Pricing Illiquid Assets

Miklós Koren and Ádám Szeidl (YE)

September 2001

Abstract

The present paper investigates the portfolio allocation decisions of an investor with infinite horizon when available financial assets differ in their degrees of liquidity. A model with risk neutral agents allows us to endogenously determine the liquidity premium. With risk averse agents, we develop a nontrivial portfolio allocation problem, which enables us to calculate the demand for an illiquid asset for any given yield premium. We calibrate and numerically simulate both models. Reasonable parameter values imply a liquidity premium of 1.7% for the risk neutral case. In the portfolio allocation problem we find that a reasonable amount of illiquidity can cause a substantial drop of demand for the asset. We are also able to calculate the price discount at which an agent would be indifferent between immediate sale and waiting for a buyer with a fundamentally justified price.

JEL Classification: *G11, G12*

Keywords: *Liquidity, Financial Markets, Asset Pricing, Portfolio Choice*

Harvard University, Department of Economics, Cambridge, Massachusetts.

E-mail: koren@fas.harvard.edu and szeidl@fas.harvard.edu.

1. Introduction*

The purpose of this paper is to examine portfolio investment decisions when available financial asset markets differ in their degrees of liquidity. It is a common observation that liquidity properties vary across financial markets. In our view liquidity is one of the fundamental determinants of portfolio investment decisions, comparable in its importance to more standard and better studied determinants, such as risk and return (consumption CAPM, see Lucas, 1978, Merton, 1971, for instance).

Throughout we assume that liquidity refers to the ease with which an asset can be sold. Thus an asset (or asset market) is liquid if trade can take place at short notice in large quantities without a substantial price change. Out of these three aspects of liquidity, that is at short notice, in large quantities and without a substantial discount, we will mostly focus on the first one. The source of illiquidity of an asset will be the difficulty of finding a trading partner immediately when the necessity of trade is realized by the agent. Since trading is not always possible, one expects that in equilibrium a less liquid asset will carry a higher yield premium relative to a more liquid one.

In this paper we maintain the view that the utility possibilities set is not time invariant, rather during certain periods consumption is more rewarding than at other times. One reason might be that consumption is not infinitely divisible, but takes place in large pieces. In that case, the availability of a large piece of good at a reasonable price might correspond to the aforementioned liquidity shock. Alternatively, the utility stemming from a given level of consumption may vary over the life cycle (marriage, children, vacation). A third argument would be that from time to time agents have a favorable private investment opportunity. In short, agents are sometimes faced with liquidity shocks (Diamond and Dybvig, 1983), and for the duration of the shock consumption has higher marginal utility.

Given the presence of liquidity shocks and the difficulty of selling illiquid assets immediately, agents will face a non-trivial trade off in their portfolio allocation decisions. Namely, holding a less liquid asset will yield a higher premium; on the other hand, it makes the agent more vulnerable to liquidity shocks, because when a shock comes, she finds it more difficult

* Prepared for the 2002 Royal Economic Society Conference in Warwick. Presenting author: Ádám Szeidl (YE). We are grateful to Péter Benczúr, Laurent Calvet, John Campbell, David Laibson, János Vincze and participants of the CEPR Transition Economics Summer Workshop for Young Academics (Portorož, June 27–July 5, 2001) and the Central European University Workshop on Liquidity of Financial Markets (Budapest, September 3, 2001) for helpful comments and suggestions. Any remaining errors are ours. This research is supported by a grant from CERGE–EI Foundation under a program of the Global Development Network. The views expressed herein are ours and have not been endorsed by CERGE–EI or the GDN.

to liquidate asset holdings. The analysis of this trade off is the main purpose of the present paper.

A prominent example of illiquid financial markets is emerging country bond markets. It is a general observation that government bonds issued by developing country governments are sold at significant discounts compared to those issued by developed country governments. Consumption based theories have difficulties in explaining such a yield premium, as the covariance of these returns with world or "rich world" consumption is negligible. An alternative explanation could be based on other characteristics of these markets, such as their relative illiquidity compared to developed country treasury markets. One study in this direction is Benczúr (2001), who empirically attempts to identify part of the premium that is caused by illiquidity on emerging country government bonds. Motivated by this study, we aim at calibrating the liquidity premium in our main model to these empirical estimates, and examine what kind of portfolio investment behavior it applies. In this sense, our approach is not dissimilar to Merton (1971), who studies the portfolio decision problem of a representative investor under uncertainty, and calibrates the model to the empirically relevant risk premium (equity premium). Insofar as stock markets are illiquid, the present approach might also have some value added concerning the equity premium puzzle (Mehra and Prescott, 1985). As to illiquid non-financial markets, we believe the most prominent example is the real estate market. Although our focus is more on financial markets, the present approach can be readily applied, indeed calibrated to real estate markets. (See Williams, 1995 for a recent asset pricing model of real assets.)

We are not aware of any other paper in the literature that studies portfolio investment under liquidity risk. However, there are a number of papers that examine the causes and consequences of market liquidity or the lack of it. Holmström and Tirole (1998) present a liquidity based asset pricing model where the intertemporal marginal rate of substitution of corporate entities governs the liquidity premium (similarly to consumption CAPM). This paper is related to our baseline model, to be presented in the next section; but it does not study portfolio investment decisions like we do in Section 3. Lippman and McCall (1986) characterize liquidity as the expected waiting time before trade provided that trade is optimal. Implicit in their setting is that trade is sometimes suboptimal; this paper makes the harsher assumption that trade is sometimes impossible. However, rational agents do not trade suboptimally, thus our impossibility requirement is only seemingly stronger than theirs. Again they do not study portfolio decisions. There is a considerable amount of literature on market microstructure, that studies the possible determinants of liquidity; Campbell, Lo and MacKinley (1997) is a good overview.

The main driving force of our model is the recurring needs for liquidity. There are several ways to formalize these needs. In a model by Baldwin and Meyer (1979), investment opportunities with different rate of return arrive at stochastic intervals. This formulation is very much similar to what we adopt in Section 2. Liquidity shocks were introduced in Diamond and Dybvig (1983) as taste shocks, since then their basic setting has become a standard in the literature. A Diamond–Dybvig type liquidity shock in period t implies that the marginal rate of substitution between that period and any other becomes infinite, so that the agent wants to consume all her wealth immediately. The liquidity shocks of the present paper are less stark then this; a liquidity shock does switch the intertemporal marginal rate of substitution, but it will always be finite (excluding zero consumption).

