Time Consistent Policy in Markov Switching Models*

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Abstract

In this paper we consider the optimal control problem of models with Markov regime shifts and forward looking agents. These models are general and flexible tools for modelling model uncertainty. An algorithm is devised to compute the solution of a rational expectations model with random parameters or regime shifts. A second algorithm computes the time consistent policy and the resulting Nash Stackelberg equilibrium. The latter algorithm can also handle the case in which the policymaker and the private sector hold different beliefs. We apply these methods to compute the optimal (nonlinear) monetary policy in a small open economy subject to random structural breaks.

1 Introduction

Uncertainty is one of the considerable problems faced by economic policymakers. It surrounds observed data, unobserved expectations and potential equilibria as well as both the structure and parameters of the economy. Even if models are subject to quantifiable risk, this can have a substantial impact on the formulation of optimal economic policies. A considerable amount of recent research has been directed at countering these and various sources of uncertainty.[1]

In this paper we focus on one such quantifiable risk, one in which the economy is subject to regime shifts with the particular regime followed being determined by a Markov process. This set up can be thought of as

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[1]See, in a completely arbitrary but recent list, Kozicki (2004); Swanson (2004); Planas and Rossi (2004).

encompassing a number of possible representations of the world. It can be viewed as a model with stochastic parameters or perhaps a model in which agents learning is characterised as a jump process. This latter set up can be particularly useful for models where bubble-like behaviour is observed. A collapsed bubble is one where sufficient agents feel it is unsustainable.

The economic policy problem is pervasive in such a world. For any model, particularly a stochastic one, we need to decide what form of policy rule we should implement and together with rational, forward-looking agents we need to consider the appropriate treatment of expectations in the optimal policy problem. In this paper we adopt a game-theoretic framework for the design of optimal policy. In particular we seek policies which are both time consistent and subgame perfect, following Fershtman (1989): Policies need to both be consistent and take into account the stochastic nature of the problem. The time consistency restriction rules out policymakers adopting policies which are ex ante likely to become sub optimal simply because time passes, and are therefore unsustainable as a description of credible a policy. Both considerations require us to consider solutions derived by dynamic programming rather than Lagrange multipliers: We need a 'rule' for agents' expectations, not a time path for future actions (Başar and Olsder, 1999).

This is particularly appropriate in our case. We adopt a recursive approach to optimal policy formulation with Markov-switching parameters. Such an approach necessarily imposes time consistency via the principle of optimality. If the model itself is subject to change, why should policymakers actions be immune? We therefore rule out potentially time inconsistent behaviour through our recursive formulation.

To do this we develop algorithms both for the solution of rational expectations models with probabalistically-driven regime changes and for the optimal time-consistent subgame-perfect control of such models. The control solution adopted in Zampolli (2005) is adapted to provide the best policy.[2] We also show how these algorithms can be modified to allow the policymakers and private agents to hold different beliefs over the probability of a regime shift. These methods are applied to a small open economy model developed by e.g. Batini and Nelson (2000) and Leitemo and Söderström (2004) to investigate structural changes in agent behaviour. These can both be characterised as a form of learning. In Appendix A we develop the same methods in a form consistent with Oudiz and Sachs (1985) rather than the semi-structural form used in the main part of the paper (see Dennis and Söderström (2002)).

Whilst our focus is on time consistency, it should be noted that the rational expectations solution we develop could be used for any arbitrary

[2] In the engineering literature, Aoki (1967) studied the control of regime-shifting models. These models are currently referred to in this literature as Markov Jump Linear Systems (MJLS). For recent contributions, see Costa et al. (2005).

policy rules, such as a Taylor rule, and the optimal time inconsistent policy could be obtained using very similar methods. There are difficulties with time inconsistent policy in this context however, as any change in policy must be in response only to news about changes in regime rather than potential welfare improvements from reneging. This means that the implications of any inherited part of policy for a new regime could be extremely bad, and policymakers would never want to carry them out even if the consequences of a loss of reputation were severe. We focus on time consistency to remove this possibility.

The paper is organised as follows. Section 2 provides the undermined coefficient model solution to a rational expectation model with regime shifts. This solution forms the basis for solving the optimal control problem which is dealt with in Section 3. Section 4 describes the small open economy model used in the application and the experiments being carried out. Section 5 describes how to simulate the model both under symmetric and asymmetric beliefs. Section 6 presents the results of the application. Section 7 concludes.

2 Undetermined coefficient model solution with regime shifts

We consider a linear rational expectations model in semi-structural form:

$$x_t = A(s_t)x_{t-1} + B(s_t)E\left[x_{t+1}|I_t\right] + C(s_t)\varepsilon_t \tag{1}$$

where x is a vector of variables that can depend on lags and leads, $A(s_t)$, $B(s_t)$ and $C(s_t)$ are stochastic matrices which will depend on regime $s_t \in \{1, 2, ...N\}$ and $E\left[\varepsilon_{t+1}|I_t\right] = 0$ is a vector of stochastic shocks with I_t the information set at time t. The dominant regime will be determined by a Markov process. This model is described as semi-structural as it distinguishes between leads and lags for each potential equation, although for longer leads and lags the model would need to be augmented. By contrast the state space form (Appendix A) requires classification of the variables by type (jump or predetermined).

The model can be solved depending on agent's expectations of future policy regimes. Let the assumed reduced-form law of motion be:

$$x_t = D(s_t)x_{t-1} + F(s_t)\varepsilon_t \tag{2}$$

where $D(\cdot)$ and $F(\cdot)$ are matrices of undetermined coefficients and we have solved out for expectations. For simplicity we assume that there are only two states. The formulae are easily generalised to the N-state case (see Appendix A, for example).

To find the unknown coefficients, first solve for the expectation:

$$\begin{aligned}
E\left[x_{t+1}|I_t\right] &= E\left[D(s_{t+1})x_t + F(s_{t+1})\varepsilon_{t+1}|I_t\right] \\
&= E\left[D(s_{t+1})|I_t\right]x_t + E\left[F(s_{t+1})|I_t\right]E\left[\varepsilon_{t+1}|I_t\right] \\
&= E\left[D(s_{t+1})|I_t\right]x_t \\
&= \left(p_{i1}D_1 + p_{i2}D_2\right)x_t \\
&\equiv \bar{D}_i x_t \\
&= \bar{D}_i\left(D_i x_{t-1} + F_i\varepsilon_t\right) \\
&= \bar{D}_i D_i x_{t-1} + \bar{D}_i F_i \varepsilon_t
\end{aligned}$$

where i denotes the regime at time t, i.e. $s_t = i$. Now plugging the above expression back into the model gives:

$$\begin{aligned}
x_t &= A_i x_{t-1} + B_i\left(\bar{D}_i D_i x_{t-1} + \bar{D}_i F_i\varepsilon_t\right) + C_i\varepsilon_t \\
&= \left(A_i + B_i\bar{D}_i D_i\right)x_{t-1} + \left(B_i\bar{D}_i F_i + C_i\right)\varepsilon_t. \qquad (3)
\end{aligned}$$

Given the assumed law of motion, $x_t = D_i x_{t-1} + F_i\varepsilon_t$, the undetermined coefficients must satisfy the following conditions:

$$D_i = A_i + B_i\bar{D}_i D_i \qquad (4)$$

$$F_i = B_i\bar{D}_i F_i + C_i \qquad (5)$$

for $i = 1, ...N$. The first set of equations are to be solved for the feedback part of the solution, D_i:

$$\begin{aligned}
D_i &= A_i + B_i\bar{D}_i D_i \\
&= A_i + B_i\left(p_{i1}D_1 + p_{i2}D_2\right)D_i.
\end{aligned}$$

So, for $i = 1$:

$$\begin{aligned}
D_1 &= A_1 + B_1\left(p_{11}D_1 + p_{12}D_2\right)D_1 \\
&= A_1 + B_1 p_{11}D_1^2 + B_1 p_{12}D_2 D_1 \\
0 &= B_1 p_{11}D_1^2 + \left(B_1 p_{12}D_2 - I\right)D_1 + A_1
\end{aligned}$$

Likewise for $i = 2$. This yields a pair of coupled equations that need to be solved simultaneously:

$$0 = B_1 p_{11}D_1^2 + \left(B_1 p_{12}D_2 - I\right)D_1 + A_1 \qquad (6)$$

$$0 = B_2 p_{22}D_2^2 + \left(B_2 p_{21}D_1 - I\right)D_2 + A_2. \qquad (7)$$

These equations can be solved iteratively, if a solution exists[3] using an appropriate solution method. Given a procedure for solving matrix quadratic equations, we can solve the linked equations sequentially. The following is a possible solution algorithm for the two-state case. It generalises easily for the multi-state model.[4]

[3] There are few proofs about the existence of solutions to such problems. We consider this to be a useful avenue for future research, as, in our experience, solution methods can fail for interesting and plausible economic models.

[4] As with the control solutions below we have implemented the solutions in Matlab™.

Algorithm 1 *Rational solution with Markov switching (two-state case). For the model (1) assume a solution of the form (2).*

1. *Select initial values for* $D^0 = \left(D_1^0,\ D_2^0\right)$.

2. *Solve quadratic equations for given values of* D^r, *obtaining a new set* D^{r+1}:

$$\begin{aligned} D_1^{r+1} &= g(B_1 p_{11},\ B_1 p_{12} D_2^r - I,\ A_1) \\ D_2^{r+1} &= g(B_2 p_{22},\ B_2 p_{21} D_1^r - I,\ A_2) \end{aligned}$$

   where $g(\cdot)$ *is a quadratic equation solver for (6) and (7). Similarly solve* F.

3. *Check convergence: if* $\left|D^{r+1}\right| < \varepsilon$ *or too many iterations stop; else repeat 2.*

There are some issues to consider. First, in the standard case the roots of the single quadratic equation can be checked and it can be established if there are determinate, indeterminate or no solutions. In our linked case this is no longer possible. If a solution exists and can be found by this procedure, we can check whether the solution is stable conditional on the other Riccati solution(s). As mentioned above, issues of existence have not been established in this class of model and time consistent policy problem.

Second, if the model incorporates the optimal policy rule, is this solution stabilising and unique? In the linear-quadratic optimal control problem, it is. What about in this non-standard case? Again, results do not currently exist, but we have so far been able to find solutions using our suggested algorithms.

Third, we can easily adapt this as a method for solving for an optimal fixed parameter policy rule. Given that we have a rational expectations solution we could simply impose a fixed parameter policy rule and optimise over the coefficients to find the best Taylor-type rule, for example. Such a policy would likely vary depending on the initial regime. A min-max approach could yield a rule that was best given any initial regime.

In the next section we turn to the optimal control problem, which relies on the reduced form solutions obtained here.

