Pareto Efficiency with Spatial Rights*

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Abstract: We investigate the problem of constructing a Pareto-efficient social welfare function that respects individual rights when preferences are defined over the location of a public facility. Restricting individual preferences to be either single-peaked or single-dipped, we find necessary and sufficient conditions for the existence of a Pareto-efficient social welfare function that respects individual rights.

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1 Introduction

Imagine that each individual in society lives at some location along a single street. In addition, imagine that a social planner is deciding where to locate a particular facility along this street. This facility may be something desirable like a library or a public swimming pool. However, it could be something undesirable like a nuclear waste plant. To complicate matters, it could be something that some in society view as desirable but others view as undesirable, examples include airports and shopping malls. Where should the planner locate the facility?

In deciding where to put it, the planner might want to take into account the preferences of people on the street. Although individual preferences are not known to the planner, he might be able to infer something about their general properties given the spatial character of the problem. For example, if the facility is desirable then it is reasonable to assume that individual preferences are single-peaked in the sense of Black (1958). Loosely speaking, this means that each individual has a most preferred location for the facility (possibly next to his house) and views locations that are further away in a particular direction from this point as less desirable than locations that are closer. Alternatively, preferences may be single-dipped in the sense of Vickrey (1960) and Inada (1964). This is a reasonable assumption if the facility is undesirable. In this case each individual has a least preferred location for the facility (again, possibly next to his house) and views locations that are further away in a particular direction from this point as more desirable than locations that are closer. Finally, the planner may conclude that the residents of the street contain a combination of both types - some with single-peaked preferences and others with single-dipped preferences.

Imagine that the question of the appropriate preference domain is settled. How would the planner then proceed? One natural approach would be to use a social welfare function - a mapping from an n-tuple of strict orderings into a strict ordering. Ideally the planner would want the social welfare function to satisfy the Pareto condition; if everyone prefers location a to b then so should society. But what other condition might be desirable in addition to Pareto? One possibility is that the planner would want the function to respect individual rights. For example, each resident may have rights over whether the facility is located to his immediate left or immediate right. We can think of these rights as "spatial" rights.[1] In such cases it seems desirable that the resident's ranking of this pair of locations determines the social ranking. This is what Sen (1970a) had in mind when he introduced rights into social choice theory. Spatial rights are the natural extension of Sen's concept of a "personal sphere" to geography.

If a social welfare function exists that satisfies the Pareto condition and respects individual rights (what Sen called "minimal liberalism") then the planner can aggregate individual preferences into a social ranking of alternative locations and choose the location for the facility that is top in this ranking (our ordering assumption guarantees that it is unique). Unfortunately when the domain of preferences is unrestricted, the number of individuals is at least two and the number of alternatives is at least three, if we give just two people rights in Sen's sense then no social welfare function exists.

Sen's theorem depends crucially on an unrestricted preference domain. In social choice theory it is often possible to obtain possibility results if we restrict the domain of preferences (Gaertner (2001)). In the economic environment described above this approach is natural.[2]

In this paper we prove three characterization theorems. Firstly we consider the case where everyone in society has single-peaked preferences. We characterize the only rights assignments that avoid Sen's "Paretian Liberal" paradox in this context. If the social planner knows the assignment of rights in society, and if (and only if) this assignment satisfies our characterizing

[1] The acronym NIMBY ("Not in my back yard") is often used to describe people who appeal to these kinds of rights.

[2] An early paper by Blau (1975) deals with domain restriction in the context of rights. Unlike the present paper Blau has a possibility result for only two individuals.

condition, then a social welfare function exists that can be used to determine the location of the facility. This function satisfies both the Pareto condition and respects individual rights for any possible configuration of single-peaked preferences. If the rights assignment does not satisfy this condition then no such function exists and the planner will have to use some other criterion in order to locate the facility. For social aggregation to be possible two different individuals cannot have rights over two different pairs of locations if the set of locations contained in between one individual's pair partially overlaps with the set of locations contained in between the other individual's pair.

We then show that the same characterizing condition holds in the case of single-dipped preferences. We conclude with a characterization for the "mixed" domain case. In this domain an individual can either have single-peaked preferences or single-dipped preferences. Social aggregation is possible in this domain only if individual rights are "nested". This means that for any two individuals with rights over two different pairs of locations, the set of locations contained in between one individual's pair and the pair itself must be a subset of the set of locations contained in between the other individual's pair. All assignments of spatial rights that satisfy this condition also satisfy our initial condition (but not vice versa). Loosely speaking, this means that as we expand the preference domain the possibility of social aggregation falls.

An important result for our theorems is Lemma 1. This result holds in the unrestricted domain. It provides a unified mathematical explanation as to why several proposed resolutions of Sen's paradox work (Wriglesworth (1985) surveys these attempts). This theme has recently been explored by Saari (1998, 2001). Compared to Arrovian social choice, there has been very little work on rights in economic environments. Exceptions are Campbell (1989) and Campbell and Kelly (1997).

2 Model

Let $N = \{1, ..., n\}$ be a finite set of individuals ($n \geq 2$). Let A be a finite set of alternatives or "locations" ($\#A = r \geq 3$). Alternatives in A will be denoted as $a, b, c... \in A$. A strict ordering is a complete, asymmetric and transitive binary relation.[3] Let $\Re$ denote the set of all strict orderings of A. Throughout this paper $> \in \Re$ represents the fixed, underlying ordering of locations (from left to right).

Each individual $i \in N$ has strict preferences $P_i \in \Re$. We write aP_ib to denote that individual i prefers alternative a to b. A preference profile $\mathbf{P}$ is an n-tuple $(P_1, ..., P_n) \in \Re^n$ where $\Re^n = \prod_{i=1}^n \Re$. The following definition is from Austen-Smith and Banks (1999, p. 94).

Definition 1 *Label A so that $a_{t+1} > a_t$ for all $t = 1, 2, ..., r-1$. An individual's preferences $P_i \in \Re$ are* single-peaked on A with respect to $>$ *if and only if there exists $t \in \{1, ..., r\}$ such that*

$$a_t P_i a_{t+1} P_i a_{t+2} P_i ... P_i a_r \ \& \ a_t P_i a_{t-1} P_i a_{t-2} P_i ... P_i a_1.$$

In the above definition a_t denotes individual i's most preferred location, i.e. $a_t = \{a \in A \mid aP_ib \ \forall b \in A\}$. Let $\Re^{SP}_{>}$ denote the set of all single-peaked preferences on A with respect to $>$. A single-peaked profile with respect to $>$ is an element of $SP = \prod_{i=1}^n \Re^{SP}_{>} \subset \Re^n$.[4]

In this context, a single-peaked profile is one in which (i) the set of locations is ordered along a left-right scale and (ii) each individual has a unique most preferred location (or *ideal point*) on this scale and his ranking of other locations falls as we move away from this point.

In the statement of our theorems the following concept is important. For all $a, b \in A$, let $B(a, b) = \{c \in A \mid a > c > b\} \cup \{c \in A \mid b > c > a\}$.

[3] The definitions of completeness, asymmetry and transitivity used in this paper come from Sen (1970b, p. 8). What we call a strict ordering, Sen calls a "strong" ordering.

[4] Since we are dealing with a one-dimensional location space and a fixed ordering of locations, this "rectangular" profile property is legitimate. See Austen-Smith and Banks (1999, p. 103).

Intuitively, $B(a,b)$ is the set of alternatives that lie in between a and b according to the ordering $>$. By construction $B(a,b) = B(b,a)$. Furthermore, $B(a,b) = \emptyset$ if a and b are next to each other in the ordering $>$.