The least standard assumption of the present paper is that trade is sometimes impossible. We tend to think about this assumption as a parable that represents one aspect of illiquidity. One expects that no asset is impossible to sell if the price discount is large enough. However, if the discount is too high, it may be optimal to wait, as in Lippman and McCall (1986). Additionally, the present framework will allow us to calculate the discount on a hypotetical immediate sale that makes the agent exactly indifferent between trading immediately and waiting for a potential counterparty with whom trade takes place at the "correct" price. In this sense the present approach is capable of calculating the price discount corresponding to instant sale due to liquidity reasons; thus it can be interpreted as a first step in endogenizing the liquidity or noise traders present in several models of financial markets (e.g., Kyle, 1985). The reason why we focus on one particular aspect of illiquidity is mainly technical. The assumption that trade is possible only occasionally makes our analysis more tractable. Nevertheless, we feel that introducing a price discount would not change qualitatively our results. Indeed, in our framework waiting is equivalent to some price discount as the above reasoning shows. Although the fact that trading is not always possible is exogenously imposed in the model, Appendix A demonstrates how it might endogenously arise. The idea is that of adverse selection; under certain informational conditions a market may fail to operate. A similar story could be incorporated into our main model too, but it would unnecessarily complicate the analysis, and our main focus is more on the consequences of illiquidity on investment behavior.

The issue of liquidity that we study here is also related to models of portfolio choice with *transaction costs*. If illiquidity means that immediate sale incurs a discount relative to the fundamentally justified price of the asset, this is technically similar to having a proportional transaction cost on trading the illiquid asset. Here we discuss some of the transaction cost literature relevant to our problem. Constantinides (1986) considers a portfolio allocation problem with two assets, one of which is traded with a proportional transaction cost. He shows that even high transaction costs have little effect on the expected yield of the "illiquid" asset because trade only occurs infrequently and in only infinitesimal amounts. Dumas and Luciano (1991) analyze a different framework, in which the investor only consumes at a terminal date, there are thus less effects shifting the portfolio balance away from the optimal. In their model, trade is even more infrequent, making the impact of transaction costs on returns small. The effect of transaction cost in these models is of second order because portfolio balance evolves continuously and the trade necessary to restore the balance to optimal is rare and small. In our model, liquidity events call for a sudden and large adjustment in the portfolio, thereby raising the loss from illiquidity. Our framework is thus similar to that of Grossman and Laroque (1990), where transaction costs of adjusting the level of durable consumption causes the investor to trade infrequently and in large amounts. In contrast to the above papers, Grossman and Laroque find that transaction costs have a substantial effect on the demand for financial assets (and, in equilibrium, on their return). Instead of real transaction costs, Gabaix and Laibson (2001) study the effect of *decision costs* on portfolio allocation.

The rest of this paper is organized as follows. Section 2 presents our baseline model of investment under illiquidity. The baseline model will involve risk neutral agents, therefore the issue of portfolio investment is extremely simple. Nevertheless, this model will be useful for two reasons. The first is that it is analytically solvable in closed form. The second is that consumption based pricing will imply an unambiguous liquidity premium on the illiquid asset, a property we shall lose when we later move to risk averse agents. This section will also present some elementary calibration results, and we find that reasonable parameter values can imply a non-negligible (1-2%) liquidity premium. Section 3, in its turn, presents a similar model with risk averse agents. Once the utility function is concave, the portfolio decision problem of the agent becomes nontrivial. Therefore we have to rely on numerical simulations. Our simulation approach is based on the iteration of the Bellman equation. We also attempt to calibrate the model with reasonable figures and study the implications. Finally, Section 4 concludes. Appendix A presents a simple model that endogenizes our assumption about trade not being always possible. Appendix B contains the details of the numerical simulation of the main model.

2. Risk Neutral Agents

We develop a simple partial equilibrium continuous-time model of asset liquidity and study the endogenously arising liquidity premium. The agents in our model are risk neutral, and maximize expected lifetime discounted utility stream

$$\mathrm{E}_0 \int_0^\infty e^{-\rho t} c_t \,\mathrm{d}t$$

where c_t is the instantenous consumption flow at time t, ρ is the rate of time preference, and E_0 denotes expectation formed at time 0. There are two assets and one consumption good (the numeraire) in this economy. The liquid asset is instantly tradeble at any point in time, and one unit of it has a market price of e^{rt} at time t (in terms of the consumption good). The model will endogenously pin down the equilibrium value of r.

The "illiquid" asset provides an infinite dividend stream of one unit of the consumption good at each time instant. This asset is illiquid in the sense that buyers are not necessarrily available at any point in time. More precisely, potential buyers of the asset arrive according to the jumps of a Poisson process. Thus a seller might be in a situation where there is no instant demand for her asset supply. However, the asset can be bought at any point in time at the endogenously determined market price p.

Additionally, during certain favorable time periods, an agent has an alternative investment opportunity, which is more attractive than the liquid asset. This opportunity again arrives at a Poisson rate, and is available for a random, exponential time period (the parameters of the three exponential distributions can differ). While it is available, this opportunity provides a dividend stream of y units of the consumption good per each unit invested. We assume that y is large enough such that the return on this investment dominates both assets in the economy.

The alternative investment opportunity is not meant to be taken literally and can be interpreted in several ways. For instance the agent may have abrupt liquidity needs from time to time due to some exogenous reason. In that case, y would represent the flow cost of not obtaining sufficient liquidity. Technically speaking, the alternative investment is a means of introducing a time varying random pattern of liquidity needs for the agent. To be consitent, we will use the favorable investment opportunity interpretation below.

The two assets in the economy are priced to the consumption good; we assume that whenever there is no alternative investment, the agent is indifferent between holding the liquid asset, holding the illiquid asset or consuming all of her wealth. Since the agent is risk-neutral, this equilibrium condition also corresponds to the possibilty of an interior equilibrium.

Intuitively, if there was no alternative investment in the model, the agent would be indifferent between consumption and investment in the liquid asset only if r was equal to ρ. However, the possibility of a favorable investment at a future date would then make the liquid asset more attractive relative to consumption; in order to compensate for this affect we expect that in equilibrium $r < \rho$. To put it differently, the liquid asset has a lower return because it provides a means of preserving wealth for future liquidity needs. The difference $\rho - r$ can be interpreted as a discount on the yield of the liquid asset. This yield discount arises from the asset being a means of storing value and having the capacity of instant liquidity provision when needed. This yield discount corresponds the liquidity discount in the example of Holmström and Tirole (1998).

In contrast, the illiquid asset cannot be sold immediately should good times come. This implies that the agent may incur an opportunity cost because of not being able to invest in the alternative when it is available, as she cannot liquidate her asset holdings. To compensate for this opportunity cost, we expect that in equilibrium the illiquid asset commands a higher return than the liquid; that is, $1/p > r$. The difference between the two, $1/p - r$, is what we will call the liquidity premium in this model.

In the following we turn to formally characterize the equilibrium and in particular derive prices and returns.

