WEAKENING THE STRONG CONVEXITY OF PREFERENCES

Christophe MULLER

School of Economics, University of Nottingham,
University Park, Nottingham NG7 2RD,
United Kingdom.
Fax: +44 0115 951 4159
Email: christophe.muller@nottingham.AC.UK

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1. Introduction

The global uniqueness, the smoothness of the demands and the negative semi-definiteness of the substitution matrix are among the main theoretical restrictions in consumer theory. The strong quasi-concavity of the utility function (SQC)[1] is a sufficient global condition for these restrictions. It is a cornerstone of consumer and general equilibrium theories. As a matter of fact, global concavity assumptions in economic models are often responsible for many crucial properties of these models. They play an essential role in that they largely express the structure of the model, without which a direct empirical approach without theory could as well be pursued. Global conditions are important because, as opposed to local conditions, one can impose or check them a priori without any knowledge of the optimal decisions. In practice, applied economists often specify their models by using functional forms for objective, constraint or decision functions, such that desirable global conditions can easily be imposed or checked.

Although theoretical properties can also be developed for demand correspondences[2], the uniqueness of the demands considerably simplifies the analysis. It enables one to separate considerations related to the inaccuracy of choices, from

[1] Arrow and Enthoven (1961), Debreu (1972, 1976) discuss these quasi-concavity assumptions.

[2] Ellis (1976).

the study of decision changes with characteristics of agents and environment. In consumer theory, the uniqueness of the demands is global. The same legitimate desire of focusing on the law of decisions is valid for general models. However, global uniqueness is not necessarily appropriate for a decision problem with several constraints. Consider for example the consumption of a person who can obtain his consumption from two distinct domestic technologies. Assume that only the production frontier is observed. Assume also that one technology is enjoyable but has low productivity, while the other technology has opposite characteristics. Then, if the arguments of the person's utility are the penibility of domestic work and consumption, she may be indifferent between two solutions corresponding each to one of these technologies. One does not wish to artificially eliminate this reasonable situation by imposing a unique global solution. Then, what is required is that the decisions are *locally* unique. The smoothness of the decision functions is also important since it allows an easy study of the comparative statics and other variational properties of decisions. All this explains why SQC or similar assumptions are fundamental in consumer economics.

The negativity of the substitution matrix is also a major theoretical restriction. In theoretical analysis, many authors[3] stress the importance of the negativity and

[3]e.g. Barten (1977), Afriat (1983), Varian (1984), Takayama (1985), Beavis and

symmetry restrictions of the Slutsky matrix in consumer theory. Other authors[4] use the negativity of the Slutsky matrix, or similar properties, to derive sufficient conditions for the law of aggregate demand, which supports the existence of the competitive equilibrium of the whole economy. In the study of price dynamics in general equilibrium, negativity restrictions or related conditions ensure globally stable equilibria[5]. In applied work, these restrictions are used to incorporate theoretical results in estimated models[6].

Unfortunately, the strong quasi-concavity of the utility function that delivers all these restrictions has few theoretical or empirical bases. The quasi-concavity of the utility is related to preference by individual of 'mixtures' of commodities to unbalanced consumption structures. However, various authors have expressed a strong dissatisfaction with the hypothesis of strict convexity of preferences (equiv-

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Dobbs (1990), El-Hodiri (1991). Shapiro and Braithwait (1979) begin their article with a quotation of Samuelson (1961): "The assumption that [the Slutsky matrix is] ... symmetrical and negative semi-definite completely exhausts the empirical implications of utility analysis. All other demand restrictions can be derived as theorems from this single assumption".

[4] e.g. Hildenbrand (1983), Grandmont (1987), Quah (1997).

[5] Khilstrom, Mas-Colell, Sonnenschein (1976).

[6] See Samuelson (1947), Kalman and Intriligator (1973), Chichilnisky and Kalman (1978), Deaton and Muellbauer (1980), Varian (1984), Chung (1994). Kodde and Palm (1987) discuss a parametric test of the negativity of the substitution matrix. In the context of cost function estimation, Gallant and Golub (1984), Diewert and Wales (1987) propose methods for imposing curvature conditions on specific flexible functional forms. The latter ones insist on the importance of imposing concavity *globally*, consistently with economic theory. Imposition or verification of the negativity in applied demand systems is common practice.

alent to the strict quasi-concavity of the utility function, itself very close to SQC)[7]. Also, experimental evidence[8] contradicts the convexity of preferences. In fact, the convexity of consumer preferences is intuitive only when comparing standard average baskets with extreme consumption choices concentrated only in a few commodities. When comparing two rather balanced commodity baskets, the intuition is somewhat lost and SQC looks rather arbitrary.

Even if we admitted SQC for the consumer case, this would be much less tolerable for other models. The presence of heterogenous arguments in the objective function may generate the possibility of different 'life styles' or strategies, which may imply nonconvexities in preferences. For example, this is the case for the fertility choice between having a large family with limited human capital, or a small family with educated and healthy members. Collective settings for aggregate household decisions[9] may also contradict the convexity of household preferences. In trade theory or in macroeconomics, a country objective function is not necessarily quasi-concave. Finally, in some models the decisions are the characteristics of contracts and there is no reason why the objective function should

[7] e.g. Kirman (1982). Other 'technical' conditions on preferences have been attacked as altering the empirical content of models (Ghirardato and Marinacci, 2001).

[8] Tversky and Kahneman (1991).

[9] Chiappori (1988), Browning and Chiappori (1998).

be quasi-concave[10] in these cases. Clearly, it is desirable to dispose of alternative conditions to the SQC.

In this paper, we provide a new global generalised concavity condition adapted to models with several constraints, possibly nonlinear. These models are used in several economic fields. The New Household Economics[11] and agricultural household models[12] involve production and budget constraints. Models with nonlinear budget constraints arising from quality effects[13], nonlinear taxation[14], productive consumption[15], nonlinear wage schedules[16], rationing[17], and nonlinear pricing by firms with monopoly power, are also characterised by nonlinear constraints. Finally, international trade theory, the study of first-best and second-best optima[18], collective household models[19], or other types of bargaining or incentive models, may include several nonlinear constraints for agents' optimal choices.