3 Optimal control

The rational solution algorithm presented above can be used as a basis for solving the optimal control problem with regime shifts and forward looking expectations. There are different equilibrium concepts one can use to come up with a solution. Here the primary concern is to find a time-consistent solution. We proceed with a closed-loop (feedback) time-consistent approach similar to Oudiz and Sachs (1985). In Appendix A we follow their state-space approach. Here we develop solutions using the so-called semi-structural form, following Dennis (2001).

Write the model (which represents the constraint of the optimal control problem) as:

$$x_t = A(s_t)x_{t-1} + B(s_t)u_{t-1} + D(s_t)E_t\left[x_{t+1}|s_t\right] + C(s_t)\varepsilon_t \tag{8}$$

where $A(s_t)$, $B(s_t)$, $C(s_t)$ and $D(s_t)$ are random matrices depending on the same Markov chain s_t, $E_t\left[x_{t+1}|s_t\right]$ is the expectation conditional on the information set available at time t which also include s_t. s_t is observable.

It is convenient to begin with the assumption that a control law exists:

$$u_t = -F(s_t)x_t$$

which is conveniently re-formulated as a function of the states and shocks. To make sure the system parameters are always a function of the same regime s_t (rather than e.g. $(s_t,\ s_{t-1})$), and to get rid of the control (that is why we are assuming that a control rule exists), it is convenient to use the augmented model:

$$\begin{aligned}\begin{bmatrix} I & 0 \\ F\left(s_t\right) & I \end{bmatrix}\begin{bmatrix} x_t \\ u_t \end{bmatrix} \quad &= \quad \begin{bmatrix} A(s_t) & B(s_t) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} D(s_t) & 0 \\ 0 & 0 \end{bmatrix} E_t\left[\begin{bmatrix} x_{t+1} \\ u_{t+1} \end{bmatrix}|s_t\right] \\ &\quad + \begin{bmatrix} C(s_t) \\ 0 \end{bmatrix}\varepsilon_t\end{aligned}$$

or (after pre-multiplying with the inverse of the left hand matrix):

$$z_t = A^{+}(s_t)z_{t-1} + D^{+}(s_t)E_t\left[z_{t+1}|s_t\right] + C^{+}(s_t)\varepsilon_t$$

where the definitions are obvious. Now that the system is one without control variables (which are incorporated into z), we can then use the solution method developed in the previous section to solve for the equilibrium law of motion for z, and hence for the expectations. Assume an equilibrium law of motion for z:

$$z_t = G_i z_{t-1} + H_i\varepsilon_t \tag{9}$$

where G_i and H_i are undetermined, and $s_t = i$ in an obvious notation. Following the steps above, one can find G_i and H_i by solving the following systems of inter-twined equations:

$$G_i \quad = \quad A_i^{+} + D_i^{+}\bar{G}_i G_i \tag{10}$$

$$H_i \quad = \quad D_i^{+}\bar{G}_i H_i + C_i^{+} \tag{11}$$

where $i = 1, 2, ..., N$ and $\bar{G}_i = \sum_{j=1}^{N} p_{ij}G_j = \sum_{j=1}^{N} p[s_{t+1} = j|s_t = i]G_j$. (10) is a system of N coupled quadratic equations in $G = (G_1, ..., G_N)$. After solving for the feedback part, the feedforward part can be easily solved as: $H_i = (I - D_i^+ \bar{G}_i)^{-1} C_i^+$.

What have we established? Subject to some feedback rule F, we have computed the law of motion of the economy (9) which is now a backward looking regime-switching VAR (where the regime is observable). Recalling the definition of z, we can rewrite the law of motion of the economy in such a way that the control actions are explicit:

$$x_t = G_{xx}(s_t)x_{t-1} + G_{xu}(s_t)u_{t-1} + H_x(s_t)\varepsilon_t \tag{12}$$

where G_{xx}, G_{xu} and H_x are matrices partitioned conformably. (12) can be used as an input into the optimal control problem with regime shifts, for which we have a solution algorithm. This takes G_{xx}, G_{xu} and the transition probability matrix P as input and returns an updated feedback rule $u_t = -F(s_t)x_t$. This is used to update the matrices A^+, D^+ and C^+ and start a new iteration of the algorithm.

So far we have characterised but not solved the control problem. This is established in the next subsection, following Zampolli (2005).

3.1 The optimal control problem with regime shifts

The policymaker's problem is to choose a decision rule for the control u_t to minimise the inter-temporal loss function:

$$\sum_{t=0}^{\infty} \beta^t r(x_t,\ u_t) \tag{13}$$

where $\beta \in (0,\ 1]$ is the discount factor and r is a quadratic form:

$$r(x_t,\ u_t) = x_t' R x_t + u_t' Q u_t \tag{14}$$

with R a $n \times n$ positive definite matrix, Q a $m \times m$ positive semi-definite matrix. The optimisation is subject to x_0, s_0 and the model of the reduced-form economy:

$$x_{t+1} = A(s_{t+1})x_t + B(s_{t+1})u_t + \varepsilon_{t+1} \qquad t \geq 0 \tag{15}$$

x is the n-vector of state variables, u is the m-vector of control variables and ε is the n-vector of mean-zero shocks with variance-covariance matrix Σ_ε. The matrices A and B are stochastic and take on different values depending on the regime or state of the world $s_t \in \{1, ..., N\}$. The regime s_t, which is observable at t,[5] is assumed to be a Markov chain with probability transition

[5] This means that the uncertainty faced by the policymaker is about where the system will be at $t + 1$, $t + 2$, and so forth. Other assumptions about timing could be made, and we discuss them further in Appendix A.

matrix[6]

$$P = [p_{ij}] \qquad i, j = 1, .., N \tag{16}$$

in which $p_{ij} = prob\{s_{t+1} = j | s_t = i\}$ is the probability of moving from state i to state j at $t+1$; and $\sum_{j=1}^{N} p_{ij} = 1$, $i = 1, ..., N$. These probabilities are assumed to be time-invariant and exogenous. The formulation (15) is general enough to capture different types of jumps or extreme changes in the economic system.

3.1.1 Solution

Solving the problem means finding a state-contingent decision rule, i.e. a rule which tells how to set the control u_t as a function of the current vector of reduced-form state variables, x_t, and the current regime s_t. Associated with each current state of the world is a Bellman equation. Therefore, solving the model requires jointly solving the following set of N *inter-twined* Bellman equations:

$$v(x_t, i) = \max_{u_t} \left\{ r(x_t, u_t) + \beta \sum_{j=1}^{N} p_{ij} E_t^{\varepsilon} \left[v(x_{t+1}, j) \right] \right\} \qquad i = 1, ..., N \tag{17}$$

where $v(x_t,\ i)$ is the continuation value of the optimal dynamic programming problem at t written as a function of the state variables x_t as well as the state of the world at t, $s_t = i$, E_t^{ε} is the expectation operator with respect to the martingale ε, conditioned on information available at t, such that $E_t^{\varepsilon}[\varepsilon_{t+1}] = 0$.

The policymaker has to find a sequence $\{u_t\}_{t=0}^{\infty}$ which maximises her current payoff $r(\cdot)$ as well as the discounted sum of all future payoffs. The latter is the expected continuation value of the dynamic programming problem and is obtained as the average of all possible continuation values at time $t+1$ weighted by the transition probabilities (16). Given the infinite horizon of the problem, the continuation values (conditioned on a particular regime) have the same functional forms.

Given the linear-quadratic nature of the problem, let us further assume that:

$$v(x_t,\ i) = x_t' V_i x_t + d_i \quad i = 1, .., N \tag{18}$$

where V_i is a $n \times n$ symmetric positive-semidefinite matrix, and d_i is a scalar. Both are undetermined. To find them, we substitute (18) into the Bellman

[6] For an introduction to Markov chain and regime switching vector autoregressive models see e.g. Hamilton (1994).

equations (17) (after using (14)) and compute the first-order conditions, which give the following set of decision rules:

$$u(x_t,\ i) = -F_i x_t \quad i = 1,..,N \tag{19}$$

where the set of F_i depend on the unknown matrices $V_i,\ i = 1,..,N$. By substituting these decision rules back into the Bellman equations (17), and equating the terms in the quadratic forms, we find a set of *inter-related* Riccati equations, which can be solved for $V_i,\ i = 1,..,N$ by iterating jointly on them, that is:

$$[V_1 \ldots V_N] = T\left([V_1 \ldots V_N]\right). \tag{20}$$

This set of Riccati equations defines a contraction over $V_1, \ldots, V_N$, the fixed point of which, $T(\cdot)$, is the solution. After lengthy matrix algebra, the resulting system of Riccati equations can be written in compact form as:

$$\begin{aligned} V_i &= R + \beta G\left[A'VA|_{s=i}\right] \\ &\quad -\beta^2 G\left[A'VB|_{s=i}\right]\left(Q + \beta G\left[B'VB|_{s=i}\right]\right)^{-1} G\left[B'VA|_{s=i}\right] \end{aligned} \tag{21}$$

where $i = 1,..,N$, and $G(\cdot)$ is a conditional operator defined as follows:

$$G\left[X'VY|_{s=i}\right] = \sum_{j=1}^{N} X_j'\left(p_{ij}V_j\right)Y_j$$

where $X \equiv A, B$; $Y \equiv A, B$. Written in this form the Riccati equations contain 'averages' of different 'matrix composites' conditional on a given state i.

Having found the set of V_i which solves (21), the matrices F_i in the closed-loop decision rules (19) are given by:

$$F_i = (Q + \beta G\left[B'VB|_{s=i}\right])^{-1}\beta G\left[B'VA|_{s=i}\right] \quad i = 1,..,N \tag{22}$$

Solving for the constant terms in the Bellman equations (17) after substitution of (19) gives $(I_N - \beta P)\, d = \beta P\Gamma$. The vector of scalars $d = [d]_{i=1,\ldots,N}$ in the value functions (18) is given by:

$$d = (I_N - \beta P)^{-1}\, \beta P\Gamma \tag{23}$$

where $\Gamma = \left[tr\left(V_i \Sigma_\varepsilon\right)\right]_{i=1,\ldots,N}$.[7]

[7] The transition law (15) can be generalised to make the variance of the noise statistics vary across states of the world, i.e.

$$x_{t+1} = A\left(s_{t+1}\right)x_t + B\left(s_{t+1}\right)u_t + C(s_{t+1})\varepsilon_{t+1}$$

Assuming $E^{\varepsilon}\left(\varepsilon_t\varepsilon_t'\right) = I$, then the covariance matrix of the white-noise additive shocks would be $\Sigma\left(s_t\right) = C\left(s_t\right)C\left(s_t\right)'$ or, to simplify notation, $\Sigma_i = C_iC_i'$ $(i = 1,..,N)$. As we note elsewher, the introduction of a state-contingent variance for the noise process does not affect the decision rules but does affect the value function.

