Generalized Sharpe Ratios and Asset Pricing in Incomplete Markets*

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Abstract

This paper draws on the seminal article of Cochrane and Saa-Requejo who pioneered option pricing based on the absence of arbitrage and high Sharpe Ratios. Our contribution is threefold:

We base the equilibrium restrictions on an arbitrary utility function, obtaining the C&S-R analysis as a special case with truncated quadratic utility. We restate the discount factor restrictions in terms of Generalised Sharpe Ratios suitable for practical applications. Last but not least, we demonstrate that for Ito processes C&S-R price bounds are invariant to the choice of the utility function, and that in the limit they tend to a unique price determined by the minimal martingale measure.

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reward for risk measure, Generalized Sharpe Ratio, optimal portfolio, duality and martingale methods, minimal martingale measure

1 Introduction

Asset pricing in incomplete markets is an intriguing problem because of the price ambiguity it has to deal with. Traditionally this ambiguity is either removed completely by assuming a representative agent equilibrium or it is acknowledged in its fullest by looking at no-arbitrage bounds. Arguably the former assumption is too strong and the latter assumption is too weak. Good-deal pricing introduces moderately strong equilibrium restrictions somewhere between the two extremes, postulating the absence of attractive investment opportunities – good deals – in equilibrium. Under the influence of CAPM and APT attractive investments became associated with high Sharpe Ratios, both in theoretical and empirical work (Ross 1976, Shanken 1992, Cochrane and Saá-Requejo 2000), but Černý and Hodges (2001) show that one can impose the good-deal restrictions with considerable generality. The generic term 'good deals' was introduced by Cochrane and Saá-Requejo (2000) (henceforth C&S-R) who were the first to successfully apply the good-deal restrictions to option pricing. The idea of C&S-R was to restrict the availability of high Sharpe Ratios at every point in time. Using the dual discount factor restrictions and backward recursion they calculated option price bounds that are based on very believable equilibrium restrictions, yet are much narrower than the corresponding super-replication bounds.

While the idea of no-good-deal restrictions as a strengthening of the no-arbitrage principle is attractive, the association of good deals with high Sharpe Ratios has its pitfalls. One limitation is related to the interpretation of equilibrium restrictions – Sharpe Ratio does not preserve the stochastic dominance ordering and as a consequence high Sharpe Ratios do not include all arbitrage opportunities. To make the equilibrium restrictions meaningful one must eliminate not just high Sharpe Ratios but also arbitrage opportunities and all the convex combinations between the two types of investments. This leaves one with a problem of determining how much of a given excess returns is pure Sharpe Ratio and how much of it belongs to an arbitrage.

Consequently one may not be quite sure whether that particular distribution of excess return is acceptable in equilibrium. The second limitation relates to the tightness of no-good-deal bounds. The objective of no-good-deal pricing is to formulate equilibrium restrictions which are more stringent than the no-arbitrage requirement. Unfortunately, in some models Sharpe Ratio bounds are only as tight as the no-arbitrage bounds. Černý and Hodges (2001) show that this problem is common to all reward-for-risk measures derived from bounded utility functions.

In this paper we show how one can address both limitations of the Sharpe Ratio analysis. Every increasing utility function defines a consistent ranking of investment opportunities if one measures good deals by, for example, the certainty equivalent of the risky investment. The first contribution of our paper is in demonstrating that the C&S-R analysis is a special case corresponding to the truncated quadratic utility. Maximization of truncated quadratic utility gives us a measure of risk which is identical to Sharpe Ratio for excess returns with small dispersion and which, for a general distribution, determines how much of the excess return should be attributed to pure arbitrage and how much to pure Sharpe Ratio. For any excess return distribution the pure Sharpe Ratio is an upward correction of the standard Sharpe Ratio capturing the true investment potential. This correction can be significant, with standard Sharpe Ratio of 2 and with lognormally distributed returns calibrated to annual data the correction is more than 100%, see Table 1.

To overcome the second limitation of Sharpe Ratios one faces two challenges. Firstly, one must be able to derive discount factor restrictions for good deals defined by unbounded utility functions. Secondly, one must link these restrictions to a Sharpe-Ratio-like risk measure. The second contribution of our paper is here; we derive the good-deal discount factor restrictions for a *general* utility function, and for standard utility functions (CARA, CRRA) we relate these restrictions to the absence of *Generalised Sharpe Ratios*. Our general result permits to prove an interesting property of C&S-R good-deal bounds: for Ito price processes the good-deal bounds are invariant to the choice of the reward-for-risk measure (utility function). The representative agent equilibrium in this case always corresponds to pricing with the *minimal martingale measure*, closely related to the *numeraire portfolio.*

1.1 Summary of the results

As an answer to 'What are the discount factor restrictions implied by standard utility functions?' we can offer the following:

1. Truncated quadratic utility

$$1 + h_Q^2(basis) \leq \mathrm{E}m^2 \leq 1 + h_Q^2 \tag{1}$$

2. Negative exponential (CARA) utility

$$\frac{1}{2}h_E^2(basis) \leq \mathrm{E}m \ln m \leq \frac{1}{2}h_E^2 \tag{2}$$

3. CRRA utility $0 < \gamma \neq 1$

$$\left(1 + h_\gamma^2(basis)\right)^{\frac{1-\gamma}{2\gamma^2}} \leq \mathrm{E}m^{1-\frac{1}{\gamma}} \leq \left(1 + h_\gamma^2\right)^{\frac{1-\gamma}{2\gamma^2}} \tag{3}$$

4. Logarithmic utility $\gamma = 1$

$$\ln\left(1 + h_1^2(basis)\right) \leq -2\mathrm{E}\ln m \leq \ln\left(1 + h_1^2\right) \tag{4}$$

where $m > 0$ is the change of measure, h_Q is the Sharpe Ratio adjusted for arbitrage, h_E, and h_γ are the generalized Sharpe Ratios generated by the CARA and CRRA utility, respectively. All variables with attribute '*basis*' refer to the market containing only basis assets (that is without focus assets to be priced).

For each of the utility functions the two inequalities are a direct consequence of the Extension Theorem, familiar from no-arbitrage pricing[1]. The

[1] For the derivation of the Extension Theorem for near arbitrage opportunities and for proofs of general properties of good-deal price bounds see Černý and Hodges (2001).

left hand side inequalities are known in financial literature, although the authors do not seem to be aware of the common principle (extension theorem) underlying all of them. These restrictions have been used to diagnose asset pricing models, and correspond to the above utility functions as follows 1. Hansen and Jagannathan (1991), Hansen, Heaton, and Luttmer (1995), 2. Stutzer (1995), 3. Snow (1991), for derivation see also Bernardo and Ledoit (1999) 4. Bansal and Lehmann (1997). The economic interpretation of the left hand side inequalities is simple: the best deal in a market containing only basis assets cannot be better than the best deal in a market including also the focus asset. The genuine no-good-deal restrictions are the right hand side inequalities, which quantify by how much can the best deal improve after the introduction of a focus asset. Here the only representative was the restriction (1) of Cochrane and Saá-Requejo (2000).

Also new is the reinterpretation of the discount factor restrictions in terms of Generalised Sharpe Ratios. These provide a scale-free measure of risk which behaves like the standard Sharpe Ratio for excess returns with small dispersion and has the same time scaling properties. In particular, this means that in spite of working with utility functions the measurement of good deals is independent of the initial level of wealth. We derive simple formulae that permit calculation of Generalized Sharpe Ratio for an arbitrary excess return X

$$h_Q^2(X) = \frac{1}{\max_\lambda \mathrm{E}\left[\max(1-\lambda X, 0)\right]^2} - 1 \tag{5}$$

$$h_E^2(X) = -2\ln\left[\min_\lambda \mathrm{E}e^{-\lambda X}\right] \tag{6}$$

$$h_\gamma^2(X) = \left[\max_\lambda \mathrm{E}\left(1+\lambda X\right)^{1-\gamma}\right]^{\frac{2\gamma}{1-\gamma}} - 1 \tag{7}$$

$$h_1^2(X) = e^{2\max_\lambda \mathrm{E}\ln(1+\lambda X)} - 1. \tag{8}$$

To obtain the standard Sharpe Ratio one simply removes the truncation at zero in (5).

The third main finding of our paper is that the instantaneous restrictions for Ito price processes coincide for all utility functions. This may not be obvious if one uses different types of reward-for-risk measures as in equations (1) - (4), but it transpires easily when viewed through the prisma of certainty equivalent gains. Denoting ν the market price of risk vector, the no-good-deal restriction becomes

$$\frac{1}{2}||\nu||^2 \leq Aa,$$

where a is the maximum excess certainty equivalent gain per unit of time and A is the coefficient of absolute risk aversion. Moreover, for small certainty equivalent gains and small Sharpe Ratios it is easy to show the equivalence

$$Aa \doteq \frac{h^2}{2}$$

whereby one immediately obtains the C&S-R restriction $||\nu||^2 \leq h^2$.

A direct consequence of the previous observation is that the no-good-deal price bounds for Ito price processes lie between the no-arbitrage super-replication bounds, see El Karoui and Quenez (1995), and a unique price determined by the *minimal martingale measure*. The minimal martingale measure has been studied previously in the finance literature in the context of mean-variance hedging in an incomplete market, see Schweizer (1991), and it has also been applied to option pricing, see Hofmann, Platen, and Schweizer (1992). Černý (1999) shows that for Ito price processes the minimal martingale measure corresponds to the numeraire portfolio of Long (1990).