In general settings[20], no *global* generalised concavity condition is known that

[10] e.g. Stiglitz and Weiss (1992).
[11] Becker (1965), Lancaster (1966).
[12] Sen (1966), Barnum and Squire (1980), Pitt and Rosenzweig (1985), Singh, Squire and Strauss (1986), Benjamin (1992).
[13] Houthakker (1952), Edlefsen (1981, 1983).
[14] Hausman (1985), Weymark (1987).
[15] Suen and Hung Mo (1994).
[16] Blomquist (1989).
[17] Madden (1991).
[18] Ben-Israel, Ben-Tal, Charnes (1977), Dixit (1985).
[19] Chiappori (1988, 1992), Browning and Chiappori (1998).
[20] Silberberg (1974), Hatta (1980), Caputo (1999) and Drandakis (2000) study problems with several constraints by using dual methods, although they do not deal with

would be as weak as possible for the negativity of the substitution matrix and the smoothness and local uniqueness of decisions. Is there such a condition and what are its properties? The aim of this paper is to answer these questions so as to improve the specification of general economic models. In Section 2, we present the general optimisation problem. In Section 3, we recall the consequences of SQC in consumer theory and we analyse a new global generalised concavity condition for optimisation programmes with several constraints. In Section 4, we study the properties of the decision functions under this condition. We provide an example of application in Section 5. Finally, we conclude in Section 6. The proofs are given in the appendix.

2. The Optimisation Problem

General behavioural models with several constraints can be represented by the following programme:

$$\max_x U(x, \theta) \quad \textit{subject to}: \; g(x, \theta) \leq 0_q, \tag{2.1}$$

global concavity conditions.

where U is the objective function, which is often assumed to be strictly quasi-concave (or strictly concave, e.g. in Varian, 1984). $x \in R^n$ is the n-dimensional vector of decision functions, $\theta \in R^p$ is the vector of parameters that may be prices and incomes as well as any characteristics of the environment or of the agent. We allow for parameters common to the objective and the constraints[21]. However, these parameters will be omitted for the presentation when they are not necessary. g is a q-dimensional vector of constraint functions. The decisions may be of any type, including possibly negative values, as for variables such as netputs or net trading positions. Positive decisions can be accounted for in the constraints. The set of choices, X, defined by the constraints, is often assumed to be convex. Appendix 1 contains the definitions of the generalised concavity notions that we use in this article, with their properties that are employed.

To be able to use the first-order Kuhn-Tucker conditions (KTC) as necessary for the existence of a solution, one must assume a constraint qualification condition. We follow the common practice of assuming that the gradient vectors of the

[21] Often, exogenous variables or random effects influence preferences as well as constraints. This is useful for applied agricultural household models (Singh, Squire and Strauss, 1986, Pitt and Rosenzweig, 1985), for evolutionary economics (Lesourne, 1993, Young, 1993) and for models in which preferences depend on random states of Nature that may also affect constraints (Viscusi and Evans, 1990). Finally, for Pareto optima, bargaining and incentives models, objective and constraints that all include utility functions, may incorporate the same common characteristics of preferences.

components of g are linearly independent. Despite their intrinsic interest, changes in regime may correspond to discrete discontinuity jumps of decisions, which would justify not paying much attention to negligible marginal substitution effects. In these situations, the negativity property as well as the smoothness of decisions lose most of their appeal as theoretical restrictions. At solutions where the strict complementarity slackness fails, comparative statics may be problematic because the set of binding constraints may change as the parameter changes, destroying the differentiability of the solutions. To avoid these problems, practitioners generally assume that non-negativity constraints would not bind, but all other constraints always bind. Moreover, researchers are often concerned only with the solutions of one specific regime of interest (one set of binding constraints). This leads us to focus on the following Lagrange conditions, which are the KTC associated with such a specific regime.

$$U_x - g_x \lambda = 0_n$$

$$g(x, \theta) = 0_q \tag{2.2}$$

where 0_q is the q-dimensional vector null and λ is the q-dimensional vector of the Lagrange multipliers. The Lagrange function associated with the problem is $L = U - \lambda' g$. We now discuss global concavity conditions for behavioural models, first by examining the link of global condition and local properties of decisions.

3. Global Concavity Conditions

Theoretical restrictions for the decisions similar to those obtained with SQC in demand theory can be obtained from the sufficient second-order conditions (SSOC) of the optimisation programme, for example in Blackorby and Diewert (1979). However, without a global concavity condition this approach involves several shortcomings. Firstly, the derived decision functions may not satisfy desirable global properties. For example, flexible functional forms used in consumer analysis have been criticised on the grounds that they did not easily allow the imposition of the convexity of preferences (Diewert and Wales, 1987). Secondly, the consistency of the local duality structures presupposes some global concavity properties (Blackorby and Diewert, 1979). For example, for the consumer problem one needs to assume that the expenditure function is concave in prices over its domain or that the direct utility function is quasi-concave over its domain. Without these global concavity conditions there is no correspondence of the respective second-order approximations of the expenditure function and of the direct utility function. Moreover, the global conditions alleviate difficulties that may arise for the coincidence of the domains of the local utility function and of the other local dual representations of preferences. Therefore, even if local approximations are useful tools, they do not permit a precise control of global properties of objective and constraints and of the consistency of the dual. Thirdly, global concavity conditions are used to incorporate decision models in general equilibria frameworks describing the economy by a unique and stable solution. On the whole, we need global generalised concavity conditions on the optimisation problem, even for obtaining desirable local properties of decision functions. This has been a fertile approach in the literature, notably to obtain local uniqueness and differential properties of decisions[22]. Besides, local uniqueness, smoothness and semi-definite negativeness of decisions are global properties when they must be satisfied a priori for the whole domain.

In the consumer problem, the only constraint (to simplify the exposition we ignore any positivity constraint) is the linear budget constraint, $p'x = m$, where x is the vector of consumption, p is the vector of prices and m is the exogenous income. The utility function U is generally assumed to be of type C^2, strictly increasing in the consumption of every commodity and strictly (or strongly) quasi-concave.

[22] e.g. Debreu (1972), Arrow and Enthoven (1961), Laroque (1981), Smale (1982), Dana (1999).

Under these assumptions, the vector of demands *x(p,m)* is derived from the Lagrange first-order conditions of the optimisation programme. From the budget constraint, one can derive the adding-up and homogeneity restrictions that are somewhat specific to the consumer problem. The other theoretical restrictions characterise the Slutsky matrix S and are discussed in Afriat (1983). The symmetry property (S is a symmetric matrix) results from the separation structure of the optimality problem. The negativity property results from the assumption of strong quasi-concavity of U (Diewert, Avriel and Zang, 1981), which implies the SSOC: S is orthogonal to the price vector and is negative definite in the hyperplane orthogonal to the price vector.