We now introduce the concept of a rights assignment.

Definition 2 *A* rights assignment $D = (D_1, D_2, ..., D_n)$ *is an n-tuple of subsets of $A \times A$. D satisfies the following properties,* (i) $\forall i \in N$ *&* $\forall a \in A$, $(a,a) \notin D_i$, (ii) $\forall i \in N$ *&* $\forall a, b \in A$, $(a,b) \in D_i \rightarrow (b,a) \in D_i$, *and* (iii) $\forall i, j \in N$, $i \neq j$, $D_i \cap D_j = \emptyset$.

Whenever $(a,b) \in D_i$ we say that individual i has been assigned rights over the pair (a,b). The above conditions are weak. Condition (i) says that no-one has rights over an alternative and itself, condition (ii) says that rights are "two-sided" and condition (iii) says that no two individuals are assigned rights over the same ordered pair. In the literature an additional condition called "coherence" is often assumed (Suzumura (1978)). A rights assignment that is coherent avoids "Gibbard's paradox" (Gibbard (1974), Farrell (1978)). Gibbard's paradox refers to the possibility that a rights assignment can generate a social preference cycle at some profile even without the Pareto principle. Although we have chosen to work with a weaker notion of rights assignment for reasons of generality, it is important to note that a rights assignment that satisfies any of our characterizing conditions is coherent.

For our definition of coherence and also in a subsequent lemma, the following concept is important (Suzumura (1983)). Let R^1 and R^2 be two binary relations. We say that R^2 is an extension of R^1 if and only if (i) $R^1 \subset R^2$ and (ii) $P(R^1) \subset P(R^2)$.[5] If R^2 is an extension of R^1 and R^2 is a strict ordering, we say that R^2 is a strict ordering extension of R^1.

[5] If R is a binary relation then $P(R) = \{(x,y) \mid (x,y) \in R \ \& \ (y,x) \notin R\}$. We have chosen to define the concept of extension in general terms. Our ordering assumption implies that $P(R) = R$.

Definition 3 *A rights assignment D is* coherent *if and only if for every $(P_1, ..., P_n) \in \wp \subseteq \Re^n$ there exists a strict ordering extension T of $\bigcup_{i \in N} \{D_i \cap P_i\}$.*

The following concept was introduced by Sen (1970a).

Definition 4 *A rights assignment D is* minimally liberal *if $\exists i, j \in N,\ i \neq j$, such that $D_i \neq \emptyset$ and $D_j \neq \emptyset$.*

A rights assignment is minimally liberal when there exists at least two individuals that have been assigned some rights.

We now define a social aggregation rule.

Definition 5 *For a given domain of preferences $\wp \subseteq \Re^n$, a* Social Aggregation Rule *(SAR) is a function $G : \wp \rightarrow \Upsilon$ where Υ is the set of all complete and asymmetric binary relations on A. A* Social Welfare Function *(SWF) is a SAR the range of which is restricted to $\Re \subset \Upsilon$.*

Writing $aG(\mathbf{P})b$ means that a is socially preferred to b at profile $\mathbf{P}$.

Definition 6 *A SAR G is* Pareto-efficient *if $\forall \mathbf{P} \in \wp,\ \forall a, b \in A,\ aP_ib\ \forall i \in N \rightarrow aG(\mathbf{P})b$.*

Definition 7 *Given any $\mathbf{P} = (P_1, ..., P_n) \in \wp$, the set of* Pareto pairs at $\mathbf{P}$ *($PP(\mathbf{P})$) is defined as $\{(a, b) \in A^2 \mid aP_ib\ \forall i \in N\}$.*

Definition 8 *A SAR G* respects a rights assignment *D if $\forall i \in N,\ \forall (a, b) \in D_i,\ \forall \mathbf{P} \in \wp,\ aP_ib \rightarrow aG(\mathbf{P})b$.*

A SAR respects a given rights assignment whenever the preferences of individuals who have been allocated rights over particular pairs of locations determine the social ranking of those locations. In the literature there is controversy over the appropriate way to formulate individual rights. We adopt the original Sen approach in this paper. Gaertner, Pattanaik and Suzumura (1992) discuss the “game form” approach. Sen (1992) contains a response.

Definition 9 *A rights assignment D is* admissible given a domain $\wp$ *if and only if there exists a Pareto-efficient SWF defined on $\wp$ that respects D.*

Sen (1970a) proved the following.

Sen's Theorem: *The only admissible rights assignments when $\wp = \Re^n$ are not minimally liberal.*

In fact the only admissible rights assignments in Sen's framework give either just one person rights, or no-one rights.

3 Single-peaked preferences

This is our first characterization theorem.

Theorem 1: *D is admissible given $\wp = SP$ if and only if $\forall i, j \in N,\ i \neq j,\ \forall (a,b) \in D_i,\ \forall (c,d) \in D_j$,*

$$c \in B(a,b) \longleftrightarrow d \in B(a,b). \qquad (*)$$

Throughout this paper we use simple diagrams to illustrate our characterizing conditions. Any assignment of spatial rights can be represented in a diagram. Figure 1 is an example of an extremely simple rights assignment. We represent the fact that an individual has rights over a particular pair of locations by drawing an arc connecting those locations. In this example individual j has rights over the pair (a, c), individual i has rights over the pair (c, e) and individual v has rights over the pair (e, g).

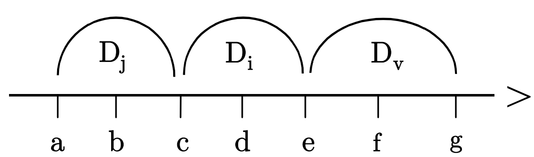

Figure 1: An assignment of spatial rights.

As we explain shortly, this assignment of rights satisfies condition (*). Therefore, in this example, a social welfare function exists that respects the Pareto principle and individual rights for any logically possible profile of single-peaked preferences. Another example of an assignment satisfying condition (*) is Figure 2.

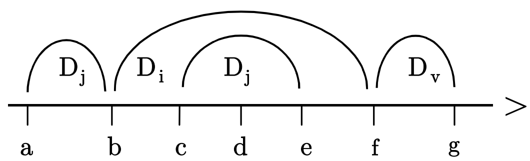

Figure 2: Another assignment satisfying (*).

We want to understand why these two examples satisfy our characterizing condition. In order to do so, we only need to understand what must be true about an assignment of spatial rights in order for condition (*) to be violated. Fortunately, there are only two possible circumstances in which condition (*) is violated. We call them “Property A” and “Property B” respectively, and illustrate them in the following diagrams.

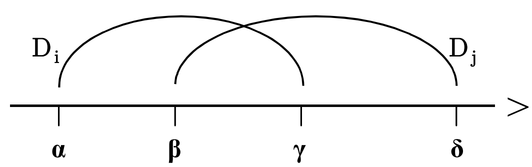

Figure 3: Property A.

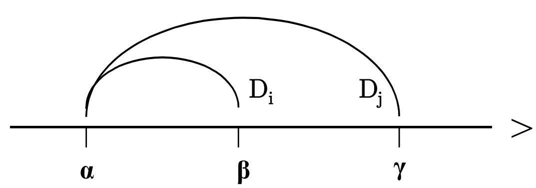

Figure 4: Property B.