The decision rules (19) depend on the uncertainty about which state of the world will prevail in the future, as reflected in the transition probabilities (16). Yet, the response coefficients (i.e. the entries in F_i) do not depend on the variance-covariance matrix Σ_ε of the zero-mean shock ε in (15). Thus, with respect to ε, *certainty equivalence* holds in that the policy rules (19) are identical to the ones obtained by assuming that *within* each regime the system behaves in a completely deterministic fashion. The noise statistics, as is clear from (23), affect the objective function.

It is interesting to note that the above solutions incorporate the standard linear regulator solutions as two special cases. First, by setting the transition matrix $P = I_N$ (i.e. N-dimensional identity matrix), one obtains the solution of N separate linear regulator problems, each corresponding to a different regime on the assumption that each regime will last forever (and no switching to other regimes occurs). This case could be useful as a benchmark to see how the uncertainty about moving from one regime to another impacts on the state-contingent rule. In other words, by setting $P = I_N$, we are computing a set of rules which will differ from ones computed with $P \neq I_N$, in that the latter will be affected by the chance of switching to another regime. Second, by choosing identical matrices (i.e. $A_i = A$, $B_i = B$), the solution obtained is trivially that of a standard linear regulator problem with a time-invariant law of transition.[8]

3.2 Complete solution

For greater clarity, the algorithm is given in steps below. It consists of two main blocks: one solve the REH model with regime shifts given a feedback rule, thereby putting it into backward looking form; the other solves the optimal control problem given the backward looking form. By iterating back and forth on these two distinct blocks the algorithm converges to a solution if one exists, perhaps with the use of some damping. The gist of the algorithm is thus to make expectation formation and optimisation consistent, through repeated iteration. It can be compared with the solutions given in Appendix A.

[8] In this case (22) reduces to:

$$F = \left(Q + \beta B'VB\right)^{-1} \beta B'VA$$

where V is the solution to the single Riccati equation:

$$V = R + \beta A'VA - \beta^2 A'VB \left(Q + \beta B'VB\right)^{-1} B'VA$$

and (23) is the constant:

$$d = \frac{\beta}{1-\beta} tr \left(V\Sigma_\varepsilon\right).$$

See e.g. Ljungqvist and Sargent (2000, pp. 56-58).

Algorithm 2 *We want to compute the optimal control of the following economy:*

$$x_t = A(s_t)\, x_{t-1} + B(s_t)\, u_{t-1} + D(s_t)\, E[x_{t+1}|I_t] + C(s_t)\, \varepsilon_t.$$

The algorithm is implemented in Matlab$^{\mathrm{TM}}$ *and uses intrinsic functions (called RSSOLVE and ISRE). The algorithm consists of the following steps:*

1. *Assume an initial control law:*

$$u_t = -F(s_t)\, x_t.$$

2. *Form the augmented system (the goal here is to get rid of the control and make sure that the stochastic matrices depend only on* s_t*, not on* $(s_t,\ s_{t-1})$*):*

$$\begin{bmatrix} I_{n_x} & 0_{n_x,n_u} \\ F(s_t) & I_{n_u} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} A(s_t) & B(s_t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} D(s_t) & 0 \\ 0 & 0 \end{bmatrix} E\left[\begin{matrix} x_{t+1} \\ u_{t+1} \end{matrix} |I_t \right] + \begin{bmatrix} C(s_t) \\ 0 \end{bmatrix} \varepsilon_t.$$

Premultiply by $\begin{bmatrix} I_{n_x} & 0_{n_x,n_u} \\ -F(s_t) & I_{n_u} \end{bmatrix}$ *(the inverse of the left hand matrix above) to get:*

$$z_t = A^{+}(s_t)\, z_{t-1} + D^{+}(s_t)\, E[z_{t+1}|I_t] + C^{+}(s_t)\, \varepsilon_t$$

where $z_t = [x_t\ u_t]'$.

3. *The augmented system can be solved by RRSOLVE, yielding the equilibrium law of motion:*

$$z_t = G(s_t)\, z_{t-1} + H(s_t)\, \varepsilon_t$$

or:

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} G_{xx}(s_t) & G_{xu}(s_t) \\ G_{ux}(s_t) & G_{uu}(s_t) \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} H_x(s_t) \\ H_u(s_t) \end{bmatrix} \varepsilon_t$$

The bottom part gives the policy rule as a function of the past states and controls.

4. *The upper part is used as an input into the optimal control toolbox:*

$$x_t = G_{xx}(s_t)\, x_{t-1} + G_{xu}(s_t)\, u_{t-1} + H_x(s_t)\, \varepsilon_t$$

5. *The optimal control obtained from ISRE is:*

$$u_t = -F(s_t)\, x_t$$

6. *Having obtained this, the next step is to check for convergence:*

$$\left\| F(s_t) - F(s_t)^{(0)} \right\| < \varepsilon$$

   If there is convergence (or too many iterations) terminate, otherwise go to the next step.

7. *Select the control law to use in the subsequent iteration:*

$$F(s_t)^{(1)} = \gamma F(s_t) + (1-\gamma) F(s_t)^{(0)}$$

   where $\gamma \in (0, 1]$ *is appropriately chosen. A combination is necessary to prevent the law of motion to move too further away from the stable one, which ensures convergence.*

We conclude this section with a number of remarks. First, this algorithm has unknown numerical properties, as with the Oudiz and Sachs (1985) method. This is a fixed point algorithm, modified to allow for a relaxation parameter γ. This substantially improves convergence properties in some cases.

Second, it is possible that the algorithm could be made both faster and more stable by iterating on the first order conditions rather than solving the optimal control problem as in Oudiz and Sachs (1985). We outline this in Appendix A. Our approach has the considerable expositional advantage that the two 'blocks' of the solution procedure are distinct. We have also found that sufficient damping has so far proved a reliable method for finding the fixed point. Indeed, it is not known if the Oudiz and Sachs (1985) procedure is at all reliable (and it can certainly be very slow) even without the modifications we propose. In practice both methods might be usefully implemented in case one fails.

Third, the algorithm solves for the time-consistent Nash-Stackelberg equilibrium. See Appendix A for a different Nash approach and Dennis (2001) for a similar one. The intrinsic difference is that the algorithm allows the policymaker to take into account the contemporaneous actions of agents in determining the optimal policy. In Appendix A, where we make a distinction between jump and predetermined variables, this can be modelled explicitly as part of the first order conditions. Here, as all variables are modelled the same, the reactions of agents are treated no differently to any predetermined behaviour.

Finally, an interesting extension to the algorithm of Section 1 is to introduce stochastic re-optimisation by the policymaker (as in Roberds (1987)): for example, if one can reformulate the problem in such a way that the Lagrange multiplier is reset to zero stochastically, then one could solve the problem using such algorithm.

4 Application

In our application we look at how optimal policy is affected if the structure of the economy might change in some specific way, and investigate probabilities that key parameters change. We outline our model here, and then the control and simulation experiments later.[9]

4.1 A small open-economy model

We apply the methods discussed above to an open economy model. Ours model embeds those of Batini and Nelson (2000) and Leitemo and Söderström (2004) and enables us to discuss stochastic changes in parameters. The model is in the tradition of New Keynesian policy models. It consists of the following equations:

1. IS curve The now-standard intertemporal IS curve is used:

$$y_t = \phi\left[(1-\theta)\, E_t y_{t+1} + \theta y_{t-1}\right] - \sigma\left(R_t - E_t \pi_{t+1}\right) + \delta q_{t-1} + v_t$$

2. Phillips Curve A forward-looking Phillips curve with inertia:

$$\pi_t = \alpha \pi_{t-1} + (1-\alpha)\, E_t \pi_{t+1} + \phi_y y_{t-1} + \phi_q q_{t-1} - \phi_q q_{t-2} + u_t$$

3. Uncovered interest parity Nominal exchange rate equation:

$$\bar{s}_t = \hat{E}_t s_{t+1} - R_t - k_t - z_t$$

4. Definition of q Real exchange rate definition:

$$q_t - q_{t-1} = s_t - s_{t-1} - \pi_t$$

5. Expectations of s

$$\hat{E}_t \bar{s}_{t+1} = \psi E_t \bar{s}_{t+1} + (1-\psi)\, s^a_{t+1,t}$$

where the operator E indicates rational expectations.

[9] The algorithms above are developed with the intent to provide new insights in the area of optimal monetary policy. As suggested by some authors, monetary policy may optimally react differently if the model changes, say in a pre-bubble and a post bubble regime. It would also be affected by the uncertainty that a bubble is not a rational bubble but reflects expectation of a higher earnings or productivity regime, and so on. One immediate application would be to compute the optimal policy which would be regime contingent in the model, perhaps to study how monetary policy should react to asset prices. Another potential interesting application is the study of how asymmetric risk about future earnings affects households' debt and saving decisions.

6. Adaptive expectations

$$s^a_{t+1,t} = \xi s^a_{t,t-1} + (1-\xi)\, s_t$$

7. IS shock

$$v_t = \rho_v v_{t-1} + e_{vt}$$

8. Phillips curve shock

$$u_t = \rho_u u_{t-1} + e_{ut}$$

9. Risk premium/non-UIP factors

$$k_t = \rho_k k_{t-1} + e_{kt}$$

In addition there are a number of definitional equations we need for our model form, which are the definition of $\bar{q}$ as well as q_{t-1} and q_{t-2}. We add two new variables R_{t-1} and R_{t-2}, necessary to add a smoothing target to the cost function, i.e. $(R_t - R_{t-1})^2$. We give further details in Appendix B.

4.2 Experiments

In this paper we conduct the following experiments. We assume that there is a structural break in some key parameters, e.g. α. We assume there is some probability P of a permanent shift up or down. We then plot selected response coefficients as a function of P.

In the graphs we plot a mixture of experiments. Firstly, we assume in a two-state model that there is a probability p that there will be a change in the coefficient, and a probability q that once it has changed regime it will switch back. The Markov matrix is given by:

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$

In the first set of experiments we assume that $q = 0$, that is once a switch has occurred there is no switch back. On the same graphs we plot a three-state problem using the Markov matrix:

$$P = \begin{bmatrix} 1-p & \frac{1}{2}p & \frac{1}{2}p \\ q & 1-q & 0 \\ q & 0 & 1-q \end{bmatrix}$$

where there is equal likelihood of two changes—which we choose to be up or down by the same amount—so we can get a handle on the certainty

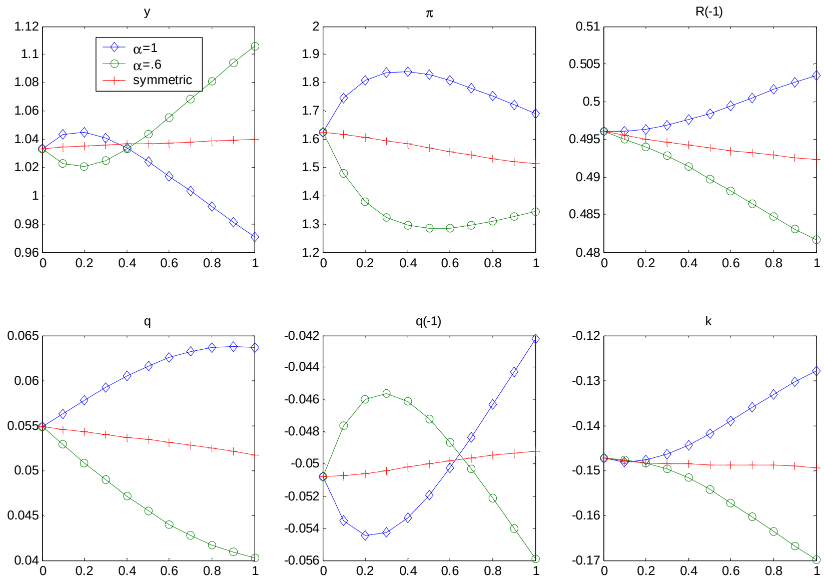

Figure 1: Effect of changes in α

equivalence of the results. This is the red (usually central) line on the graphs.