1.2 Literature

The systematic exploration of the theory and applications of good-deal pricing does not have a very long history. Likely reason is that the specification of a good-deal equilibrium requires preference-based assumptions to rank the good deals, and therefore any such technique is considered inferior to the preference-free arbitrage pricing. However, in the recent years we came to appreciate that the market is inherently incomplete, due to both non-traded sources of risk and transaction costs. With this fact in mind the good-deal restrictions appear more natural, not only because they will provide tighter price bounds but also because the width of the price bounds can be seen as an important characteristic of the market equilibrium.

The good-deal pricing technique of Cochrane and Saá-Requejo (C&S-R) dates back to 1996 when it appeared as a working paper. C&S-R limit the maximum available Sharpe Ratio in *every period*, something we refer to as *instantaneous good-deal restrictions*, and based on this assumption they find option price bounds among others in a model with stochastic volatility, see also Cochrane and Saá-Requejo (1999).

Hodges (1998) argues that Sharpe Ratio pricing is implicitly supported by a truncated quadratic utility and he suggests to base the calculation of option price bounds on an alternative reward-for-risk measure, a *generalized Sharpe Ratio* generated by the negative Exponential utility (E-SR). Instead of instantaneous restrictions, he imposes one bound on the maximum E-SR for an investment running over the whole time horizon. Since the asset price bounds can be obtained by solving a dynamic hedging problem, with this type of equilibrium restrictions one does not require the dual discount factor restrictions.

Černý and Hodges (2001) discuss the theoretical properties of good-deal price bounds. The main tool of their analysis is the extension theorem already known from arbitrage pricing. They show that by considering certainty equivalent gains from an increasing utility function one can in fact build a general no-good-deal pricing theory of which representative agent equilibrium and no-arbitrage pricing are the two limiting cases. The analyses of C&S-R and Hodges appear as special cases. Černý and Hodges prove that in finite state models only price bounds generated by unbounded utility functions are always tighter than the no-arbitrage bounds.

Bernardo and Ledoit (2000) propose to base the definition of good deals on the gain-loss ratio. This reward-for-risk measure cannot be represented by certainty equivalent gain, for the following reason. In the language of utility functions, the gain-loss ratio is based on the Domar-Musgrave utility. With a piecewise linear utility in a frictionless market the certainty equivalent gain from a risky investment is either zero or plus infinity, hence the good deals are ranked by changing the *shape* of the utility function. This approach has theoretical advantages compared to using bounded utility functions and the discount factor restrictions are similar in nature to those mentioned above

$$L_{basis} \leq \frac{ess \sup m}{ess \inf m} \leq L,$$

where L denotes the maximum gain-loss ratio in the market. Alas, the gain-loss does not work well in Ito process environment with continuous trading where typically $L_{basis} = +\infty$, as in, for example, the standard Black-Scholes model. The application of gain-loss ratio calls for alternative models of asset returns.

1.3 Organization of the paper

The second section sets up a discrete time intertemporal model of asset prices, briefly explains how the one-period problem arises within the multiperiod framework, and derives the one-period good-deal discount factor restrictions for several frequently used utility functions. The third section discusses the links between the certainty equivalent gains, (generalized) Sharpe Ratios, and the corresponding discount factor restrictions. As a *practical example* in section four we find option price bounds in a trinomial tree using the logarithmic utility, demonstrating that good-deal price bounds generated by mild equilibrium restrictions are significantly tighter than the no-arbitrage bounds.

Section five translates the discrete time results into the Ito process framework and derives the instantaneous restrictions on the market price of risk. Section six shows how the instantaneous good-deal restrictions imply restrictions on investment opportunities in the long run. Section seven explores the limiting cases of the instantaneous good-deal price bounds, and section eight concludes.

2 No-good-deal restrictions in discrete time

2.1 The relationship between one-period and multi-period model

The following is a standard setup from mathematical finance literature. Let us have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t=0}^T)$ with E_t denoting the expectation conditional on the information at time t. There are n risky securities with $\mathbb{R}^n$-valued processes p and δ denoting their price and dividends respectively in money terms. Suppose that there is *short-term riskless borrowing* at a bounded rate r_t, that is an agent can borrow one unit of the numeraire in period t at the known rate r_t and repay $(1 + r_t)$ units of the numeraire in the next period. It is convenient to define a cumulative return on one unit of the numeraire invested into the bank account at the beginning and thereafter rolled over until time t

$$\begin{aligned} R_0 &= 1 \\ R_t &= \prod_{i=0}^{t-1}(1 + r_i), \end{aligned}$$

Let θ be an $\mathbb{R}^n$-valued portfolio process for 'risky securities' and let ϑ be a scalar process describing the amount of money invested in the bond. If an agent uses self-financing strategies his wealth $w_t = \theta_t p_t + \vartheta_t$ evolves over time as follows

$$w_t = (1 + r_{t-1})w_{t-1} + \theta_{t-1}[p_t + \delta_t - (1 + r_{t-1})p_{t-1}] \ . \tag{9}$$

No arbitrage means that there is a strictly positive $\mathcal{F}_t$-measurable variable[2]

[2] The variable $m_{t|t-1}$ can be visualised as the ratio between one-step risk-neutral probabilities and one-step objective probabilities at every node of a multinomial tree at time $t-1$. The ratio $\frac{m_{t|t-1}}{1+r_{t-1}}$ is known under a score of names: Intertemporal Marginal Rate of Substitution, stochastic discount factor, pricing kernel, or state price density.

Since there are finitely many securities the marketed subspace is finite dimensional and then by Theorem 6 in Clark (1993) a strictly positive valuation operator exists which is nothing else than the conditional change of measure $m_{t|t-1}$.

$m_{t|t-1}$ with $\mathrm{E}_{t-1} m_{t|t-1} = 1$ such that

$$\mathrm{E}_{t-1} m_{t|t-1} [p_t + \delta_t - (1 + r_{t-1}) p_{t-1}] = 0 \ ,$$

that is with artificial probabilities defined by $m_{t|t-1}$ the discounted wealth process is a martingale between $t-1$ and t

$$\mathrm{E}_{t-1} m_{t|t-1} w_t = (1 + r_{t-1}) w_{t-1} \ .$$

Now if we define unconditional change of measure m_T as

$$m_T = m_{1|0} \times m_{2|1} \times \ldots \times m_{T|T-1}$$

then from the law of iterated expectations $\mathrm{E}_0 m_T = 1$ and we can define a new probability measure Q

$$\frac{dQ}{dP} = m_T \ .$$

It is useful to note that the density process m_t

$$m_t \equiv \mathrm{E}_t m_T = m_{1|0} \times m_{2|1} \times \ldots \times m_{t|t-1}$$

is related to the conditional change of measure as follows

$$m_{t|t-1} = \frac{m_t}{m_{t-1}} \tag{10}$$

It follows from the construction of Q that the discounted gain process

$$G_t = \frac{p_t}{R_t} + \sum_{i=1}^{t} \frac{\delta_i}{R_i}$$

and the discounted wealth process

$$\frac{w_t}{R_t} = \frac{w_0}{R_0} + \sum_{i=1}^{t} \theta_i \triangle G_i$$

are Q-martingales.

2.2 One-period utility based no-good-deal restrictions

It is natural to measure the attractiveness of a self-financing investment by the certainty equivalent of the resulting wealth w_t relative to the wealth of a riskless investment into the bank account, that is $(1+r_{t-1})w_{t-1}$. Namely we can say that there is no good deal of size a from period $t-1$ to period t if the certainty equivalent of the risky investments exceeds the riskless investment by less than a, that is

$$\max_{\theta_{t-1}} \mathrm{E}_{t-1} U\left[(1+r_{t-1})(w_{t-1}+\theta_{t-1}R_{t-1}\triangle G_t)\right] < U[(1+r_{t-1})w_{t-1}+a] \ , \tag{11}$$

having substituted for w_t from equation (9). The value of parameter a is considered fixed. Section 3 discusses the link between the certainty equivalent gain a and Sharpe Ratios. It turns out that a is not a scale-free measure and therefore it is not particularly suitable for practical applications. However, parameter a allows us to formulate and solve the pricing problem for *any utility function,* therefore formulation (11) is the most convenient at this point.

In the rest of this section we will proceed in 2 steps. First we will explain how to find the highest a attainable in a complete market. In the second step we will show how, with the help of an extension theorem, this information can be used to find the no-good-deal price of an arbitrary focus asset. The second step will in a natural way lead to the dual discount factor restrictions.