For general models, SQC is no longer necessarily appropriate and may be weakened. Weakest conditions for properties of optimal solutions play important roles in nonlinear programming[23] and in economics[24]. We search for a generalised *global* concavity condition that is as weak as possible and implies the local uniqueness and smoothness of decision functions and the negativity of the generalised substitution matrix. Its specification is inspired from the SSOC.

The necessary (respectively sufficient) local second-order conditions corre-

[23] Avriel (1977), Hirriart-Urruty (1996).
[24] Debreu (1983).

spond to the local negative semi-definiteness (respectively the local negative definiteness) of the Hessian matrix of the Lagrange function with respect to decisions at the optimum (respectively, at the optimum for directions in the tangent space to the constraints). To our knowledge, they have never been interpreted in terms of global properties of objective and constraint functions. Moreover, because each constraint function and objective function is separately specified in economic models and sometimes separately estimated from different datasets, we look for a condition that can be explicitly expressed in terms of these functions, rather than in terms of the Lagrange function. Next, we recall the definition of the strong quasi-concavity.

Definition 3.1. *Let U be a directionally differentiable real function defined over a convex subset X of R^n. U is called* ***Strongly Quasi-Concave*** *over X if and only if*

$$[x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}v \in X, D_v U(x^0) = 0] \Rightarrow$$

$$[\exists \varepsilon > 0, \exists \alpha > 0, \varepsilon < \bar{t}, \forall t \in [0, \varepsilon],\ U(x^0 + tv) < U(x^0) - \alpha t^2],$$

where D_v denotes the directional derivative operator in direction v.

Equivalent notions have been used[25]. If U is twice differentiable, an equivalent

[25] Dhrymes (1967), Barten, Lempers and Kloek (1969), Newman (1969), Ginsberg

definition is the following (Diewert, Avriel and Zang, 1981).

Definition 3.2. *Let U be a twice differentiable real function defined over a convex subset X of R^n. U is*

Strongly Quasi-Concave (SQC) if and only if $\forall (x, y) \in X^2$, such that $x \neq y$, $(\nabla U(x)(y - x) = 0) \Rightarrow (y - x)' \nabla^2 U(x)(y - x) < 0$.

We now introduce a new notion of generalised concavity by changing the premises of this definition.

Definition 3.3. *The twice differentiable objective function U subject to a differentiable vector of constraint functions g,*

*is called "**Constraint – Strongly Quasi – Concave**" (CSQC) with respect to g if and only if U is twice differentiable on X convex subset of R^N, and*

$$\begin{aligned} \forall (x, y) &\in X^2 \ \textit{such that} \ x \neq y \ \text{and} \ g(x, \theta) = 0_q, \\ (\nabla g(x)(y - x) &= 0 \ \textit{and} \ \nabla U(x)(y - x) = 0) \Longrightarrow \\ ((y - x)' \, \nabla^2 U(x) \, (y - x) &< 0) \end{aligned} \tag{3.1}$$

(1973), McFadden (1978). See also Barten and Böhm (1982) for a discussion of the properties of strongly quasi-concave utility functions in consumer theory.

CSQC must be checked for all points x satisfying the constraints, in particular for x non-optimal. This is a crucial requirement, since, at the stage of model specification, the actual optimum may be unknown. Although it is related to SSOC as we shall show in Section 4.1, the definition of the CSQC differs by several elements. First, it is global. Second, an orthogonality condition involving the gradient ∇U intervenes in the premises of CSQC and not in that of SSOC. Third, the negativity condition in the conclusion of CSQC is for $\nabla^2 U$, whereas it is for $\nabla^2 L$ in SSOC. Also, CSQC differs from SQC by the presence in the premises of the choice set and of orthogonality conditions with respect to the constraint gradients.

The fact that the constraints intervene in CSQC is more natural than it may seem at first sight, because both objective and constraints characterise the optimisation problem and they should therefore be considered together. Hotelling (1935), for consumer theory with a linear budget constraint, affirms that "*only the portions of the indifference curves that are convex to the origin can be regarded as possessing any importance since the others are essentially unobservable*". Novshek (1980) states that "*the second-order conditions impose constraints on the curvature of level sets for f relative to the curvature of level sets for g [here f describes the objective and g describes the constraints]. The absolute properties of [*$f_{ij}(x)$*]*

(positive definite, negative definite, corresponding to a saddle point, etc.) are unimportant. The properties of [$f_{ij}(x)$] relative to [$g^i_j(x)$] are important." These quotes are consistent with the definition of CSQC in which the curvature properties of U are considered relatively to g.

The tangent subspace at x to the constraints and to the indifference hypersurfaces (i.e. the subspace orthogonal to the gradients of functions $g^i(i = 1, ..., q)$ and U) generates a hypersurface, as x varies along the frontier of the constraints. This hypersurface is generally nonlinear and is neither a hyperplane as in consumer theory with one linear constraint, nor a sub-vector space as when considering only local conditions with several constraints. It is along this hypersurface that the negativity of $\nabla^2 U$ is imposed by CSQC.

The sole consideration of the frontier of the constraints in the definition of CSQC is motivated by the search for as weak a condition as possible. Indeed, it is generally useless to incorporate restrictions occurring at points that are never reached at the equilibrium because they are not at the frontier, since the objective function is generally specified as increasing in its arguments. We could define another condition by considering specific families of objective and constraint vectors with parameter θ varying and still obtain similar results. To save space, we do not discuss these variants.

Some functions U and g may yield an empty set of directions corresponding to the premises of the CSQC definition. When that is the case, essentially no arbitrary restriction of generalised concavity is imposed on the optimisation problem. However, this seems unlikely to happen in models of interest. In order to characterise CSQC by the shape of the graph of the objective function, we need an additional definition.

Definition 3.4. *$f : X$ convex set $\subset R \to R$ attains a **strong local maximum** (SLM) at $t_0 \in X$ if and only if $\exists\, \alpha > 0, \exists\, \varepsilon > 0, \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon] \cap X,\ f(t) \leq f(t_0) - \alpha(t - t_0)^2$.*

Intuitively, a function attains a SLM when its curvature to the origin at the maximum is at least as strong as that of a quadratic function. Definition 3.1 shows that a strongly quasi-concave function U is such that for directions v orthogonal to ∇U in x^0, $h(t) \equiv U(x^0 + t\, v)$ attains a SLM at $t = 0$. We now show that CSQC can also be characterised in terms of SLM ins some directions.

Proposition 3.5. *Let U be a twice differentiable function over X. Then,*

[U CSQC over X] if and only if

[($x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}\, v \in X, \forall i = 1, \ldots, q, g^i(x^0) = 0$ and $\nabla g^i(x^0)'v = 0$ and $\nabla U(x^0)'v = 0) \Rightarrow h(t) \equiv U(x^0 + t\, v)$ attains a SLM at $t = 0$].