In Figure 3, individual i has rights over the pair (α, γ) and individual j has rights over the pair (β, δ). Condition (*) is violated in this example since $\beta \in B(\alpha, \gamma)$ and $\delta \notin B(\alpha, \gamma)$. In Figure 4, individual i has rights over the pair (α, β) and individual j has rights over the pair (α, γ). Condition (*) is violated in this example since $\beta \in B(\alpha, \gamma)$ and $\alpha \notin B(\alpha, \gamma)$.[6]

A social welfare function exists that respects both the Pareto principle and individual rights for any logically possible profile of single-peaked preferences if and only if (i) the rights assignment does not exhibit Property A and (ii) the rights assignment does not exhibit Property B. This explains why social aggregation is possible in the examples represented by Figures 1 and 2. In these examples, Property A and Property B do not occur at any point in the assignment of spatial rights. If Property A or Property B do occur in some assignment of spatial rights, then no social welfare function exists that respects both the Pareto principle and individual rights. For any such assignment, a social preference cycle must be generated at some single-peaked profile.

We now prove Theorem 1. We first prove necessity. Without loss of generality, assume that $\exists i, j \in N,\ i \neq j,\ \exists\,(a, b) \in D_i,\ \exists\,(c, d) \in D_j$ such that

$$c \in B(a, b) \ \&\ d \notin B(a, b). \tag{**}$$

Given (**), there are four possibilities.

Case 1: $a > c > b > d$ (and the symmetric case $d > b > c > a$). Consider the preference profile $\overline{\mathbf{P}} \in SP$ defined as follows. Preferences for individual i are: $c\overline{P}_i a\overline{P}_i b\overline{P}_i d$. Preferences for individual j are: $b\overline{P}_j d\overline{P}_j c\overline{P}_j a$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\overline{P}_i, \overline{P}_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\overline{P}_i$ or $\overline{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SP$. Because $(a, b) \in D_i$ and $a\overline{P}_i b$, we have $aF(\overline{\mathbf{P}})b$ (i). Since F is Pareto-efficient and $b\overline{P}_k d\ \forall k \in N$, we have $bF(\overline{\mathbf{P}})d$ (ii). Given that $(c, d) \in D_j$ and $d\overline{P}_j c$, we have $dF(\overline{\mathbf{P}})c$ (iii). Since $c\overline{P}_k a\ \forall k \in N$ we have

[6] Property B is also satisfied if individual i has rights over the pair (γ, β) and individual j has rights over the pair (α, γ).

$cF(\overline{\mathbf{P}})a$ (iv). However, (i) to (iv) imply that $F(\overline{\mathbf{P}}) \notin \Re$; a contradiction.

Case 2: $d > a > c > b$ (and the symmetric case $b > c > a > d$). Consider the preference profile $\widehat{\mathbf{P}} \in SP$ defined as follows. Preferences for individual i are: $c\widehat{P}_i b\widehat{P}_i a\widehat{P}_i d$. Preferences for individual j are: $a\widehat{P}_j d\widehat{P}_j c\widehat{P}_j b$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\widehat{P}_i, \widehat{P}_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\widehat{P}_i$ or $\widehat{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SP$. Because $(a, b) \in D_i$ and $b\widehat{P}_i a$, we have $bF(\widehat{\mathbf{P}})a$ (i). Since F is Pareto-efficient and $a\widehat{P}_k d\ \forall k \in N$, we have $aF(\widehat{\mathbf{P}})d$ (ii). Given that $(c, d) \in D_j$ and $d\widehat{P}_j c$, we have $dF(\widehat{\mathbf{P}})c$ (iii). Since $c\widehat{P}_k b\ \forall k \in N$ we have $cF(\widehat{\mathbf{P}})b$ (iv). However, (i) to (iv) imply that $F(\widehat{\mathbf{P}}) \notin \Re$; a contradiction.

Case 3: $d = a > c > b$ (and the symmetric case $b > c > d = a$). Consider the preference profile $\widetilde{\mathbf{P}} \in SP$ defined as follows. Preferences for individual i are: $c\widetilde{P}_i b\widetilde{P}_i d = a$. Preferences for individual j are: $d = a\widetilde{P}_j c\widetilde{P}_j b$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\widetilde{P}_i, \widetilde{P}_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\widetilde{P}_i$ or $\widetilde{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SP$. Because $(d = a, b) \in D_i$ and $b\widetilde{P}_i d = a$, we have $bF(\widetilde{\mathbf{P}})d = a$ (i). Since F is Pareto-efficient and $c\widetilde{P}_k b\ \forall k \in N$, we have $cF(\widetilde{\mathbf{P}})b$ (ii). Given that $(c, d = a) \in D_j$ and $d = a\widetilde{P}_j c$, we have $d = aF(\widetilde{\mathbf{P}})c$ (iii). However, (i) to (iii) imply that $F(\widetilde{\mathbf{P}}) \notin \Re$; a contradiction.

Case 4: $a > c > b = d$ (and the symmetric case $b = d > c > a$). Consider the preference profile $\mathbf{P}' \in SP$ defined as follows. Preferences for individual i are: $cP'_i aP'_i b = d$. Preferences for individual j are: $b = dP'_j cP'_j a$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $P'_i, P'_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either P'_i or P'_j.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SP$. Because $(a, b = d) \in D_i$ and $aP'_i b = d$, we have $aF(\mathbf{P}')b = d$ (i). Since F is Pareto-efficient and $cP'_k a$ $\forall k \in N$, we have $cF(\mathbf{P}')a$ (ii). Given that $(c, b = d) \in D_j$ and $b = dP'_j c$, we have $b = dF(\mathbf{P}')c$ (iii). However, (i) to (iii) imply that $F(\mathbf{P}') \notin \Re$; a contradiction. This concludes the proof of necessity.

In order to prove sufficiency we make use of the following concept.

Definition 10 *A chain $S \subseteq A \times A$ is a set of ordered pairs*

$$\left\{(x^1, y^1), ..., (x^\mu, y^\mu), ..., (x^s, y^s)\right\}$$

$(s \geq 2)$ *such that $x^1 = y^s$ and $x^\mu = y^{\mu-1}$ for all $\mu = 2, ..., s$.*

Let O denote the set of all such chains. The following lemma holds in the unrestricted domain $\Re^n$.

Lemma 1: *If D is not admissible given $\wp = \Re^n$ then a chain $S \in O$ exists at some $\mathbf{P}^* \in \Re^n$ such that* (i) $\forall (z, w) \in S$, $(z, w) \in \bigcup_{i \in N} D_i \cup PP(\mathbf{P}^*)$, *and* (ii) $\forall i \in N$ *&* $\forall (z, w) \in S \cap D_i$, $\mathbf{P}^*$ *is such that* $zP^*_i w$.

Proof: Assume that $\forall \mathbf{P} \in \Re^n$ there does not exist a chain with the desired properties. We prove that D is admissible.

To do this we construct a Pareto-efficient social welfare function F that respects D. Take any $\overline{\mathbf{P}} \in \Re^n$ and define the following binary relation:

$$\forall (a, b) \in A^2, \left[(a, b) \in \bigcup_{i \in N} D_i \;\&\; a\overline{P}_i b\right] \longleftrightarrow (a, b) \in H_1(\overline{\mathbf{P}}).$$

(1) Take a pair $(a, b) \in H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$. If any pair in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ has b as its first element, say (b, c), then let $(a, c) \in H_2(\overline{\mathbf{P}})$ unless (a, c) is already in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$. Repeat this process for every other ordered pair in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$. If $H_2 = \emptyset$ then the procedure stops, if not then proceed to (2). (2) Take a pair $(c, d) \in H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}}) \cup H_2(\overline{\mathbf{P}})$. If any

pair in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}}) \cup H_2(\overline{\mathbf{P}})$ has d as its first element, say (d, e), then let $(c, e) \in H_3(\overline{\mathbf{P}})$ unless (c, e) is already in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}}) \cup H_2(\overline{\mathbf{P}})$. Repeat this process for every other ordered pair in $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}}) \cup H_2(\overline{\mathbf{P}})$. If $H_3 = \emptyset$ then the procedure stops, if not then continue "expanding" the set $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ in this manner until we reach some $z \in Z^+ \geq 2$ at which $H_z(\overline{\mathbf{P}}) = \emptyset$.[7] We must reach such a point given the finiteness of A.