Suppose there is a complete market with a unique conditional change of measure $m_{t|t-1}$, our aim is to find the maximum certainty equivalent gain $a(m_{t|t-1})$ in this market. Instead of looking for the optimal investment strategy θ_{t-1} we will use an elegant trick, due to Pliska (1986), of searching for the optimal distribution of wealth, subject to the budget constraint dictated by the state prices $m_{t|t-1}$

$$\max_{\theta_{t-1}} \mathrm{E}_{t-1} U\left[(1+r_{t-1})(w_{t-1}+\theta_{t-1}R_{t-1}\triangle G_t)\right] = \max_{\substack{w_t \\ \text{s.t.}\mathrm{E}_{t-1}\ m_{t|t-1}w_t=(1+r_{t-1})w_{t-1}}} \mathrm{E}_{t-1}U(w_t),$$

whereby we obtain

$$\max_{\substack{w_t \\ \text{s.t.}\mathrm{E}_{t-1}\ m_{t|t-1}w_t=(1+r_{t-1})w_{t-1}}} \mathrm{E}_{t-1}U(w_t) \triangleq U[(1+r_{t-1})w_{t-1}+a(m_{t|t-1})]. \tag{12}$$

The maximization problem (12) is standard, it can be solved for a finite state model and this solution carries over to an infinite state space subject to certain integrability condition which we assume to be satisfied. Since there is just one linear constraint one solves (12) using unconstrained maximization over all states with a Lagrange multiplier

$$\max_{\substack{w_t(\omega) \\ \omega\in\Omega}} U[w_t(\omega)] - \lambda m_{t|t-1}(\omega) w_t(\omega)$$

The first order conditions give

$$U'(w_t(\omega)) = \lambda m_{t|t-1}(\omega)$$

Denoting $I(.)$ the inverse function to the marginal utility $U'(.)$ we obtain

$$w_t = I(\lambda m_{t|t-1})$$

and from the restriction $\mathrm{E}_{t-1}\ m_{t|t-1} w_t = R_{t-1} w_{t-1}$ we can recover the value of λ.

As an example let us apply the above procedure to the negative exponential utility. The time subscripts and the conditional nature of expectations are ommited. First we find the inverse of the marginal utility

$$\begin{aligned} U(w) &= -e^{-Aw} \\ U'(w) &= Ae^{-Aw} \\ I(x) &= -\frac{1}{A}\ln\frac{x}{A}. \end{aligned}$$

Optimal wealth is then

$$\hat{w} = I(\lambda m) = -\frac{1}{A}\ln\frac{\lambda m}{A}.$$

We recover the Lagrange multiplier from the budget constraint and plug this value back into the expression for optimal wealth

$$\begin{aligned} \mathrm{E}m\hat{w} &= Rw_0 \\ \lambda &= Ae^{-ARw_0 - \mathrm{E}_0 m \ln m} \\ \hat{w} &= Rw_0 + \mathrm{E}\frac{m}{A}\ln\frac{m}{A} - \frac{1}{A}\ln\frac{m}{A}. \end{aligned}$$

Finally, we recover the certainty equivalent of the optimal risky investment

$$\begin{aligned} U(\hat{w}) &= -me^{-ARw_0 - \mathrm{E}m\ln m} \\ U^{-1}\left[\mathrm{E}U(\hat{w})\right] - Rw_0 &= \frac{1}{A}\mathrm{E}m\ln m = a(m). \end{aligned}$$

2.2.1 Discount factor restriction in good-deal pricing

We have just seen how one calculates the maximum attainable $a(m)$ in a complete market. The crucial link between the complete and incomplete market is provided by the *extension theorem*[3] which asserts that any incomplete market without good deals can be embedded in a complete market that has no good deals. Let us denote the certainty equivalent of the best deal attainable in the market containing only the basis assets a_{basis}. Two observations follow from the extension theorem. The best deal in the completed market cannot be worse than the best deal in the original market containing only basis assets. On the other hand, for any $\varepsilon > 0$ there is no good deal of size $a_{basis} + \varepsilon$ in the market containing just basis assets. Consequently, by extension theorem there must be a completion with a pricing kernel for which $a(m) < a_{basis} + \varepsilon$. By letting $\varepsilon \to 0$ we obtain

$$a_{basis} = \inf_m a(m)$$

where m must price correctly all basis assets. This argument has been inspired by 'fictitious completions' of Karatzas, Lehoczky, Shreve, and Xu (1991).

The restrictions of the type $a_{basis} \leq a(m)$ are well known in financial economics, where they have been employed to test different asset pricing models[4]. We are, however, primarily interested in the pricing implications

[3] Interestingly, both the idea of Sharpe Ratio restrictions and the use of the extension theorem can be traced back to Ross, see Ross (1976) pg. 354 and the appendix of Ross (1978).

[4] See Stutzer for CARA utility, Bansal and Lehmann for log utility. Snow, and Bernardo and Ledoit discuss the CRRA utility. In all these cases the discount factor restrictions are derived ad hoc from the Jensen's inequality.

of the extension theorem. Suppose that we want to find all prices of a focus asset that do not provide good deals of size a in the enlarged market. From the extension theorem all such prices must be supported by pricing kernels for which $a(m) \leq a$. This is the dual no-good-deal discount factor restriction.

For example, the discount factor restrictions for the CARA utility read

$$Aa_{basis} \leq \mathrm{E}_{t-1} m_{t|t-1} \ln m_{t|t-1} \leq Aa \tag{13}$$
$$\mathrm{E}_{t-1} m_{t|t-1} \triangle G_t = 0$$

In conclusion, the market including both basis and focus assets does not provide deals better than a, as measured by CARA utility, if (and only if) the focus assets are priced with pricing kernels consistent with basis assets and satisfying the restriction (13).

2.3 State price restrictions

Below we summarize the no-good-deal restrictions on the conditional change of measure $m_{t|t-1}$ for standard utility functions. The derivation proceeds as explained above.

1. Truncated quadratic utility $U(w) = -(\bar{w} - w)^2$; $w < \bar{w}$ and $U(w) = 0$; $w \geq \bar{w}$

$$\left(\frac{1}{1 - A((1 + r_{t-1})w_{t-1})a_{basis}} \right)^2 < \mathrm{E}_{t-1} m^2_{t|t-1} < \left(\frac{1}{1 - A((1 + r_{t-1})w_{t-1})a} \right)^2 \tag{14}$$

2. Negative exponential utility $U(w) = -e^{-Aw}$

$$Aa_{basis} < \mathrm{E}_{t-1} m_{t|t-1} \ln m_{t|t-1} < Aa \tag{15}$$

3. Power (isoelastic) utility $U(w) = \frac{w^{1-\gamma}}{1-\gamma}$; $\gamma < 1, w > 0$

$$\left(1 + \frac{a_{basis}}{(1 + r_{t-1})w_{t-1}} \right)^{\frac{1}{\gamma}-1} < \mathrm{E}_{t-1} m_{t|t-1}^{1-\frac{1}{\gamma}} < \left(1 + \frac{a}{(1 + r_{t-1})w_{t-1}} \right)^{\frac{1}{\gamma}-1} \tag{16}$$

4. Logarithmic utility $U(w) = \ln w, w > 0$

$$\ln\left(1+\frac{a_{basis}}{(1+r_{t-1})w_{t-1}}\right) < -\mathrm{E}_{t-1}\ln m_{t|t-1} < \ln\left(1+\frac{a}{(1+r_{t-1})w_{t-1}}\right). \tag{17}$$

$A((1+r_{t-1})w_{t-1})$ stands for the coefficient of absolute risk aversion evaluated at point $(1+r_{t-1})w_{t-1}$.

3 Comparing Sharpe Ratio with certainty gains

Having derived the state price restrictions (14)-(17) the task changes into interpreting the state price bounds as reward for risk measures, preferably ones that are close in nature to Sharpe Ratio. Note that if one uses a as the measure of attractiveness then one has to specify the coeffcient of absolute risk-aversion in restrictions (14)-(17). It turns out that for small Sharpe Ratios there is an unambiguous link between Sharpe Ratios and certainty equivalent gains, which we describe next.

Excess return X with a small Sharpe Ratio h gives a maximum certainty equivalent gain

$$a = \frac{h^2}{2A(w_t)} + o(h^2) \tag{18}$$

where $A(w) = -\frac{U''(w)}{U'(w)}$ is the absolute coefficient of risk-aversion and $\lim_{h^2\to 0}\frac{o(h^2)}{h^2} = 0$. To keep technicalities at minimum we assume that X has bounded support and that the utility function is sufficiently differentiable. From the Taylor expansion we obtain

$$\mathrm{E}U(w_0+\lambda X) = U(w_0) + U'(w_0)\lambda \mathrm{E}X + \frac{1}{2}U''(w_0)\lambda^2\mathrm{E}X^2 + o(\lambda^2\mathrm{E}X^2)$$

and after maximization with respect to λ we obtain

$$\lambda = \frac{U'(w_0)\mathrm{E}X}{U''(w_0)\mathrm{E}X^2} + o(\frac{\mathrm{E}X}{\mathrm{E}X^2}) \tag{19}$$

$$\max_{\lambda}\mathrm{E}U(w_0+\lambda X) = U(w_0) - \frac{1}{2}\frac{[U'(w_0)\mathrm{E}X]^2}{U''(w_0)\mathrm{E}X^2} + o(\frac{(\mathrm{E}X)^2}{\mathrm{E}X^2})\ . \tag{20}$$

Without loss of generality we can assume that $X\frac{\mathrm{E}X}{\mathrm{E}X^2}$ is small for all values of X so that the Taylor series approximation can be made arbitrarily precise. At the same time, for a small certainty equivalent gain we can write

$$U(w_0 + a) = U(w_0) + U'(w_0)a + o(a), \tag{21}$$

and the comparison of (20) and (21) gives the desired result (18).