2.1. SOLUTION

We determine the equilibrium by means of dynamic programming. First let us solve for the equilibrium value of r, the return on the liquid asset. Let V^{L} (L for "liquid") denote the lifetime utility stemming from holding one unit of the liquid asset, given that there is no current alternative investment opportunity. In other words, this is the utility along the optimal consumption plan of an agent who has a single unit of the liquid asset. Since the utility function is homogenous of degree one, holding several units of the asset yields proportionally higher utility.

Similarly, V^{E} (E for "enjoy") denotes the lifetime value of holding one unit of the alternative asset, given that it is available. This value also has the homogeneity property in the quantity of the asset.

We have the following Bellman equations.

$$\rho V^{\mathrm{L}} = rV^{\mathrm{L}} + \mu(V^{\mathrm{E}} - V^{\mathrm{L}}) \tag{1a}$$

$$\rho V^{\mathrm{E}} = y + \lambda(V^{\mathrm{L}} - V^{\mathrm{E}}) \tag{1b}$$

The first equation says that the subjective loss from holding the asset for one more instant should equal the instantenous capital gain on the asset (rV^{L}) plus the expected gain from being able to buy the alternative asset. The alternative opportunity arrives with instantenous probability μ (that is, μ is the parameter of the exponential distribution) and yields a total utility of V^{E}. The second equation can be interpreted similarly. In that case the dividend y is assumed to be a flow of utility, in particular, it cannot be reinvested. The hazard rate corresponding to the termination of the alternative opportunity is denoted by λ.

In equilibrium the agent is indifferent whether to hold the liquid asset or sell it and consume the proceeds immediately. This implies that

$$V^{\mathrm{L}} = 1. \tag{2}$$

Equations (1a), (1b) and (2) can be solved for V^{L}, V^{E} and r. The formula for r is

$$r = \rho - \frac{\mu(y-\rho)}{\lambda+\rho}. \tag{3}$$

As long as $y > \rho$, that is, the alternative investment is preferred to immediate consumption, we have $r < \rho$. This corresponds to the yield discount on the liquid asset anticipated in the previous section. The comparative statics of the discount is straightforward. An increase in μ

corresponds to more frequent arrival of the alternative, hence a larger discount. An increase in λ shortens the duration of the alternative, thus lowering the discount. Finally, an increase in y raises the discount.

Let us now turn to the illiquid asset. Again, we make use of dynamic programming. The owner of an illiquid asset can find herself in three possible states. The first is when she holds the asset without having the alternative. Let V^{N} (N for "no alternative") denote her value in this state per unit of asset held. The second state is when she has an alternative but is unable to liquidate her asset as there is no potential buyer. Her value in this state V^{W} (W for "wait") per unit of asset. The third state corresponds to holding the alternative asset. The agent's value in this states is V^{E} per unit of the alternative asset. As the reader may expect, this value will be the same as the V^{E} defined in (1b). Finally, let p denote the time invariant price of the illiquid asset in terms of the consumption good.

The Bellman equations are as follows.

$$\rho V^{\mathrm{N}} = 1 + \mu(V^{\mathrm{W}} - V^{\mathrm{N}}) \tag{4a}$$

$$\rho V^{\mathrm{W}} = 1 + \eta(pV^{\mathrm{E}} - V^{\mathrm{W}}) + \lambda(V^{\mathrm{N}} - V^{\mathrm{W}}) \tag{4b}$$

$$\rho V^{\mathrm{E}} = y + \lambda(V^{\mathrm{N}}/p - V^{\mathrm{E}}) \tag{4c}$$

The first equation says that the instantenous dividend plus the expected gain in value due to the arrival of the alternative should compensate the agent for holding the asset for another instant. The second equation refers to the state when the agent is waiting for a buyer. In that case the last two terms on the right hand side represent the expected gains from finding a buyer (with instantenous probability η) respectively the expected loss of losing the alternative (with probability λ). Since one unit of the illiquid asset buys p units of the alternative, we have pV^{E} in the second term. The third equation can interpreted similarly.

Our equilibrium condition is that when there is no alternative, the agent is indifferent between buying one unit of the illiquid asset and consuming the price. This implies

$$p = V^{\mathrm{N}}. \tag{5}$$

Solving the system of equation (4a)-(4c) and (5) implies that the equilibrium value of p is

$$p = \frac{(\lambda+\rho)(\eta+\lambda+\mu+\rho)}{\rho(\eta+\lambda+\rho)(\lambda+\mu+\rho) - \eta\mu y}. \tag{6}$$

The yield premium on the illiquid asset, $1/p - r$, equals

$$\pi \equiv \frac{1}{p} - r = \frac{\mu(y-\rho)(\lambda+\mu+\rho)}{(\lambda+\rho)(\eta+\lambda+\mu+\rho)}. \tag{7}$$

Note that as long as $y > \rho$, the liquidity premium is positive. In the following, we will assume that this holds.

The comparative statics are as follows. An increase in y makes the alternative more attractive thereby raising the liquidity premium. An increase in μ corresponds to more frequent arrival of the alternative and hence raises the premium. A higher λ represents shorter expected duration of the alternative investment opportunity, thus lowering the gains from liquidity. Therefore the premium is smaller. A rise in η means that buyers arrive more often, thus the premium decreases. Also note that we have

$$\lim_{\eta \to \infty} \pi = 0, \tag{8}$$

that is, if a buyer is available immediately then there is no premium.

2.2. CALIBRATION

To get a sense of magnitudes, we tried to calibrate the model. The following parameter values will serve as a benchmark. Time is measured in years.

ρ	μ	λ	η	y
0.1	2	26	120	1.27

These values imply $r = 0.010$ and $\pi = 0.017$, that is, a real rate of return of approximately 1% and a liquidity premium of 1.7%. The parameter values can be interpreted as follows.

A discount rate of $\rho = 0.1$ leads to a discount factor of 0.90, which seems like a fair number for one year. The arrival rate of the alternative, $\mu = 2$, means that liquidity events occur every six months on average. A λ of 26 means that the expected duration of the alternative is two weeks. For $\eta = 120$, buyers arrive every three days on average. Finally, $y = 1.27$ implies that whenever the liquidity event occurs, a one-day delay in satisfying the liquidity needs leads to a loss equivalent to 0.3% of the amount required.

The next step is to check the sensitivity of the results. Our strategy is the following. Out of the five parameters, we fix three at the benchmark values and vary the fourth one. The fifth is chosen such that the implied real interest rate, $r = 0.01$. The implicitly determined fifth parameter will always be y.

Figures 1 through 4 display the results of our calibration. We find that even a small amount of illiquidity implies a non negligible liquidity premium. This finding seems to be robust across parameters.