We begin by assuming that all changes are expected to be permanent ($q = 0$). In Figure 1 we show the effect of a change in α from the central case of 0.8. An anticipated fall requires a more aggressive response to the output gap for example, but only past some critical point. In Figure 2 we show the same effect on σ. A similar pattern emerges, but with no marked switching effect on the real exchange rate and output coefficients. Figure 3 illustrates an almost perfect certainty equivalence result for changes to the exchange rate pass through coefficient, as the red line is near horizontal.

However if we consider changes to ϕ_y a different picture emerges (Figure 4). Here complicated tradeoffs between coefficients occur. This seems particularly true of the coefficients on the real exchange rate and the inflation rate. In Figure 5 changes to φ have small and predictable effects.

As ϕ_y seems an important parameter we plot this for different assumptions about q. Figure 6 refers to the case q =0.5, and Figure 7 refers to q =0.25. The pattern of trade-offs in coefficients seems to be preserved.

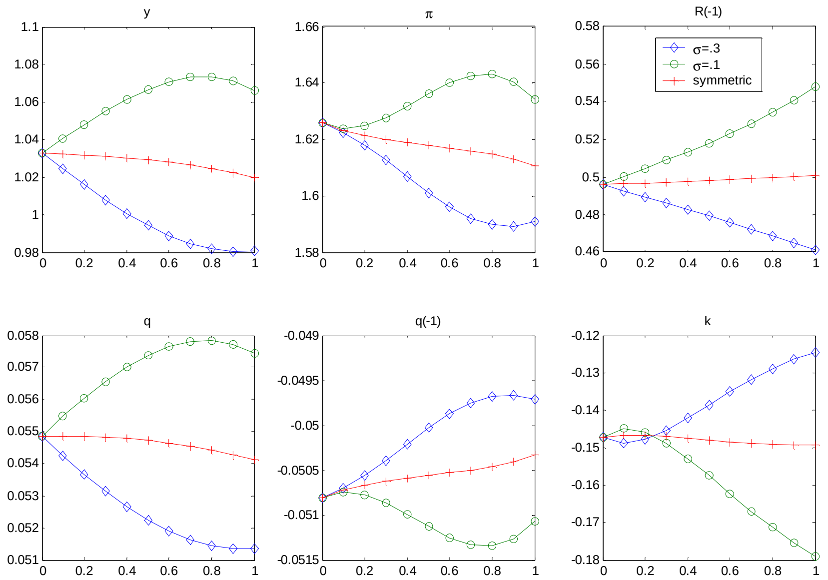

Figure 2: Effect of changes in σ

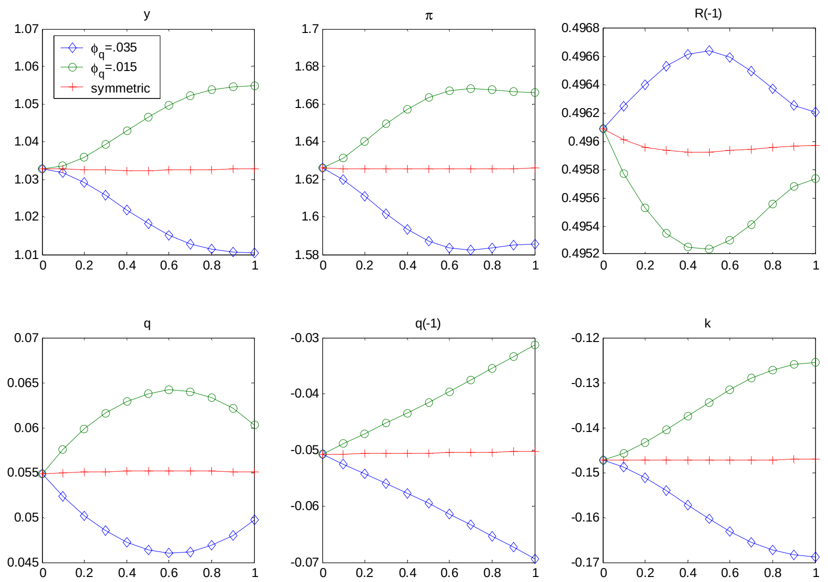

Figure 3: Effect of changes in ϕ_q

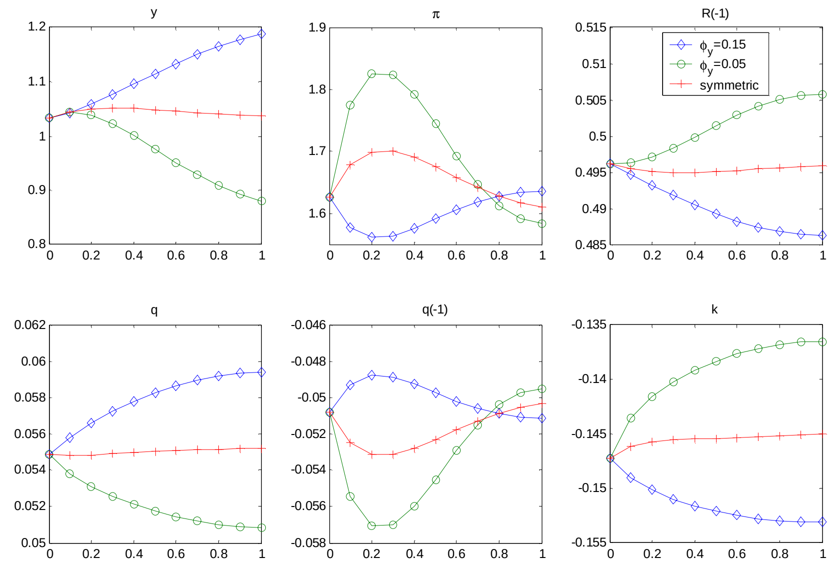

Figure 4: Effect of a change in ϕ_y

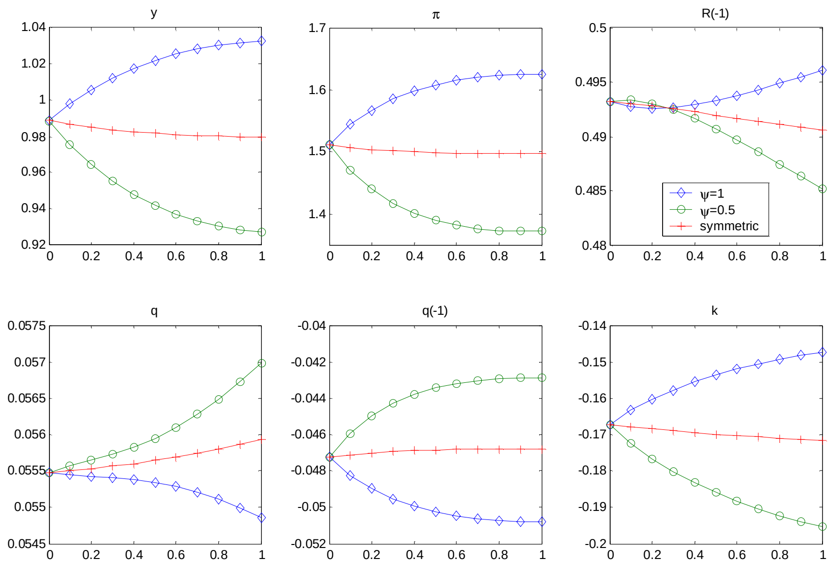

Figure 5: Effect of changes in φ

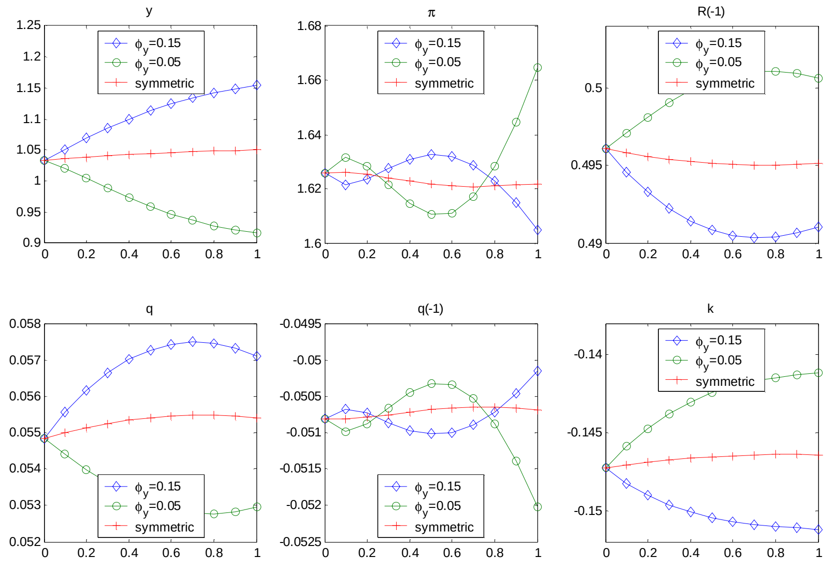

Figure 6: Effect of changes in ϕ_y, $q = 0.5$

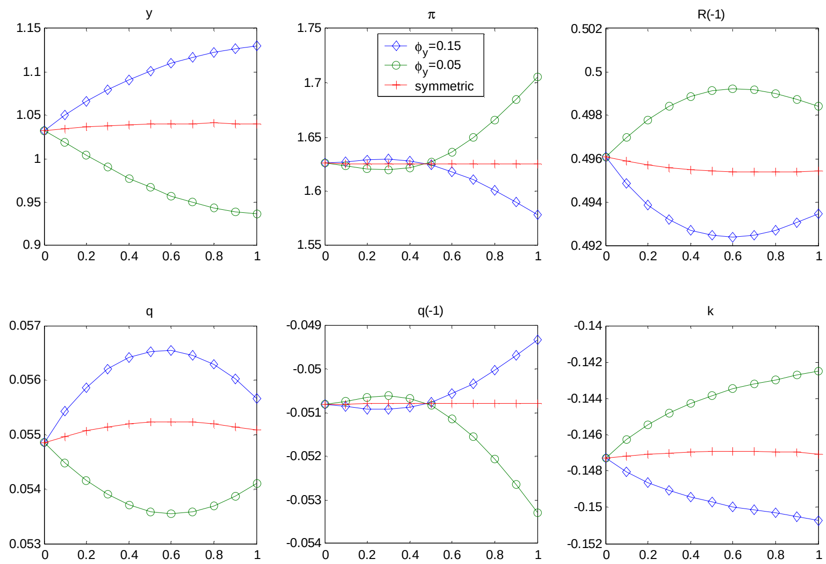

Figure 7: Effect of changes in ϕ_y, $q = 0.2$

5 Simulating the model under symmetric and asymmetric beliefs

The above indicates how we calculate optimal policies. It has built into it assumptions about agent and policymaker perceptions about each other's behaviour. Consider the following. Our control algorithm solves a fixed point problem, which can be succinctly represented as follows:

1. The policy maker (cb) computes policy u as a function of the probability P and the private sector's (ps) expectations E_{ps}, that is:

   $$u_{cb} = u(P,\ E_{ps}).$$

2. In turn, the private sector forms expectations E_{ps} as a function of the probability P and the policy rule u_{cb}, that is:

   $$E_{ps} = E\left(P, u_{cb}\right).$$

3. Hence, $u_{cb} = u\left(P, E_{ps}\right) = u\left(P, E\left(P, u_{cb}\right)\right)$. The algorithm solves for the fixed point u_{cb}. It is assumed that P is the true probability governing the transition across regimes.