The convexity of a function $f(x)$ is equivalent to the convexity of its epigraph, $E_p(f)$, the set of couples (x, y) where $y \geq f(x)$. The following proposition characterises the epigraph in the case of CSQC.

Proposition 3.6. *Let the function U, from X convex set of R^n to R, be CSQC with respect to the vector of constraints g. We consider the hypersurface limiting the epigraph of U:*

$$\partial E_p(U) = \{(x_1,\ldots,x_n, U(x_1,\ldots,x_n)) \,|\, (x_1,\ldots,x_n) \in X\},$$

and we define a "g–U-admissible" direction at x, as $d \in R^n$ such that $\nabla g^i(x)' d = 0$, for all i and $\nabla U(x)' d = 0$ (i.e. a direction of the domain generated by the tangent space to the constraints and to the indifference hypersurfaces). Then,

(a) *the frontier of the epigraph of U is strictly below all its tangent hyperplanes, in any g–U-admissible direction at the frontier of constraints;*

(b) the curvature (to the origin) of the frontier of the epigraph of U in any g-U-admissible direction at the frontier of constraints is strictly positive;

(c) the dimension of the subspace spanned by the ∇g^i $(i = 1, \ldots, q)$, which is q because of the constraint qualification condition, is greater than the number of positive or null eigenvalues of $\nabla^2 U$ at the frontier of constraints.

Condition (c) clearly illustrates that the more constraints the less restrictive is CSQC. In a von Neumann-Morgenstern framework, CSQC is related to risk aversion in the domains of choices defined by the constraints and by the indifference hypersurfaces. It is worth noting, as the next proposition shows, that CSQC utility functions can be ordinal and therefore that they correspond to relevant restrictions on the representation of preferences.

Proposition 3.7. *CSQC is an ordinal property of the preferences.*

In the next section, we examine the consequences of CSQC for the decisions.

4. The Properties of the Decisions Functions under CSQC

4.1. The Link of CSQC with SQC and Second-Order Conditions

We now turn to the link of CSQC with SQC and SSOC, not only because of the intrinsic interest of these conditions, but also because SSOC is a convenient intermediate to derive some properties of decisions. First, CSQC and SQC can be ranked.

Proposition 4.1.

If function U is strongly quasi-concave, then it is CSQC.

The reciprocal proposition is not true because, even at the optimum, $\nabla U' v = 0$ does not generally imply $\nabla g^{i'} v = 0$ for all i. When there are several constraints, CSQC is a weaker condition than the strong quasi-concavity, because it is associated with a local curvature that is strictly positive only in a sub-space of dimension $n-q$ or generally $n-q-1$, and only at the frontier of the constraints, while this curvature must be strictly positive in a whole hyperplane for U strongly quasi-concave. CSQC is an assumption that does not locally impose any structure on the preferences in a subspace of dimension q at least, and therefore globally in a large domain. Moreover, out of the frontier of the constraints, CSQC is tantamount to the absence of restrictions[26]. We now turn to the relationship between the CSQC of the Lagrange function and the CSQC of the objective function, as a first step towards the second-order conditions of optimality.

Proposition 4.2.

If g is quasi-convex, the CSQC of the utility function implies the CSQC of the Lagrange function associated with the optimisation programme, whether calculated with optimal or non-optimal Lagrange multipliers.

[26] The case of families of constraint and utility functions with θ varying instead of given a priori, leads to obvious generalisations and stronger global conditions, although still weaker than SQC. These generalisations do not change the core of our results.

The CSQC of the Lagrange function is important because it characterises the shape of the optimisation problem under CSQC of the objective function. However, the Lagrange function cannot be directly used to impose structural conditions, because of its dependence on a priori unknown Lagrange multipliers. The following proposition shows that CSQC ensures that the second-order conditions are satisfied when the choice set is convex.

Proposition 4.3. *If g is quasi-convex, then U CSQC implies both the necessary and the sufficient local second order conditions of optimality.*

The contention that g is quasi-convex is little restrictive since it is equivalent to assume that the choice set is convex. By contrast with the necessary second-order conditions, the SSOC are not necessary the consequences of any optimisation programme and a condition at least as strong as CSQC is necessary to obtain them. Locally, SSOC implies the negativity of the generalised substitution matrix (Pauwels, 1979). However, without global conditions one would have to know the optimum to a priori check SSOC in a tractable way. We are now ready to examine the properties of the decision functions under CSQC.

4.2. The Properties of the Decision Functions

In consumer theory, the budget set is bounded, therefore compact in finite dimension. This implies that there is always a solution to the maximisation of an upper semi-continuous utility function. In general models, the choice set inside a given regime is defined as $C = \{x \in R^n, g(x, \theta) = 0_q\}$ and is no longer necessarily bounded. Firstly, we consider a non-empty choice set C to avoid absurd situations. Secondly, the problem optimum is given by the KTC and corresponds to a tangential contact point of the constraint frontier with an indifference hypersurface. CSQC yields strict curvatures that seems to geometrically imply the existence of the optimum. Surprisingly, this is not the case[27] and additional assumptions are necessary to guarantee the existence of the optimum. Since U and g are continuous, if one assumes that the decisions belong to a bounded set Ξ, then the choice set C is compact and is not empty by hypothesis, and an upper semi-continuous utility function has a maximum in this set. Another possibility is to assume that U is coercive, upper semi-continuous on a closed feasible set with at least one point where U is finite.

[27] Indeed, in the following example there is no optimum. Let be a 2-dimensional vector of decisions $x = (x_1, x_2)\prime \in R^2$, and a constraint vector of one dimension described by the equation: $x_1 + x_2 = 1$. Assume that the objective function U is strictly increasing and differentiable in x_1 and x_2, and CSQC. Because the constraint is a unique line, this is obtained with U strongly quasi-concave. But there exist families of strongly convex indifference curves that satisfy these conditions and are asymptotically tangent to the constraint when x_1 goes to $+\infty$ or x_2 goes to $-\infty$. In that case, there is no optimum since the contact point is at the infinite.

We now present a characterisation of CSQC in terms of a bordered Hessian, similarly to consumer theory with strong quasi-concavity (Barten & Böhm, 1982). The non-singularity of another related bounded Hessian will be necessary to the derivation of properties of decisions.