3. Portfolio Choice

With risk neutral agents, the problem of portfolio choice is extremely simple: in equilibrium agents have to be indifferent to holding the two different assets otherwise the demand for the less preferred asset becomes zero. If the consumer is risk averse, however, then the portfolio choice becomes non-trivial. In particular, for every price differential (or, equivalently, yield premium) there is a well-defined relative demand for the two assets.[1] Then the relative supply is needed to pin down the price difference. In this section we derive the relative demand for the liquid and the illiquid assets.

We adopt the standard constant relative risk aversion (CRRA) utility function, $C^{1-\theta}/(1-\theta)$, because it will allow us to abstract from the wealth level of the consumer and concentrate on the relative holdings of the two assets. Then the value function of the consumer's maximization problem is

$$V(B_0, M_0, \chi_0) = \max_{\{C_t\}} \mathrm{E}_0 \int_0^\infty \chi_t^\theta \frac{C_t^{1-\theta}}{1-\theta} \exp(-\rho t)\, \mathrm{d}t, \tag{9}$$

where B_0 is the amount of bond, M_0 is the amount of cash held at time 0, C_t is total consumption at time t, and χ_t is a time-dependent taste parameter, $\chi_t = 1$ at "normal" times and $\chi_t = \gamma > 1$ at liquidity events. $\gamma > 1$ means that the marginal utility of consumption is higher in case of liquidity events than at normal times so the consumer wishes to consume more than at normal times. In the following we will denote $V(B, M, 1)$ by $V^{\mathrm{N}}(B, M)$ (N for "normal") and $V(B, M, \gamma)$ by $V^{\mathrm{E}}(B, M)$ (E for "event").

The evolution of state variables is given by

$$\begin{aligned} \mathrm{d}M &= [(r+\pi)B + rM - C]\, \mathrm{d}t + S, \\ \mathrm{d}B &= -S. \end{aligned}$$

During a $\mathrm{d}t$ length of time, the bond pays a coupon of $r+\pi$ per unit, and the cash pays interest r (π is the "liquidity premium"). Only cash holdings can be consumed, otherwise there would be no liquidity problem. At times of trade, the agent can buy and sell both cash and bond but cannot borrow or short-sell any of the assets ($M, B > 0$). This means the agents are liquidity constrained and cannot consume more than the amount of cash at hand. The amount of bond sold is denoted by S ($\gtrless 0$). Similarly to the framework in the previous section, trading opportunities arrive at an exponential rate, η.

Note that the bond is assumed to pay the coupon in cash. This ensures that the amount of bond between trades is constant, which will simplify the analysis.

1. We will refer to the liquid asset as "cash" and to the illiquid asset as "bond." The reader should bear in mind the alternative interpretations outlined in the introduction.

As in the previous section, we assume that liquidity events ($\chi_t = \gamma$) arrive according to a Poisson process with arrival rate μ, and end with a rate λ. Then we have the following Bellman equation for the two states.

$$\begin{aligned}\rho V^{\mathrm{N}}(B,M)\,\mathrm{d}t = \max_{C<M,S} \frac{C^{1-\theta}}{1-\theta}\,\mathrm{d}t + V_M^{\mathrm{N}}(B,M)[(r+\pi)B+rM-C]\,\mathrm{d}t \quad &(10a)\\ +\eta\,\mathrm{d}t\left[V^{\mathrm{N}}(B-S,M+S)-V^{\mathrm{N}}(B,M)\right]&\\ +\mu\,\mathrm{d}t\left[V^{\mathrm{E}}(B,M)-V^{\mathrm{N}}(B,M)\right]&\\ \rho V^{\mathrm{E}}(B,M)\,\mathrm{d}t = \max_{C<M,S} \gamma^{\theta}\frac{C^{1-\theta}}{1-\theta}\,\mathrm{d}t + V_M^{\mathrm{E}}(B,M)[(r+\pi)B+rM-C]\,\mathrm{d}t \quad &(10b)\\ +\eta\,\mathrm{d}t\left[V^{\mathrm{E}}(B-S,M+S)-V^{\mathrm{E}}(B,M)\right]&\\ +\lambda\,\mathrm{d}t\left[V^{\mathrm{N}}(B,M)-V^{\mathrm{E}}(B,M)\right]&\end{aligned}$$

where V_M^k is $\frac{\partial}{\partial M}V^k(B,M)$ in state k.

In optimum, the subjective loss from waiting another instant is perfectly offset by the flow utility from consumption and the expected capital gain. The amount of cash changes by $(r+\pi)B+rM-C$ at any instant of time changing the value function by $V_M\,\mathrm{d}M$.[2] With instantenous probability η, a trade occurs, and the portfolio is rebalanced optimally. With probability μ, the state switches from normal to liquidity event state from which it switches back with probability λ. Since the change in cash is of order $\mathrm{d}t$, we can ignore it as $\mathrm{d}t \to 0$ in the case of trade and the two state changes, because the probability of these events is of order $\mathrm{d}t$, too.

It is straightforward to show that both value functions are homogenous of degree $1-\theta$. Let us introduce the following notations: $m = M/B$, $c = C/B$, $s = S/B$, $v_{\mathrm{N}}(m) = V^{\mathrm{N}}(1,m)$, and $v_{\mathrm{E}}(m) = V^{\mathrm{E}}(1,m)$. For this we need that B is nonzero, which is shown by the following argument.

Suppose that the consumer never buys any bonds when she is faced with a trade opportunity, that is, $B_t \equiv 0$. The the problem is a modified "eat-the-pie" problem, in which the amount of cash is consumed gradually over the infinite horizon. Because zero consumption is prohibitively undesirable (recall that $u'(c) \to \infty$ as $c \to 0$) and the there is no other source of income, there is always an amount of cash left over, $C < M$. Then the loss from buying a small amount of bond is only of second order, because it does not constrain the consumer in her consumption possibilities. The gain, however, is of first order, since the bond yields a positive premium over cash. Hence it is not optimal to hold no bond.

By the homogeneity property, we have that

$$\begin{aligned} V(B,M) &= v(m)B^{1-\theta}\\ V_M(B,M) &= v'(m)B^{-\theta}\end{aligned}$$

2. Recall that M is of bounded variation and thus no higher order terms enter.

for both value functions. Factoring out $B^{1-\theta}$, we get the following Bellman equation for the cash–bond share.

$$
\begin{aligned}
(\rho+\eta+\mu)v_{\mathrm{N}}(m)B^{1-\theta} = \max_{c<1,s} \frac{c^{1-\theta}}{1-\theta}B^{1-\theta} + v'_{\mathrm{N}}(m)[r(1+m)+\pi-c]B^{1-\theta} \\
+\eta v_{\mathrm{N}}(m^*_{\mathrm{N}})(1-s)^{1-\theta}B^{1-\theta} + \mu v_{\mathrm{E}}(m)B^{1-\theta}
\end{aligned}
\tag{11}
$$

where m^*_{N} denotes the optimal cash–bond ratio after trade:

$$
m^*_{\mathrm{N}} = \frac{M+S}{B-S},
$$

After rewriting the traded amount,

$$
S = B\frac{m^*-m}{m^*+1},
$$

we get the following maximum problem given that trade is possible:

$$
\max_{m^*} v(m^*)\left(\frac{m+1}{m^*+1}\right)^{1-\theta} B^{1-\theta} \equiv \max_{m^*} \frac{v(m^*)}{(m^*+1)^{1-\theta}}(M+B)^{1-\theta}.
$$

Maximum is characterized by the following first order condition:

$$
v'(m^*)(m^*+1)+(\theta-1)v(m^*) = 0. \tag{12}
$$

Observe that $M+B$ (nominal wealth) is constant in a market trade, so the optimal m^*_{N} does not depend on M or B, it is a constant.