We prove that $H(\overline{\mathbf{P}}) \equiv \bigcup_{h=1}^{z-1} H_h(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ is a strict partial ordering. First of all, we prove that $H(\overline{\mathbf{P}})$ is transitive. If not then $\exists a, b, c \in A$ such that $aH(\overline{\mathbf{P}})bH(\overline{\mathbf{P}})cH(\overline{\mathbf{P}})a$. Consider $(a, b) \in H(\overline{\mathbf{P}})$. If $(a, b) \in H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ there is a sequence of elements $\{(x^\mu, y^\mu)\}_{\mu=1}^{s^*}$ $(s^* \geq 1)$ of $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ with the property that $x^1 = a$, $y^{s^*} = b$ and if $s^* > 1$ then $x^\mu = y^{\mu-1}$ for all $\mu = 2, ..., s^*$. If $(a, b) \notin H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ there is a sequence of elements $\{(x^\mu, y^\mu)\}_{\mu=1}^{s^{**}}$ $(s^{**} \geq 2)$ of $H_1(\overline{\mathbf{P}}) \cup PP(\overline{\mathbf{P}})$ with the property that $x^1 = a$, $y^{s^{**}} = b$ and $x^\mu = y^{\mu-1}$ for all $\mu = 2, ..., s^{**}$. Identical reasoning holds for (b, c) and (c, a). If we connect these three sequences together we form a chain S with the following properties: $\forall (z, w) \in S$, $(z, w) \in \bigcup_{i \in N} D_i \cup PP(\overline{\mathbf{P}})$ and $\forall i \in N$ *&* $\forall (z, w) \in S \cap D_i$, the profile $\overline{\mathbf{P}}$ is such that $z\overline{P}_i w$. This contradicts our assumption that no such chain exists. Therefore $H(\overline{\mathbf{P}})$ is transitive. Our assumptions about P_i and D_i guarantee that $H(\overline{\mathbf{P}})$ is asymmetric.

We have proved that $H(\overline{\mathbf{P}})$ is a strict partial ordering. Since $\overline{\mathbf{P}}$ is arbitrary then $\forall \mathbf{P} \in \Re^n$, $H(\mathbf{P}) \equiv \bigcup_{h=1}^{z-1} H_h(\mathbf{P}) \cup PP(\mathbf{P})$ is a strict partial ordering. From Szpilrajn (1930) there is a strict ordering extension $J(\mathbf{P}) \in \Re$ of $H(\mathbf{P})$. Consider the SWF F defined as: $\forall \mathbf{P} \in \Re^n$, $F(\mathbf{P}) = J(\mathbf{P})$. F exists so D is admissible. ■

This result (which can be strengthened to an "if and only if" statement) provides a unified mathematical explanation as to why several proposed resolutions of Sen's paradox work. They work by removing from any profile the possibility of a chain satisfying conditions (i) and (ii).[8]

[7] Z^+ is the set of positive integers.

[8] An example is the "conditional Pareto principle" proposed by Sen (1976) and developed by Suzumura (1978).

We now perform the following operation on S at $\mathbf{P}^*$. For all $(a,b),(b,c) \in S$ such that $(a,b),(b,c) \in PP(\mathbf{P}^*)$ there exists a chain S' at $\mathbf{P}^*$ with fewer elements defined by $S' = (a,c) \cup S \backslash \{(a,b),(b,c)\}$. Let $\overline{S}$ ($\#\overline{S} \geq 3$) denote the chain at $\mathbf{P}^*$ with the smallest number of elements (or one of *the* chains at $\mathbf{P}^*$ with the smallest number of elements). This "minimal" chain possesses the following useful property:

$$\forall (a,b) \in \overline{S} \cap PP(\mathbf{P}^*), (b,c) \in \overline{S} \rightarrow (b,c) \in \bigcup_{i \in N} D_i.$$

To prove sufficiency we show that if condition (*) holds then no such chain can be constructed at any profile in the domain SP. Lemma 1 then implies that D is admissible. To do this we establish two intermediate results.

To motivate these results we give an example of a chain (Figure 5). In this example alternatives in the chain are linked together by arcs and the arrows indicate the direction of preference at the particular profile. The underlying ordering of alternatives is $a > b > c$. Individual i has rights over the pair (a,b) and aP_ib. Individual j has rights over the pair (b,c) and bP_jc. Everyone in society prefers c to a and so (c,a) is a Pareto pair (denoted PP).

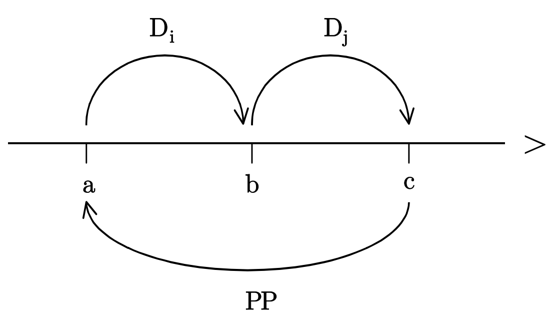

Figure 5: A chain.

However, a chain like this cannot exist. Since individual i prefers a to b it cannot be the case that he also prefers c to a. If so then his preferences are not single-peaked on A with respect to $>$.

The following lemma generalizes this example.

Lemma 2 (Non-contraction property): *Let* $\mathbf{P}^* \in SP$ *and assume that* (i) *a chain* $\overline{S}$ *exists at* $\mathbf{P}^*$ *and* (ii) *condition* (*) *holds. Take any two contiguous elements of* $\overline{S}$, (a,b) *&* (b,c). *Then* $c \notin B(a,b)$.

Proof: We prove by contradiction. Take $(a,b) \in \overline{S}$ and without loss of generality assume that $a > c > b$. There are four possibilities.

Case 1: $(a,b) \in \overline{S} \cap PP(\mathbf{P}^*)$ & $(b,c) \in \overline{S} \cap D_i$ for some $i \in N$. Since $(a,b) \in \overline{S} \cap PP(\mathbf{P}^*)$, it must be that $aP_j^*b \; \forall j \in N$. Since $(b,c) \in \overline{S} \cap D_i$ for some $i \in N$, bP_i^*c. Therefore, $aP_i^*bP_i^*c$ but then $P_i^* \notin \Re_>^{SP}$; a contradiction. Case 2: $(a,b) \in \overline{S} \cap D_i$ for some $i \in N$ & $(b,c) \in \overline{S} \cap PP(\mathbf{P}^*)$. Since $(a,b) \in \overline{S} \cap D_i$, it must be that aP_i^*b. Since $(b,c) \in \overline{S} \cap PP(\mathbf{P}^*)$, it must be that bP_j^*c for all $j \in N$. Therefore, $aP_i^*bP_i^*c$ but then $P_i^* \notin \Re_>^{SP}$; a contradiction. Case 3: $(a,b) \in \overline{S} \cap D_i$ for some $i \in N$ & $(b,c) \in \overline{S} \cap D_j$ for some other $j \in N$. However, this rights assignment violates condition (*). Case 4: $(a,b), (b,c) \in \overline{S} \cap D_i$ for some $i \in N$. This means that $aP_i^*bP_i^*c$ but then $P_i^* \notin \Re_>^{SP}$; a contradiction. ■

The example in Figure 5 violates the non-contraction property. The chain $\overline{S}$ contracts "inwards" when we move from c to a to b.

