium, known as the rolling risk premium. The problem is that two of these components (the real interest rate and the rolling risk premium) are unobservable. The usual response, in practice, is to assume that the underlying real interest rate is constant and to ignore the rolling risk premium. As both economic constraints imposed by this approach are counter-intuitive and have been rejected by a large body of research we propose an alternative response.

The key contribution of this paper is to take account of the rolling risk premium by using an estimate obtained from a stochastic discount (SDF) model of the term structure. Further, we allow for a time-varying *ex ante* real interest rate. We show that this considerably improves the inflation forecasts of the Fisher equation. This approach is prompted by the findings of Balfoussia and Wickens (2004) who used an SDF model to show how term premia are related to inflation and how it is possible to obtain time series estimates of term premia based on observed macroeconomic variables.

Previous work by Remolona, Wickens and Gong (1998) took a related approach. They estimated a rolling inflation risk premium using an affine two-factor Cox-Ingersoll-Ross pricing model of the U.K. nominal yield curve and a one-factor model of the real (indexed-linked) yield curve. This was an improvement over the break-even approach of Barr and Campbell (1997) and Deakon and Derry (1994) which simply subtracted indexed from nominal yields to obtain an estimate of expected future inflation, as it takes into account not only the real risk premium but also the inflation risk premium. Nevertheless, the significance of the risk premium in the Fisher equation is not directly examined in this literature.[1] Shome, Smith and Pinkerton (1988) are the first to theoretically model a time-varying risk premium in the Fisher equation. However, their univariate framework does not allow its direct estimation, obliging them to use survey data instead. Evans and Wachtel (1992) generate a risk premium proxy in a preliminary step and subsequently include it in univariate estimation of the Fisher equation, but do not model the conditional covariance of bond returns with the pricing kernel. The advantage of the approach adopted in this paper is that by jointly modelling the term structure and macroeconomic variables in a stochastic discount factor model we obtain a less restrictive specification of the risk premium than affine term structure models while allowing its direct inclusion in the Fisher equation.

[1] Moreover, the implementation of such models for the US is difficult since index-linked debt has only recently been introduced.

The paper is structured as follows. In Section 2 we construct the theoretical model proposed. We first discuss the SDF asset pricing framework and the Fisher equation in some detail. Subsequently, we propose a multivariate estimation framework, which models the term structure of interest rates in a manner consistent with the SDF theory, while at the same time generating and including a time varying risk component in the Fisher equation. The econometric methodology, set out in Section 3, modifies the multi-variate GARCH model to fit our specification. The data are described in Section 4. Section 5 presents the estimates obtained when this methodology is applied to the U.S. economy, re-examining the Fisher equation at horizons of up to one year. Finally, in Section 6 we present our conclusions.

# 2 Theoretical Framework

## 2.1 Notation and basic concepts

We use the following notation. $P_{n,t}$ is the price of an $n$-period, zero-coupon (pure discount) default-free bond at $t$, where $P_{0,t}= 1$ as the pay-off at maturity is 1. $R_{n,t}$ is the yield to maturity of this bond, where the one-period, risk-free rate $R_{1,t}= s_t$. The return to holding an $n$-period bond for one period from $t$ to $t+1$ is $h_{n,t+1}$. It follows that

$$P_{n,t} = \frac{1}{[1 + R_{n,t}]^n}$$

and

$$1 + h_{n,t+1} = \frac{P_{n-1,t+1}}{P_{n,t}} \tag{1}$$

If $p_{n,t} = \ln P_{n,t}$ then, taking logs

$$h_{n,t+1} \simeq p_{n-1,t+1} - p_{n,t} = nR_{n,t} - (n-1)R_{n-1,t+1} \tag{2}$$

In the absence of arbitrage opportunities, after adjusting for risk, investors are indifferent between holding an $n$-period bond for one period and holding a risk-free 1-period bond. In the absence of default, the risk is due to the price of the bond next period being unknown this period.

$$E_t[h_{n,t+1}] = s_t + \rho_{n,t} \tag{3}$$

where $\rho_{n,t}$ is the risk, or term, premium on an $n$-period bond at time $t$.

## 2.2 A general equilibrium SDF model of the term structure

The SDF model relates the price of an $n$-period zero-coupon bond in period $t$ to its discounted price in period $t+1$ when it has $n-1$ periods to maturity. Thus

$$P_{n,t} = E_t[M_{t+1}P_{n-1,t+1}]$$

where $M_{t+1}$ is a stochastic discount factor, or pricing kernel. It follows that

$$E_t[M_{t+1}(1 + h_{n,t+1})] = 1$$

and for $n = 1$,

$$(1 + s_t)E_t[M_{t+1}] = 1$$

If $P_{n,t}$ and $M_{t+1}$ are jointly log-normally distributed and $m_{t+1} = \ln M_{t+1}$ then

$$p_{n,t} = E_t(m_{t+1}) + E_t(p_{n-1,t+1}) + \frac{1}{2}V_t(m_{t+1}) + \frac{1}{2}V_t(p_{n-1,t+1}) + Cov_t(m_{t+1}, p_{n-1,t+1}) \tag{4}$$

and as $p_{o,t} = 0$,

$$p_{1,t} = E_t(m_{t+1}) + \frac{1}{2}V_t(m_{t+1}) \tag{5}$$

Subtracting (5) from (4) and re-arranging gives the no-arbitrage equation

$$E_t(p_{n-1,t+1}) - p_{n,t} + p_{1,t} + \frac{1}{2}V_t(p_{n-1,t+1}) = -Cov_t(m_{t+1}, p_{n-1,t+1}) \tag{6}$$

Using (1) this can be re-written in terms of holding-period returns as

$$E_t(h_{n,t+1} - s_t) + \frac{1}{2}V_t(h_{n,t+1}) = -Cov_t(m_{t+1}, h_{n,t+1}) \tag{7}$$

This is the fundamental no-arbitrage condition for an $n$-period bond[2] which must be satisfied to ensure there are no arbitrage opportunities across the term structure. The term on the right-hand side is the term premium and $\frac{1}{2}V_t(h_{n,t+1})$ is the Jensen effect. Comparing equations (3) and (7), we note the SDF model implies that

$$\rho_{n,t} = -\frac{1}{2}V_t(h_{n,t+1}) - Cov_t(m_{t+1}, h_{n,t+1})$$

Empirical work on the term structure can be distinguished by the choice of $\rho_{n,t}$ and the discount factor $m_t$. The expectations hypothesis assumes that $\rho_{n,t} = 0$ but this is rejected by a vast amount of evidence. We shall therefore assume that the risk premium is non-negligible and hence must be explicitly modelled.

To complete the specification, it is necessary to specify $m_t$. Assuming joint log-normality of the excess returns and the factors, $m_t$ is a linear function of the underlying factors. The SDF model does not specify which factors to use. Different formulations of the SDF model for the term structure are discussed and tested in Balfoussia and Wickens (2004). The best known SDF model is the general equilibrium consumption-based capital asset pricing model (C-CAPM) based on power utility. For nominal returns C-CAPM defines $m_{t+1}$ as

$$m_{t+1} \simeq \ln\beta - \sigma\frac{\Delta C_{t+1}}{C_t} - \pi_{1,t+1} \tag{8}$$

where $C_t$ is real consumption, $\pi_{1,t+1}$ is the rate of inflation[3] between periods $t$ and $t+1$, $\beta$ is the discount factor for computing the present value of current and future utility and $\sigma$ is the coefficient of relative risk aversion. The no-arbitrage condition for an $n$-period bond is then obtained from equation (7) as

$$E_t(h_{n,t+1} - s_t) = -\frac{1}{2}V_t(h_{n,t+1}) + \sigma Cov_t(\frac{\Delta C_{t+1}}{C_t}, h_{n,t+1}) + Cov_t(\pi_{1,t+1}, h_{n,t+1}) \tag{9}$$

[2] Arbitrage opportunities are excluded if and only if there exists a unique positive stochastic discount factor $M_{t+1}$ that prices all assets (Cochrane 2001). In the models presented in this paper a positive discount factor is used to price bonds of all maturities included in each estimation.

[3] Throughout this paper we denote the rate of inflation realised during the single period from time $t$ to $t+1$ by $\pi_{1,t+1}$. We use $\pi_{n,t+1}$ to denote the inflation rate between times $t$ and $t+n$. Hence $n$ is the horizon over which inflation is measured.

The right hand side of expression (9) is the risk premium $\rho_{n,t}$. This implies that the greater the predicted covariation of the risky return with consumption growth and inflation, the higher the risk premium. In other words, assets are being priced in accordance to the insurance they offer against adverse movements in consumption.