These expectations are determined by the various agent's perceived values for P. All, some or none of these beliefs may be accurate. We can simulate the model under a variety of assumptions about perceived values for P.

5.1 A number of cases

Policy and expectations can be set under different assumptions than above. Assumptions regarding what each agent believes or knows about the world, the transition probabilities and the other agent's decision problem. There are a number of cases that we consider, which are not exhaustive.

The first case we consider is one in which all agents share the same beliefs about the probability matrix P (as well as everything else) but such beliefs may be wrong. Let us indicate these beliefs with $\hat{P}$. The problem can now be characterised by the pair of decision rules:

$$\begin{aligned} u_{cb} &= u(\hat{P},\ E_{ps}) \\ E_{ps} &= E(\hat{P},\ u_{cb}) \end{aligned}$$

The problem is solved as before: $u_{cb} = u(\hat{P},\ E_{ps}) = u\left(\hat{P}, E\left(P, u_{cb}\right)\right)$ with the difference being the different probability matrix $\hat{P}$. Once u_{cb} and E_{ps} have been found, they can be substituted out from the true model, obtaining a reduced form. This reduced form is the same as obtained under $\hat{P}$.

However, it needs to be simulated under the true (but unknown to agents) value of P. One can compare responses under $\hat{P}$ and P to gauge the possible errors involved in selecting $\hat{P} \neq P$. If P is genuinely unknown, one can compute the losses corresponding to the probability pairing $\left(\hat{P},\ P\right)$, where $\hat{P}$ are the probabilities chosen by agents and P is the realisation of the true probability. The losses can inform the selection of as 'optimal' $\hat{P}$ that minimises risk. For example, it can be selected using a min-max criterion or some other criterion. Operationally this requires that the policymaker is believed by all other agents in their assessment of the probability, so the policymaker can influence expectations through this channel. In this circumstance the policymaker seeks to modify expectations to its advantage, that of increased robustness. This can be seen as a way of manipulating agents that is akin to time inconsistency, but in effect as long as beliefs about the true probability never change then agents are never fooled and there is no incentive to renege.

The second case is one in which the private sector correctly perceives P and perfectly knows the policy rule adopted by the policymaker. The policymaker, on the other hand, has beliefs $\hat{P}$, which in general differ from the true P, and also believes that the public shares those beliefs and hence forms expectations according to $E\left(\hat{P},\ u_{cb}\right)$, i.e.:

$$u_{cb} = u\left(\hat{P},\ E\left(\hat{P},\ u_{cb}\right)\right).$$

As the public correctly perceives P *and* the beliefs of the policymaker:

$$E_{ps} = E\left(P,\ u_{cb}\right) = E\left(P,\ u\left(\hat{P},\ E\left(\hat{P},\ u_{cb}\right)\right)\right).$$

To find the equilibrium solution, one needs to find the fixed point in $u_{cb} = u\left(\hat{P},\ E\left(\hat{P},\ u_{cb}\right)\right)$, which is done using the standard algorithm. Then u_{cb} is substituted out from the true model. The solution algorithm for forward looking models with regime shifts will contextually compute the expectations $E_{ps} = E(P,\ u_{cb})$ based on the true P as well as the policy u_{cb} computed in the previous step.

A third case is one in which the policymaker and the private sector do not share the same beliefs but perfectly understand each other's beliefs and decisions. Namely:

$$\begin{aligned} u_{cb} &= u(\hat{P},\ E_{ps}) \\ E_{ps} &= E(\bar{P},\ u_{cb}) \end{aligned}$$

where in general $\hat{P} \neq \bar{P}$. Both $\hat{P}$ and $\bar{P}$ may also be different from the true P. The standard algorithm needs to be modified to allow computation of this case. If an equilibrium exists, we can designate it the 'known disagreement' equilibrium. A special case of this is a variation of case two illustrated above: the policymaker chooses policy $u_{cb} = u\left(\hat{P},\ E_{ps}\right)$ knowing that the public has knowledge of the true probability matrix P, i.e. $E_{ps} = E\left(P,\ u_{cb}\right)$.

A fourth case is one in which a disagreement is unknown to both players:

$$\begin{aligned} u_{cb} &= u\left(\hat{P},\ E\left(\hat{P},\ u_{cb}\right)\right) \\ E_{ps} &= E\left(\bar{P},\ u\left(\bar{P},\ E_{ps}\right)\right). \end{aligned}$$

The standard algorithm can be run twice to solve for u_{cb} and for E_{ps} separately. Then, u_{cb} and E_{ps} need to be substituted out from the true model to find the reduced form associated with this case.

There are, of course, many other cases which can be considered. Each agent may form beliefs not only about the true model but also about the other agent's beliefs about the true model, beliefs about his own beliefs, beliefs about his own beliefs over the other beliefs, and so on ad infinitum. This problem of infinite regress is not dealt with here. It is also clear that there could be considerable value to private information, as in Morris and Shin (2002). We do not further consider the strategic advantages that may accrue here.

5.2 Learning

When simulating the model under the previous cases we implicitly assume that agents do not learn through time. This is clearly not realistic but there are two ways of defending the approach. First, the simulations help us inform about the choice of P, and therefore we are actually learning from them. Second, we could extend the algorithm to allow for passive learning. In other words, agents updates their probabilities using (for example) a Bayesian scheme in every period, but they make decision assuming that these probabilities will not change in the future. This is in some ways realistic: not all agents are so rational as to anticipate the way they will learn in the future, i.e. know the law of motion of the probabilities. In this case of passive learning, Bayesian techniques can be used to update the probabilities period by period, and the above algorithm can be used to compute the policymaker's instrument choice as well as the private sector's expectations of future variables. A more sophisticated algorithm may record the evolution of the probabilities and estimate a law of motion for them. Thus the policymaker will need to solve a more sophisticated control problem in which he has to allow for future variation in the probabilities.

6 Simulation results

We plot a variety of responses in the following graphs.

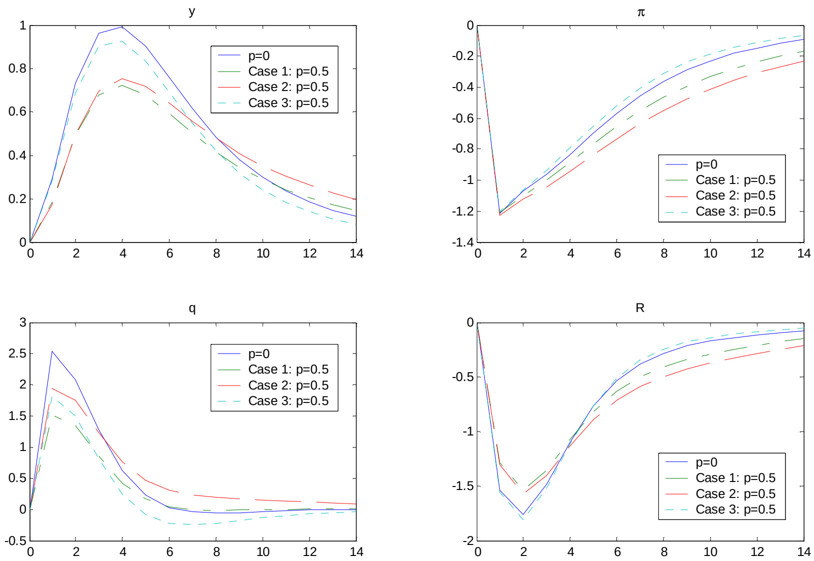

Figure 8: α goes from 0.8 to 0.6 with $p = 0.5$

- Case 1: both agents incorporate uncertainty as well as each other reactions.
- Case 2: only the central bank factors in uncertainty while the private sector does not and assumes regime 1 persists forever.
- Case 3: the central bank has a certainty equivalent rule, which is understood by the public, but the public factors in the probability of a regime shift

In each of the graphs the blue line is the 'certainty equivalent' policy, so that $p = 0$.

We concentrate on a break in α, as before possibly falling from 0.8 to 0.6. In Figure 8 we show a supply shock of unity and the assumption that $p = 0.5$. Here the responses of the output gap, inflation, the real exchange rate and interest rates are shown for each of the scenarios above. In Figure 9 we show the interest rate responses for this and other shocks. In Figure 10 we repeat the analysis for $p = 0.25$. It is clear that the perceptions of the various players can matter a great deal.