In conclusion, one could replace Aa in expressions (14)-(17) with $\frac{h^2}{2}$. Naturally, this is not the only transformation that satisfies the asymptotic property (18). For example, for small values of h^2 we have

$$h^2 = \frac{h^2}{1+h^2} + o(h^2) = e^{h^2} - 1 + o(h^2),$$

and indeed we might equally well replace Aa with any other function $f(h^2)$ as long as f is continuously differentiable around 0 with $f(0) = 0$ and $f'(0) = 1$. The rest of this section describes how the ambiguity in choosing the function $f(h^2)$ is resolved for negative exponential, truncated quadratic and CRRA utility.

3.1 Sharpe Ratio and negative exponential utility

Interestingly, there is a special case where the relationship $h^2 = 2A(w_t)a$ holds for large certainty equivalent gains. By inverting the no-good-deal restriction (11) for negative exponential utility with an arbitrary random excess return X one obtains

$$-\frac{1}{A}\ln\left[-\max_{\lambda} -\mathrm{E}e^{-A\lambda X}\right] \leq a.$$

Hodges (1998) points out that for a *normally distributed* excess return X we have

$$\frac{1}{2}h^2(X) = -\ln\left[-\max_{\lambda} -\mathrm{E}e^{-\lambda X}\right] \tag{22}$$

where $h(X)$ is a standard Sharpe Ratio, and consequently he uses equation (22) to define the *generalized Sharpe Ratio* h_E for an *arbitrarily distributed* excess return. The maximum E-Sharpe Ratio is hence again related to the maximum certainty equivalent gain through (18)

$$\frac{1}{2}h_E^2 = Aa.$$

and one can write the state price restriction (15) in a more intuitive form

$$\mathrm{E}_{t-1} m_{t|t-1} \ln m_{t|t-1} \leq \frac{1}{2}h_E^2$$

3.2 Sharpe Ratio and truncated quadratic utility

The truncated quadratic utility provides another case where we can relatively easily compare non-infinitesimal Sharpe Ratios and certainty equivalent gains. One can easily show that quadratic utility maximization for a single asset with excess return X is related to the Sharpe Ratio of that asset $h(X)$ as follows.

$$\left(\frac{1}{1 - A((1+r_{t-1})w_{t-1})a}\right)^2 = 1 + h^2(X). \tag{23}$$

Intuitively, maximization of truncated quadratic utility will give the same result if the excess return does not reach the points where the quadratic has negative marginal utility. A simple calculation reveals that an excess return X with an upper bound $x_{\max} \equiv \operatorname{ess\,sup} X$ and Sharpe Ratio $h(X)$ is related to the maximum certainty equivalent gain a as stated in (23) provided that

$$h^2(X) < \frac{\mathrm{E}X}{x_{\max} - \mathrm{E}X} \tag{24}$$

when $\mathrm{E}X > 0$. For $\mathrm{E}X < 0$ the value $x_{\max}$ in condition (24) has to be replaced with $x_{\min} \equiv \operatorname{ess\,inf} X$. We learn two things from this result. Firstly, the comparison of (23) and the no-good-deal restriction (14) tells us that the truncated quadratic utility leads to the same state price restriction as the Sharpe Ratio analysis whenever the dispersion of the excess return is small, more precisely whenever condition (24) is satisfied. Secondly, it tells us how to re-define the Sharpe Ratio for excess returns that do not meet condition (24) in order to obtain a state price restriction which *formally* coincides with

(1). Looking at equation (23) the new generalized Sharpe Ratio is defined by the truncated quadratic utility as follows

$$h_Q^2(X) \equiv \frac{1}{\max_\lambda \mathrm{E}\left[\max(1-\lambda X, 0)\right]^2} - 1. \tag{25}$$

The maximization (25) can be reformulated more intuitively. When condition (24) is satisfied (25) gives the standard Sharpe Ratio. When (24) is not satisfied we will actually increase the Sharpe Ratio by throwing away some money in the states with highest excess return. More specifically, we can replace the original excess return distribution X with a distribution *capped* at value $x_{\max}$. Initially $x_{\max}$ is set at plus infinity and condition (24) is not satisfied. By lowering $x_{\max}$ we increase the Sharpe Ratio of the capped distribution and make the difference between the LHS and the RHS of condition (24) smaller. The Sharpe Ratio reaches its maximum just when the condition (24) applied to the capped distribution is met. This maximum value is the Generalised Sharpe Ratio h_Q. Mathematical justification of the argument above is given in Appendix A.

Table 1 shows that standard Sharpe Ratio of 2.0 may seriously underestimate the true investment potential if the excess returns have high dispersion, whereas at the value of 0.5 this difference is negligible. The table shows arbitrage-adjusted Sharpe Ratios h_Q against standard Sharpe Ratios for log-normally distributed returns. Because the returns are unbounded from above, the standard Sharpe Ratio is not an appropriate measure of risk. The difference between the GSR and SR is reported in the last column. The necessary calculations are given in the Appendix A.

3.3 CRRA Generalized Sharpe Ratios

A priori it is not very clear what reward for risk measure to consider for a general CRRA utility. Recall that the duality between pricing kernels and certainty equivalent gains in this case reads

$$\mathrm{E}_{t-1} m_{t|t-1}^{1-\frac{1}{\gamma}} = \left(1 + \frac{a}{(1+r_{t-1})w_{t-1}}\right)^{\frac{1}{\gamma}-1}, \tag{26}$$

$\bar{R}$	R	σ	h	h_Q	**%error** $\frac{h_Q-h}{h}$
1.04	1.02	0.04	0.5	0.502	**0.5%**
1.06	1.02	0.08	0.5	0.503	**0.6%**
1.18	1.02	0.16	0.5	0.512	**2.4%**
1.04	1.02	0.02	1.0	1.085	**8.5%**
1.06	1.02	0.04	1.0	1.093	**9.3%**
1.18	1.02	0.08	1.0	1.140	**14.0%**
1.06	1.02	0.02	2.0	3.675	**83.7%**
1.10	1.02	0.04	2.0	3.824	**91.2%**
1.34	1.02	0.08	2.0	4.845	**142.3%**

Table 1: $\bar{R}$ expected risky return, R risk-free return, σ return volatility, h standard Sharpe Ratio, h_Q arbitrage-adjusted Sharpe Ratio

it might therefore seem natural to define $g = \frac{a}{(1+r_{t-1})w_{t-1}}$ as the scale-free measure called 'certainty equivalent growth'. This is fine, except that for the same excess return X one will obtain different values of $g(X)$ for different γ, even asymptotically as g tends to zero. In order to achieve some kind of uniformity across γ it is helpful rewrite (26) using the coefficient of absolute risk aversion for CRRA utility

$$\frac{a}{(1+r_{t-1})w_{t-1}} = \frac{A\left((1+r_{t-1})w_{t-1}\right)a}{\gamma},$$

and the asymptotic relationship $Aa = \frac{h^2}{2}$ which yields

$$\mathrm{E}_{t-1} m_{t|t-1}^{1-\frac{1}{\gamma}} = \left(1 + \frac{h_\gamma^2}{2\gamma}\right)^{\frac{1}{\gamma}-1}. \tag{27}$$

By virtue of (18) all the generalised Sharpe Ratios h_γ have the same asymptotic behaviour for small values. It remains to check the consistency of this definition with the definition of the standard Sharpe Ratio, for which the duality is

$$\mathrm{E}_{t-1} m_{t|t-1}^2 = 1 + h_Q^2.$$

Recall that quadratic utility has $\gamma = -1$, substituting this value into equation (27) we obtain

$$\mathrm{E}_{t-1} m_{t|t-1}^2 = \left(1 - \frac{h_{-1}^2}{2}\right)^{-2},$$

and it is clear that $h_{-1} \neq h_Q$ even though asymptotically they are the same. Fortunately, there is an easy way out to achieve $h_{-1} = h_Q$. It is enough to realise that asymptotically

$$\left(1 + \kappa \frac{h_\gamma^2}{2\gamma}\right)^{\frac{1}{\kappa}\left(\frac{1}{\gamma}-1\right)} = \left(1 + \frac{h_\gamma^2}{2\gamma}\right)^{\left(\frac{1}{\gamma}-1\right)} + o(h_\gamma^2)$$

for all κ. There are many choices of κ, for example $\kappa = -2$ or $\kappa = 2\gamma$, such that $h_{-1} = h_Q$. A good way to pinpoint the 'right' value of κ is to look at the time scaling properties of the standard Sharpe Ratio and to compare it with the time scaling properties of the Generalised Sharpe Ratio h_γ, see section 6. It turns out that one needs $\kappa = 2\gamma$. The discount factor restrictions then become

$$\left(1 + h_{\gamma basis}^2\right)^{\frac{1-\gamma}{2\gamma^2}} < \mathrm{E}_{t-1} m_{t|t-1}^{\frac{\gamma-1}{\gamma}} < \left(1 + h_\gamma^2\right)^{\frac{1-\gamma}{2\gamma^2}} \tag{28}$$

$$\frac{1}{2}\ln\left(1 + h_{1basis}^2\right) < -\mathrm{E}_{t-1} \ln m_{t|t-1} < \frac{1}{2}\ln\left(1 + h_1^2\right). \tag{29}$$