As condition (9) is required to hold for all bonds, it provides a set of restrictions across the term structure that guarantee the absence of arbitrage opportunities between bonds of different maturities. However, Balfoussia and Wickens (2004) found that these restrictions were not satisfied for the US term structure and that a more general model was required, in which the log discount factor depended on the time to maturity. As our primary aim is to assess the forecasting ability of the term structure with respect to inflation, we extend their specification to include inflation over different horizons in the future. The log stochastic discount factor for forecasting inflation at the $n$-month horizon shall take the form

$$m_{n,t+1} \simeq a_n + b_{n,C}\frac{\Delta C_{t+1}}{C_t} + b_{n,\pi}\pi_{n,t+1} \tag{10}$$

where $\pi_{n,t+1}$ denotes realised inflation between period $t$ and period $t+n$. This specification remains in line with the C-CAPM intuition, while allowing us to jointly model inflation over an $n$-period horizon.[4] The resulting asset pricing equation is

$$E_t(h_{n,t+1} - s_t) = -\frac{1}{2}V_t(h_{n,t+1}) + b_{n,C}Cov_t(\frac{\Delta C_{t+1}}{C_t}, h_{n,t+1}) + b_{n,\pi}Cov_t(\pi_{n,t+1}, h_{n,t+1}) \tag{11}$$

where the right-hand side of the expression is a measure of the risk premium $\rho_{n,t}$. This shall be endogenously generated in our estimation and will form the basis of the risk premium component to be included in the Fisher equation.

[4] The use of $\pi_{n,t+1}$ instead of $\pi_{1,t+1}$ allows us to model inflation over different horizons in our multivariate framework. Although strictly not exact, such deviations from the theory-implied SDF specification are very common in the literature. Examples can be found in recent research on affine term structure models, much of which uses year-on-year growth rates of macroeconomic variables as their observable factors, despite the fact that the term structure data used are typically monthly. See for instance Ang and Piazzesi (03), DeWachter, Lyrio and Maes (04) and Diebold, Rudebusch and Aruoba (04).

## 2.3 The Fisher equation under uncertainty

### 2.3.1 The $n$-period Fisher equation

The Fisher (30) equation is a simple decomposition of the nominal interest rate into a real rate and an expected inflation component. For the short rate $s_t$ it takes the following, familiar form

$$s_t = r_t + E_t[\pi_{1,t+1}] \tag{12}$$

where $r_t$ is the one-period underlying real interest rate and $E_t[\pi_{1,t+1}]$ is the one period ahead inflation expectation, conditional on information available at time $t$. We seek a corresponding relation for the yield to maturity on an $n$-period bond.[5]

The relation between the yield to maturity and the short rate is obtained from equations (2) and (3). Eliminating the holding-period yield gives

$$E_t[h_{n,t+1}] = E_t[nR_{n,t} - (n-1)R_{n-1,t+1}] = s_t + \rho_{n,t} \tag{13}$$

This gives an unstable difference equation for $R_{n,t}$

$$R_{n,t} = \frac{n-1}{n} E_t[R_{n-1,t+1}] + \frac{1}{n}(s_t + \rho_{n,t})$$

It can be solved forwards to give the following intuitive decomposition of the yield to maturity into the expected value of average current and future short rates and the expected value of the average risk premium until maturity on that bond

$$R_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} E_t s_{t+i} + \frac{1}{n}\sum_{i=0}^{n-1} E_t \rho_{n-i,t+i} \tag{14}$$

Using the Fisher equation (12) to eliminate the short rate from equation (14) we obtain the $n$-period Fisher equation

$$R_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} E_t r_{t+i} + \frac{1}{n}\sum_{i=1}^{n} E_t \pi_{1,t+i} + \frac{1}{n}\sum_{i=0}^{n-1} E_t \rho_{n-i,t+i} \tag{15}$$

[5] In early work (see Fama (75), Barthold and Dougan (86) for instance) the Fisher equation was often directly extended to bonds of any maturity. However, in an uncertain world with risk averse agents, Fisher's original proposition is only accurate for the short rate which, by assumption, is taken to be risk free. Hence we draw on term structure theory in order to derive a more general version of the Fisher equation for the $n$-period bond, which will allow for risk compensation.

The yield to maturity on an $n$-period bond is therefore equal to the expected value of the average real interest rate and future inflation over the next $n$-periods plus a risk premium, known as the rolling risk premium. In other words, we have decomposed the yield into three components: a real, a nominal and a risk component, all of which are functions of the time to maturity.

Inflation prediction from the term structure consists of recovering the forward-looking inflation component in equation (15) which can be re-written as

$$E_t \pi_{n,t+1} = R_{n,t} - \psi_{n,t} - \omega_{n,t} \tag{16}$$

where $E_t \pi_{n,t+1} = \frac{1}{n} \sum_{i=1}^{n} E_t \pi_{1,t+i}$ is the average expected inflation between times $t$ and $t+n$, $\psi_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} E_t r_{t+i}$ denotes the average expected real rate and $\omega_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} E_t \rho_{n-i,t+i}$ is the rolling risk premium of an $n$-period bond at time $t$.[6] [7] Although extracting expectations is, in general, not an easy task, the additional complication in this case is that the last two terms in equation (16) are unobservable *ex ante* as well as *ex post*.

It is common to assume that the average real interest rate is constant over time plus risk neutrality implying that $\omega_{n,t} = 0$. The resulting forecasting equation for inflation is

$$\pi_{n,t+1} = a + bR_{n,t} + \eta_{n,t} \tag{17}$$

where the constant $a$ replaces both the average real rate and the risk premium component of equation (16) and $\eta_{n,t}$ is an error term. We use Chow's test to examine the stability of this constrained form of the $n$-period Fisher equation over our sample period. Table 1 reports two test statistics for Chow's test, the F-statistic and the likelihood ratio.[8] The hypothesis of coefficient

[6] Note that the decomposition of equation 16 involves the average of expected one-period real interest rates which are by assumption risk free, while both real and nominal risk are captured by the rolling risk premium.

[7] Variations of equation 16 are occasionally referred to in the literature as the "inverse generalised Fisher equation".

[8] The principle of the Chow test is to fit the equation separately for each of two or more subsamples, in order to examine whether there are significant differences in the estimated equations. On the basis of the structural breaks commonly found in the literature and of the evolution of the *ex post* real rate over our sample period (see Figure 1) we select as breakpoints the two months corresponding to the adoption and the abandonment of strict money-base targeting by the Fed, that is October 1979 and October 1982. The test statistic is based on a comparison of the sum of squared residuals obtained by fitting a single equation to the entire sample with that obtained when separate equations are fitted to each subsample of the data. It is distributed as an $F$-distribution, a significant difference

stability is decisively rejected at all horizons, suggesting that either or both of these common assumptions are unrealistic. Recent research confirms this conclusion.

### 2.3.2 The real rate component

There is substantial empirical evidence that a constant alone cannot adequately capture the real rate component in the Fisher equation. Mishkin (1990) and Caporale and Pittis (1998) find the *ex ante* US real interest rate to be significantly time-varying. Malliaropulos (2000), Evans and Lewis (1995) and Garcia and Perron (1996) report evidence of structural breaks in its mean while, in different contexts, Chen (2001) and Shrestha, Chen and Lee (2002) also draw the same conclusion. Moosa and Kwiecien (1999) demonstrate for the US that relaxing the assumption of a constant real interest rate renders the Fisher equation a more accurate and efficient forecasting tool. Consequently we must allow for a time-varying *ex ante* real rate in the Fisher equation, if we are to ensure it is not misspecified.

The only observable measure of the underlying real rate is the *ex post* real interest rate. This differs from the *ex ante* real interest rate by the errors involved in predicting both the real rate and inflation. Nevertheless, assuming that agents are rational and hence that their errors are independent and have a zero mean, we can use a smoothed function of the *ex post* real interest rate as a proxy. We use the realised, *ex post* 1-month real interest rate which, by assumption, is risk free.[9] As this is likely to be much more volatile than the *ex ante* 1-month real interest rate, let alone the 3, 6 or 12-month averages we are actually proxying for, we take the two-year moving average of this variable instead. In this way we are using information on the changes in the trend of the underlying *ex ante* real interest rate, while not allowing the month-to-month shocks and

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indicating a structural change in the underlying relationship. We also report a second test statistic for this test, the likelihood ratio statistic. This is based on the comparison of the restricted and unrestricted maximum of the log-likelihood function and, in this test, has an asymptotic distribution with 4 degrees of freedom under the null hypothesis of no structural change. Maddala (92) offers a textbook discussion of Chow's various specification tests.