Solving for optimal consumption requires the following first order condition (assuming that the liquidity constraint is not binding)

$$
\begin{aligned}
c^{-\theta} &= v'_{\mathrm{N}}(m), \\
\gamma^{\theta}c^{-\theta} &= v'_{\mathrm{E}}(m),
\end{aligned}
$$

or

$$
\begin{aligned}
c^*_{\mathrm{N}} &= v'_{\mathrm{N}}(m)^{-1/\theta}, \\
c^*_{\mathrm{E}} &= \gamma v'_{\mathrm{E}}(m)^{-1/\theta}.
\end{aligned}
$$

Note that unless v is linear, the optimal consumption depends on the cash–bond ratio nontrivially.

Substituting in the optimal m^* and c^*, and dividing through by $B^{1-\theta}$ yields

$$\begin{aligned}(\rho+\eta+\mu)v_N(m) &= \frac{\theta}{1-\theta}v_N'(m)^{1-1/\theta}+v_N'(m)\left[r(1+m)+\pi\right] \\ &\quad +\eta v_N(m_N^*)\left(\frac{m+1}{m_N^*+1}\right)^{1-\theta}+\mu v_E(m), \end{aligned} \tag{13a}$$

$$\begin{aligned}(\rho+\eta+\lambda)v_E(m) &= \frac{\theta}{1-\theta}\gamma v_E'(m)^{1-1/\theta}+v_E'(m)\left[r(1+m)+\pi\right] \\ &\quad +\eta v_E(m_E^*)\left(\frac{m+1}{m_E^*+1}\right)^{1-\theta}+\lambda v_N(m). \end{aligned} \tag{13b}$$

(13a) and (13b) constitute a system of two first-order ordinary differential equations. The family of solutions is thus two-dimensional. We have altogether four unknown constants: the two constants of integration and the two m^*s. The two first order conditions in (12) provide two boundary conditions, and the following limit property gives the remaining two.

$$\lim_{m\to 0} v_N'(m) = \lim_{m\to 0} v_E'(m) = \infty.$$

Given all the parameter values, we can numerically solve for the relative demands for the assets, m_N^* and m_E^*.[3] We anticipate that $m_E^* > m_N^*$, since consumption is higher under a liquidity event and the consumer needs more cash to finance it. In this case, when a liquidity shock occurs, the consumer wishes to sell some of her bonds to provide the higher cash–bond ratio. Because trade is not immediately possible, the agent consumes less than optimal in the beginning of the liquidity event. This makes her worse off, so she demands a yield premium for holding the illiquid asset. Alternatively, for a given premium, she holds less of the asset. As a limiting case, if trade occurs instantly ($\eta \to \infty$), the liquidity premium is zero, or, to put it differently, a positive premium implies infinite relative demand for the bond.

To obtain an alternative measure of illiquidity, let us turn to the case of a selling *discount*. So far we have measured illiquidity as the average time of waiting until trade. It is important to note that earlier trade is not necessarily impossible, it may be simply *suboptimal*. If selling at short notice involves a substantial price discount (either because the market is thin relative to the size of the sell order, or because uninformed buyers believe that the sell order is a result of an adverse fundamental shock, see Appendix A below) then the seller may find it optimal to wait until she can trade at a more favorable price (by placing the order gradually, or waiting for a more informed buyer).

Once we have solved for the value functions, we are able to calculate the discount at which the consumer would not trade in a liquidity event even if immediate trade was possible. Assume that the agent holds a portfolio optimal for normal times (m_N^*) when the liquidity shock

3. Appendix B describes the numerical technique used. Even though we apply a discrete-time iteration procedure, we find it simpler to show the algebra in continous time.

occurs. Then her utility is

$$v_{\mathrm{E}}(m^*_{\mathrm{N}})B^{1-\theta}.$$

Suppose she can sell the bond instantly at price $p' \leq 1$. Her utility from doing so is

$$v_{\mathrm{E}}(m^*_{\mathrm{E}})(B-S)^{1-\theta},$$

where the optimal amount of sale, S, is given by

$$S = B\frac{m^*_{\mathrm{E}} - m^*_{\mathrm{N}}}{m^*_{\mathrm{E}} + p'},$$

so the overall utility is

$$v_{\mathrm{E}}(m^*_{\mathrm{E}})\left(\frac{m^*_{\mathrm{N}} + p'}{m^*_{\mathrm{E}} + p'}\right)^{1-\theta} B^{1-\theta}.$$

The consumer is reluctant to sell if

$$v_{\mathrm{E}}(m^*_{\mathrm{E}})\left(\frac{m^*_{\mathrm{N}} + p'}{m^*_{\mathrm{E}} + p'}\right)^{1-\theta} < v_{\mathrm{E}}(m^*_{\mathrm{N}}),$$

or

$$p' < \tilde{p} \equiv \frac{m^*_{\mathrm{E}}\, v_{\mathrm{E}}(m^*_{\mathrm{E}})^{1/(\theta-1)} - m^*_{\mathrm{N}}\, v_{\mathrm{E}}(m^*_{\mathrm{N}})^{1/(\theta-1)}}{v_{\mathrm{E}}(m^*_{\mathrm{E}})^{1/(\theta-1)} - v_{\mathrm{E}}(m^*_{\mathrm{N}})^{1/(\theta-1)}}.$$

So if the price is below $\tilde{p}$, the consumer is better off waiting an expected $1/\eta$ amount of time till it recovers to p. Throughout the calibration of the model, we will also report the discount, $(1-\tilde{p})$, at which the agent is unwilling to sell immediately. This enables us to get a better picture of how much illiquidity the agent really faces when deciding on the timing of sale.

There is a clear link between the *volatility* of the price of the asset and its liquidity. In particular, if the price of the bond is highly volatile, the consumer may wish to wait for a better price and hence delay consumption after a liquidity shock. By incorporating the seller's search for an optimal price in the model, we would be able to investigate the effect of volatility on the liquidity premium. We do not address this question in the present paper but it is a candidate for future research.