Comparing (28) with (16) and using the definition of certainty equivalent gain we obtain the computational definition of CRRA Sharpe Ratio for a given excess return X

$$1 + h_\gamma^2(X) = \left[\max_\lambda \mathrm{E}\left(1 + \lambda X\right)^{1-\gamma}\right]^{\frac{2\gamma}{1-\gamma}},$$

with a special case for log-utility

$$1 + h_1^2(X) = e^{2\max_\lambda \mathrm{E}\ln(1+\lambda X)}.$$

3.4 Comparing Generalized Sharpe Ratios

It is instructive to compare the restrictions on the risk-neutral probabilities generated by the two generalized Sharpe Ratios h_E and h_Q . We do this

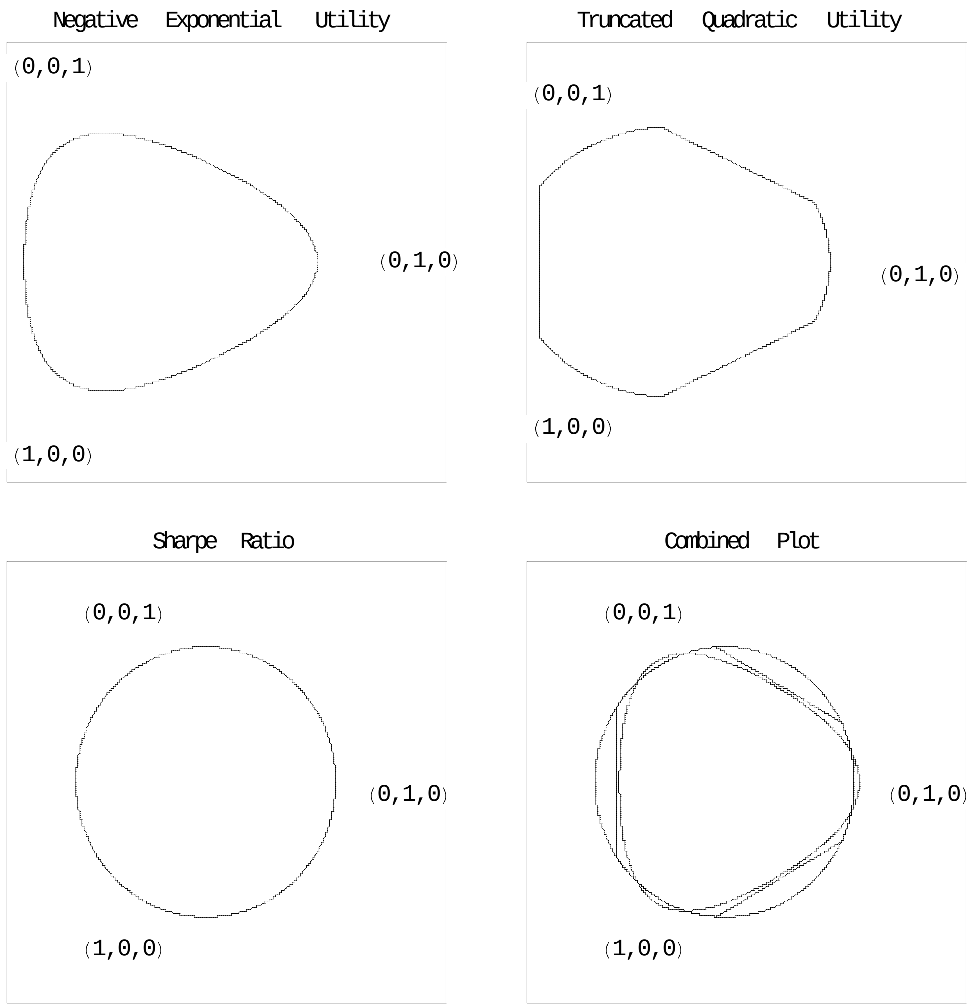

Figure 1: No-good-deal risk-neutral probabilities consistent with the maximum Sharpe Ratios $h_E = h_Q = h = 0.85$

graphically[5] for a model with three states and uniform objective probabilities $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The shaded triangle corresponds to no-arbitrage risk-neutral probabilities. The no-good-deal risk-neutral probabilities lie *inside* the oval boundaries. Note that the sets of no-good-deal risk-neutral probabilities implied by the two generalized Sharpe Ratios match very closely, especially for small and large Sharpe Ratio values. For $h \geq 0.85$ the no-good-deal bounds on the risk-neutral probabilities are *not strictly inside* the no-arbitrage trian-

[5] The graphs are produced in Mathematica. The code is available from the author on request.

gle. This is a consequence of using bounded utility functions, and it means that even for finite values of h we can obtain no-good deal price bounds which are *as wide as* the no-arbitrage super-replication bounds.

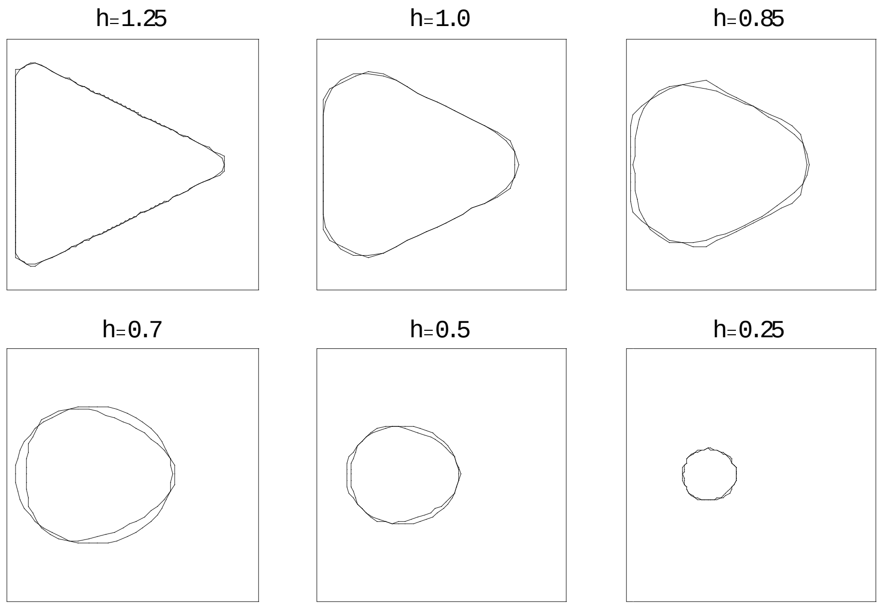

Figure 2: No-good-deal risk-neutral probabilities consistent with various maximum levels of Generalised Sharpe Ratios h_E, h_Q .

It is instructive to compare numerical values of Generalised Sharpe Ratios. The table below summarises GSR-s in a trinomial tree with $p_1 = p_2 = 0.421, p_3 = 0.158$ and excess return $X_1 = 32.75\%, X_2 = -15\%, X_3 = -47.32\%$. The numbers are chosen to give expected excess return of 15% with volatility 30%. The Sharpe Ratios are evaluated numerically using formulae (5)-(8).

4 Pricing with Logarithmic Utility: An Example

Consider a model with a constant risk-free rate $r = 10\%$ p.a. where the expected rate of return on the stock is 15% p.a. and annual volatility is 20%. The stock price moves in a recombining trinomial lattice calibrated to the

Risk measure	**Value**
h	0.500
h_Q	0.500
h_E	0.502
h_1	0.531
h_2	0.539
h_{10}	0.537

Table 2: The value of the standard Sharpe Ratio, arbitrage-adjusted Sharpe Ratio and Generalised Sharpe Ratios in a recombining trinomial tree

stated volatility and expected return with upstep $u = 1.15$. Each time period represents one month and stock returns are by assumption independent. Our aim is to price a 5-month at the money European call option with strike price $K = 100$. The calibrated objective probabilities of movement in the lattice are $p_1 = 0.1188, p_2 = 0.8343, p_3 = 0.0469$ for the upstep, middle and downstep respectively.

We assume that the above model is a true representation of stock price movements rather than an approximation to a diffusion model. Then, in the absence of other securities, the market is incomplete and the no-arbitrage price of the option is not unique. More specifically, the risk-neutral probabilities $q = (q_1, q_2, q_3)$ have one free parameter, and satisfy

$$\begin{pmatrix} q_1(\alpha) \\ q_2(\alpha) \\ q_3(\alpha) \end{pmatrix} = \begin{pmatrix} 0.324 \\ 0.334 \\ 0.342 \end{pmatrix} + \alpha \begin{pmatrix} 1.435 \\ -3.085 \\ 1.65 \end{pmatrix}$$

with $-0.207 < \alpha < 0.108$ parametrizing the range of no-arbitrage pricing kernels.

The maximum logarithmic Sharpe Ratio in the absence of the option can be found by *minimizing*[6] the central expression in equation (17)

$$\min_{\substack{\alpha \\ -0.207<\alpha<0.108}} \quad -\sum_{i=1}^{3} p_i \ln \frac{q_i(\alpha)}{p_i}$$

which gives $\hat{\alpha} = -0.165$, $\hat{q} = (0.0877, 0.8427, 0.0696)$ and $-\sum_{i=1}^{3} p_i \ln \frac{q_i(\hat{\alpha})}{p_i} = 0.00923$. From expression (29) the basis logarithmic Sharpe Ratio is

$$h_{1basis} = \sqrt{\exp\left(-2\sum_{i=1}^{3} p_i \ln \frac{q_i(\hat{\alpha})}{p_i}\right) - 1} = 0.136$$

monthly, equivalent to 0.473 per annum.