[9] In contrast, the *ex post* real interest rates of longer maturities would include a non-negligible risk premium component associated with perceived real risk in the economy. This would contaminate our primary effort to model and estimate the rolling risk premium, especially since our rolling risk premium measure will include a conditional consumption covariance term which can be thought of as a direct measure of the real risk premium. Hence, the *ex post* 1-month real interest rate is preferable, irrespective of the prediction horizon.

expectation errors included in the ex post real interest rate to enter our estimation. We include the first lag of this variable in the Fisher equation, a quantity known at time $t$. Hence our real rate proxy, which we denote by $\tilde{\psi}_t$ is[10]

$$\tilde{\psi}_t = \frac{1}{24}\sum_{l=1}^{24}(s_{t-l} - \pi_{1,t-l}) \tag{18}$$

### 2.3.3 The rolling risk premium

Empirical research allowing for a time-varying risk premium in the Fisher equation is limited, having emerged only after the rejection of the "pure" expectations hypothesis of the term structure in the 1980s and early 1990s.[11] In a notable early paper, Shome, Smith and Pinkerton (1988) use survey data as well as time series forecasts to establish the significance of modelling the risk premium in the Fisher equation. Evans and Wachtel (1992) confirm their results while, more recently, Evans (2003) finds that the presence of time-varying risk premia in the term structure makes inferences regarding expected inflation based on the classic Fisher equation very unreliable, the link between the current term structure and expectations of future inflation only approaching the implied relation at very long horizons.

However, the asset pricing literature, *albeit* in a different context, offers abundant evidence on this issue. Recent research on term structure dynamics unequivocally rejects risk-neutrality and maintains that bond risk premia are not only significant but also highly time-varying. See for example Cochrane and Piazzesi (2005), Duffee (2002), Piazzesi (2003) and Tzavalis (2003). This financial information has several interesting implications for macroeconomists, *inter alia* that it potentially casts doubt on much of the empirical work on the Fisher equation. Based on this evidence, to omit or constrain the rolling risk premium to a constant could lead to substantial

[10] Since this proxy will be used in all Fisher equation estimations, irrespective of the horizon examined, the subscript $n$ is henceforth omitted from the notation.

[11] Indeed, even in recent research the information content of the term structure on future inflation is often defined simply as the ability of the slope of the yield curve to predict changes in inflation. For example, Mishkin (95), Mishkin and Simon (95), Caporale and Pittis (98) and Malliaropulos (00) among others choose not to model the risk premium at all, thus essentially assuming risk neutrality on behalf of the investors. Crowder and Hoffman (96) and Shrestha, Chen and Lee (02) discuss the significance of a time-varying risk premium, but nevertheless set it to a constant.

biases of the Fisher equation estimates.

We hence assume investors to be risk-averse and the rolling risk premium $\omega_{n,t}$ to vary with $n$ and over time. Consequently we need to estimate $\omega_{n,t}$. As $\omega_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} E_t\rho_{n-i,t+i}$, this in turn requires estimates of $\rho_{n-i,t+i}$. In principle, these may be obtained from equation (11) which provides a measure of $\rho_{n,t}$.

The problem, in practice, is that it would be necessary to estimate equations for the holding-period for each maturity up to $n$. Apart from the fact that data do not exist for yields of each maturity at every period of time and would therefore need to be interpolated using estimates of the yield curve, even if such data were used this would entail an intractable estimation problem due to the large number of equations that would need to be estimated simultaneously for reasonable values of $n$. We would also have to form expectations of future conditional covariance terms. As a result, we adopt a alternative, but closely related, approach in this paper.

The term premia $\rho_{n-i,t+i}$ involve the variables $Cov_t(\frac{\Delta C_{t+1+i}}{C_{t+i}}, h_{n-i,t+i+1})$ and $Cov_t(\pi_{t+i+1}, h_{n-i,t+i+1})$.[12] As we are unable to contemporaneously estimate all of these and having established in Balfoussia and Wickens (2004) that the $\rho_{n,t}$ for different $n$ are highly correlated, we instead estimate $\omega_{n,t}$ directly using

$$\tilde{\omega}_{n,t} = \vartheta_{n,C} Cov_t(\frac{\Delta C_{t+1}}{C_t}, h_{n,t+1}) + \vartheta_{n,\pi} Cov_t(\pi_{n,t+1}, h_{n,t+1}) \qquad (19)$$

In effect, we are assuming that the average value of $Cov_t(\frac{\Delta C_{t+1+i}}{C_{t+i}}, h_{n-i,t+i+1})$ can be expressed as a linear function of $Cov_t(\frac{\Delta C_{t+1}}{C_t}, h_{n,t+1})$ and that the average value of $Cov_t(\pi_{t+i+1}, h_{n-i,t+i+1})$ may be approximated by $Cov_t(\pi_{n,t+1}, h_{n,t+1})$.

As the two conditional covariance terms are unobservable, we estimate the $n$-period Fisher equation jointly with the term structure equation (11) of the corresponding maturity. In this way, by integrating the Fisher equation within the stochastic discount factor theory, we are able

[12] There is also a Jensen term, resulting from the assumption of log-normality. However, based on evidence by Shome, Smith and Pinkerton (88) who found it insignificant and by Balfoussia and Wickens (04) who find its size to be negligible for the term structure, we chose to ignore it in the construction of the rolling risk premium.

to generate estimates for the two conditional covariances that are consistent with our general equilibrium term structure model.

# 3 Econometric methodology

Our aim is to estimate the Fisher equation including the risk premium term, while jointly modelling the term structure. Hence, we must model the joint distribution of the macroeconomic variables, i.e. inflation and consumption, jointly with the excess holding-period returns in such a way that the mean of the conditional distribution of inflation is allowed to include the appropriate risk premium terms. The conditional means of both inflation and the excess holding-period returns involve selected time-varying second moments of the joint distribution. We therefore require a specification of the joint distribution that admits a time-varying variance-covariance matrix. We use the Ding and Engle (1994) vector-diagonal multivariate GARCH-in-mean model, while appropriately adjusting the in-mean equations to our inflation specification.[13]

Let $\mathbf{x}_{t+1} = (h_{n,t+1} - s_t,\, h_{k,t+1} - s_t, \pi_{n,t+1}, \dfrac{\Delta C_{t+1}}{C_t})'$ and $\mathbf{y}_t = (R_{n,t},\, \tilde{\psi}_{n,t})'$, where $k > n$. Our model can be written

$$\mathbf{x}_{t+1} = \boldsymbol{\alpha} + \Gamma\mathbf{x}_t + \Theta\mathbf{y}_t + \mathbf{B}\mathbf{g}_t + \boldsymbol{\varepsilon}_{t+1}$$

where

$$\boldsymbol{\varepsilon}_{t+1} \mid I_t \sim D[0, \mathbf{H}_{t+1}]$$

$$\mathbf{g}_t = vech\{\mathbf{H}_{t+1}\}$$

The *vech* operator converts the lower triangle of a symmetric matrix into a vector. The distribution is the multivariate log-normal distribution. The specification of $\mathbf{H}_t$ is

$$\mathbf{H}_t = \mathbf{H}_0(\mathbf{ii}' - \mathbf{aa}' - \mathbf{bb}') + \mathbf{aa}' * \Sigma_{t-1} + \mathbf{bb}' * \mathbf{H}_{t-1}$$

[13] For a review of multivariate GARCH models see Bollerslev, Chou and Kroner (1997) and for a discussion of their use in financial models see Flavin and Wickens (1998) and Smith and Wickens (2002).

where $\mathbf{i}$ is a vector of ones, $*$ denotes element by element multiplication (the Hadamard product) and $\Sigma_{t-1} = \boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}'_{t-1}$. This is a special case of the diagonal Vech model, in which each conditional covariance depends only on its own past values and on surprises. The restrictions implicitly imposed by this parameterisation of the multivariate GARCH process guarantee positive-definiteness and also substantially reduce the number of parameters to be estimated, thus facilitating computation and convergence. Stationarity conditions are imposed. The estimation is performed using quasi-maximum likelihood.

Following the $n$-period Fisher equation (16) the inflation specification will be[14]

$$\pi_{n,t+1} = \delta_{n,R} R_{n,t} + \delta_{n,\psi}\tilde{\psi}_{n,t} + \delta_{n,\omega}\tilde{\omega}_{n,t} + \varepsilon_{\pi_n,t} \tag{20}$$

where $\tilde{\psi}_{n,t}$ is the real rate proxy as defined in equation (18) and $\tilde{\omega}_{n,t}$ is the rolling risk premium proxy as defined in equation (19). The $4 \times 2$ matrix $\Theta$ is appropriately constrained so that the regression form of the $n$-period Fisher equation is

$$\pi_{n,t+1} = \delta_{n,R} R_{n,t} + \delta_{n,\psi}\tilde{\psi}_{n,t} + \zeta_{n,C} Cov_t\left(\frac{\Delta C_{t+1}}{C_t}, h_{n,t+1} - s_t\right) + \zeta_{n,\pi} Cov_t(\pi_{n,t+1}, h_{n,t+1} - s_t) + \varepsilon_{\pi_n,t} \tag{21}$$

where $\zeta_{n,C} = \delta_{n,\omega}\vartheta_{n,C}$ and $\zeta_{n,\pi} = \delta_{n,\omega}\vartheta_{n,\pi}$.