Lemma 3 (Non-intersection property): *Let* $\mathbf{P}^* \in SP$ *and assume that* (i) *a chain* $\overline{S}$ *exists at* $\mathbf{P}^*$ *and* (ii) *condition* (*) *holds. Take any two elements of* $\overline{S}$, (a,b) *&* (c,d), *where all four alternatives are distinct. Then none of the following conditions can hold*: (I) $a > d > b > c$ (*and the symmetric case* $c > b > d > a$), (II) $b > c > a > d$ (*and the symmetric case* $d > a > c > b$).

Proof: We prove by contradiction. Without loss of generality assume that $a > d > b > c$. Note that it cannot be that $(a,b) \in \overline{S} \cap D_i$ for some $i \in N$ and $(d,c) \in \overline{S} \cap D_j$ for some $j \in N$ with $i \neq j$, since (*) would be violated. In the other four cases: (1) $(a,b), (c,d) \in \overline{S} \cap D_i$ for some $i \in N$, (2) $(a,b), (c,d) \in \overline{S} \cap PP(\mathbf{P}^*)$, (3) $(a,b) \in \overline{S} \cap D_i$ for some $i \in N$ and

$(c, d) \in \overline{S} \cap PP(\mathbf{P}^*)$, and (4) $(a, b) \in \overline{S} \cap PP(\mathbf{P}^*)$ and $(c, d) \in \overline{S} \cap D_i$ for some $i \in N$, it holds for individual $i \in N$ that $aP_i^* b$ & $cP_i^* d$. However, this cannot be true if $a > d > b > c$ because the fact that P_i^* is single-peaked implies that $aP_i^* b \rightarrow aP_i^* c$ and $cP_i^* d \rightarrow cP_i^* a$. This contradicts the asymmetry of P_i^*. ■

Figure 6 illustrates the non-intersection property.

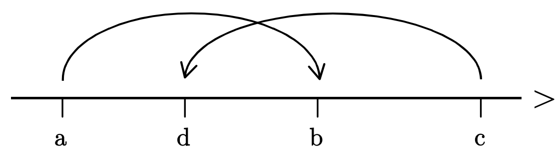

Figure 6: Non-intersection property.

No chain $\overline{S}$ can exhibit the "pattern" illustrated in Figure 6.

To complete the sufficiency part of the proof assume that (i) a chain $\overline{S}$ exists at $\mathbf{P}^* \in SP$, and (ii) condition (*) holds. First of all, we identify the location $a^* \in A$ in $\overline{S}$ that is "farthest to the left" in terms of the ordering $>$. Define the set $K(\overline{S})$ as follows,

$$K(\overline{S}) = \left\{c \in A \mid \exists x \in A \text{ such that either } (c, x) \in \overline{S} \text{ or } (x, c) \in \overline{S}\right\}.$$

$K(\overline{S})$ denotes the set of locations that form the chain $\overline{S}$. Let $a^* \in K(\overline{S})$ denote the location with the following property: $a^* > b \ \forall b \in K(\overline{S}) \backslash \{a^*\}$. This location exists by construction. Since $\#K(\overline{S}) \geq 3$ then $\exists b, c \in K(\overline{S})$ such that $(c, a^*), (a^*, b) \in \overline{S}$. By the non-contraction property, it must be the case that $a^* > c > b$. In addition, it cannot be the case that $(b, c) \in \overline{S}$ or the non-contraction property is violated. Intuitively, this means that the chain $\overline{S}$ cannot be "closed" by moving back to location c once we reach location b. Hence $\exists d \in A$ such that $(b, d) \in \overline{S}$. Note that by the non-contraction property $d \notin B(a^*, b)$. Therefore $b > d$. However, in order for

$\overline{S}$ to exist $\exists(t_1, t_2) \in \overline{S}$ such that $b > t_1$ and $t_2 \in B(a^*, b)$. This violates the non-intersection property. Therefore $\overline{S}$ cannot exist and the proof is complete.

4 Single-dipped preferences

We now consider single-dipped preferences.

Definition 11 *Label A so that $a_{t+1} > a_t$ for all $t = 1, 2, ..., r-1$. An individual's preferences $P_i \in \Re$ are* single-dipped on A with respect to $>$ *if and only if there exists $t \in \{1, ..., r\}$ such that*

$$a_1 P_i a_2 P_i a_3 P_i ... P_i a_t \text{ \& } a_r P_i a_{r-1} P_i a_{r-2} P_i ... P_i a_t.$$

In the above definition a_t denotes individual i's least preferred location, i.e. $a_t = \{a \in A \mid bP_i a \ \forall b \in A\}$. Let $\Re_>^{SD}$ denote the set of all single-dipped preferences on A with respect to $>$. A single-dipped profile with respect to $>$ is an element of $SD = \prod_{i=1}^{n} \Re_>^{SD} \subset \Re^n$.

In this context, a single-dipped profile is one in which (i) the set of locations is ordered along a left-right scale and (ii) each individual has a unique least preferred location on this scale and his ranking of other locations rises as we move away from this point. These preferences are natural in an environment where the facility to be located is undesirable.

The only admissible rights assignments within this domain are again those characterized by condition (*).

Theorem 2: *D is admissible given $\wp = SD$ if and only if $\forall i, j \in N$, $i \neq j$, $\forall (a, b) \in D_i$, $\forall (c, d) \in D_j$,*

$$c \in B(a, b) \longleftrightarrow d \in B(a, b). \tag{*}$$

The necessity part of Theorem 2 mirrors that of Theorem 1. Without loss of generality, assume that $\exists i, j \in N,\ i \neq j,\ \exists\, (a,b) \in D_i,\ \exists\, (c,d) \in D_j$ such that

$$c \in B(a,b) \text{ \& } d \notin B(a,b). \tag{**}$$

Given (**), there are four possibilities.

Case 1: $a > c > b > d$ (and the symmetric case $d > b > c > a$). Consider the preference profile $\overline{\mathbf{P}} \in SD$ defined as follows. Preferences for individual i are: $d\overline{P}_i b\overline{P}_i a\overline{P}_i c$. Preferences for individual j are: $a\overline{P}_j c\overline{P}_j d\overline{P}_j b$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\overline{P}_i, \overline{P}_j \in \Re^{SD}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\overline{P}_i$ or $\overline{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SD$. Because $(a,b) \in D_i$ and $b\overline{P}_i a$, we have $bF(\overline{\mathbf{P}})a$ (i). Since F is Pareto-efficient and $a\overline{P}_k c\ \forall k \in N$, we have $aF(\overline{\mathbf{P}})c$ (ii). Given that $(c,d) \in D_j$ and $c\overline{P}_j d$, we have $cF(\overline{\mathbf{P}})d$ (iii). Since $d\overline{P}_k b\ \forall k \in N$ we have $dF(\overline{\mathbf{P}})b$ (iv). However, (i) to (iv) imply that $F(\overline{\mathbf{P}}) \notin \Re$; a contradiction.