Proposition 4.4. *Let be*

$$H = \begin{bmatrix} U_{xx} & g_x^1 & \cdots & g_x^q \\ g_x^{1\prime} & 0 & & 0 \\ \vdots & & 0 & \\ g_x^{q\prime} & 0 & & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} L_{xx} & g_x^1 & \cdots & g_x^q \\ g_x^{1\prime} & 0 & & 0 \\ \vdots & & 0 & \\ g_x^{q\prime} & 0 & & 0 \end{bmatrix}.$$

(a) If U is CSQC, then H is non-singular at any solution of the KTC.

(b) If U is CSQC and g is quasi-convex, then J is non-singular at any solution of the KTC.

The non-singularity of H characterises CSQC at the optimum. The non-singularity of J enables us to use the implicit function theorem under CSQC when g is quasi-convex, and we exploit it in the next proposition.

Proposition 4.5. *Let $(x_0, \lambda_0, \theta_0)$ be such that the KTC of Problem 2.1 are satisfied with U CSQC and g quasi-convex. Then,*

(a) $\exists V_0$ *open neighbourhood of* θ_0, $\forall V \subset V_0$, *open and connected neighbourhood of* θ_0, *there is a unique function* h: $V \to R^{2n}$, *such that*

$(x_0, \lambda_0) = h(\theta_0)$ *and* $\forall \theta \in V, \exists (x,\lambda) \in R^{2n}$, $(x, \lambda) = h(\theta)$

and $KTC[h(\theta), \theta] = 0$, *where* $KTC[.]$ *is the vector of functions defining the equations in the KTC.*

(b) h is of type C^1 *in* V *and its derivative is*

$$h'(\theta) = -\left[D_{x,\lambda} KTC[\, h(\theta),\ \theta]\right]^{-1} \circ \left[D_\theta KTC[\, h(\theta),\ \theta]\right].$$

(c) If moreover, U and g are of type C^{p+1} *in a neighbourhood of* $(x_0, \lambda_0, \theta_0)$, *then h is of type* C^p *in a neighbourhood of* θ_0.

(d) If moreover, U and g are analytic in a neigbourhood of $(x_0, \lambda_0, \theta_0)$, *then h is analytic in a neigbourhood of* θ_0.

The first component of h defines the vector decision functions. Proposition 4.5 proves the local uniqueness and smoothness of the decision functions. It also justifies the usual calculus of the derivatives of the decision functions. We now discuss the negativity property.

Proposition 4.6. *Under CSQC and the convexity of the choice set, the generalised substitution matrix, S, is negative semi-definite and is negative definite in* *the tangent space to the constraints.*

The negative semi-definiteness of matrix S at the optimum is related to the local stability of the equilibrium that is ensured when matrix L_{xx} is negative definite in the tangent space to the constraints. At the optimum, CSQC jointly with the quasi-convexity of g is the weakest available global condition for the negativity of the substitution matrix.

The definition of CSQC suggests to verify the semi-definite negativity of $\nabla^2 U$ in a limited domain. Therefore, as opposed to SSOC, this does not require the knowledge of all decisions and all multipliers, or verifications over the whole spaces of decisions, multipliers and directions. Clearly, a priori checking SSOC for complex problems is not tractable. By contrast, because firstly multipliers need not be considered, and secondly decisions and directions need be checked only in reduced domains, checking CSQC may be practically possible. Proposition 4.4 (a) provides a necessary condition on a bordered determinant that could be used for the test. However, this test would be only valid for the solutions of the KTC, which may be hard to calculate. The test of CSQC can also be directly implemented by using a grid of the domain of decisions, and for each knot of this grid by calculating all the eigen-values of $\nabla^2 U$ in the $(U - g)-$admissible directions and verifying that they are all negative. Statistical tests are available (Kodde and Palm, 1987) to account for approximations done in the model or with the grid. Often, a grid may not be necessary if the functional forms used for U and g lead to simplifications. We give an example of this in the next section.

On the whole, when the KTC are difficult to solve, checking CSQC is likely to be much less difficult than directly checking the SSOC. It may also be easier to test than SQC because the domain to explore for this test is much smaller. The following example makes it clear.

5. An Example

We use as an illustrative example the case of an autarchic community subject to two different production functions with externalities in the preferences. The optimisation programme is the following.

$Max \;\; U \equiv x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$

subject to $x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} x_4^{\beta_4} = \mu \quad (\equiv G(x_1, x_2, x_3, x_4))$

and $x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} x_4^{\gamma_4} = \nu \quad (\equiv H(x_1, x_2, x_3, x_4))$,

where the x_i $(i = 1, ..., 4)$ are consumptions of goods that may also be inputs or outputs of the two production processes, μ and ν describe production constants.

The $\alpha_i s$ can be normalised for example with $\sum_i \alpha_i = 1$. The $\alpha_i s, \beta_i s$ and $\gamma_i s$ can be of either sign, so that they allow the distinction between inputs and outputs in the production, but also the possibility of externalities in consumption. We assume that the investigation of these externalities is the main interest and that therefore the possibility of some $\alpha_i < 0$, e.g. because of pollution is crucial. Imposing SQC would imply all $\alpha_i > 0$ and is therefore not useful in this case. By contrast, imposing CSQC is possible and relatively easy as we shall show.

G and H describe two implicit production functions. For example, some leisure indicator may appear as an argument of the utility function with a positive α_i and as an argument of the production functions with positive β_i and γ_i (to express in a synthetic way that increasing leisure reduces labour input[28]). The chosen specification of the production for the example is particular since it is Kohli input non-joint. We can think, for example, of a community producing two types of outputs in a joint way, e.g. on the same site and where the inputs cannot be distinguished for each output type. Naturally, different specifications are possible, but we only want to develop an example that is easy to manipulate. This 'Treble Cobb-Douglas model' is appropriate for this purpose.

[28] Alternatively, the leisure can be related to labour input by defining it as $T - x_1$, where T is the total available time. This more accurate specification of the link labour-leisure yields more complicated expressions, and we do not develop it for the sake of the simplicity of the exposition.

To fix ideas we could consider, for example, that x_1 and x_2 are some directly consumed outputs; x_3 is a polluting by-product in one technology that may cause negative externalities for the other technology and for the consumption; x_4 is the household leisure. Such assumptions would determine the signs of the parameters of the model. Other interpretations may correspond to different signs. Let us leave this question of the coefficient signs open since it can always be introduced later in the analysis if needed. By contrast with SQC, CSQC provides us with enough flexibility to deal with the question of interest.