Having established in Balfoussia and Wickens (2004) that it is important to adequately represent the yield curve when modelling the term structure, we include in each estimation not only the excess return on the bond of maturity equal to the horizon $n$ examined, but also a second one, of medium to long maturity $k$. For the 3-month horizon forecast bonds of 3 and 24 months to maturity are included; for the 6-month horizon bonds of 6 and 24 months to maturity are included while for the 12-month horizon the maturities included are of 12 and 60 months. The conditional means of the two excess holding period returns included in each estimation are restricted to satisfy the condition (11). Real consumption growth is specified as an AR(1) process.

[14] The inclusion of a constant could capture a possible constant element of the rolling risk premium, thus distorting our estimates of the parameters and the risk premium itself. Since we have incorporated a real rate proxy as an exogenous variable in our estimation, we set the constant term of the Fisher equation to its theoretical value of zero.

We are interested in testing the Fisher equation's validity and predictive power. We estimate our model for the 3, 6 and 12 month horizon. For each horizon $n$ we want to test $\delta_{n,R} = 1$ and $\delta_{n,\omega} = 0$ individually and jointly. Since $\delta_{n,\omega}$, $\vartheta_{n,C}$ and $\vartheta_{n,\pi}$ cannot not be separately identified, we shall instead test the hypotheses $\delta_{n,R} = 1$, $\zeta_{n,C} = 0$ and $\zeta_{n,\pi} = 0$. According to the Fisher equation we expect to accept the first hypothesis and reject the following two. We also expect the inclusion of the proxy $\tilde{\omega}_{n,t}$ to improve the explanatory power of the Fisher equation. Finally, an integral implication of the theory is that the real rate should contribute negatively to the Fisher equation. Hence, as a test of the appropriateness of our real rate proxy, we shall be testing not only its significance, but also whether indeed $\delta_{n,\psi} < 0$.

# 4 Data

The complete sample is monthly, from January 1970 to December 1998. Inflation is the 3, 6 and 12-month ahead realised *ex-post* growth rate of the consumer price index for all urban consumers. Until 1991, the term structure data are those of McCulloch and Kwon.[15] This dataset was extended until 1998 by Bliss using the same technique. Excess holding-period returns are taken in excess of the one-month risk-free rate provided by K. French.[16] [17]

The consumption measure used in this work is the month-on-month growth rate of total real personal consumption. Our sample has 345 observations for the 3-month horizon, 342 for the 6-month horizon and 336 for the 12-month horizon. All data are expressed in annualised percentages.

Descriptive statistics for our dataset are presented in Table 2. We see that, though the mean inflation increases with the horizon, the standard deviation decreases. This is to be expected, since the longer the period over which we take the growth rate, the smoother the series will be. The average yield curve is upward sloping. Average excess holding-period returns are positive

[15] See McCulloch and Kwon (1993). We do not use the complete term structure dataseries available by McCulloch and Kwon because no real personal consumption data was available for earlier dates.

[16] In constructing holding-period returns we use the change in $n$-period yields $\Delta R_{n,t+1}$ instead of $R_{n-1,t+1} - R_{n,t}$, since $R_{n-1,t+1}$ is not available. This a common approximation in the literature.

[17] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/

for all maturities and increase with time to maturity. Like most financial data, excess holding-period returns exhibit excess skewness and kurtosis, particularly for short maturities. Fitting a univariate GARCH(1,1) to them indicates there is also significant heteroskedasticity.[18] Their unconditional variances increase with maturity, as do the absolute unconditional covariances of the excess returns with the macroeconomic variables.

Table 3 reports the unconditional sample correlations of the series used in this paper. The unconditional correlations of the macroeconomic variables with the excess returns are negative and for each macroeconomic variable they increase in absolute value with the maturity of the bond.

# 5 Empirical Results

All results are reported in Tables 4 to 7. Row 1 of Table 4 reports our estimates of the Fisher equation (21) for the 3-month horizon, when no risk premium term has been included. The coefficient $\delta_{3,R}$ of the yield is estimated at 0.745 and is highly significant. It has the correct sign and is closer to its theoretical value of 1 than the coefficient obtained in the corresponding preliminary OLS estimation of Table 1 and than those typically reported in univariate estimations in the literature. Hence, the inclusion of the real rate proxy and the GARCH specification of the error appear to help reduce the bias usually observed in estimations of the Fisher equation. Nevertheless, the yield coefficient remains significantly different from the expected value of 1. The real rate proxy coefficient $\delta_{3,\psi}$ is estimated at $-0.448$, implying, as we would expect, that the real rate component is subtracted from the yield in order to extract the inflation expectation. With only 41% of the inflation variance explained,[19] the fit of this equation is poor, though once again offering a higher explanatory power than typically reported in the literature for the usual Fisher equation specification.

[18] Estimation results are available upon request.

[19] Since no constant has been included in our estimations, $R$-squared may be biased. Hence, we report instead the share of inflation's variance which is explained by each specification.

The second row reports the corresponding estimates obtained once the risk premium terms, i.e. the estimated conditional covariances of the 3-month excess holding-period return with each of the two macroeconomic variables have been included in the Fisher equation. The estimated coefficient of the 3-month yield $\delta_{3,R}$ is now 0.95 and again highly significant. It still has the expected sign and, though statistically it remains significantly different from its theoretical value of unity, it is much closer to it now than it was before the inclusion of the two second moments. Hence, it seems that the inclusion of the risk premium proxy reduces the bias observed in the standard specification of the Fisher equation. The real rate proxy coefficient $\delta_{3,\psi}$ still has the correct sign, estimated at $-0.733$. The coefficients of the conditional covariances included in the mean are reported in the following two columns. The coefficient of the inflation covariance $\zeta_{3,\pi}$ is estimated to be $-0.158$, significant at the 10% level. The coefficient $\zeta_{3,C}$ of the consumption covariance is much smaller at 0.054 and statistically insignificant. However, their inclusion is jointly highly significant, as demonstrated by the likelihood ratio test comparing the two specifications. Its value is 21.8, much higher than the critical value of 3.84 for one constraint at a 5% significance level. The share of variance explained is now substantially higher than before, at 77%.

Finally, row 3 reports the results of our null estimation. As above, the risk premium terms have been included and the coefficient of the yield has now been constrained to its theoretical value of unity. The real rate proxy coefficient $\delta_{3,\psi}$ is very close to its previous value, estimated at $-0.675$. The conditional covariance terms now have positive estimated coefficients and are much more significant individually. The inflation covariance coefficient $\zeta_{3,\pi}$ is estimated at 0.158 and the consumption covariance coefficient $\zeta_{3,C}$ has increased to 1.37. Finally, the share of variance explained by this specification of the Fisher equation has increased further to 80%. Nevertheless, the likelihood ratio criterion clearly rejects this specification against the previous, less restricted one of row 2, the test statistic of 20.3 being much higher than the critical value of 5.99 for two constraints at a 5% significance level.

Figure 2 provides a graphical representation of the fit of all three estimations. Both specifications including a risk premium proxy offer a remarkably better fit than the first estimation, while the null provides a marginally better fit than the specification allowing the yield coefficient unconstrained.

Tables 5 and 6 present estimates of the corresponding estimations for the 6-month and 1-year horizons. Our conclusions are broadly similar. The estimated yield coefficient is very close to its theoretical value of 1 once the risk premium has been proxied for, essentially alleviating the bias previously observed. The explanatory power of the Fisher equation also increases substantially at all horizons. Furthermore, although we cannot accept the hypothesis that $\delta_{n,R} = 1$ for the two shortest maturities, the null cannot be rejected against the less constrained alternative at the 1-year horizon, once the time-varying rolling risk premium proxy has been included.

The estimated coefficient of the real rate proxy $\delta_{n,\psi}$ is negative and highly significant in all estimations. It is interesting that, as the horizon increases, this coefficient decreases in absolute value, deviating from its theoretical value of $-1$. Indeed, given that it is based on the *ex post* 1-month real rate, our real rate proxy should, by construction, be more appropriate, in terms of magnitude, for shorter horizons.

Figures 3 and 4 provide graphical representations of the fit of the three estimations for the 6-month and 1-year horizons respectively. The explanatory power of the Fisher equation clearly increases once the risk premium proxy has been included. In view of the reduction of bias of the $\delta_{n,R}$ estimates and the improvement in the goodness of fit, we conclude the Fisher equation provides a much better predictor of future inflation once the risk premium is taken into account by including an appropriate risk premium proxy.

Our results confirm those of Shome, Smith and Pinkerton (1988) and Evans and Wachtel (1992) who also draw on the C-CAPM in their models. Shome, Smith and Pinkerton (1988) use survey data and a rolling regression technique to construct estimates of the conditional moments, which they find to be highly significant in the Fisher equation. Evans and Wachtel (1992) essentially extend the Shome, Smith and Pinkerton (1988) model to allow for taste shocks to utility. They first estimate a bivariate ARCH model of consumption and inflation to obtain estimates of the conditional moments, which they then substitute in a GMM estimation of the Fisher equation. They also conclude the risk premium is statistically significant.