3.1. NUMERICAL SOLUTION

We solve the above optimization problem numerically as described in the Appendix. Here we only report the solution for a few parameter values, more detailed calibration results and sensitivity check will follow.

The following parameter values serve as a benchmark. Although we use daily numbers in the iteration, here we report annualized values.

ρ	μ	λ	η	γ	θ	r	π
0.1	2	26	120	4	0.9	0	0.01

Recall the interpretation of the parameters from Section 2.2. A discount rate of $\rho = 0.1$ leads to a discount factor of 0.90. The arrival rate, $\mu = 2$, means that liquidity events occur every six months on average. A λ of 26 means that the expected duration of the liquidity shock is two weeks. For $\eta = 120$, buyers arrive every three days on average. $\gamma = 4$ means that at a liquidity event the marginal propensity to consume is four times higher than at normal times. $\theta = 0.9$ implies relatively little risk aversion and a high degree of intertemporal substitution. $r = 0$ and $\pi = 0.01$ mean that the real rate of interest is zero and the bond commands a one-percent liquidity premium.

These values imply $m_{\mathrm{N}}^{*} = 0.098$, $m_{\mathrm{E}}^{*} = 0.161$, and $(1-\tilde{p}) = 0.028$. That is, the consumer holds 8.9% ($= m_{\mathrm{N}}^{*}/(1+m_{\mathrm{N}}^{*})$) of her wealth in cash at normal times, and 13.9% during liquidity events. This confirms are conjecture that the arrival of a taste shock raises the optimal cash–bond ratio. The consumer consumes roughly 10% of her *liquid* wealth each day. This number does not change substantially with the actual cash–bond ratio. This implies that total consumption responds substantially to the amount of liquid asset holdings, and is about twice as high during liquidity events than at normal times. The discount at which the investor is indifferent between immediate sale and waiting is 2.8%. This seems not to be an unreasonable magnitude: an immediate sell order from a large investor in an illiquid market might result in such a discount.

If we increase the measure of illiquidity to an average five days ($\eta = 72$), the results change in the anticipated direction. The optimal cash–bond ratios are $m_{\mathrm{N}}^{*} = 0.148$ and $m_{\mathrm{E}}^{*} = 0.247$, implying that the agent holds 12.9% of her total wealth in cash during normal times, and 19.8% at liquidity shocks, that is, she demands less of the illiquid asset in both states. Such a decrease in the liquidity of the bond lowers its demand by 5–7% for a given nominal wealth and a given yield premium. The optimal consumption is around 8% of liquid wealth, meaning that the consumer saves more so that she can offset the increased illiquidity of the bond. The discount threshold is now 3.8%; selling at this discount or waiting an average five days are equally bad for the asset holder.

Tables 1 through 3 display the calibration results for a number of parameter values. The first column reports the parameter varied, all the other parameters are fixed at the benchmark value. We report the share of liquid assets in the portfolio during normal times and liquidity events, the discount at which the consumer is indifferent to immediate sale, and the share of the 1% liquidity premium that is explained by liquidity events. This number is calculated as follows. When a liquidity shock occurs, the consumer is willing to sell her illiquid asset at the given discount. She does not sell all of her bonds, only a fraction required to set the optimal cash–bond balance. Waiting for the sale to take place is equivalent to suffering a loss of the given discount for the above fraction of bonds. This loss is incurred μ times a year on average. In the last column of the tables, this annual average loss is reported in percentage points. The remainder of the 1% premium compensates the loss from everyday, small trades. This number gives us a sense of how much the taste shocks are relevant in explaining the liquidity premium.

Table 1 shows that and increasing waiting time (decreasing η) reduces the demand for the illiquid asset. We also find that the more illiquid the asset is, the higher fraction of the liquidity premium is explained by the taste shocks. Table 2 demonstrates that the role of taste shocks moves very closely with their size. There is an unexpected finding, however. We see that, contrary to our first intuition, higher liquidity shocks decrease the demand for the liquid assets. This is because the intertemporal elasticity of substitution is quite large (recall the $\theta = 0.9 < 1$). The consumer is willing to redistribute a large amount of consumption from normal times to times of liquidity events. She consumes very little during normal times and hence does not need much liquid wealth. We also see this effect at work when varying the frequency of taste shocks (see Table 3). In order to get rid of the dominance of the substitution effect, we will need to consider lower (and more realistic) values of intertemporal substitution.

4. Conclusion

We developed two simple models of portfolio decision under illiquidity. The first model implied an endogenously arising liquidity premium of 1.7% for reasonable parameter values. The second model enabled us to investigate the portfolio allocation decision of a risk averse agent with illiquid assets. Preliminary calibration suggests that if the average time to sell is three days, a mildly risk averse agent holds 8.9% of her wealth in liquid assets during normal times and 13.9% during liquidity events. If waiting time increases to five days then these ratios go up to 12.9% and 19.8%, respectively, meaning that the demand for the illiquid asset decreases by 5–7%. The price discount that makes the asset holder indifferent between immediate sale and waiting is 2.8% for an average waiting of three days and 3.8% for five days. A thorough calibration exercise is to be accomplished. Still, the preliminary results seem reasonable, and they reinforce our claim that a small variation in illiquidity can cause substantial change in the optimal portfolio allocation.

Directions for further research include incorporating fundamental risk into the model in order to study how the interaction of illiquidity and uncertainty modifies portfolio investment decisions and whether any of the two might be a dominant factor. Additionally, a fully satisfactory model might allow for endogenously arising illiquidity (but see Appendix A).

Appendix

A Asymmetric Information and Illiquidity

Here we develop a simple matching model with informational asymmetry that endogenously implies the impossibility of asset trade at certain dates. The idea is that buyers sometimes cannot distinguish between sale implied by liquidity reasons and sale implied by weak fundamentals. In that case they might be reluctant to buy at a high price; and at a low price the seller is not interested in trading, thus the market collapses.

Time is discrete, agents are risk neutral. The seller has all of her wealth invested into one unit of the illiquid asset, and wishes to sell it to a buyer. Potential buyers arrive at each date. Buyers may or may not have full information about the asset. A fully informed buyer knows exactly the value of the asset (0 or 1 for the buyer). An uninformed buyer has the following prior. She believes that with probability p_0 the seller has a liquidity shock and the asset is good, i.e., it has a continuation payoff of 1 to the buyer. With probability $1-p_0$ the asset is bad, and has a continuation payoff of 0. Hence the informed buyer is willing to buy at any price less than or equal to the true value of the asset, and the uninformed buyer is willing to buy at any price less than or equal to p_0. The probability that a given buyer is informed is α.