To decide which discount factors are admissible in equilibrium after the option is introduced, we must decide what level of Sharpe Ratio constitutes a good deal. One can either target an absolute level of Sharpe Ratio, say 2, or use a relative measure saying that the introduction of the option should not allow Sharpe Ratios in excess of c times the basis Sharpe Ratio, that is only those risk-neutral probabilities are admissible which satisfy

$$-\sum_{i=1}^{3} p_i \ln \frac{q_i(\alpha)}{p_i} \leq \ln(1 + (c\, h_{1basis})^2). \tag{30}$$

We take $c = 2$ and find numerically

$$-0.188 \leq \alpha \leq -0.133. \tag{31}$$

The admissible risk-neutral probabilities are a convex combination of vectors q_L and q_U corresponding to the lower and upper bound for α in (31),

$$q_L = \begin{pmatrix} 0.1330 \\ 0.7455 \\ 0.1215 \end{pmatrix} \quad q_U = \begin{pmatrix} 0.0541 \\ 0.9149 \\ 0.0310 \end{pmatrix}. \tag{32}$$

With this range of discount factors we can price our option, bearing in mind that at every node of the lattice we have to keep track of the highest and

[6] Alternatively, one can solve the primal portfolio problem

$$\max_{\beta} \mathrm{E} \ln \left[\beta R + (1-\beta) R^f\right]$$

where R is the risky return and R^f is the risk-free return.

t=0	t=1	t=2	t=3	t=4	t=5
					101.14
					101.14
					101.14
				75.31	74.90
				75.31	74.90
				75.31	74.90
			52.90	52.49	52.09
			52.90	52.49	52.09
			52.90	52.49	52.09
		33.49	33.06	32.66	32.25
		33.47	33.06	32.66	32.25
		33.46	33.06	32.66	32.25
	17.39	16.68	16.00	15.41	15.00
	16.91	16.38	15.87	15.41	15.00
	16.68	16.25	15.82	15.41	15.00
6.76	5.82	4.76	3.51	1.99	0.00
5.16	4.34	3.45	2.45	1.31	0.00
3.60	2.96	2.28	1.57	0.81	0.00
	1.12	0.66	0.26	0.00	0.00
	0.56	0.31	0.11	0.00	0.00
	0.24	0.12	0.04	0.00	0.00
		0.03	0.00	0.00	0.00
		0.01	0.00	0.00	0.00
		0.00	0.00	0.00	0.00
			0.00	0.00	0.00
			0.00	0.00	0.00
			0.00	0.00	0.00
				0.00	0.00
				0.00	0.00
				0.00	0.00
					0.00
					0.00
					0.00

Figure 3: Option price bounds with $c = 2$.

lowest no-good-deal price C_t^H and C_t^L

$$
\begin{aligned}
C_t^H &= \frac{\max(\mathrm{E}_t^{q_L} C_{t+1}^H, \mathrm{E}_t^{q_U} C_{t+1}^H)}{1.00407} \\
C_t^L &= \frac{\min(\mathrm{E}_t^{q_L} C_{t+1}^L, \mathrm{E}_t^{q_U} C_{t+1}^L)}{1.00407} \\
C_5^H &= C_5^L = (S_5 - K)^+
\end{aligned}
$$

The results are reported in a spreadsheet (figure 3) with the middle price being the unique price which would result from taking $c = 1$. This price *coincides with representative equilibrium price* of the option for a representative agent with logarithmic utility of terminal wealth.

It is interesting to note that at $t = 4$ the option is a redundant asset in all states but one. The effect of this state, however, spreads quickly and at $t = 2$ the option is not redundant in any state. The option price bounds for different values of c are summarized in table 3. The value $c = +\infty$ corresponds to the no-arbitrage (super-replication) bounds.

basis multiple	$c = 1$	$c = 2$	$c = 4$	$c = +\infty$
max monthly SR	$h = 0.136$	$h = 0.272$	$h = 0.375$	$h = +\infty$
implied max annual SR	$h = 0.498$	$h = 1.17$	$h = 4.67$	$h = +\infty$
C_0^L	5.16	3.70	2.01	2.01
C_0^H	5.16	6.76	10.87	13.94

Table 3: No-good-deal option price bounds

4.1 Graphical representation of good-deal state prices

The good-deal discount factors corresponding to different values of c are displayed in figure 4. The triangle contains all no-arbitrage risk-neutral probabilities for the three states, with the objective probability corresponding to the point P. The risk-neutral probabilities which give less than 4 times the basis logarithmic Sharpe Ratio, that is those which satisfy equation (30) with $c = 4$, are contained in the oval area[7] σ_2 and those which only give double of the basis Sharpe Ratio rate are within the smaller oval area σ_1. The segment A_1A_2 contains all the no-arbitrage risk-neutral probabilities which are consistent with the stock returns, and among those probabilities segments B_1B_2 and C_1C_2 represent the good-deal risk-neutral probabilities consistent with $c = 4$ and $c = 2$ respectively.

[7] Note that, unlike in the case of bounded utility functions, the no-good-deal state prices derived from the log utility are *strictly inside* the no-arbitrage triangle for all $c < +\infty$. Consequently, the no-good-deal price bounds are strictly sharper than the no-arbitrage price bounds for all $c < +\infty$.

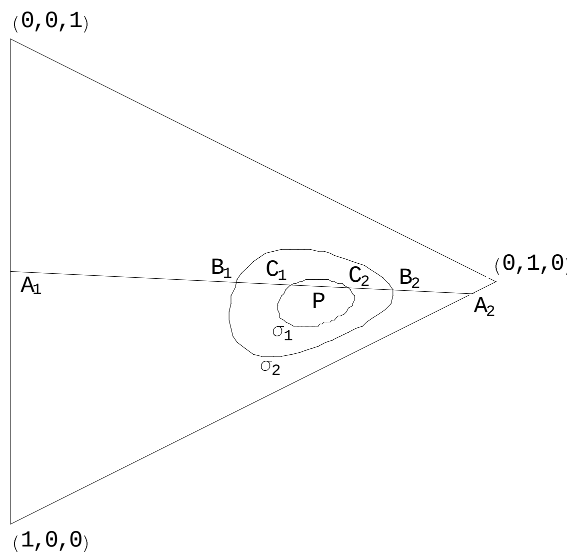

Figure 4: Admissible good-deal risk-neutral probabilities. Points C_1 and C_2 correspond to q_L and q_H from equation (32).

5 Continuous time Brownian motion setting

In continuous time we have

$$R_t = \exp\left(\int_0^t r_t dt\right),$$

the self-financing condition is written as

$$d\frac{w_t}{R_t} = d\frac{p_t}{R_t} + \frac{\delta_t}{R_t} dt, \tag{33}$$

and the discounted gain process is

$$G_t = \frac{p_t}{R_t} + \int_0^t \frac{\delta_s}{R_s} ds.$$

Suppose that the discounted gain process is an Ito process of the form

$$dG_t = \mu_t dt + \sigma_t dB_t$$

where B_t is an s-dimensional Brownian motion.

The trick of risk-neutral pricing is to write dG_t as

$$dG_t = \sigma_t(\nu_t dt + dB_t)$$

and then set

$$d\tilde{B}_t = \nu_t dt + dB_t \ .$$

The process ν_t is known as the *market price of risk.* It is a known result that the density process[8] m_t for the unconditional change of measure m_T under which $\tilde{B}_t$ is a martingale[9] is given as

$$m_t = \exp\left[-\frac{1}{2}\int_0^t ||\nu_s||^2 ds - \int_0^t \nu_s dB_s\right]. \tag{34}$$

By analogy to equation (10) we have

$$m_{t+dt|t} = \frac{m_{t+dt}}{m_t} = \exp[-\frac{1}{2}||\nu_t||^2 dt] \exp[-\nu_t dB_t]. \tag{35}$$

that is the conditional change of measure is a lognormal variable.

5.1 Instantaneous no-good-deal restrictions

The propositions below summarize one of the main findings of the paper.

Proposition 1 *The market price of risk ν_t does not admit Sharpe ratio of more than $h\sqrt{dt}$ between time t and $t+dt$ if and only if*

$$||\nu_t||^2 \leq h^2 \tag{36}$$

Proposition 2 *The market price of risk ν_t does not admit certainty equivalent gain of more than adt for a utility function U from time t until time $t+dt$ if and only if*

$$\frac{1}{2}||\nu_t||^2 \leq A(w_t)a \tag{37}$$

where $A(w_t) = -\frac{U''(w_t)}{U'(w_t)}$ is the coefficient of absolute risk aversion.

[8] The density process m_t and the discount factor Λ_t used in Cochrane and Saá-Requejo are related through $\Lambda_t = \frac{m_t}{R_t}$.

[9] Note also that the no-good-deal restrictions on the market-price-of-risk process ν_t derived in the following sections guarantee that the Novikov condition

$$\mathrm{E}_0 \exp\left[\int_0^T ||\nu_t||^2 dt\right] < +\infty$$

is satisfied and hence the density process m_t is a martingale as required.

Proof The proofs are stated in Appendix B. ■

Since our analysis was performed for small Sharpe Ratios and small certainty gains it is natural that the bounds in restrictions (36) and (37) correspond via (18). On the other hand, it is hard to see how one can use the intuition (18) to correctly derive the restriction (37) directly without referring to the ideas in Appendix B.