We now take a closer look at the estimated coefficients of the conditional covariance terms included in the different Fisher equation specifications. The coefficients of the inflation covariance component of the risk premium increase with the horizon. For example, in the null specification, the inflation covariance coefficients are estimated at 0.158, 0.587 and 0.891 for the 3, 6 and 12-month horizons respectively. The opposite pattern appears in the consumption covariance coefficients. Furthermore, while the inflation covariance is generally highly significant, especially as the horizon increases, the consumption covariance is often insignificant. All three estimations indicated by the likelihood ratio criterion confirm the conclusion of *inter alia* Balfoussia and Wickens (2004) and Ang and Piazzesi (2003) that inflation seems to be the dominant source of risk in the term structure.

One might also argue that the estimated conditional covariance coefficients have the expected sign. These are positive for both macroeconomic variables.[20] Given the negative unconditional correlation of the excess returns with both macroeconomic variables, this would imply that risk, whether associated with the nominal or the real component of the stochastic discount factor, individually generates a positive premium which is subtracted from the yield as implied by the Fisher equation (16).[21] Hence, our results are overall in line with our a priori intuition.

Figure 5 plots the aggregate contribution of the risk premium proxy terms to the Fisher equation in both specifications and at all horizons. Indeed, their contribution is almost entirely negative. Moreover, the evolution of the estimated rolling risk premia through time can be related to major macroeconomic events and shocks of the period. They are much higher and more volatile during the 1979-82 period, which corresponds to the "monetary experiment" of the Fed. They

[20] There is an exception of two terms which, however are not significantly negative.

[21] This is in contrast to the estimates presented in Balfoussia and Wickens (04), where the estimated coefficient of the consumption covariance had the opposite sign than expected.

are also relatively high during the first oil crisis, and in the early 1980s, when inflation volatility was still relatively high. Throughout the 1990s, a decade largely marked by Greenspan's success in maintaining a low and stable level of inflation, the estimated rolling risk premia become lower and increasingly stable.

The rolling risk premium seems to increase in magnitude with the horizon over which we are predicting. Indeed, yields on bonds of longer maturities should include a higher risk premium component. Nevertheless it is notable that one of the two plots for the 3-month horizon is very close to zero during periods of relative stability, possibly indicating that the 3-month T-bill is almost risk-free. An additional explanation may be that, at such a short maturity, much of the volatility is due to noise, thus rendering our effort to decompose the yield into its different components more difficult. A final interesting observation is that, while at the 3-month horizon the rolling risk premia estimated for each of the two specifications of the generalised Fisher equation do not coincide, they gradually converge as the horizon increases. This reinforces our inference from the likelihood ratio criterion which leads us to reject the null hypothesis of $\delta_{n,R} = 1$ for the two shortest maturities while accepting it for the 1-year horizon.

Table 6 reports one representative complete set of estimated parameters corresponding to the 1-year horizon specification of the $n$-period Fisher equation where the yield coefficient is unconstrained.[22] The coefficients of the conditional covariances in the excess return equations are highly significant. They are negative and significantly so, implying once again a positive risk premium associated with both the real and the inflation component of the stochastic discount factor. Further, neither macroeconomic variable's covariance coefficient changes notably as maturity increases. This is an important point, as the equality of the covariance coefficients across the yield curve is implicit in the no-arbitrage assumption. It is a substantial improvement over Balfoussia and Wickens (2004) who, in a similar setup find these coefficients decrease along the yield curve, possibly implying that the specification for the SDF or inflation used in this paper is

[22] The complete estimates for all specifications are available upon request.

superior. Figure 6 plots the excess holding-period return risk premia $\rho_{12,t}$ and $\rho_{60,t}$ generated by the two term structure equations of this estimation. Despite the slightly modified SDF specification, these are very similar to the ones obtained in Balfoussia and Wickens (2004), and explain a very high share of the excess holding-period return variance, 19% and 16% for the 1-year and 5-year maturities respectively. The constant and first lag of total personal consumption growth in the corresponding equation are highly significant, as are the ARCH and GARCH estimates. The dynamic structure of the conditional variance-covariance matrix, depends largely on the lagged conditional covariance matrix and much less on lagged innovations.

# 6 Conclusions

In this paper we have shown how it is possible to integrate the workhorse of inflation prediction, the Fisher equation, with the stochastic discount factor theory. This allows us to develop a multivariate framework in which to jointly model the term structure and inflation, while endogenously generating a suitable rolling risk premium proxy for each inflation prediction horizon. The inclusion of this risk component in the Fisher equation is highly significant, substantially improving the predictive power of the Fisher equation at all horizons while reducing the bias of the estimated yield coefficient. We conclude that the Fisher equation provides a sound theory and a useful modelling tool for inflation, once the risk component has been appropriately taken into account.

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| Table 1: Stability tests of the Fisher equation | | | | | |
|---|---|---|---|---|---|
| Prediction Horizon | Constant | $R_{n,\,t}$ | $R^2$ | Chow breakpoint test[1] | |
| | | | | *F*-statistic | Likelihood ratio |
| 3 month horizon | 0.510<br>*1.09* | 0.694<br>*10.97* | 0.26 | 69.19 * | 205.91 * |
| 6 month horizon | 1.120<br>*2.47* | 0.582 #<br>*9.81* | 0.22 | 104.40 * | 276.25 * |
| 12 month horizon | 2.091<br>*4.37* | 0.433 #<br>*7.14* | 0.13 | 151.16 * | 349.80 * |

Notes

1. Chow's breakpoint test fits the equation separately for each subsample and examines whether there are significant differences in the estimated equations. A significant difference indicates a structural change in the relationship. The two breakpoint dates used in all tests are October 1979 and October 1982. We report two test statistics for the test.
2. * denotes rejection of the hypothesis of coefficient stability for our sample period.
3. # denotes rejection of the hypothesis that the yield coefficient is equal to 1 at the 5% significance level.
4. *t*-statistics are below the estimated parameters in italics.

| Table 2: Descriptive statistics | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Mean | Median | Maximum | Minimum | Std. Dev. | Skewness | Kurtosis |
| $\pi_{3,\,t+1}$ | 5.28 | 4.13 | 18.95 | -3.60 | 3.70 | 1.03 | 3.79 |
| $\pi_{6,\,t+1}$ | 5.28 | 4.32 | 16.26 | -0.18 | 3.40 | 1.12 | 3.64 |
| $\pi_{12,\,t+1}$ | 5.31 | 4.25 | 14.76 | 1.10 | 3.19 | 1.14 | 3.39 |
| $s_t$ | 6.80 | 6.04 | 17.46 | 2.55 | 2.85 | 1.27 | 4.85 |
| $R_{3,\,t}$ | 6.85 | 6.08 | 16.00 | 2.78 | 2.72 | 1.18 | 4.36 |
| $R_{6,\,t}$ | 7.10 | 6.43 | 16.51 | 2.88 | 2.74 | 1.14 | 4.22 |
| $R_{12,\,t}$ | 7.34 | 6.85 | 16.35 | 3.09 | 2.68 | 1.06 | 3.94 |
| $R_{24,\,t}$ | 7.64 | 7.07 | 16.15 | 3.80 | 2.52 | 1.07 | 3.85 |
| $R_{60,\,t}$ | 8.04 | 7.53 | 15.70 | 4.35 | 2.31 | 1.09 | 3.65 |
| $h_{3,\,t+1} - s_t$ | 0.07 | 0.01 | 8.84 | -7.47 | 1.46 | 0.38 | 13.51 |
| $h_{6,\,t+1} - s_t$ | 0.35 | 0.19 | 24.81 | -14.40 | 3.19 | 1.23 | 15.79 |
| $h_{12,\,t+1} - s_t$ | 0.65 | 0.74 | 47.24 | -33.23 | 6.72 | 0.56 | 12.50 |
| $h_{24,\,t+1} - s_t$ | 1.07 | 1.02 | 82.48 | -71.01 | 12.60 | 0.20 | 11.30 |
| $h_{60,\,t+1} - s_t$ | 1.77 | 2.17 | 111.49 | -128.11 | 26.02 | -0.22 | 6.08 |
| $\Delta \ln C_{t+1}$ | 3.52 | 3.35 | 29.87 | -25.59 | 7.16 | 0.14 | 4.76 |
| Real rate proxy | 1.43 | 1.35 | 6.90 | -3.31 | 2.58 | 0.09 | 2.21 |

Notes

1. All series are in annualised percentages.
2. $\pi_{3,\,t+1}$ is the 3-month ahead, annualised *ex-post* change in CPI inflation, $\pi_{6,\,t+1}$ the same for a 6-month horizon etc.
3. The real rate proxy is the lagged 2-year moving average of the *ex-post* 1-month real interest rate