Case 2: $d > a > c > b$ (and the symmetric case $b > c > a > d$). Consider the preference profile $\widehat{\mathbf{P}} \in SD$ defined as follows. Preferences for individual i are: $d\widehat{P}_i a\widehat{P}_i b\widehat{P}_i c$. Preferences for individual j are: $b\widehat{P}_j c\widehat{P}_j d\widehat{P}_j a$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\widehat{P}_i, \widehat{P}_j \in \Re^{SD}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\widehat{P}_i$ or $\widehat{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SD$. Because $(a,b) \in D_i$ and $a\widehat{P}_i b$, we have $aF(\widehat{\mathbf{P}})b$ (i). Since F is Pareto-efficient and $b\widehat{P}_k c\ \forall k \in N$, we have $bF(\widehat{\mathbf{P}})c$ (ii). Given that $(c,d) \in D_j$ and $c\widehat{P}_j d$, we have $cF(\widehat{\mathbf{P}})d$ (iii). Since $d\widehat{P}_k a\ \forall k \in N$ we have $dF(\widehat{\mathbf{P}})a$ (iv). However, (i) to (iv) imply that $F(\widehat{\mathbf{P}}) \notin \Re$; a contradiction.

Case 3: $d = a > c > b$ (and the symmetric case $b > c > d = a$). Consider the preference profile $\widetilde{\mathbf{P}} \in SD$ defined as follows. Preferences for individual i are: $d = a\widetilde{P}_i b\widetilde{P}_i c$. Preferences for individual j are: $b\widetilde{P}_j c\widetilde{P}_j d = a$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\widetilde{P}_i, \widetilde{P}_j \in \Re_{>}^{SD}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\widetilde{P}_i$ or $\widetilde{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SD$. Because $(d = a, b) \in D_i$ and $d = a\widetilde{P}_i b$, we have $d = aF(\widetilde{\mathbf{P}})b$ (i). Since F is Pareto-efficient and $b\widetilde{P}_k c$ $\forall k \in N$, we have $bF(\widetilde{\mathbf{P}})c$ (ii). Given that $(c, d = a) \in D_j$ and $c\widetilde{P}_j d = a$, we have $cF(\widetilde{\mathbf{P}})d = a$ (iii). However, (i) to (iii) imply that $F(\widetilde{\mathbf{P}}) \notin \Re$; a contradiction.

Case 4: $a > c > b = d$ (and the symmetric case $b = d > c > a$). Consider the preference profile $\mathbf{P}' \in SD$ defined as follows. Preferences for individual i are: $b = dP'_i aP'_i c$. Preferences for individual j are: $aP'_j cP'_j b = d$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $P'_i, P'_j \in \Re_{>}^{SD}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either P'_i or P'_j.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = SD$. Because $(a, b = d) \in D_i$ and $b = dP'_i a$, we have $b = dF(\mathbf{P}')a$ (i). Since F is Pareto-efficient and $aP'_k c$ $\forall k \in N$, we have $aF(\mathbf{P}')c$ (ii). Given that $(c, b = d) \in D_j$ and $cP'_j b = d$, we have $cF(\mathbf{P}')b = d$ (iii). However, (i) to (iii) imply that $F(\mathbf{P}') \notin \Re$; a contradiction. This concludes the proof of necessity.

Given any $P \in \Re$, let us define the set $I(P)$ as follows,

$$\forall (a, b) \in A^2, (a, b) \in P \rightarrow (b, a) \in I(P).$$

$I(P)$ is a complete, asymmetric and transitive binary relation. Define the function $\overline{I} : \Re^n \rightarrow \Re^n$ as follows,

$$\forall \mathbf{P} \in \Re^n,\ \overline{I}(\mathbf{P}) = (I(P_1), I(P_2), ..., I(P_n))\,.$$

We now prove sufficiency. From Theorem 1, we know that when condition (*) holds, D is admissible given $\wp = SP$ (i.e. a SWF $F : SP \rightarrow \Re$ exists that it is both Pareto-efficient and respects D). Consider the following SWF $G : SP \rightarrow \Re$,

$$\forall \mathbf{P} \in SP,\ G(\mathbf{P}) = I(F(\mathbf{P})).$$

Since F exists, G must exist too. Now define the SWF $H : SD \rightarrow \Re$ as follows,

$$\forall \mathbf{P} \in SD,\ H(\mathbf{P}) = G(\overline{I}(\mathbf{P})) = I(F(\overline{I}(\mathbf{P}))).$$

The function H exists. We prove that H is Pareto-efficient. Consider any $\mathbf{P}' \in SD$ such that $aP_i'b\ \forall i \in N$. Since F is Pareto-efficient it must be the case that $bF(\overline{I}(\mathbf{P}'))a$. By definition $aG(\overline{I}(\mathbf{P}'))b$ and so $aH(\mathbf{P}')b$. H is Pareto-efficient. It only remains to prove that H respects the rights assignment D. Take any $\mathbf{P}'' \in SD$, any individual $i \in N$ and any pair of alternatives $(a, b) \in D_i$. Suppose that $aP_i''b$. Note that individual i prefers b to a in the profile $\overline{I}(\mathbf{P}'')$. Moreover, we know that the SWF F exists and respects D. Therefore it must be that $bF(\overline{I}(\mathbf{P}''))a$. By definition of H, $H(\mathbf{P}'') = I(F(\overline{I}(\mathbf{P}'')))$ and so it must be the case that $aH(\mathbf{P}'')b$. Individual i's rights over the pair (a, b) are respected. The proof is complete.

5 Mixed domain

We now consider the larger domain $MD = \prod_{i=1}^{n}\{\Re_{>}^{SP} \cup \Re_{>}^{SD}\}$. This is an interesting domain to study and is appropriate when considering location problems within "divided societies" like Northern Ireland. An example is the Ormeau Road in Belfast where conflict occurs periodically. The residents of the Lower Ormeau Road are mainly Catholics while the residents of the Upper Ormeau Road are mainly Protestants. The two communities are separated by the Ormeau Bridge which runs across the River Lagan. Imagine that a social planner wants to locate a police station somewhere along this street. Where should he put it? Northern Irish Catholics tend to view the police as an instrument of the British state, an entity whose presence in Ireland they regard as undesirable. On the other hand, Northern Irish Protestants consider themselves to be British and believe that historically the police have protected them from terrorism. In such an example, it is reasonable to assume that the Catholics of the Ormeau Road have single-dipped preferences (the further away the better) whilst the Protestants have single-peaked preferences (the closer the better).

In such a domain when does a rights-respecting, Pareto-efficient social welfare function exist? The answer is provided in Theorem 3 and illustrated in Figure 7. Social aggregation is possible in this domain only if individual rights are "nested". This means that for any two individuals with rights over two different pairs of locations, the set of locations contained in between one individual's pair and the pair itself must be a subset of the set of locations contained in between the other individual's pair.

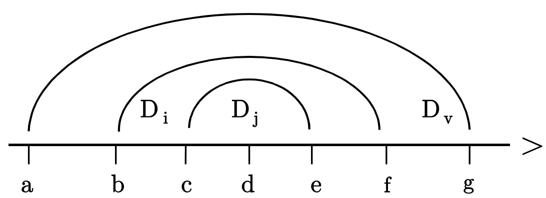

Figure 7: Nested rights.