6 Time scaling of maximum attainable Sharpe Ratio

An interesting question is how the instantaneous no-good-deal restrictions affect availability of high Sharpe Ratios over a longer time horizon[10]. Following the discussion in the previous section we will limit our attention to E-SR and Q-SR restrictions, for which we have respectively

$$\begin{aligned} \mathrm{E}_{t-1} m_{t|t-1} \ln m_{t|t-1} &\leq \frac{1}{2} h_E^2 \\ \mathrm{E}_{t-1} m_{t|t-1}^2 &\leq 1 + h_Q^2 \end{aligned}$$

Proposition 3 *If the maximum E-Sharpe Ratio attainable over a short period dt is $h_E\sqrt{dt}$ then the maximum attainable E-SR over T periods is $h_E\sqrt{T}$.*

Proof The best attainable deal over time interval $[0, T]$ is bounded from above by

$$\mathrm{E}_0 m_T \ln m_T.$$

This expression can be written equivalently as

$$\mathrm{E}_0 \left[m_T \ln m_{T-\triangle t} + m_{T-\triangle t} \frac{m_T}{m_{T-\triangle t}} \ln \frac{m_T}{m_{T-\triangle t}} \right]$$

[10] It is of course plausible that the actual bound on the long run Sharpe Ratios is *lower* than the one implied by the instantaneous Sharpe Ratio restrictions.

and using the law of iterated expectations we have

$$
\begin{gathered}
\mathrm{E}_0 m_T \ln m_T = \\
= \mathrm{E}_0 \left[\mathrm{E}_{T-\triangle t} m_T \ln m_{T-\triangle t} + m_{T-\triangle t} \mathrm{E}_{T-\triangle t} m_{T|T-\triangle t} \ln m_{T|T-\triangle t} \right] = \\
< \mathrm{E}_0 \left[m_{T-\triangle t} \ln m_{T-\triangle t} + m_{T-\triangle t} \frac{1}{2} h_E^2 \triangle t \right] = \\
= \frac{1}{2} h_E^2 \triangle t + \mathrm{E}_0 m_{T-\triangle t} \ln m_{T-\triangle t}
\end{gathered}
$$

By induction then

$$
\mathrm{E}_0 m_T \ln m_T < \frac{1}{2} h_E^2 T
$$

■

Proposition 4 *If the maximum Q-Sharpe Ratio attainable over a short period dt is $h_Q \sqrt{dt}$ the maximum attainable SR over T periods is $\sqrt{\exp[h_Q^2 T] - 1}$.*

Proof The best attainable deal over time interval $[0, T]$ is determined by

$$
\begin{gathered}
\mathrm{E}_0 m_T^2 = \mathrm{E}_0 m_{\triangle t|0}^2 m_{2\triangle t|\triangle t}^2 \ldots m_{T-\triangle t|T-2\triangle t}^2 m_{T|T-\triangle t}^2 = \\
= \mathrm{E}_0 m_{\triangle t|0}^2 \mathrm{E}_{\triangle t} m_{2\triangle t|\triangle t}^2 \ldots \mathrm{E}_{T-2\triangle t} m_{T-\triangle t|T-2\triangle t}^2 \mathrm{E}_{T-\triangle t} m_{T|T-\triangle t}^2 < \\
< (1 + h_Q^2 \triangle t)^{\frac{T}{\triangle t}} \to \exp[h_Q^2 T]
\end{gathered}
$$

■

Figure 5 compares the long time horizon Sharpe Ratio restrictions implied by the maximum instantaneous Sharpe Ratio equal to 1. The instantaneous E-Sharpe Ratio provides a sharper bound on the attractiveness of a long term investment. One has to bear in mind, however, that there may be other economic forces that limit the long run Sharpe Ratios.

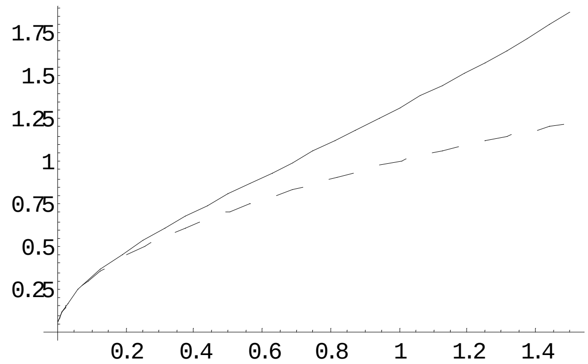

Figure 5: Maximum E-Sharpe Ratio and Q-Sharpe Ratio implied by instantaneous restrictions as a function of investment horizon. Instantaneous ratio limit set equal to 1.

7 Limiting cases of good-deal price bounds

From the identity $\sigma_t \nu_t = \mu_t$ it follows that the market price of risk has a unique decomposition

$$\begin{aligned} \nu &= \eta + \psi \\ \eta &= \sigma^*(\sigma\sigma^*)^{-1}\mu \\ \eta^*\psi &= 0 \end{aligned}$$

From here we can see that

$$||\nu||^2 = ||\eta||^2 + ||\psi||^2 \tag{38}$$

and η can be naturally called the minimal market price of risk[11]. The minimal market price of risk naturally defines the minimal martingale measure via (34).

The following proposition asserts that the good-deal price bounds obtained from instantaneous state price restrictions lie between the unique price determined by the minimal martingale measure and the no-arbitrage super-replication bounds.

[11] The minimal market price of risk defines the minimal martingale measure, see Schweizer (1991).

Proposition 5 *Consider a contingent claim C_T and let us denote $C_{NA}^{\min}$ and $C_{NA}^{\max}$ respectively its no-arbitrage price bounds, $C_{NGD}^{\min}(h)$ and $C_{NGD}^{\max}(h)$ respectively its no-good-deal price bounds corresponding to maximum instantaneous Sharpe Ratio h, and C_0 its price determined by the minimal martingale measure. Then*

$$C_{NA}^{\min} \leq C_{NGD}^{\min}(h) \leq C_0 \leq C_{NGD}^{\max}(h) \leq C_{NA}^{\max}$$

and

$$\begin{aligned}
\lim_{h \searrow ||\eta||} C_{NGD}^{\min}(h) &= \lim_{h \searrow ||\eta||} C_{NGD}^{\max}(h) = C_0 \\
\lim_{h \to \infty} C_{NGD}^{\min}(h) &= C_{NA}^{\min} \\
\lim_{h \to \infty} C_{NGD}^{\max}(h) &= C_{NA}^{\max}
\end{aligned}$$

Proof The relationship between good-deal price bounds and no-arbitrage price bounds can be read off from Theorem 3.1.1 of El Karoui and Quenez (1995). As for the relationship with the minimal martingale measure, the martingale representation theorem under the minimal martingale measure allows us to write the contingent claim C_T uniquely as

$$\begin{aligned}
C_T &= C_0 + \int_0^T \vartheta_t dG_t + \int_0^T \lambda_t dB_t, \\
\lambda_t \sigma_t^* &= 0.
\end{aligned}$$

Using the Ito formula we find the expectation of C_T under an arbitrary equivalent martingale measure

$$\mathrm{E}_0 m_T C_T = C_0 - \mathrm{E}_0 \int_0^T m_t \lambda_t^* \psi_t dt,$$

where

$$\begin{aligned}
dm_t &= -m_t(\eta_t + \psi_t) dB_t \\
m_0 &= 1.
\end{aligned}$$

Consequently the lower no-good-deal price bound is obtained as

$$C_{NGD}^{\min}(h) = \min_{||\psi_t||^2 \leq h_t^2 - ||\eta_t||^2} C_0 - \mathrm{E}_0 \int_0^T m_t \lambda_t^* \psi_t dt \leq C_0.$$

At the same time as $h_t \searrow ||\eta_t||$ we have $||\psi_t|| \to 0$ and $C^{\min}_{NGD}(h) \to C_0$. Analogous argument applies to the upper bound. ■

It is interesting to note that the minimal martingale measure has already been used to price non-redundant claims under stochastic volatility in Hofmann, Platen, and Schweizer (1992).

8 Conclusions

The paper provides a generalization of the incomplete market pricing technique of Cochrane and Saá-Requejo (2000) to good deals defined by an arbitrary (increasing) smooth utility function. We have derived the corresponding discount factor restrictions and linked these restrictions to the availability of Sharpe Ratios and Generalized Sharpe Ratios. We have established general properties of good-deal price bounds for Ito price processes.

Appendix A – Arbitrage-adjusted Sharpe Ratio

Suppose the excess return X has a piecewise absolutely continuous cumulative distribution function F. From (25)

$$h_Q(X) = \frac{1}{\max_\lambda \int_{-\infty}^{\frac{1}{\lambda}}(-1+2\lambda x - \lambda^2 x^2)dF(x)} - 1 \text{ for } \lambda > 0. \tag{A.1}$$

Let us examine the maximization in the denominator. The integral is well defined as long as $\int_{-\infty}^{0} x^2 dF(x)$ is finite, thus a necessary and sufficient condition for its existence is finite variance of $X_- \triangleq -\min(X, 0)$. Let us now calculate the formal derivatives with respect to λ

$$\frac{\partial}{\partial\lambda}\int_{-\infty}^{\frac{1}{\lambda}}(-1+2\lambda x-\lambda^2x^2)dF(x) = 2\int_{-\infty}^{\frac{1}{\lambda}}(x-\lambda x^2)dF(x) \tag{A.2}$$

$$\frac{\partial^2}{\partial\lambda^2}\int_{-\infty}^{\frac{1}{\lambda}}(-1+2\lambda x-\lambda^2x^2)dF(x) = -2\int_{-\infty}^{\frac{1}{\lambda}}x^2dF(x). \tag{A.3}$$

By §7.3 Theorem 11 in Widder (1989) the interchanges of differentiation and integration are warranted. Equation (A.2) implies that with $\int_{-\infty}^{+\infty} x dF(x) > 0$

(A.1) attains global maximum at $\lambda^* > 0$. When $\int_{-\infty}^{+\infty} x dF(x) < 0$ truncation proceeds from the other end, formally we apply the procedure above to $-X$.