Now consider Figure 11. This simulation assumes break in α jumping down to 0.6 from 0.8. There is an initial negative inflationary shock and then the break in α occurs in period 3 (with probability 50% we would expect the breaks to concentrate mostly in period 2 and 3). You can see that not

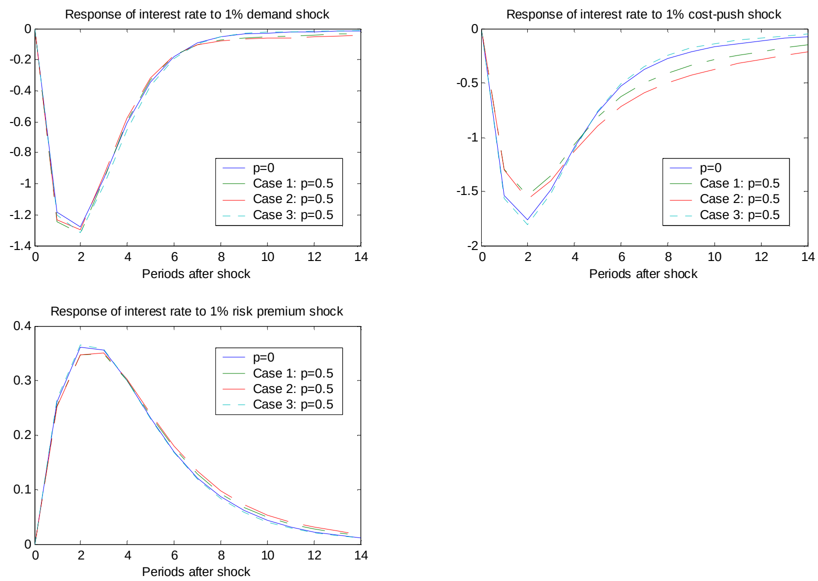

Figure 9: Interest rate responses

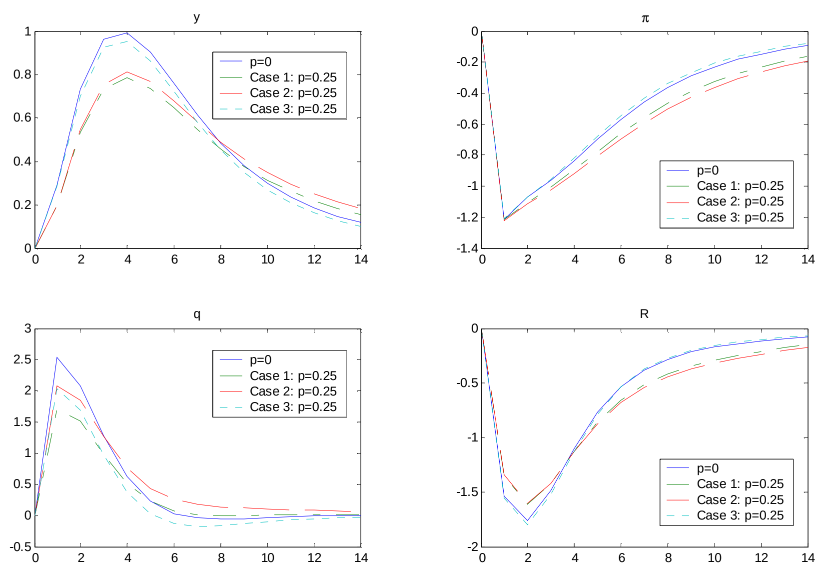

Figure 10: α goes from 0.8 to 0.6 with $q = 0.25$

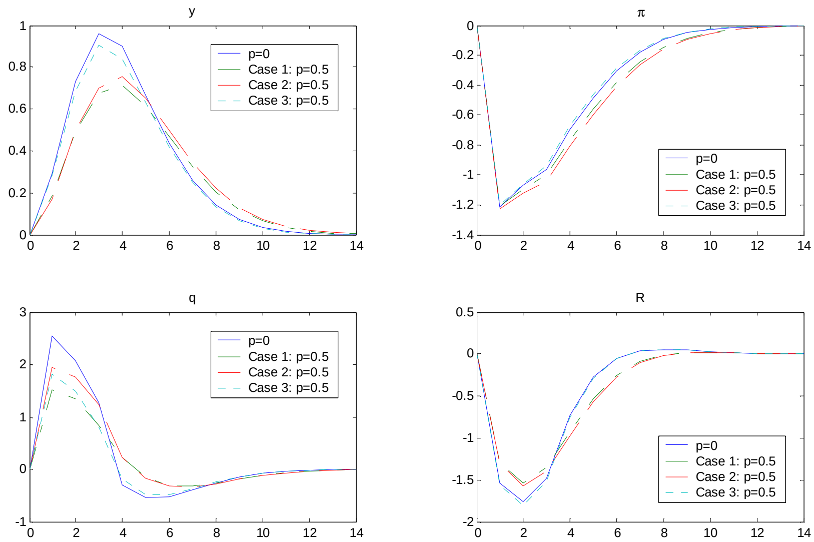

Figure 11: Negative inflation shock, break to α in period 3, $p = 0.5$

taking into account uncertainty produces a somewhat ‘bumpier’ economy. Note that when the break occurs, in all cases the policymaker can observe the break and switches to the same policy rule. However, because the system is at that point in a different state following the different policies, the responses follow different paths from that point onwards, though all converging in the long run towards equilibrium. In Figure 12 we reduce the probability to 0.25. What does this imply? Policy should be loosened less in response to a negative shock but should then return more gradually to neutral stance.

7 Conclusions

In this paper we have investigated optimal time-consistent monetary policy when the model is subject to regime shifts driven by Markov processes. We have barely scratched the surface of the control and simulation experiments that can be carried out. We in general find that policies are more cautious with this form of uncertainty. Recall that we are considering time consistent policies. If the monetary authorities are unable to affect expectations at all it may be that they would do almost nothing.

We have tried out a number of possible simulation scenarios. As the main source of uncertainty here is the Markov process and not the model (we know all the alternative models or parameterisations) and indeed how likely we are to switch between them. It is an interesting problem to extend

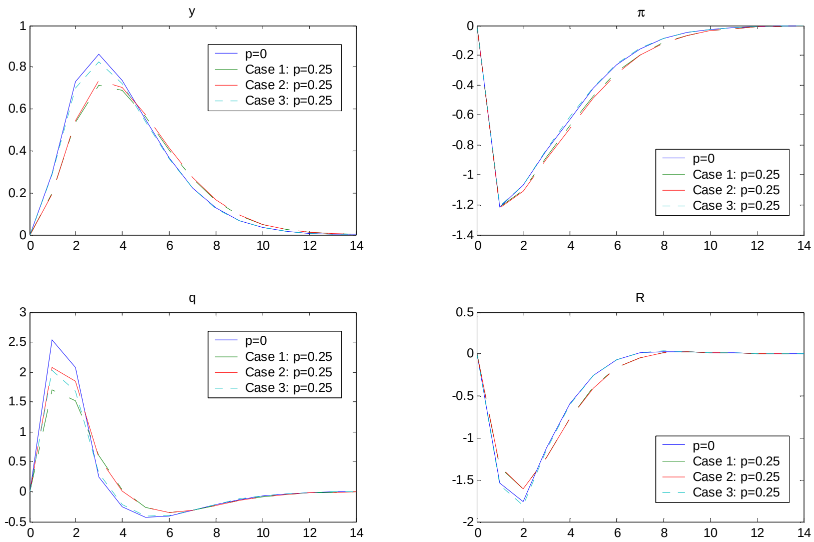

Figure 12: Negative inflation shock, break to α in period 3, $p = 0.25$

this model to where we are uncertain about the Markov process and model the learning over that rather than behavioural parameters directly.

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A State space solutions

In this Appendix we detail an alternative method for the calculation of explicitly time consistent policies. In the next section we consider the rational expectations solution, perhaps conditional on a given policy rule. We then consider the dynamic programming solution as a generalisation of the Oudiz and Sachs (1985) procedure. As the problem is certainty equivalent we only discuss the deterministic case.

A.1 A generalised rational expectations solution

Define a rational expectations model in state space as:

$$\begin{bmatrix} z_{t+1} \\ E\left[x_{t+1}|I_t\right] \end{bmatrix} = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix}. \tag{24}$$

We seek a solution of the form:

$$x_t = -N^i z_t \tag{25}$$

where we recognise that there may be a change in regime of some sort. For two possible regimes this means that $E\left[x_{t+1}|I_t\right] = -(p_{i1}N^1 + p_{i2}N^2)E\left[z_{t+1}|I_t\right]$ for a model in 'state i' or, more generally, for l possible regimes:

$$E\left[x_{t+1}|I_t\right] = -\left(\sum_{j=1}^{l} p_{ij}N^j\right) E\left[z_{t+1}|I_t\right] \tag{26}$$

for the i^{th} regime. Using this in the model (24) gives:

$$-\left(\sum_{j=1}^{l} p_{ij}N^j\right)\left(A_{11}^i z_t + A_{12}^i x_t\right) = A_{21}^i z_t + A_{22}^i x_t \tag{27}$$

implying:

$$\begin{aligned} &-\left(\left(\sum_{j=1}^{l} p_{ij}N^j\right) A_{12}^i + A_{22}^i\right) x_t \\ &\quad = \left(\left(\sum_{j=1}^{l} p_{ij}N^j\right) A_{11}^i + A_{21}^i\right) z_t \end{aligned} \tag{28}$$

or:

$$\begin{aligned} x_t &= -\left(\tilde{N}^i A_{12}^i + A_{22}^i\right)^{-1}\left(\tilde{N}^i A_{11}^i + A_{21}^i\right) z_t \\ &= -N^i z_t. \end{aligned}$$

where $\tilde{N}^i = \sum_{j=1}^{l} p_{ij} N^j$. We can develop an iteration based on this as:

$$\begin{aligned}
\tilde{N}^1_{k+1} &= \sum_{i=1}^{l} p_{1i} N^i_{k+1} \\
N^1_k &= \left(\tilde{N}^1_{k+1} A^1_{12} + A^1_{22}\right)^{-1} \left(\tilde{N}^1_{k+1} A^1_{11} + A^1_{21}\right) \\
\tilde{N}^2_{k+1} &= \sum_{i=1}^{l} p_{2i} N^i_{k+1} \\
N^2_k &= \left(\tilde{N}^2_{k+1} A^2_{12} + A^2_{22}\right)^{-1} \left(\tilde{N}^2_{k+1} A^2_{11} + A^2_{21}\right) \\
&\vdots \\
\tilde{N}^l_{k+1} &= \sum_{i=1}^{l} p_{li} N^i_{k+1} \\
N^l_k &= \left(\tilde{N}^l_{k+1} A^l_{12} + A^l_{22}\right)^{-1} \left(\tilde{N}^l_{k+1} A^l_{11} + A^l_{21}\right)
\end{aligned}$$

which continues until convergence. Thus in equilibrium for the i^{th} regime we get:

$$-\left(\sum_{j=1}^{l} p_{ij} N^j\right) \left(A^i_{11} - A^i_{12} N^i\right) = \left(A^i_{21} - A^i_{22} N^i\right) \tag{29}$$

as the solution to the i^{th} linked Riccati-type equation.[10]

A number of remarks should be made. First, in common with Oudiz and Sachs (1985) we assume that $\left(\tilde{N}^i_{k+1} A^i_{12} + A^i_{22}\right)$ is non-singular. This is almost always the case in our experience. Secondly, if the model is instead:

$$\begin{bmatrix} E^i_{11} & E^i_{12} \\ E^i_{21} & E^i_{22} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ E\left[x_{t+1} | I_t\right] \end{bmatrix} = \begin{bmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{30}$$

we can develop an equivalent iteration assuming that E^i_{11} and A^i_{22} are non-singular. Indeed, the semi-structural form model can be written:

$$\begin{bmatrix} I & 0 \\ 0 & B^i \end{bmatrix} \begin{bmatrix} x_t \\ E\left[x_{t+1} | I_t\right] \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^i & I \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_t \end{bmatrix} \tag{31}$$

which conforms to those restrictions. Finally, if the regimes are all the same then the solution reduces down to:

$$-N\left(A_{11} - A_{12} N\right) = \left(A_{21} - A_{22} N\right) \tag{32}$$

which could be solved using the method of Blanchard and Kahn (1980) or iteratively as above.