**Table 3: Sample correlations**

| | $h_{3,t+1} - s_t$ | $h_{6,t+1} - s_t$ | $h_{12,t+1} - s_t$ | $h_{24,t+1} - s_t$ | $h_{60,t+1} - s_t$ | $\pi_{3,t+1}$ | $\pi_{6,t+1}$ | $\pi_{12,t+1}$ | $\Delta \ln C_{t+1}$ |
|---|---|---|---|---|---|---|---|---|---|
| $h_{3,t+1} - s_t$ | 1.00 | | | | | | | | |
| $h_{6,t+1} - s_t$ | 0.87 | 1.00 | | | | | | | |
| $h_{12,t+1} - s_t$ | 0.73 | 0.95 | 1.00 | | | | | | |
| $h_{24,t+1} - s_t$ | 0.64 | 0.88 | 0.96 | 1.00 | | | | | |
| $h_{60,t+1} - s_t$ | 0.54 | 0.76 | 0.87 | 0.94 | 1.00 | | | | |
| $\pi_{3,t+1}$ | -0.04 | -0.12 | -0.17 | -0.21 | -0.24 | 1.00 | | | |
| $\pi_{6,t+1}$ | -0.08 | -0.15 | -0.19 | -0.23 | -0.25 | 0.92 | 1.00 | | |
| $\pi_{12,t+1}$ | -0.07 | -0.12 | -0.15 | -0.18 | -0.20 | 0.85 | 0.94 | 1.00 | |
| $\Delta \ln C_{t+1}$ | -0.02 | -0.05 | -0.07 | -0.06 | -0.09 | -0.16 | -0.12 | -0.12 | 1.00 |

**Table 4: The Fisher equation**
**Estimation results for inflation prediction at the 3-month horizon**

| Constraints imposed | $R_{3,t}$ | Real rate proxy | Risk premium proxy components[1] | | Likelihood ratio tests[2] | Share of variance[3] |
|---|---|---|---|---|---|---|
| | | | $\pi_{3,t+1}$ | $\Delta \ln C_{t+1}$ | | |
| No risk premium terms | 0.745 * <br> *65.72* | -0.448 <br> *-13.28* | - | - | 21.8 # | 0.41 |
| Incl. risk premium terms | 0.950 * <br> *53.90* | -0.733 <br> *-17.41* | -0.158 <br> *-1.59* | 0.054 <br> *1.03* | | 0.77 |
| Null | 1.000 | -0.675 <br> *-15.29* | 0.158 <br> *1.81* | 1.370 <br> *4.53* | 20.3 # | 0.80 |

Notes

1. The covariance of the 3-month excess holding-period return with $\pi_{3,t+1}$ and $\Delta \ln C_{t+1}$ respectively
2. Likelihood ratio tests: The row title corresponds to the constrained specification which is tested against the unconstrained model including risk premium terms in the Fisher equation.
   # denotes rejection of the restriction(s) tested, at the 5% significance level.
3. Share of inflation variance explained in each estimation
4. t-statistics are below the estimated parameters in italics.
5. * denotes rejection of the hypothesis of coefficient equality to unity, using a t-test at the 5% significance level.

**Table 5: The Fisher equation**
**Estimation results for inflation prediction at the 6-month horizon**

| Constraints imposed | $R_{6,\,t}$ | Real rate proxy | Risk premium proxy components[1]: $\pi_{6,\,t+1}$ | Risk premium proxy components[1]: $\Delta \ln C_{t+1}$ | Likelihood ratio tests[2] | Share of variance[3] |
|---|---|---|---|---|---|---|
| No risk premium terms | 0.680 * | -0.354 | - | - | 70.1 # | 0.39 |
| | *84.52* | *-15.20* | | | | |
| Incl. risk premium terms | 0.901 * | -0.660 | 0.672 | -0.043 | | 0.74 |
| | *29.31* | *-15.01* | *5.51* | *-0.96* | | |
| Null | 1.000 | -0.608 | 0.587 | 0.836 | 57.8 # | 0.97 |
| | | *-9.76* | *9.03* | *10.41* | | |

Notes

1. The covariance of the 6-month excess holding-period return with $\pi_{6,\,t+1}$ and $\Delta \ln C_{t+1}$ respectively
2. Likelihood ratio tests: The row title corresponds to the constrained specification which is tested against the unconstrained model including risk premium terms in the Fisher equation.
   # denotes rejection of the restriction(s) tested, at the 5% significance level.
3. Share of inflation variance explained in each estimation
4. t-statistics are below the estimated parameters in italics.
5. * denotes rejection of the hypothesis of coefficient equality to unity, using a t-test at the 5% significance level.

**Table 6: The Fisher equation**
**Estimation results for inflation prediction at the 1-year horizon**

| Constraints imposed | $R_{12,\,t}$ | Real rate proxy | Risk premium proxy components[1]: $\pi_{12,\,t+1}$ | Risk premium proxy components[1]: $\Delta \ln C_{t+1}$ | Likelihood ratio tests[2] | Share of variance[3] |
|---|---|---|---|---|---|---|
| No risk premium terms | 0.768 * | -0.535 | - | - | 35.0 # | 0.61 |
| | *75.15* | *-21.24* | | | | |
| Incl. risk premium terms | 0.971 | -0.514 | 0.856 | 0.023 | | 0.81 |
| | *34.81* | *-8.45* | *8.70* | *1.25* | | |
| Null | 1.000 | -0.530 | 0.891 | 0.021 | 0.5 | 0.83 |
| | | *-8.57* | *9.20* | *1.20* | | |

Notes

1. The covariance of the 12-month excess holding-period return with $\pi_{12,\,t+1}$ and $\Delta \ln C_{t+1}$ respectively
2. Likelihood ratio tests: The row title corresponds to the constrained specification which is tested against the unconstrained model including risk premium terms in the Fisher equation.
   # denotes rejection of the restriction(s) tested, at the 5% significance level.
3. Share of inflation variance explained in each estimation
4. t-statistics are below the estimated parameters in italics.
5. * denotes rejection of the hypothesis of coefficient equality to unity, using a t-test at the 5% significance level.

**Table 7: Complete estimation results**
**Risk premium terms included**
**Inflation prediction horizon: 12 months     Bond maturities: 12 & 60 months**

**Conditional mean equations**

| Equation | | Constant | Yield | Own lag | Real Rate Proxy | Conditional Covariances[2] | | | Share of variance[3] |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\pi_{12,\,t+1}$ | $\Delta \ln C_{t+1}$ | |
| Excess return equations | $h_{12,\,t+1} - s_t$ | 0.000 | 0.000 | 0.000 | 0.000 | $h_{12,\,t+1} - s_t$ | -1.015 | -0.212 | 0.19 |
| | | . | . | . | . | | *-7.02* | *-3.76* | |
| | $h_{60,\,t+1} - s_t$ | 0.000 | 0.000 | 0.000 | 0.000 | $h_{60,\,t+1} - s_t$ | -1.063 | -0.233 | 0.16 |
| | | . | . | . | . | | *-6.75* | *-3.62* | |
| Macro variable equations | $\pi_{12,\,t+1}$ | 0.000 | 0.971 | 0.000 | -0.514 | $h_{12,\,t+1} - s_t$ | 0.856 | 0.023 | 0.81 |
| | | . | *34.81* | . | *-8.45* | | *8.70* | *1.25* | |
| | $\Delta \ln C_{t+1}$ | 3.550 | 0.000 | -0.239 | 0.000 | | - | - | 0.06 |
| | | *7.80* | . | *-4.61* | . | | . | . | |

**Conditional variance equations**

| Equation | | ARCH | GARCH | Long run variance-covariance matrix[4] | | | |
|---|---|---|---|---|---|---|---|
| Excess return equations | $h_{12,\,t+1} - s_t$ | -0.348 | 0.912 | 46.701 | 156.832 | -0.767 | -4.806 |
| | | *-15.54* | *76.38* | . | . | . | . |
| | $h_{60,\,t+1} - s_t$ | -0.239 | 0.949 | 156.832 | 694.723 | -3.010 | -23.151 |
| | | *-10.95* | *107.04* | . | . | . | . |
| Macro variable equations | $\pi_{12,\,t+1}$ | 0.243 | 0.966 | -0.767 | -3.010 | 4.404 | 0.909 |
| | | *8.14* | *102.38* | . | . | . | . |
| | $\Delta \ln C_{t+1}$ | 0.234 | 0.727 | -4.806 | -23.151 | 0.909 | 50.079 |
| | | *5.13* | *6.63* | . | . | . | . |

*Notes*

1. *t*-statistics are below the estimated parameters in italics.
2. The coefficients are of the conditional covariances between the variables defined by the column and row of each cell. An exception is the second in-mean covariance included in the Fisher equation. The coefficient reported is not that of the covariance between the two macroeconomic variables but that between $\Delta \ln C_{t+1}$ and the 1-year excess holding period return.
3. Share of the dependent variable's variance explained in each estimation
4. A consistent estimator of the long-run variance covariance matrix, to which $H_0$ is subsequently fixed, is obtained by estimating a standard homoskedastic VAR estimator for the whole system. Hence no t-statistics are reported for $H_0$.