Restrictions of a non-empty and convex choice set can be easily imposed by specifying inequality constraints on combinations of the parameters β_i and γ_i. To simplify the exposition we omit these conditions. For the same reason of ease of exposition, we assume that the two technologies are jointly used for the cases of interest and we omit the positivity constraints for the x_i $(i = 1, \ldots, 4)$. Naturally, all these constraints may be important in practice and several regimes should be examined to solve a complete applied problem. For the sake of the argument of the present example we focus here on the regime where the positivity constraints are not binding. By calculating the first and second derivatives of U, G and H and simplifying them, it is easy to see that CSQC for this model can be written as follows:

$$\begin{pmatrix} x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}x_4^{\beta_4} & = \mu \\ x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}x_4^{\gamma_4} & = \nu \\ \alpha_1\frac{U}{x_1}u_1 + \alpha_2\frac{U}{x_2}u_2 + \alpha_3\frac{U}{x_3}u_3 + \alpha_4\frac{U}{x_4}u_4 = 0 \\ \beta_1\frac{\mu}{x_1}u_1 + \beta_2\frac{\mu}{x_2}u_2 + \beta_3\frac{\mu}{x_3}u_3 + \beta_4\frac{\mu}{x_4}u_4 = 0 \\ \gamma_1\frac{\nu}{x_1}u_1 + \gamma_2\frac{\nu}{x_2}u_2 + \gamma_3\frac{\nu}{x_3}u_3 + \gamma_4\frac{\nu}{x_4}u_4 = 0 \end{pmatrix} \Longrightarrow$$

$$\begin{gathered} (\alpha_1-1)\alpha_1\frac{U}{(x_1)^2}(u_1)^2 + (\alpha_2-1)\alpha_2\frac{U}{(x_2)^2}(u_2)^2 \\ +(\alpha_3-1)\alpha_3\frac{U}{(x_3)^2}(u_3)^2 + (\alpha_4-1)\alpha_4\frac{U}{(x_4)^2}(u_4)^2 \\ +2\alpha_1\alpha_2\frac{U}{x_1x_2}u_1u_2 + 2\alpha_1\alpha_3\frac{U}{x_1x_3}u_1u_3 + 2\alpha_1\alpha_4\frac{U}{x_1x_4}u_1u_4 \\ +2\alpha_2\alpha_3\frac{U}{x_2x_3}u_2u_3 + 2\alpha_2\alpha_4\frac{U}{x_2x_4}u_2u_4 + 2\alpha_3\alpha_4\frac{U}{x_3x_4}u_3u_4 < 0, \end{gathered}$$

where $u_1, \ldots, u_4$ are the coordinates of the vector u of the directions for the quadratic form. The conclusion of the CSQC condition is therefore of the type: for all x and u_4, $U.u(u_4)'\Omega.u(u_4) < 0$, where u is the vector of u_i $(i = 1, ..., 4)$ and Ω is the matrix of the quadratic form in u.

The last three equations of the premises can be used to eliminate u_1, u_2, u_3 by expressing them as functions on u_4. Because of the substitution, only two constraints remain in the premises and a scalar inequality in the conclusion. The latter is therefore convenient as a condition to impose or to check. After calculation, one obtains the following condition:

$$\frac{U}{\delta^2(x_4)^2}\{\alpha_1\left[\alpha_4(-\beta_2\gamma_3+\beta_3\gamma_2)+\alpha_2(\beta_4\gamma_3-\gamma_4\beta_3)+\alpha_3(\gamma_4\beta_2-\beta_4\gamma_2)\right]^2$$

$$+\alpha_2\left[\alpha_4(\beta_1\gamma_3-\beta_3\gamma_1)+\alpha_3(\beta_4\gamma_1-\gamma_4\beta_1)+\alpha_1(\gamma_4\beta_3-\beta_4\gamma_3)\right]^2$$

$$+\alpha_3\left[\alpha_4(-\beta_1\gamma_2+\beta_2\gamma_1)+\alpha_1(\beta_4\gamma_2-\gamma_4\beta_2)+\alpha_2(\gamma_4\beta_1-\beta_4\gamma_1)\right]^2$$

$$+\frac{\alpha_4}{x_4}\left[\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)+\alpha_2(\gamma_1\beta_3-\beta_1\gamma_3)+\alpha_3(\beta_1\gamma_2-\gamma_1\beta_2)\right]^2\}>0,$$

where $\delta = \alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)+\alpha_2(\gamma_1\beta_3-\beta_1\gamma_3)+\alpha_3(\beta_1\gamma_2-\gamma_1\beta_2)$.

The obtained condition is obviously a global condition on parameters and decisions. Note that with the chosen functional forms the constraints in the premises do not interfere with the verification of this conclusion, since any positive value of x_4 is possible in these constraints.

Thus, CSQC in the 'Treble Cobb-Douglas' model yields a simple expression in terms of: (1) values of parameters: $\alpha_i, \beta_i, \gamma_i$ $(i = 1, ..., 4)$, which can be obtained for example from the estimation of demand functions, supply functions and production functions; and (2) values of observable decision x_4. The expression, which is a bound on the value of x_4, is very easy to check when the parameter values are known. It can also be used to improve the efficiency of the model estimation. Finally, conditionally on the knowledge of the values of x_4, the condition can be tested as a joint nonlinear constraint on the parameters.

Let us now see what happens with SSOC. It is clear that the simplifying substitutions are not available and therefore the negativity of the Hessian matrix L_{xx} (often unobservable) must be checked for all values of the u_i $(i = 1, ..., 4)$ and all values of decisions. That is generally a very heavy task that is not normally undertaken for general specifications even by using a grid. This may explain why this verification is most of the time replaced by the imposition of much stronger global restrictions like the strong concavity. The verification of SQC generally brings about the same difficulties. In general, because only one substitution can be done by using the premises of the condition, SQC is likely to be much more difficult to check than CSQC that characterises a smaller set of directions and decisions. However, in the case of Cobb-Douglas type utility, the condition for SQC is known and simple. It implies for all $i : 1 > \alpha_i > 0$. Not only this condition covers a smaller set of parameter values but it does not allow the study of the role of externalities.

The example shows how CSQC can be used in applied work, firstly to weaken the global condition to impose, and secondly to facilitate its verification. In this example it is possible to distinguish easily cases where CSQC is satisfied or not.