[10] See Blake (2004) for a discussion of the types of Riccati equations used in rational expectations solutions.

A.2 Control

Let the control model in state space be:

$$\begin{bmatrix} z_{t+1} \\ E\left[x_{t+1}|I_t\right] \end{bmatrix} = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1^i \\ B_2^i \end{bmatrix} u_t. \tag{33}$$

We can apply the solutions of the previous section to yield:

$$\begin{aligned} x_t &= -\left(\tilde{N}^i A_{12}^i + A_{22}^i\right)^{-1}\left(\tilde{N}^i A_{11}^i + A_{21}^i\right) z_t \\ &\quad -\left(\tilde{N}^i A_{12}^i + A_{22}^i\right)^{-1}\left(\tilde{N}^i B_1^i + B_2^i\right) u_t \\ &= -J^i z_t - K^i u_t. \end{aligned} \tag{34}$$

For a given feedback rule, say $u_t = -F^i z_t$, then:

$$\begin{aligned} x_t &= -(J^i - K^i F^i) z_t \\ &= -N^i z_t. \end{aligned} \tag{35}$$

Now consider the discounted quadratic objective function:

$$C_t = \frac{1}{2}\sum_{t=0}^{\infty} \beta^t \left(z_t' Q z_t + u_t' R u_t\right). \tag{36}$$

More generally we would consider a cost function of the form:

$$C_t = s_t' \tilde{Q} s_t + 2u_t' \tilde{U} s_t + u_t' \tilde{R} u_t \tag{37}$$

where $s_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$ and we assign costs to the jump variables and covariances. We use (36) to reduce the amount of algebra without changing the essential message. Algebra for the complete cost case is available on request. (36) is minimised subject to (33) and a time consistency restriction. We next sketch a solution in the standard case and then for the Markov switching case.

A.2.1 Standard time consistent policies

The 'standard' Oudiz and Sachs (1985) dynamic programming solution is obtained from the following. Write the value function as:

$$V_t = \frac{1}{2} z_t' S_t z_t = \min_{u_t} \frac{1}{2}\left(z_t' Q z_t + u_t' R u_t\right) + \frac{\beta}{2} z_{t+1}' S_{t+1} z_{t+1}. \tag{38}$$

Note that the first line of the model is:

$$z_{t+1} = A_{11} z_t + A_{12} x_t + B_1 u_t \tag{39}$$

which we substitute in as the constraint. We can obtain the following derivatives:

$$\frac{\partial V_t}{\partial u_t} = \tilde{R}u_t + \beta B_1' S_{t+1} z_{t+1} \tag{40}$$

$$\frac{\partial V_t}{\partial x_t} = \beta A_{12}' S_{t+1} z_{t+1} \tag{41}$$

$$\frac{\partial x_t}{\partial u_t} = -K \tag{42}$$

with the last obtained from (34), our time consistency restriction. This reflects intra-period leadership with respect to private agents, so can be seen as reflecting Stackelberg behaviour. Using (34) we can also write (39) as:

$$z_{t+1} = (A_{11} - A_{12}J)z_t + (B_1 - A_{12}K)u_t. \tag{43}$$

We can use (40)–(42) and (43) to obtain the first order condition:

$$\begin{aligned}
\frac{\partial V_t}{\partial u_t} + \frac{\partial V_t}{\partial x_t}\frac{\partial x_t}{\partial u_t} &= \left(R + \beta(B_1' - K'A_{12}')S_{t+1}(B_1 - A_{12}K)\right) u_t \\
&\quad + \beta(B_1' - K'A_{12}')S_{t+1}(A_{11} - A_{12}J)z_t \\
&= 0 \\
\Rightarrow u_t &= -\beta\left(R + \beta(B_1' - K'A_{12}')S_{t+1}(B_1 - A_{12}K)\right)^{-1} \\
&\quad \times \left((B_1' - K'A_{12}')S_{t+1}(A_{11} - A_{12}J)\right) z_t \\
&= -F_S z_t
\end{aligned} \tag{44}$$

with the subscript emphasising the Stackelberg equilibrium. The value function can be written:

$$\begin{aligned}
z_t' S_t z_t &= z_t' \left(Q + F_S' R F_S + \beta(A_{11}' - J'A_{12}' - F_S'(B_1' - K'A_{12}'))\right. \\
&\quad \left. \times S_{t+1}(A_{11} - A_{12}J - (B_1 - A_{12}K)F_S)\right) z_t
\end{aligned}$$

implying:

$$S_t = Q + F_S' R F_S + \beta(A_{11}' - N'A_{12}' - F_S' B_1')S_{t+1}(A_{11} - A_{12}N - B_1 F_S) \tag{45}$$

where $N = J - KF_S$.

Note we could assume that $\partial x_t / \partial u_t = 0$, the Nash assumption, and instead obtain:

$$\begin{aligned}
\frac{\partial V_t}{\partial u_t} &= \left(R + \beta B_1' S_{t+1}(B_1 - A_{12}K)\right) u_t \\
&\quad + \beta B_1' S_{t+1}(A_{11} - A_{12}J)z_t = 0 \\
\Rightarrow u_t &= -\beta\left(R + \beta B_1' S_{t+1}(B_1 - A_{12}K)\right)^{-1} \\
&\quad \times \left(B_1' S_{t+1}(A_{11} - A_{12}J)\right) z_t \\
&= -F_N z_t
\end{aligned} \tag{46}$$

with associated Riccati equation:

$$S_t = Q + F_N' R F_N + \beta(A_{11}' - N' A_{12}' - F_N' B_1') S_{t+1} (A_{11} - A_{12} N - B_1 F_N)$$

with $N = J - KF_N$ now. This gives us a second time consistent equilibrium to investigate.

A.2.2 Markov switching models

We now turn to the case with random matrices. We modify the value function for the i^{th} regime to:

$$V_t^i = \min_{u_t} \frac{1}{2} \left(z_t' Q z_t + u_t' R u_t \right) + \beta E_t \hat{V}_{t+1}^i \tag{47}$$

where we need to make some assumption about $\hat{V}_{t+1}^i$. In common with what went before we will weight the forward value function by the probability that it comes to pass. However, the information set assumed will determine the exact form.

In either case the required modification is very simple, and it is easy to see that one possibility is to replace the last term with the probability weighted values of the alternative future value functions to give:

$$\frac{1}{2} z_t' S_t^i z_t = \min_{u_t} \frac{1}{2} \left(z_t' Q z_t + u_t' R u_t \right) + \frac{\beta}{2} z_{t+1}^{i\prime} \tilde{S}_{t+1}^i z_{t+1}^i. \tag{48}$$

where:

$$z_{t+1}^i = (A_{11}^i - A_{12}^i J^i) z_t + (B_1^i - A_{12}^i K^i) u_t$$

and $\tilde{X}^i = \sum_{j=1}^l p_{ij} X^j$ for any X, the same as the weight scheme we had before for the expectations generating mechanism. In so doing we are assuming that the policymaker identifies the regime that they currently face but is uncertain about any future one. If uncertainty extended to the current regime, then the optimisation problem would be:

$$\frac{1}{2} z_t' S_t^i z_t = \min_{u_t} \frac{1}{2} \left(z_t' Q z_t + u_t' R u_t \right) + \frac{\beta}{2} \tilde{z}_{t+1}' \tilde{S}_{t+1}^i \tilde{z}_{t+1} \tag{49}$$

where:

$$\tilde{z}_{t+1} = (\tilde{A}_{11}^i - \tilde{A}_{12}^i \tilde{J}) z_t + (\tilde{B}_1^i - \tilde{A}_{12}^i \tilde{K}) u_t$$

as policymakers would only know the previous policy regime, i, and the transition probabilities from that regime and so must 'average' the models to give the anticipated state.

What do all other agents expect? The equilibrium policy is one where agents expectations of the future policy is consistent with the assumed probabilities. Thus the value of (35) calculated to determine expectations is (in equilibrium) consistent with the policy actually followed, although we can modify this by having differing perceived probabilities across the policymaker and other agents. In fact, it is only across probabilities that we allow agents to differ in what they expect. Note that when this happens there is no intrinsic time inconsistency, as we discuss above, but rather this may lead to an inferior (or possibly superior) outcome. One of the advantages to the semi-structural form of the main text is that this is much more easily seen due to the fixed point nature of the solution.

Given expectations we need to determine that policy. For the first, Stackelberg, case, the first order condition then yields:

$$\begin{aligned}
\frac{\partial V_t^i}{\partial u_t} + \frac{\partial V_t^i}{\partial x_t}\frac{\partial x_t}{\partial u_t} &= \left(R + \beta(B_1^{i\prime} - K^{i\prime} A_{12}^{i\prime})\tilde{S}_{t+1}^i (B_1^i - A_{12}^i K^i)\right) u_t \\
&\quad + \beta(B_1^{i\prime} - K^{i\prime} A_{12}^{i\prime})\tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i J^i) z_t = 0 \\
\Rightarrow u_t &= -\beta \left(R + \beta(B_1^{i\prime} - K^{i\prime} A_{12}^{i\prime})\tilde{S}_{t+1}^i (B_1^i - A_{12}^i K^i)\right)^{-1} \\
&\quad \times \left((B_1^{i\prime} - K^{i\prime} A_{12}^{i\prime})\tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i J^i)\right) z_t \\
&= -F_S^i z_t.
\end{aligned}$$

Substituting into the value function we have the following Ricatti-type equation for regime i:

$$S_t^i = Q + F_S^{i\prime} R F_S^i + \beta(A_{11}^{i\prime} - N^{i\prime} A_{12}^{i\prime} - F_S^{i\prime} B_1^{i\prime})\tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i N^i - B_1^i F_S^i)$$

where $N^i = J^i - K^i F_S^i$.