If we realize that $\frac{1}{\lambda} = x_{\max}$ the first order condition implies

$$x_{\max} \int_{-\infty}^{x_{\max}} x dF(x) = \int_{-\infty}^{x_{\max}} x^2 dF(x), \tag{A.4}$$

which can be restated in terms of the capped distribution as follows

$$\begin{aligned} x_{\max} \left[\int_{-\infty}^{x_{\max}} x dF(x) + x_{\max}(1 - F(x_{\max})) \right] &= \int_{-\infty}^{x_{\max}} x^2 dF(x) + x_{\max}^2 (1 - F(x_{\max})), \\ x_{\max} \mathrm{E} \min(X, x_{\max}) &= \mathrm{E} \left[\min(X, x_{\max}) \right]^2 . \end{aligned}$$

The same trick can be used to show that (A.1) is in fact equal to the Sharpe Ratio of the capped distribution.

Our task now is to evaluate (A.4) for a lognormally distributed return. Let us write

$$X = e^{\mu + \sigma Z} - e^r,$$

where Z is standard normal variable, r is risk-free rate of return, expected risky return is $e^{\mu + \frac{s^2}{2}}$ and the variance of risky return is $e^{2\mu + 2s^2} - e^{2\mu + s^2}$. We first recall an auxiliary result

$$\int_{-\infty}^{z_{\max}} e^{\alpha + \beta z} d\Phi(z) = e^{\alpha + \frac{\beta^2}{2}} \Phi(z_{\max} - \beta)$$

which follows easily by direct integration or by referring to Black-Scholes formula. We apply this result repeatedly with

$$z_{\max} = \frac{\ln(x_{\max} + e^r) - \mu}{\sigma}$$

to obtain

$$\begin{aligned} \int_{-\infty}^{x_{\max}} x dF_X(x) &= \int_{-\infty}^{z_{\max}} (e^{\mu + \sigma z} - e^r) d\Phi(z) = \\ &= e^{\mu + \frac{\sigma^2}{2}} \Phi\left(z_{\max} - \sigma\right) - e^r \Phi(z_{\max}) \\ \int_{-\infty}^{x_{\max}} x^2 dF(x) &= \int_{-\infty}^{z_{\max}} (e^{\mu + \sigma z} - e^r)^2 d\Phi(z) = \\ &= e^{2\mu + 2\sigma^2} \Phi\left(z_{\max} - 2\sigma\right) - 2 e^{r + \mu + \frac{\sigma^2}{2}} \Phi\left(z_{\max} - \sigma\right) + e^{2r} \Phi(z_{\max}). \end{aligned}$$

The first order condition therefore reads

$$x_{\max}\left[e^{\mu+\frac{\sigma^2}{2}}\Phi\left(z_{\max}\right)-e^{r}\Phi(z_{\max})\right]=$$
$$=e^{2\mu+2\sigma^2}\Phi\left(z_{\max}-2\sigma\right)-2e^{r+\mu+\frac{\sigma^2}{2}}\Phi\left(z_{\max}-\sigma\right)+e^{2r}\Phi(z_{\max}).$$

To solve it one has to perform a straightforward numerical search over $x_{\max}$.

Appendix B

Recall that the conditional change of measure for Ito gain process is of the form

$$m_{t+dt|t}=\exp[-\frac{1}{2}||\nu_t||^2dt]\exp[-\nu_t dB_t]. \tag{B.1}$$

It is useful to mention the following elementary result[12],

$$\mathrm{E}_t\exp[-\nu_t dB_t]=\exp[\frac{1}{2}||\nu_t||^2dt] \tag{B.2}$$

Further by differentiating both sides with respect to ν_t we obtain auxiliary results

$$\mathrm{E}_t dB_t\exp[-\nu_t dB_t] = -\nu_t^* dt\exp[\frac{1}{2}||\nu_t||^2dt] \tag{B.3}$$
$$\mathrm{E}_t dB_t dB_t^T\exp[-\nu_t dB_t] = \left[Idt+\nu_t\nu_t^*(dt)^2\right]\exp[\frac{1}{2}||\nu_t||^2dt] \tag{B.4}$$

From the aforementioned identities it follows for example that

$$\mathrm{E}_t m_{t+dt|t}=\exp[-\frac{1}{2}||\nu_t||^2dt]\exp[\frac{1}{2}||\nu_t||^2dt]=1+o(dt), \tag{B.5}$$

as one would expect from the conditional change of measure, and that

$$\mathrm{E}_t m_{t+dt|t}^2=\exp[-||\nu_t||^2dt]\exp[2||\nu_t||^2dt]=1+||\nu_t||^2dt+o(dt). \tag{B.6}$$

Note that the formulae (B.2)-(B.4) are *exact* since ν_t is by assumption $\mathcal{F}_t$-measurable.

Proof of Proposition 1 We can write the restriction (14) as

$$\mathrm{E}_t m_{t+dt|t}^2\leq 1+h^2dt$$

[12] It is in fact formula for the moment generating function of a multivariate standard normal variable.

and evaluate the left hand side using the expression (B.6) to obtain

$$\begin{aligned} 1 + ||\nu_t||^2 dt &\leq 1 + h^2 dt \\ ||\nu_t||^2 &\leq h^2 \end{aligned}$$

■

In deriving the instantaneous no-good-deal restrictions it is natural to take the expression for the conditional change of measure (B.1), substitute it into the good-deal restrictions (14)-(17) and evaluate the resulting expectations using formulae (B.2)-(B.4). Having obtained our results in this way the first time around, we observed that the good-deal restrictions depend on the utility function only through the coefficient of absolute risk aversion. To show this result in full generality the style of the proof has to be change compared to that of Proposition 1.

Proof of Proposition 2 Define process z_τ as follows

$$z_\tau \equiv m_{t+\tau|t} = 1 - \int_t^{t+\tau} z_s \nu_s dB_s,$$

that is z_τ represents the conditional change of measure starting at time t. We know that the optimal wealth satisfies

$$w_{t+dt} = I(\lambda z_{dt})$$

and that λ is found from the condition

$$\mathrm{E}_t z_{dt} I(\lambda z_{dt}) = (1 + r_t dt) w_t + o(dt)$$

Using the Ito's lemma we find

$$\mathrm{E}_t z_{dt} I(\lambda z_{dt}) = I(\lambda) + \left[\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda)\right] ||\nu_t||^2 dt + o(dt)$$

and hence

$$(1 + r_t dt) w_t = I(\lambda) + \left[\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda)\right] ||\nu_t||^2 dt + o(dt) \tag{B.7}$$

Now we use the Ito's formula again to find $\mathrm{E}_t U(w_{t+dt})$

$$\begin{aligned} \frac{d^2}{dz^2} U[I(\lambda z)] &= \lambda^2 I'(\lambda z) + \lambda^3 z I''(\lambda z) \\ \mathrm{E}_t U(w_{t+dt}) &= U[I(\lambda)] + \frac{1}{2}\left[\lambda^2 I'(\lambda) + \lambda^3 I''(\lambda)\right] ||\nu_t||^2 dt + o(dt) \end{aligned} \tag{B.8}$$

The good-deal restriction is

$$\mathrm{E}_t U(w_{t+dt}) \leq U[(1 + r_t dt)w_t + adt]$$

Substituting from expression (B.7) and using Taylor expansion we obtain

$$\mathrm{E}_t U(w_{t+dt}) \leq U[I(\lambda)] + U'[I(\lambda)] \left\{ \left[\lambda I'(\lambda) + \frac{1}{2}\lambda^2 I''(\lambda) \right] ||\nu_t||^2 + a \right\} dt + o(dt)$$

Finally, substitution for $\mathrm{E}_t U(w_{t+dt})$ from equation (B.8) shows that the good-deal restriction becomes

$$-\frac{1}{2}\lambda I'(\lambda)||\nu||^2 \leq a + O(dt). \tag{B.9}$$

Differentiating both sides of the identity $U'[I(\lambda)] = \lambda$ we obtain

$$\begin{aligned} U''[I(\lambda)]I'(\lambda) &= 1 \\ -\lambda I'(\lambda) &= -\frac{U'[I(\lambda)]}{U''[I(\lambda)]}. \end{aligned}$$

Since equation (B.7) implies $I(\lambda) = w_t + O(dt)$ we have

$$-\lambda I'(\lambda) = \frac{1}{A(w_t)} + O(dt)$$

and the good-deal restriction (B.9) is shown to be of the form

$$\frac{1}{2}||\nu_t||^2 \leq A(w_t)a.$$

■

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