Note that the rights assignments that satisfy this new characterizing condition are a strict subset of those satisfying condition (*). As we enlarge the domain of preferences the set of admissible rights assignments shrinks. Of course, when the domain is unrestricted (Sen's case) this set contains only non-minimally liberal assignments. Surprisingly, "local" assignments of spatial rights (such as the assignment represented in Figure 1) do not survive this expansion in the domain.

Theorem 3: *D is admissible given $\wp = MD$ if and only if $\forall i, j \in N$, $i \neq j$, $\forall (a, b) \in D_i$, $\forall (c, d) \in D_j$,*

$$B(a,b) \cup \{a,b\} \subseteq B(c,d) \text{ or } B(c,d) \cup \{c,d\} \subseteq B(a,b). \qquad (***)$$

We first prove necessity. Note that any rights assignment that fails to satisfy condition (*) also fails to satisfy condition (***). Therefore, we can appeal to the necessity part of Theorem 1 to prove that in these cases no Pareto-efficient social welfare function exists that respects D given $\wp = MD$. Therefore, we only need to consider those assignments that satisfy condition (*) but do not satisfy condition (***). There are two cases to consider.

Case 1: $a > b > c$ (and the symmetric case $c > b > a$) with $(a, b) \in D_i$ and $(b, c) \in D_j$. Consider the preference profile $\overline{\mathbf{P}} \in MD$ defined as follows. Preferences for individual i are: $c\overline{P}_i a\overline{P}_i b$. Preferences for individual j are: $b\overline{P}_j c\overline{P}_j a$. In this example, individual i's preferences are single-dipped on A with respect to $>$ and individual j's preferences are single-peaked on A with respect to $>$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\overline{P}_i \in \Re^{SD}_{>}$ and $\overline{P}_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\overline{P}_i$ or $\overline{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = MD$. Because $(a, b) \in D_i$ and $a\overline{P}_i b$, we have $aF(\overline{\mathbf{P}})b$ (i). Since F is Pareto-efficient and $c\overline{P}_k a$ $\forall k \in N$, we have $cF(\overline{\mathbf{P}})a$ (ii). Given that $(b, c) \in D_j$ and $b\overline{P}_j c$, we have $bF(\overline{\mathbf{P}})c$ (iii). However, (i) to (iii) imply that $F(\overline{\mathbf{P}}) \notin \Re$; a contradiction.

Case 2: $a > b > c > d$ (and the symmetric case $d > c > b > a$) with $(a, b) \in D_i$ and $(c, d) \in D_j$. Consider the preference profile $\widehat{\mathbf{P}} \in MD$ defined as follows. Preferences for individual i are: $d\widehat{P}_i a\widehat{P}_i b\widehat{P}_i c$. Preferences for individual j are: $b\widehat{P}_j c\widehat{P}_j d\widehat{P}_j a$. In this example, individual i's preferences are single-dipped on A with respect to $>$ and individual j's preferences are single-peaked on A with respect to $>$. Preferences over the remaining alternatives (if any) are constrained only by the requirement that $\overline{P}_i \in \Re^{SD}_{>}$ and $\overline{P}_j \in \Re^{SP}_{>}$. The preferences of the remaining individuals (again, if any) are assumed to be identical to either $\overline{P}_i$ or $\overline{P}_j$.

Since D is admissible there exists a Pareto-efficient SWF F that respects D given $\wp = MD$. Because $(a, b) \in D_i$ and $a\widehat{P}_i b$, we have $aF(\widehat{\mathbf{P}})b$ (i). Since F is Pareto-efficient and $b\widehat{P}_k c$ $\forall k \in N$, we have $bF(\widehat{\mathbf{P}})c$ (ii). Given that

$(c, d) \in D_j$ and $c\widehat{P}_j d$, we have $cF(\widehat{\mathbf{P}})d$ (iii). Since $d\widehat{P}_k a$ $\forall k \in N$ we have $dF(\widehat{\mathbf{P}})a$ (iv). However, (i) to (iv) imply that $F(\widehat{\mathbf{P}}) \notin \Re$; a contradiction. This concludes the proof of necessity.

To prove sufficiency we show that if condition (***) holds then no chain $\overline{S}$ can be constructed at any profile in the domain MD. Lemma 1 then implies that D is admissible.

First of all, we prove that any $\overline{S}$ is such that $\#\overline{S} \geq 4$. Assume not. The only alternative is that $\#\overline{S} = 3$. Since individual preferences are transitive it must be the case that $\exists i, j \in N$ with $i \neq j$ and $\exists (a, b), (b, c) \in \overline{S}$ such that $(a, b) \in D_i$ and $(b, c) \in D_j$. However, any such chain on A violates condition (***) given $>$ and so $\#\overline{S} \geq 4$. Identical reasoning suggests that if a minimal chain $\overline{S}$ exists and condition (***) is satisfied then $(a, b) \in D_i$ implies that $(b, c) \notin D_j$, where $i \neq j$ and where (a, b) and (b, c) denote two contiguous elements of $\overline{S}$. Therefore, if any element of $\overline{S}$ belongs to an individual's D_i set then the next element of $\overline{S}$ must either be a Pareto pair or another element of D_i.

Figure 8 is an example of a chain that satisfies condition (***). The underlying ordering of alternatives is $a > b > c > d$. Individual i has rights over the pair (b, c) and $cP_i b$. Individual j has rights over the pair (a, d) and $aP_j d$. Everyone in society prefers b to a and d to c.

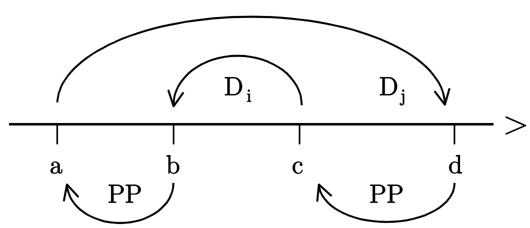

Figure 8: A chain satisfying (***).

However, a chain like this cannot exist. At the above profile we have $bP_j aP_j dP_j c$. This is inconsistent with person j having either single-peaked or single-dipped preferences on A with respect to $>$.

The following lemma generalizes this example.

Lemma 4 (Non-containment property): *Let* $\check{\mathbf{P}} \in MD$ *and assume that* (i) *a chain* $\overline{S}$ *exists at* $\check{\mathbf{P}}$ *and* (ii) *condition* (***) *holds. Take any three contiguous elements of* $\overline{S}$, $(a, b), (b, c)$ *&* (c, d). *Then none of the following conditions can hold*: (I) $b > a > d > c$ (*and the symmetric case* $c > d > a > b$), (II) $b > d > a > c$ (*and the symmetric case* $c > a > d > b$).

Proof: We prove by contradiction. Without loss of generality assume that $b > a > d > c$. There are five possible cases that do not violate condition (***): (1) $(a, b) \in \overline{S} \cap PP(\check{\mathbf{P}})$ and $(b, c), (c, d) \in \overline{S} \cap D_i$ for some $i \in N$, (2) $(a, b), (c, d) \in \overline{S} \cap PP(\check{\mathbf{P}})$ and $(b, c) \in \overline{S} \cap D_i$ for some $i \in N$, (3) $(a, b), (b, c), (c, d) \in \overline{S} \cap D_i$ for some $i \in N$, (4) $(a, b), (c, d) \in \overline{S} \cap D_i$ for some $i \in N$ and $(b, c) \in \overline{S} \cap PP(\check{\mathbf{P}})$, and (5) $(a, b), (b, c) \in \overline{S} \cap D_i$ for some $i \in N$ and $(c, d) \in \overline{S} \cap PP(\check{\mathbf{P}})$. In each case, $\check{P}_i \notin \Re^{SP}_{>} \cup \Re^{SD}_{>}$. ■

The example in Figure 8 violates the non-containment property. In this example location b and location c are "contained" in between location a and location d.