Another example where CSQC could play an important role is the collective household model in Browning and Chiappori (1998). Indeed, in this paper the individual utilities for each household member are strongly concave, a technical hypothesis which one may want to relax. If members are 'egoistic', i.e. the utility of a member depends only on her own consumption and not on other members' consumption, then it can be shown that SQC and CSQC are essentially identical in this model. By contrast, when members are 'altruistic', i.e. the utility of a member depends on consumptions of all household members, CSQC is generally strictly weaker than SQC. The optimisation programme for a given member can be easily rewritten with the household budget constraint and the levels of other members' utilities as constraints. Then, it is easy to see that CSQC in that case is a much weaker condition that SQC of each member's utility function. Therefore, there exist members' utility functions that provide the uniqueness of the demand functions of the collective household, without satisfying SQC.

In general, beyond the examined examples, with the above method of calculation, cases can be identified where CSQC is not satisfied and SQC is. In particular, new values of parameters are allowed with CSQC as compared to SQC. Since parameters are here to describe additional flexibility, it is likely that additional functional forms are allowed with CSQC. Indeed, imposing SQC on a functional form for U restricts this form to be positively curved in directions and domains that are not relevant for solving the optimisation problem. CSQC allows more general functional forms by avoiding these unjustified restrictions.

6. Conclusion

In general models, the strong quasi-concavity of the objective function, which is sufficient for theoretical properties of demands in consumer theory, is often arbitrary and weaker global concavity conditions are desirable. We propose a new global concavity condition, the constraint-strong quasi-concavity of the objective function (CSQC) that implies, for models with several nonlinear constraints, the local uniqueness and the smoothness of the decision functions as well as the negativity of the generalised substitution matrix when used jointly with the convexity of the choice set. CSQC is weaker than the strong quasi-concavity and is parsimonious because it is strictly based on what is required globally for the negativity of the generalised substitution matrix. Indeed, it does not restrict the curvature of the objective function in directions that are not compatible with the constraints or not compatible with the augmentation of the objective level. Moreover, CSQC is often easier to check or to impose on specific models than the strong quasi-concavity. Finally, using CSQC allows the extension of the set of possible functional forms as compared with the strong quasi-concavity, thereby increasing modelling flexibility.

Several extensions of this paper are possible. Firstly, in consumer theory, the negativity of the Slutsky matrix is related to revealed preferences axioms (Kihlstrom, Mas-Colell and Sonnenschein, 1976). We conjecture that analog results could be derived by limiting the decisions to the constrained choice set (as in Chavas and Cox, 1993). Such results would express the curvature properties embodied in the CSQC hypothesis. Secondly, since imposing strong concavity globally compromises the flexibility of usual flexible functional forms for cost functions and utility functions, one could investigate if in the presence of several constraints, imposing only CSQC permits to conserve the flexibility of such functional forms. Finally, game theory problems and equilibria problems seem likely to be fertile application fields of CSQC.

Appendices:

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Appendix 1: Definitions of the generalised concavity notions

The generalised convexity properties for a function f are given by the corresponding generalised concavity properties of -f.

Let X be a convex set of R^N.

f : X -> R, a differentiable function, is called concave if and only if

$\forall (x, y) \in X^2,\ f(y) - f(x) \leq \nabla f(x)(y - x).$

f : X -> R, a differentiable function, is called strictly concave if and only if

$\forall (x, y) \in X^2,\ f(y) - f(x) < \nabla f(x)(y - x).$

f : X -> R, a differentiable function, is called quasi-concave if and only if

$\forall (x, y) \in X^2, (f(x) \leq f(y)) \Rightarrow \nabla f(x)(y - x) \geq 0.$

This is equivalent by contraposition to

$\forall (x, y) \in X^2, (\nabla f(x)(y - x) < 0) \Rightarrow (f(x) > f(y)).$

Moreover, if f is quasi-concave and twice differentiable then

$\forall (x, y) \in X^2$, such that $x \neq y, (\nabla f(x)(y - x) = 0) \Rightarrow$

$(y - x)' f_{xx}(x)(y - x) \leq 0.$

f : X -> R, is called strictly quasi-concave if and only if

$\forall (x, y) \in X^2, \forall \lambda \in]0, 1[,\ (f(y) \leq f(x)) \Rightarrow f(x) < f((1 - \lambda)x + \lambda y).$

Then, f is strictly quasi-concave and, if, moreover, it is twice differentiable, it satisfies the same second-order condition as any quasi-concave function.

f : X -> R, a twice differentiable function, is called strongly quasi-concave if and only if

$\forall (x, y) \in X^2$, such that $x \neq y$,

$(\nabla f(x)(y - x) = 0) \Rightarrow (y - x)' f_{xx}(x)(y - x) < 0$.

Then, f is strictly quasi-concave.

f : X -> R, a twice differentiable function, is called constraint-strongly quasi-concave (CSQC) with respect to g, where g : X -> R^m is a differentiable vector function, if and only if

$\forall (x, y) \in X^2$ such that $x \neq y$ and $g(x) = 0_q, (\nabla g(x)(y - x) = 0$ and $\nabla U(x)(y - x) = 0) \Rightarrow$

$(y - x)' f_{xx}(x)(y - x) < 0$.

Appendix 2: Proofs.

Proof of Proposition 3.5:

$\Longrightarrow$] Let U be CSQC over X, let $x^0 \in X, v'v = 1, \bar{t} > 0, x^0 + \bar{t}v \in X$, $\forall i = 1, \dots q$, $g^i(x^0) = 0_q$, $\nabla g^i(x^0)'v = 0$, $\nabla U(x^0)'v = 0$.

Let $h(t) \equiv U(x^0 + tv)$. We calculate $j(t) \equiv h(t) - h(0) = U(x^0 + tv) - U(x^0)$.

A second order Taylor expansion of $U(x^0 + tv)$ gives

$j(t) = (t^2/2)v'\nabla^2 U(x^0)v + t^2\varepsilon(t)$ because $\nabla U(x^0)v = 0$.

U CSQC implies that for x^0 and v such that $\forall i = 1, \dots, q$, $g^i(x^0) = 0_q$, $\nabla g^i(x^0)'v = 0$ and $\nabla U(x^0)'v = 0$, we have $v'\nabla^2 U(x^0)v < 0$. Then, since $\nabla^2 U$ is continuous by hypothesis, $j(t) < 0$ when t is small enough. Therefore, $h(t)$ attains a SLM at $t = 0$ for any $0 < \alpha < Min\{-\frac{1}{2}v'\nabla^2 U(x^0)v : v'v = 1\}$.

$\Longleftarrow$] Let be x$^0 \in X$, $v'v = 1$, $\bar{t} > 0$, $x^0 + \bar{t}v \in X$, $(\forall i = 1, \dots, q, g^i(x^0) = 0$ and $\nabla g^i(x^0)'v = 0) \Rightarrow h(t) \equiv f(x^0 + t\,v)$ attains a SLM at $t = 0$.