Assume now that the seller has in fact a liquidity shock, and the asset is good. The liquidity shock is modeled as a flow cost of q for the seller each time period she is unable to trade the asset. Once trade takes place, the seller realizes a continuation payoff equal to the price, and the flow liquidity cost goes away. If p is the current price on the market and $v(p)$ is the value function of the seller, then we have the following Bellman equation

$$v(p) = \max\left\{-q + \beta\,\mathrm{E}\,v(p'),\ p\right\}$$

where β is the discount factor of the seller, E is the expectations operator and p' is the price prevailing next period. We assume that if an uninformed buyer arrives then the price is p_0 (it cannot be more than that) and if an informed buyer arrives then the price is p_1, with $p_1 > p_0$. Since both prices are positive, it is always worth it for the seller to sell if the high-price (informed) buyer comes. Hence $v(p_1) = p_1$, and the Bellman equation implies that

$$v(p_0) = \max\left\{-q + \beta\left(\alpha p_1 + (1-\alpha)v(p_0)\right),\ p_0\right\}.$$

Now the market with the uninformed buyer collapses if

$$-q + \beta\left[\alpha p_1 + (1-\alpha)v(p_0)\right] > p_0.$$

In that case $v(p_0)$ is determined by

$$v(p_0) = -q + \beta\left[\alpha p_1 + (1-\alpha)v(p_0)\right]$$

so that

$$v(p_0) = \frac{-q + \beta\alpha p_1}{1 - \beta(1-\alpha)}.$$

Then the condition that the market with the uninformed buyer collapses can be rewritten as

$$\frac{-q + \beta\alpha p_1}{1 - \beta(1-\alpha)} > p_0. \tag{14}$$

This is a condition on the underlying parameters of the model and the price p_1. As long as it holds, the market with the uninformed buyer collapses. Thus we have that even though buyers arrive each period, trade will not necessarily take place, only if the buyer has private information. Assuming uniform random matching and a large population, informed buyers will arrive at a Poisson rate, with parameter determined by the percentage of informed buyers in the population, α.

The assumption that the uninformed buyer charges exactly p_0 is not necessary. She will not pay more, and at this price it is already not worth it for the seller to trade; hence no trade will take place at any other price either. Condition (14) therefore guarantees that no trade takes place on the market with the uninformed buyer whatever the price may be.

We have not pinned down p_1, the price that prevails on the market with an informed buyer. As long as $1 > p_1 > p_0$ and condition (14) holds, it is mutually beneficial for both parties to trade. The price can be determined by some sort of bargaining process, which determines how to distribute the gains from trade between the two agents. For the purposes of the present example it is not necessary to specify the details.

This very simple model demonstrates how trade might become endogenously impossible. The structure of this model is such that it could be incorporated into the models in the main part of the paper. However we see no point in doing that as our main focus is more to study the implications of illiquidity in this paper.

The idea of this model is essentially adverse selection (Akerlof, 1970). The applications of informational asymmetries to financial markets is a recurrent theme in the literature. In particular the fact that not fully informed traders use price and/or volume as a signal of fundamentals has been explored in a number of different models, see e.g., Kyle (1985) or Genotte and Leland (1990).

B Numerical Technique

In the numerical solution, we use a discrete-time version of (11) and its liquidity-event counterpart. We iterate the following Bellman operator on an arbitrary value function until convergence. The resulting value function is a fixed point of the Bellman operator and hence a solution of the Bellman equation.

$$
\begin{aligned}
v_{\mathrm{N}}^{(k+1)}(m) \equiv \max_{c<m} \frac{c^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\Big\{ &(1-\eta)(1-\mu)v_{\mathrm{N}}^{(k)}(m+r(1+m)+\pi-c) \\
&+(1-\eta)\mu v_{\mathrm{E}}^{(k)}(m+r(1+m)+\pi-c) + \eta(1-\mu)v_{\mathrm{N}}^{(k)}(m_{\mathrm{N}}^{*})\left(\frac{(1+r)(1+m)+\pi-c}{m_{\mathrm{N}}^{*}+1}\right)^{1-\theta} \\
&+\eta\mu v_{\mathrm{E}}^{(k)}(m_{E}^{*})\left(\frac{(1+r)(1+m)+\pi-c}{m_{\mathrm{E}}^{*}+1}\right)^{1-\theta}\Big\} \\
v_{\mathrm{E}}^{(k+1)}(m) \equiv \max_{c<m} \gamma^{\theta}\frac{c^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\Big\{ &(1-\eta)(1-\lambda)v_{\mathrm{E}}^{(k)}(m+r(1+m)+\pi-c) \\
&+(1-\eta)\lambda v_{\mathrm{N}}^{(k)}(m+r(1+m)+\pi-c) + \eta(1-\lambda)v_{\mathrm{E}}^{(k)}(m_{\mathrm{E}}^{*})\left(\frac{(1+r)(1+m)+\pi-c}{m_{\mathrm{E}}^{*}+1}\right)^{1-\theta} \\
&+\eta\lambda v_{\mathrm{N}}^{(k)}(m_{N}^{*})\left(\frac{(1+r)(1+m)+\pi-c}{m_{\mathrm{N}}^{*}+1}\right)^{1-\theta}\Big\}
\end{aligned}
$$

We have chosen day as the unit of time, ensuring that the discrete and the continuous-time case are approximately equivalent. Even at such a short time horizon we could not abstract from terms of order $\mathrm{d}t^2$, this is why the discrete version of the Bellman equation is much more complicated.

Once m_{N}^{*} and m_{E}^{*} are given, the above operator is a contraction mapping by Blackwell's theorem (see Theorem 3.3 in Stokey and Lucas, 1989), and thus there is a unique fixed point to which the iteration procedure converges.

We apply a grid search technique to determine m_{N}^{*} and m_{E}^{*}. For any pair of candidate cash–bond ratios, we iterate the value functions until convergence, and check if m_k^{*} indeed maximizes $v_k(m)/(m+p)^{1-\theta}$ (see (12)) for both $k = \mathrm{N},\mathrm{E}$. If it does, we stop, if not, we pick another pair of m^{*}s.

References

Akerlof, G. A. (1970). The market for 'lemons': Quality uncertainty and the market mechanism, *Quarterly Journal of Economics* **84**(3): 488–500.

Baldwin, C. Y. and Meyer, R. F. (1979). Liquidity preference under unvertainty, *Journal of Financial Economics* **7**: 347–374.

Benczúr, P. (2001). Identifying sovereign bond risks, Manusript. National Bank of Hungary.

Campbell, J. Y., Lo, A. and MacKinley, A. C. (1997). *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.

Constantinides, G. M. (1986). Capital market equilibrium with transaction costs, *Journal of Political Economy* **94**(4): 842–862.

Diamond, D. W. and Dybvig, P. H. (1983). Bank runs, deposit insurance and liquidity, *Journal of Political Economy* **91**(3): 401–419.