**Figure 1: The Real Rate Proxy**

**(the 2-year moving average of the *ex post* 1-month real rate)**

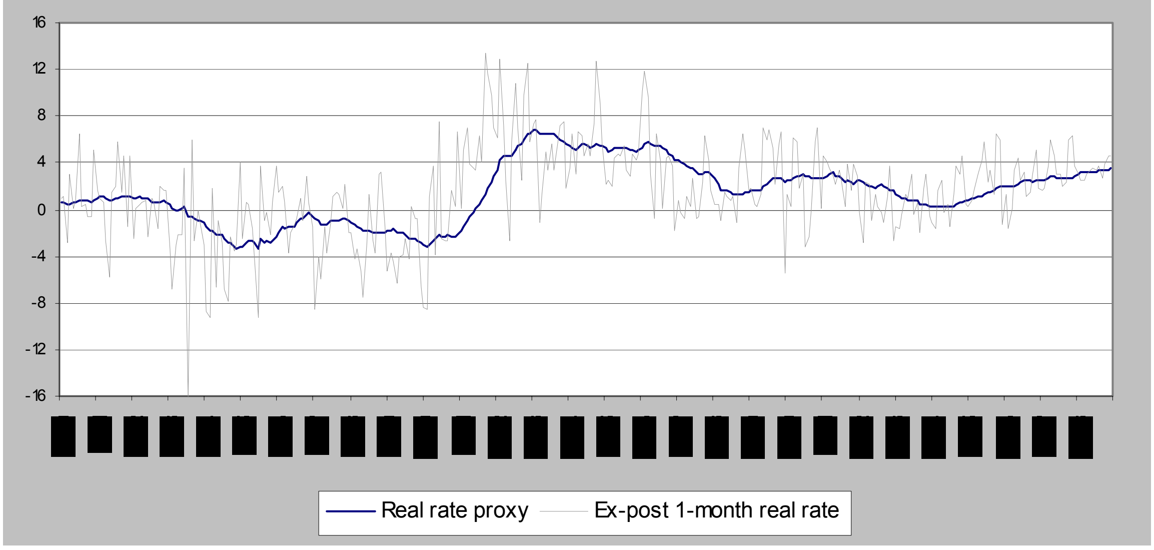

**Figure 2: Fit to 3-month Ahead Inflation**

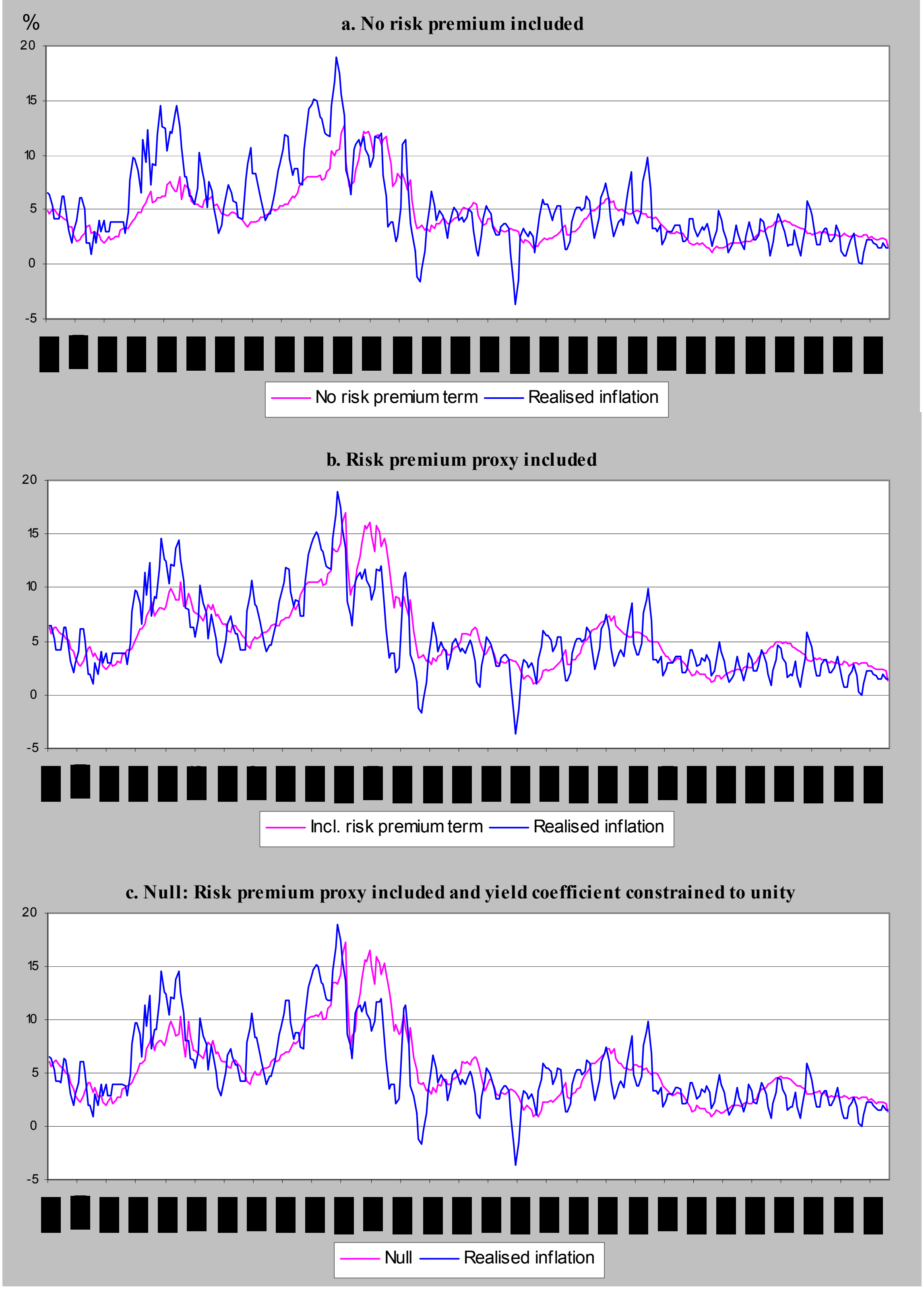

**Figure 3: Fit to 6-month Ahead Inflation**

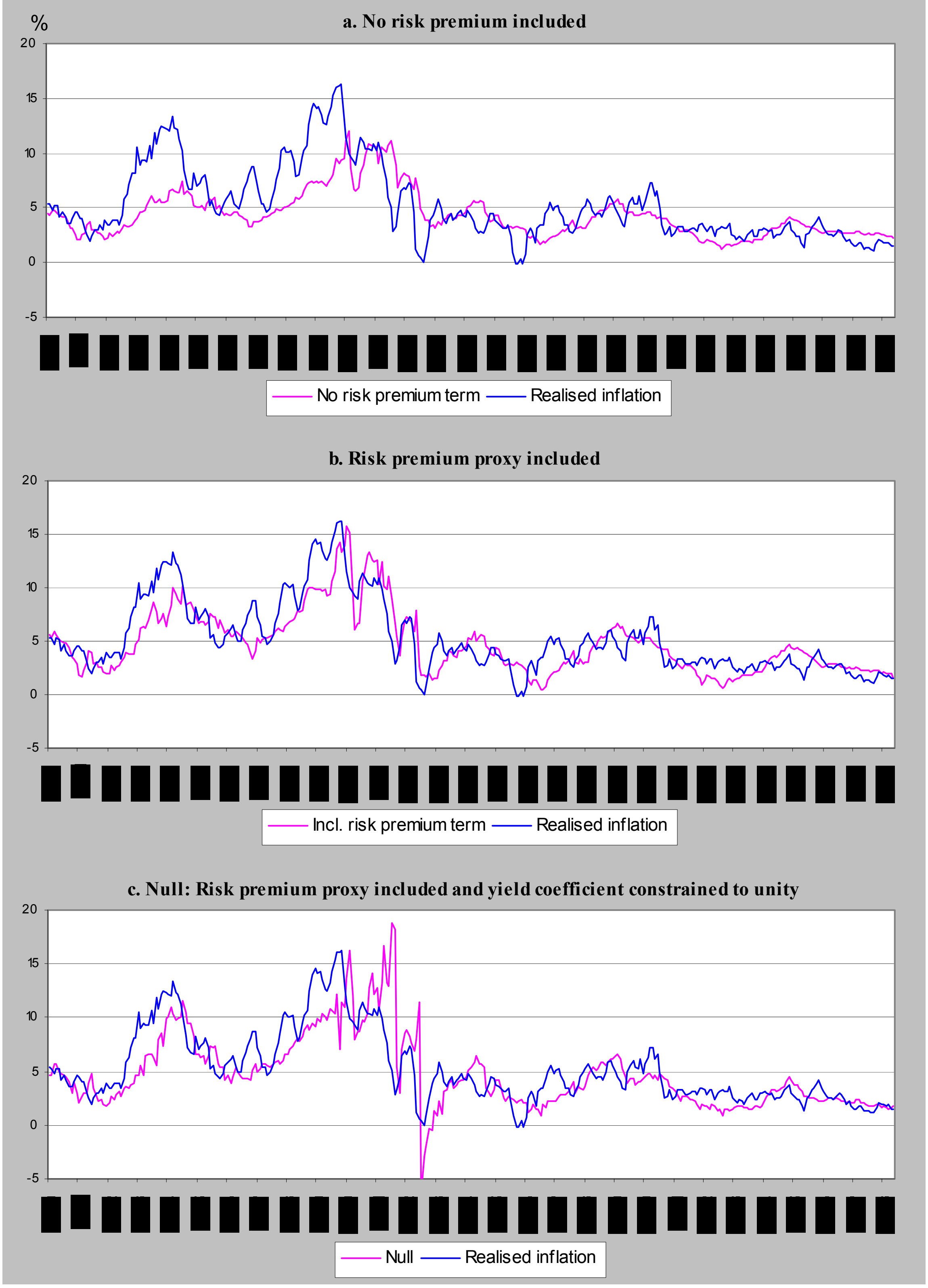