In the second case, we get the Stackelberg solution:

$$\begin{aligned}
\frac{\partial V_t^i}{\partial u_t} + \frac{\partial V_t^i}{\partial x_t}\frac{\partial x_t}{\partial u_t} &= \left(R + \beta(\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime})\tilde{S}_{t+1}^i (\tilde{B}_1^i - \tilde{A}_{12}^i K^i)\right) u_t \\
&\quad + \beta(\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime})\tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i J^i) z_t = 0 \\
\Rightarrow u_t &= -\beta \left(R + \beta(\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime})\tilde{S}_{t+1}^i (\tilde{B}_1^i - \tilde{A}_{12}^i K^i)\right)^{-1} \\
&\quad \times \left((\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime})\tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i J^i)\right) z_t \\
&= -F_S^i z_t
\end{aligned}$$

with:

$$S_t^i = Q + F_S^{i\prime} R F_S^i + \beta(A_{11}^{i\prime} - N^{i\prime} A_{12}^{i\prime} - F_S^{i\prime} B_1^{i\prime})\tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i N^i - B_1^i F_S^i).$$

There is an open question as to which solution should be used. The Stackelberg case is almost always used (our semi-structural form admits no other). However, it implies a degree of leadership over the private sector, which we could interpret as commitment. This may be appropriate for some policymakers, but may be questionable for the monetary authority. It is an empirical question as to wheter there is value to such commitments.

A.3 Iterative schemes

Consider the Stackelberg equilibrium with current state information for every participant. A possible solution scheme is shown in Table 1. We can develop Nash solutions by deleting the relevant part of the policy rules. The resulting modified algorithm in Table 2.

The 'no current information for the policymaker' solutions involve probability averaging the matrices A_{11}, A_{12} and B_1 in the recursions for F and S. The resulting algorithms are given in Tables 3 and 4. Note that this involves different data sets for agents and policymakers, emphasised by the lack of the tilde's over the system matrices in the equations determining J and K.

We need to note the termination rules that we should observe. In the tables we merely terminate when the period count reaches 0. We would normally terminate iteration before this if the matrices have converged to a steady state. In general, without the stochastic matrices, we would stop when $\text{abs}(\max(N_{t+1} - N_t)) < \epsilon$ and $\text{abs}(\max(S_{t+1} - S_t)) < \epsilon$ for some small ϵ. This does not work for the stochastic matrix case, as the future values are always probability weighted, so we need to store N and S between iterations separately.

Table 1: FBS

$S^i_T = \bar{S}$, $N^i_T = \bar{N}$, `for` $i = 1, ..., l.$

`for` $t = T-1,\ 0$

`for` $i = 1,\ l$

$$
\begin{aligned}
\tilde{N}^i_{t+1} &= \textstyle\sum_{j=1}^{l} p_{ij} N^j_{t+1} \\
\tilde{S}^i_{t+1} &= \textstyle\sum_{j=1}^{l} p_{ij} S^j_{t+1} \\
J^i &= \left(\tilde{N}^i_{t+1} A^i_{12} + A^i_{22}\right)^{-1} \left(\tilde{N}^i_{t+1} A^i_{11} + A^i_{21}\right) \\
K^i &= \left(\tilde{N}^i_{t+1} A^i_{12} + A^i_{22}\right)^{-1} \left(\tilde{N}^i_{t+1} B^i_{1} + B^i_{2}\right) \\
F^i_S &= \beta \left(R + \beta (B^{i\prime}_1 - K^{i\prime} A^{i\prime}_{12}) \tilde{S}^i_{t+1} (B^i_1 - A^i_{12} K^i)\right)^{-1} \\
&\qquad \times \left((B^{i\prime}_1 - K^{i\prime} A^{i\prime}_{12}) \tilde{S}^i_{t+1} (A^i_{11} - A^i_{12} J^i)\right) \\
N^i_t &= J^i - K^i F^i_S \\
S^i_t &= Q + F^{i\prime}_S R F^i_S + \beta (A^{i\prime}_{11} - N^{i\prime}_t A^{i\prime}_{12} - F^{i\prime}_S B^{i\prime}_1) \\
&\qquad \times \tilde{S}^i_{t+1} (A^i_{11} - A^i_{12} N^i_t - B^i_1 F^i_S)
\end{aligned}
$$

`endfor`

`endfor`

Table 2: FBN

$S_T^i = \bar{S}$, $N_T^i = \bar{N}$, `for` $i = 1, ..., l$.

`for` $t = T-1,\ 0$

  `for` $i = 1,\ l$

$$
\begin{aligned}
\tilde{N}_{t+1}^i &= \textstyle\sum_{j=1}^l p_{ij} N_{t+1}^j \\
\tilde{S}_{t+1}^i &= \textstyle\sum_{j=1}^l p_{ij} S_{t+1}^j \\
J^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i A_{11}^i + A_{21}^i\right) \\
K^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i B_1^i + B_2^i\right) \\
F_N^i &= \beta\left(R + \beta B_1^{i\prime} \tilde{S}_{t+1}^i (B_1^i - A_{12}^i K^i)\right)^{-1} B_1^{i\prime} \tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i J^i) \\
N_t^i &= J^i - K^i F_N^i \\
S_t^i &= Q + F_N^{i\prime} R F_N^i + \beta(A_{11}^{i\prime} - N_t^{i\prime} A_{12}^{i\prime} - F_N^{i\prime} B_1^{i\prime}) \\
&\qquad \times \tilde{S}_{t+1}^i (A_{11}^i - A_{12}^i N_t^i - B_1^i F_N^i)
\end{aligned}
$$

  `endfor`

`endfor`

Table 3: FBS, policy information lag

$S_T^i = \bar{S}$, $N_T^i = \bar{N}$, $\tilde{A}_{11}^i = \sum_{j=1}^l p_{ij} A_{11}^j$, $\tilde{A}_{12}^i = \sum_{j=1}^l p_{ij} A_{12}^j$
and $\tilde{B}_1^i = \sum_{j=1}^l p_{ij} B_1^j$ `for` $i = 1, ..., l$.

```
for t = T − 1, 0
  for i = 1, l
```

$$
\begin{aligned}
\tilde{N}_{t+1}^i &= \sum_{j=1}^l p_{ij} N_{t+1}^j \\
\tilde{S}_{t+1}^i &= \sum_{j=1}^l p_{ij} S_{t+1}^j \\
J^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i A_{11}^i + A_{21}^i\right) \\
K^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i B_1^i + B_2^i\right) \\
F_S^i &= \beta \left(R + \beta(\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime}) \tilde{S}_{t+1}^i (\tilde{B}_1^i - \tilde{A}_{12}^i K^i)\right)^{-1} \\
&\quad \times \left((\tilde{B}_1^{i\prime} - K^{i\prime} \tilde{A}_{12}^{i\prime}) \tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i J^i)\right) \\
N_t^i &= J^i - K^i F_S^i \\
S_t^i &= Q + F_S^{i\prime} R F_S^i + \beta(\tilde{A}_{11}^{i\prime} - N_t^{i\prime} \tilde{A}_{12}^{i\prime} - F_S^{i\prime} \tilde{B}_1^{i\prime}) \\
&\quad \times \tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i N_t^i - \tilde{B}_1^i F_S^i)
\end{aligned}
$$

```
  endfor
endfor
```

Table 4: FBN, policy information lag

$S_T^i = \bar{S},\ N_T^i = \bar{N},\ \tilde{A}_{11}^i = \sum_{j=1}^{l} p_{ij} A_{11}^j,\ \tilde{A}_{12}^i = \sum_{j=1}^{l} p_{ij} A_{12}^j$ and $\tilde{B}_1^i = \sum_{j=1}^{l} p_{ij} B_1^j$ `for` $i = 1, ..., l.$

`for` $t = T-1,\ 0$
  `for` $i = 1,\ l$

$$
\begin{aligned}
\tilde{N}_{t+1}^i &= \textstyle\sum_{j=1}^{l} p_{ij} N_{t+1}^j \\
\tilde{S}_{t+1}^i &= \textstyle\sum_{j=1}^{l} p_{ij} S_{t+1}^j \\
J^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i A_{11}^i + A_{21}^i\right) \\
K^i &= \left(\tilde{N}_{t+1}^i A_{12}^i + A_{22}^i\right)^{-1} \left(\tilde{N}_{t+1}^i B_1^i + B_2^i\right) \\
F_S^i &= \beta \left(R + \beta \tilde{B}_1^{i\prime} \tilde{S}_{t+1}^i (\tilde{B}_1^i - \tilde{A}_{12}^i K^i)\right)^{-1} \tilde{B}_1^{i\prime} \tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i J^i) \\
N_t^i &= J^i - K^i F_S^i \\
S_t^i &= Q + F_S^{i\prime} R F_S^i + \beta (\tilde{A}_{11}^{i\prime} - N_t^{i\prime} \tilde{A}_{12}^{i\prime} - F_S^{i\prime} \tilde{B}_1^{i\prime}) \\
&\quad \times \tilde{S}_{t+1}^i (\tilde{A}_{11}^i - \tilde{A}_{12}^i N_t^i - \tilde{B}_1^i F_S^i)
\end{aligned}
$$

  `endfor`
`endfor`

B Model in semi-structural form

The model can be written:

$$Hx_t = Ax_{t-1} + Bu_{t-1} + DE_t[x_{t+1}|I_t] + C\varepsilon_t$$

where the x, u and ε vectors are defined as:

	x_t
1	y_t
2	π_t
3	$\bar{s}_t$
4	q_t
5	$\hat{E}_t\bar{s}_{t+1}$ or $\hat{E}_t s_{t+1}$
6	$s^a_{t+1,t}$
7	v_t
8	u_t
9	k_t
10	$\bar{q}_t$
11	$\bar{q}_{t-1}$
12	q_{t-1}
13	q_{t-2}
14	R_{t-1}
15	R_{t-2}
16	c
17	z_t
18	b_t
19	s_t

	u_t
1	R_t

	ε_t
1	e_{vt}
2	e_{ut}
3	e_{kt}
4	e_{zt}

The parameters, similar to Batini and Nelson (2000) and Leitemo and Söderström (2004), were set as $\phi = 0.9$, $\theta = 0.7$, $\sigma = 0.2$, $\delta = 0.05$, $\alpha = 0.8$, $\phi_y = 0.1$, $\phi_q = 0.025$, $\psi = 1$ (full rationality unless stated otherwise; in examining learning we set the updating parameter to $\xi = 0.1$), $\rho_v = 0$, $\rho_u = 0$ and $\rho_k = 0.753$ consistent with a small open economy. The shock variances were set as $\sigma_v = 1\%$, $\sigma_u = 0.5\%$, and $\sigma_k = 0.92\%$.

Finally, the policymaker's preferences were set (in the main case) to be $\beta = 1$, $\lambda_y = 1$, $\lambda_\pi = 2$ and $\lambda_{\Delta R} = 0.1$.

B.1 Loss function

Period function:

$$x_t'Rx_t + u_t'Qu_t + 2x_t'Wu_t$$

$$
\begin{aligned}
R &= \begin{bmatrix} \lambda_y & 0 & 0 & \cdots & 0 \\ 0 & \lambda_\pi & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \\
Q &= \left[\lambda_{\Delta R}\right], \\
W &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\lambda_{\Delta R} \\ 0 \end{bmatrix}
\end{aligned}
$$

This implies that $2x_t' W u_t = 2\left(-R_{t-1}\lambda_{\Delta R}\right) R_t = -2\lambda_{\Delta R} R_t R_{t-1}$.