Dumas, B. and Luciano, E. (1991). An exact solution to a dynamic portfolio choice problem under transaction costs, *Journal of Finance* **46**(2): 577–595.

Gabaix, X. and Laibson, D. (2001). The 6D bias and the equity premium puzzle, Forthcoming in NBER Macroeconomics Annual.

Genotte, G. and Leland, H. (1990). Market liquidity, hedging, and crashes, *American Economic Review* **80**(5): 999–1021.

Grossman, S. J. and Laroque, G. (1990). Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods, *Econometrica* **58**(1): 25–51.

Holmström, B. and Tirole, J. (1998). LAPM: A liquidity-based asset-pricing model, *NBER Working Paper* **#6673**.

Kyle, A. S. (1985). Continuous auctions and insider trading, *Econometrica* **53**(6): 1315–1336.

Lippman, S. A. and McCall, J. J. (1986). An operational measure of liquidity, *American Economic Review* **76**(1): 43–55.

Lucas, R. E. (1978). Asset prices in an exchange economy, *Econometrica* **46**(6): 1429–1445.

Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle, *Journal of Monetary Economics* **15**: 145–161.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* **3**: 373–413.

Pagano, M. (1989a). Endogenous market thinness and stock price volatility, *Review of Economic Studies* **56**: 269–288.

Pagano, M. (1989b). Trading volume and asset liquidity, *Quarterly Journal of Economics* pp. 255–274.

Stokey, N. L. and Lucas, R. E. (1989). *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, MA.

Williams, J. T. (1995). Pricing real assets with costly search, *Review of Financial Studies* **8**(1): 55–90.

η	$M^N/(M^N + pB^N)$	$M^E/(M^E + pB^E)$	Discount	Premium
180	6.0%	9.9%	2.7%	0.22
120	8.9%	13.9%	2.8%	0.31
72	12.9%	19.8%	3.8%	0.60

Table 1: The effect of illiquidity on asset demand

γ	$M^N/(M^N + pB^N)$	$M^E/(M^E + pB^E)$	Discount	Premium
2	9.9%	12.9%	1.2%	0.08
4	8.9%	13.9%	2.8%	0.31
8	6.9%	13.9%	4.7%	0.70

Table 2: The effect of size of taste shocks on asset demand

μ	$M^N/(M^N + pB^N)$	$M^E/(M^E + pB^E)$	Discount	Premium
0.5	10.9%	19.8%	1.4%	0.28
1	9.9%	16.8%	2.0%	0.31
2	8.9%	13.9%	2.8%	0.31

Table 3: The effect of frequency of liquidity events on asset demand

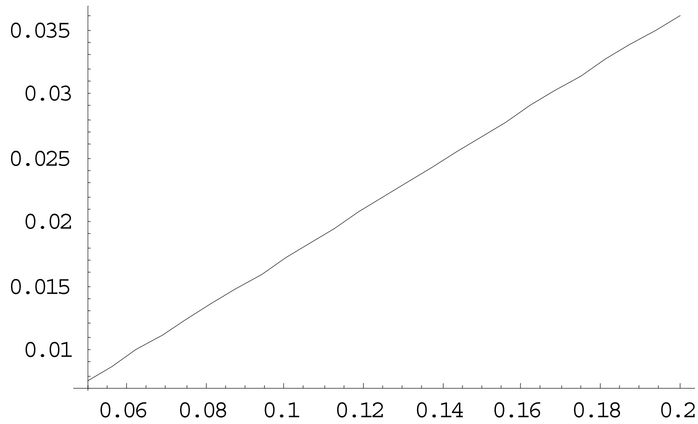

Figure 1: Premium as a function of ρ

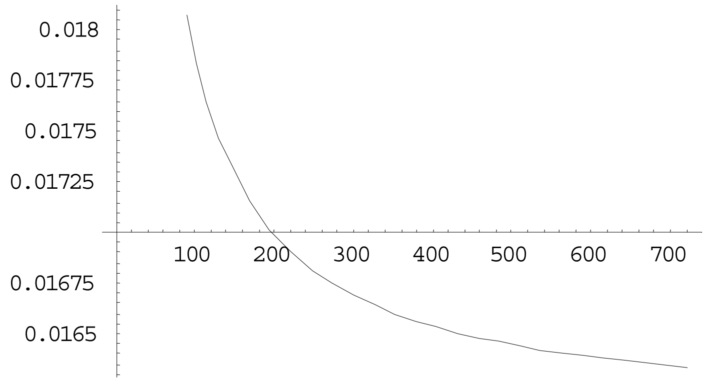

Figure 2: Premium as a function of expected time until alternative (days $= 365/\mu$)

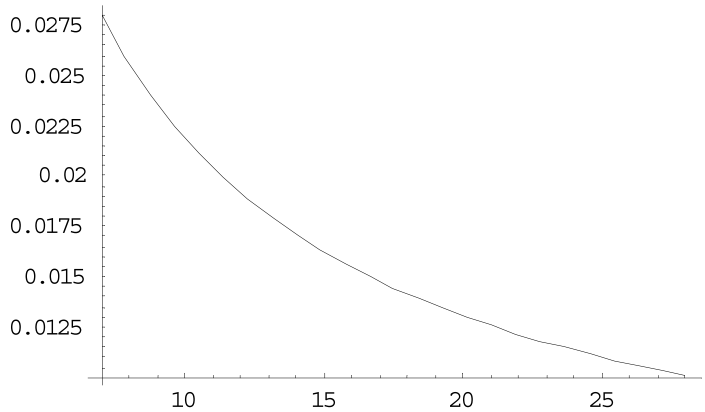

Figure 3: Premium as a function of expected duration of alternative (days $= 365/\lambda$)

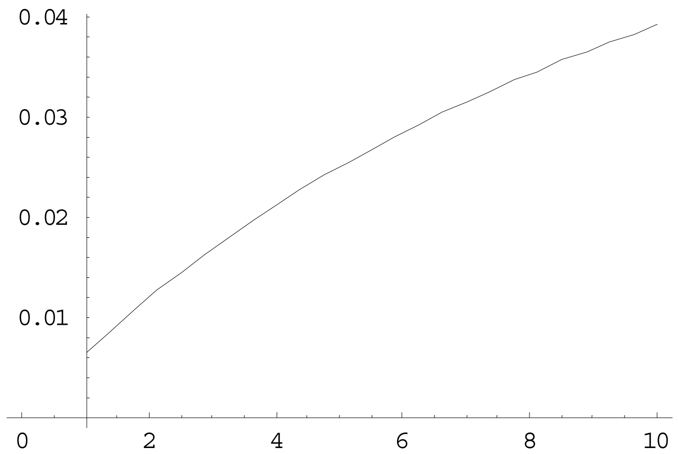

Figure 4: Premium as a function of expected time until first buyer (days $= 365/\eta$)