**Figure 4: Fit to 12-month Ahead Inflation**

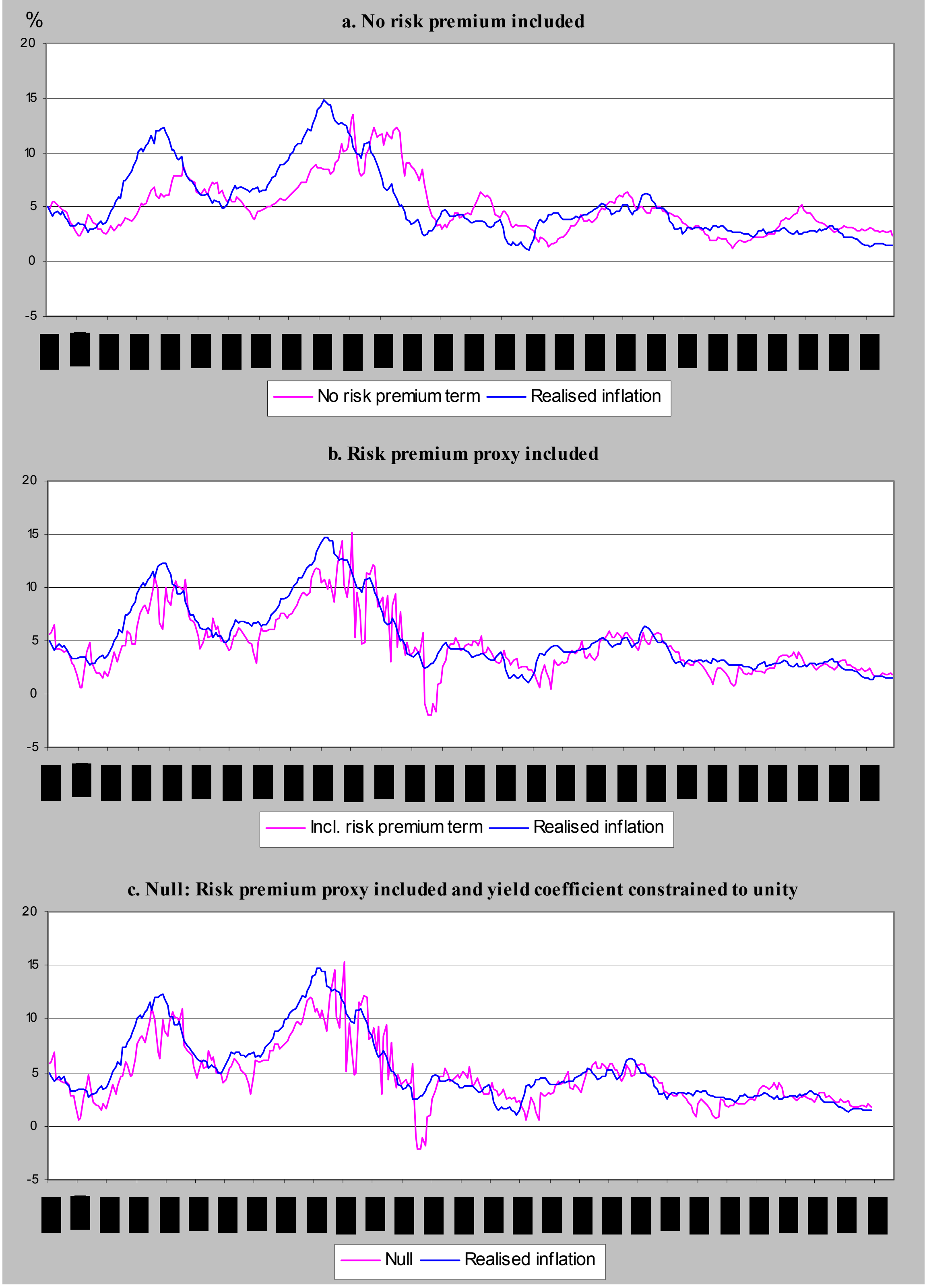

**Figure 5: The Risk Premium Proxy Contribution**

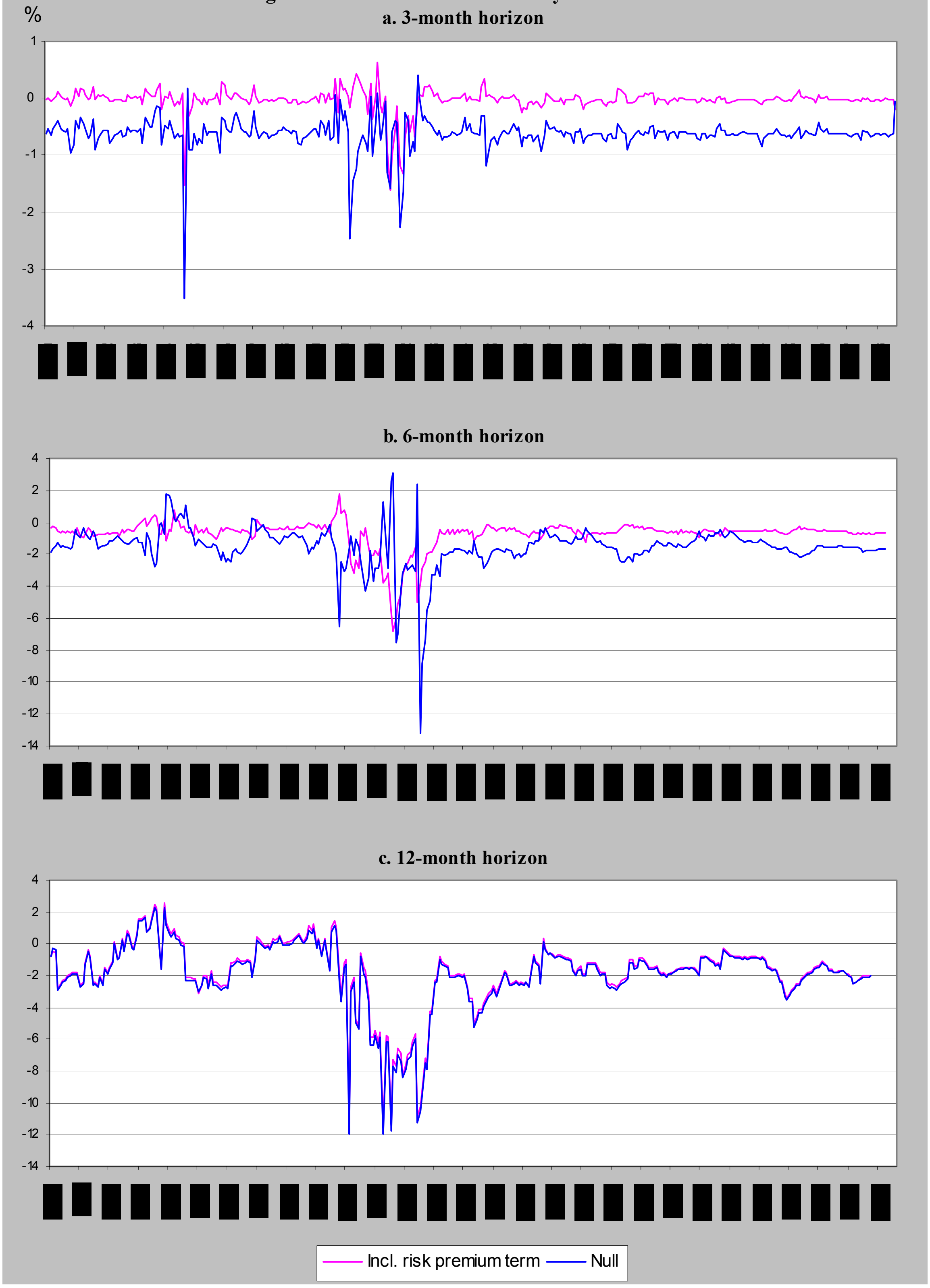

**Figure 6: Excess Holding-Period Return Risk Premia**
**12-month Inflation Prediction Horizon - Risk premium term included**