To complete the sufficiency part of the proof assume that (i) a chain $\overline{S}$ exists at $\check{\mathbf{P}} \in MD$, and (ii) condition (***) holds. As before, we identify the location $a^* \in A$ in $\overline{S}$ that is "farthest to the left" in terms of the ordering $>$. Let $a^* \in K(\overline{S})$ denote the location with the following property: $a^* > b \; \forall b \in K(\overline{S}) \backslash \{a^*\}$. This location exists by construction. Since $\#K(\overline{S}) \geq 4$ then $\exists c, b, d \in K(\overline{S})$ such that $(c, a^*), (a^*, b), (b, d) \in \overline{S}$. There are six possible cases and we deal with each in turn.

Case 1: $a^* > c > b > d$. We prove that $(d, c) \notin \overline{S}$. Intuitively, this means that the chain $\overline{S}$ cannot be "closed" by moving back to location c once we reach location d. Given that any chain $\overline{S}$ must involve at least two individuals exercising their rights, if $(d, c) \in \overline{S}$ then condition (***) is violated. Note that the definition of $\overline{S}$ and condition (***) imply that $\exists i \in N$ such that $c\check{P}_i a^* \check{P}_i b$ and so $\check{P}_i \in \Re^{SP}_{>}$. Individual i's most preferred location is an element of $B(a^*, b)$. Furthermore, the definition of $\overline{S}$ and condition (***) imply that either $(c, a^*) \in \overline{S} \cap D_i$ or $(a^*, b) \in \overline{S} \cap D_i$. In order for $\overline{S}$ to exist $\exists (t_1, t_2) \in \overline{S}$ such that $b > t_1$ and $t_2 \in B(a^*, b)$. Since (***) is satisfied it must be the case that either $(t_1, t_2) \in \overline{S} \cap D_i$ or $(t_1, t_2) \in \overline{S} \cap PP(\check{\mathbf{P}})$. In both cases we have $t_1 \check{P}_i t_2$. However, this contradicts the fact that $\check{P}_i \in \Re^{SP}_{>}$.[9] Therefore $\overline{S}$ cannot exist in this case.

Case 2: $a^* > c > d > b$. By the non-containment property $\overline{S}$ cannot exist.

Case 3: $a^* > b > c > d$. In this case, the definition of $\overline{S}$ and condition (***) imply that $\exists i$ such that $c \check{P}_i a^* \check{P}_i b \check{P}_i d$. However, this means that $\check{P}_i \notin \Re^{SP}_{>} \cup \Re^{SD}_{>}$ and so $\overline{S}$ cannot exist.

Case 4: $a^* > b > d > c$. By the non-containment property $(d, c) \notin \overline{S}$. This property also implies that $\exists e \in A$ such that $(e, c) \in \overline{S}$ and $c > e$. Note that the definition of $\overline{S}$ and condition (***) imply that $\exists i$ such that $c \check{P}_i a^* \check{P}_i b$ and so $\check{P}_i \in \Re^{SD}_{>}$. Individual i's least preferred location is an element of $B(a^*, c)$. Furthermore, the definition of $\overline{S}$ and condition (***) imply that either $(c, a^*) \in \overline{S} \cap D_i$ or $(a^*, b) \in \overline{S} \cap D_i$. In order for $\overline{S}$ to exist $\exists (t_1, t_2) \in \overline{S}$ such that $t_1 \in B(a^*, c)$ and $c > t_2$. Since (***) is satisfied it must be the case that either $(t_1, t_2) \in \overline{S} \cap D_i$ or $(t_1, t_2) \in \overline{S} \cap PP(\check{\mathbf{P}})$. In both cases we have $t_1 \check{P}_i t_2$. However, this contradicts the fact that $\check{P}_i \in \Re^{SD}_{>}$. Therefore $\overline{S}$ cannot exist in this case.

Case 5: $a^* > d > b > c$. By the non-containment property $(d, c) \notin \overline{S}$. This property also implies that $\exists e \in A$ such that $(e, c) \in \overline{S}$ and $c > e$. Note that the definition of $\overline{S}$ and condition (***) imply that $\exists i$ such that $c \check{P}_i a^* \check{P}_i b$ and so $\check{P}_i \in \Re^{SD}_{>}$. Individual i's least preferred location is an element of $B(a^*, c)$. Furthermore, the definition of $\overline{S}$ and condition (***) imply that either $(c, a^*) \in \overline{S} \cap D_i$ or $(a^*, b) \in \overline{S} \cap D_i$. In order for $\overline{S}$ to exist $\exists (t_1, t_2) \in \overline{S}$ such that $t_1 \in B(a^*, c)$ and $c > t_2$. Since (***) is satisfied it must be the case that either $(t_1, t_2) \in \overline{S} \cap D_i$ or $(t_1, t_2) \in \overline{S} \cap PP(\check{\mathbf{P}})$. In

[9] It is false that individual i prefers t_1 to t_2 since this is false at every logically possible peak.

both cases we have $t_1 \check{P}_i t_2$. However, this contradicts the fact that $\check{P}_i \in \Re^{SD}_{>}$. Therefore $\overline{S}$ cannot exist in this case.

Case 6: $a^* > d > c > b$. By the non-containment property $\overline{S}$ cannot exist.

This completes the proof of Theorem 3. One consequence of this theorem is that no admissible, minimally liberal assignment of spatial rights exists if $\#A = 3$.

6 Extensions

We have given necessary and sufficient conditions for the existence of a Pareto-efficient social welfare function that respects individual rights under three different assumptions about the preference domain. Although we have restricted the preference domain, our restrictions are quite natural in the economic environment under consideration. Furthermore, these domains are the most widely used restrictions in social choice theory.

Although single-peaked and single-dipped domains have been studied extensively in the literature, they have never been considered in the context of individual rights. This is surprising since the idea that people have spatial rights is a philosophically attractive one. When a social planner has to decide where to locate a public facility on a street, spatial rights are the natural extension of Sen's concept of a "personal sphere".

One conclusion of this paper is that in the single-peaked and single-dipped domains the set of rights assignments that satisfy our characterizing condition is surprisingly rich. There are many possible assignments of spatial rights under which social aggregation is possible. However, in the mixed domain only nested assignments permit aggregation. This is a much more restrictive class of assignments.

It is possible to extend this paper in a number of ways. We have assumed throughout that both individual and social preferences are strict, and this is probably too strong. Another possibility is to consider other profile restrictions.[10] It would also be interesting to consider a continuum of alternatives and a location space of higher dimension.

Another possibility is to prove a general characterization theorem.[11] Our theorems work because our characterizing conditions describe when there can be, or cannot be, cycles in pairwise comparisons. However, it should be possible to identify directly the conditions on profiles that produce such cycles and the conditions that do not. This would enable us to characterize all domains admitting Pareto-efficient social welfare functions that respect individual rights.[12] Help in this direction comes from the work of Saari (1995, 2000, 2001). For example, Saari (2000) shows that cycles in pairwise comparisons are caused by profile components coming from what he terms "Condorcet n-tuples". By removing this portion of a profile, cycles cannot occur. We leave these issues for future work.

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[10] On this see Ward (1965) and Saari and Valognes (1999).

[11] This was suggested to us by the referee.

[12] In the Arrovian tradition this was accomplished by Kalai and Muller (1977).

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