Then, $\exists\, \alpha > 0$,

$$j(t) + \alpha\, t^2 \leq 0. \tag{1}$$

Besides, a second order Taylor expansion of $U(x^0 + t\,v)$ about x^0 yields $U(x^0 + t\,v) = U(x^0) + t^2\, v'\, \nabla^2 U(x^0)\, v + t^2\, \varepsilon(t)$ where $\varepsilon(t) \to 0$ when $t \to 0$.

Therefore,

$$j(t) = t^2\, v'\, \nabla^2 f(x^0)\, v + t^2\, \varepsilon(t) \tag{2}$$

because $v' \nabla U(x^0) = 0$. Eqs. 1 and 2 imply that $v' \nabla^2 U(x^0)\, v \leq -\alpha - \varepsilon(t)$, which gives for t small enough $v' \nabla^2 U(x^0)\, v < 0$, which proves that U is CSQC. QED.

Proof of Proposition 3.6:

(a) Let d be a $(g - U)-$admissible direction at x at the frontier of the constraints, then from a Taylor expansion of U at x, we have

$U(x+d) = U(x) + \nabla U(x)'\, d + \frac{1}{2}\, d' \nabla^2 U(x)\, d + \|d\|^2 \varepsilon(d)$ where $\lim \varepsilon(d) = 0$ when $\|d\| \to 0$. Because U is CSQC we have at the frontier of the constraints $d' \nabla^2 U(x)\, d < 0$. Choosing α small enough shows that the hypersurface $\partial E_p(U)$ is strictly below all its tangent hyperplane in $(g - U)-$admissible directions at the frontier of the constraints.

(b) is deduced from the fact that the curvature of the epigraph in direction d at x can be associated with $-d' \nabla^2 U(x) d$ with a positive factor of proportionality to adjust for the local metric of the hypersurface. Under CSQC all points of the frontier of the constraints are 'elliptic' for U in any $g - U-$admissible direction, while they may be 'parabolic or hyperbolic' in the whole space.

(c) is a direct consequence of the definition of CSQC and of the fact that since $\nabla^2 U$ is symmetric there exists an orthogonal basis of eigenvectors of $\nabla^2 U$ whose first q vectors generate the subspace spanned by the ∇g^i (theorem of the incomplete base). The orthogonality condition with respect to ∇U generally enables one to add an unity from the number of possible positive or null eigenvalues of $\nabla^2 U$, although not for an optimum since in that case the KTC are satisfied and ∇U is a linear combination of the ∇g^i.

Proof of Proposition 3.7:

$V = FoU$ gives $\nabla V = (F'oU).\nabla U$,

and $\nabla^2 V = (F'oU)\nabla^2 U + (F''oU)\nabla U \nabla U'$.

Then, with $F' > 0$, and moreover $\forall\, i = 1, \ldots, q$, $\nabla g^{i'} v = 0$ and $\nabla U' v = 0$ at the frontier of constraints implies $v' \nabla^2 U v < 0$, we have $v' \nabla^2 V v < 0$. Therefore, V is CSQC. QED.

Proof of Proposition 4.1:

The premises of the definition of CSQC imply that of the definition of SQC. QED.

Proof of Proposition 4.2:

We first give the proof for one constraint function g only. U is CSQC and g is quasiconvex. Then, for all x and y such that $x \neq y, g(x, \theta) = 0_q, \nabla g(x)(x-$

$y) = 0$ and $\nabla U(x)(x-y) = 0$, we have $(x-y)' \nabla^2 U(x)(x-y) < 0$ and $(x-y)'\nabla^2 g(x)(x-y) \geq 0$. Then, for all vectors $\lambda \geq 0$, for all x and y such that $g(x,\theta) = 0_q, \nabla g(x)(x-y) = 0$ and $\nabla U(x)(x-y) = 0$, we have $(x-y)'\nabla^2 L(x,\lambda)(x-y) < 0$ and the Lagrange function is CSQC for any vector of nonnegative multipliers. In particular, this result is true for optimal solution and with optimal Kuhn-Tucker multipliers. The extension of the proof to several inequality constraints is straightforward because the Hessian matrix of a linear combination of functions is the linear combination of the Hessian matrices of these functions. QED.

Proof of Proposition 4.3: Consequence of Proposition 4.2, applied at an optimal solution. QED.

Proof of Proposition 4.4:

(a) Assume that H is singular. Then, $\exists\, z \in R^n$, $\exists\, r \in R^q$, such that

$$U_{xx}.z + \sum_{i=1}^{q} r_i.g_x^i = 0 \tag{3}$$

$$(z', r')' \neq 0_{n+q} \tag{4}$$

$$g_x^{i\prime}.z = 0, \forall i = 1, \ldots, q \tag{5}$$

$z = 0$ and $r \neq 0$ is impossible since $\sum_i r_i.g_x^i = 0$, is a system of n equations with q unknown variables (r_i, $i = \ldots, q$), which implies $r_i = 0$ for all i because of the hypothesis of constraint qualification (∇g is full rank).

$z \neq 0$ is also impossible because from eqs. 3 and 5 we would obtain $z'.U_{xx}.z = 0$ and $g_x^{i\prime}.z = 0, \forall i = 1, \ldots, q$, in contradiction to CSQC for a solution of the KTC, $\nabla U \in< \nabla g)$. Therefore, no non null vector $(z', r')'$ exists such that $(z', r').H = 0$, which implies that H is non singular.

(b) The proof is similar to that of (a), taking advantage of the fact that $g_x^{i\prime}.z = 0, \forall i = 1, \ldots, q$ implies $z'.g_{xx}^{i\prime}.z \geq 0, \forall i = 1, \ldots, q$.

Proof of Proposition 4.5:

The system describing the KTC, has $n+q$ equations whose vector function is denoted KTC[.], and $2n+p$ variables *(x, λ, θ)*. Because ∇U, ∇g and g are of type C^1, KTC[.] is of type C^1. Since R^{2n+p} is an open set (this is as well the case if the 'regime' of interest is defined by strict inequality for non-binding constraints) and since $|J|$, which is the Jacobian determinant associated with the KTC for their solution in (x, λ), is non-singular at a solution of the KTC when U is CSQC and g is quasiconcave, we can apply the theorem of implicit functions. This generates all the results of the proposition. QED.

Proof of Proposition 4.6: CSQC implies SSOC and SSOC implies S negative semidefinite in the tangent space to the constraints and orthogonal to the constraint gradients (Pauwels, 1979).