Pricing American Options on Jump-Diffusion Processes using Fourier-Hermite Series Expansions

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Abstract

This paper presents a numerical method for pricing American call options where the underlying asset price follows a jump-diffusion process. The method is based on the Fourier-Hermite series expansions of Chiarella, El-Hassan and Kucera (1999), which we extend to allow for Poisson jumps, in the case where the jump sizes are log-normally distributed. The series approximation is applied to both European and American call options, and algorithms are presented for calculating the option price in each case. Since the series expansions only require discretisation in time to be implemented, the resulting price approximations require no asset price interpolation, and are demonstrated to produce both accurate and efficient solutions when compared with alternative methods, such as numerical integration and the method of lines.

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1 Introduction

Modern financial markets rely heavily on mathematical models and numerical methods when pricing financial derivative securities. In particular, the celebrated models of Black and Scholes (1973) and Merton (1973) for European options have been accepted as a standard fundamental pricing theory from which all other option pricing models have since evolved. One purpose of such evolutions has been to try and find alternative models for financial asset returns that better capture observed market-price activity. There exists considerable evidence, such as Jarrow and Rosenfeld (1984), Ball and Torous (1985), Jorion (1988), Ahn and Thompson (1992) and Bates (1996), demonstrating that observed stock prices and foreign exchange rates are better modelled by jump-diffusion processes, rather than the pure-diffusion process originally suggested. While Merton (1976) offers a closed-form solution for European options under jump-diffusion dynamics, one must apply numerical methods when addressing the American option pricing problem. At present there exist few numerical solution alternatives for the American option under jump-diffusion, and only some of these display sufficiently high levels of computational efficiency whilst retaining the required accuracy. The purpose of this paper is to extend the Fourier-Hermite series expansion method of Chiarella, El-Hassan and Kucera (1999) to the jump-diffusion case, and demonstrate that this numerical approach can offer a highly efficient alternative to existing methods in the task of pricing American call options.

The problem of pricing American options within the Black-Scholes framework remains a contemporary research topic. The earliest exploration of American call option pricing was by McKean (1965), who assumes pure-diffusion dynamics for the underlying asset, and uses an incomplete Fourier transform approach to derive the integral equations for both the price and early exercise boundary. Kim (1990) was the first to verify McKean's results in light of the Black-Scholes risk-neutral pricing model, by taking the limit of the Geske and Johnson (1984) compound option approach as the number of early exercise dates increases without bound. Further confirmation is supplied by Carr, Jarrow and Myneni (1992), along with detailed economic interpretations for a range of American put price representations. Jacka (1991) further contributed to the pure-diffusion analysis by proving the existence and uniqueness of both the price and free boundary for an American put.

Merton (1976) is the first primary example demonstrating how the Black-Scholes model can be extended to consider asset returns following jump-diffusion dynamics, in the case where the market price of jump risk is assumed to be fully diversifiable. The corresponding free boundary problem for American options under these dynamics is presented by Pham (1997), in which he allows the market price of jump-risk to be non-zero, and uses probability arguments to derive the integral equations for the price and free boundary of the American put. Gukhal (2001) generalises Kim's compound option method to cater for Merton's jump-diffusion model. Chiarella and Ziogas (2004) demonstrate how McKean's incomplete Fourier transform method can be used to derive the integral equations for American calls under jump-diffusion. They also present a method for solving these equations via numerical integration based on an iterative generalisation of the techniques used in solving Volterra intergal equations, which are demonstrated by Charella and Ziogas (2003) in the case of an American strangle portfolio under pure-diffusion.

While iterative numerical integration can be used to solve the integral equations of the American call pricing problem, the method is computationally cumbersome. Several alternative methods have been explored, with a view to finding a method that offers more efficiency for the same level of accuracy. Amin (1993) extends the binomial tree model to demonstrate the impact jump-diffusion has on the free boundary and option price when compared with a pure-diffusion model. This idea is further extended by Wu and Dai (2001) in the form of a multi-nomial tree. By considering the American option problem as a variational inequality, Zhang (1997) is able to apply a finite difference method. Carr and Hirsa (2002) also use finite differences, by applying Crank-Nicolson to the partial-integro differential equation for the American put. Mullinaci (1996) uses a discrete time solution for the underlying stochastic differential equation, leading to explicit formulae for the Snell envelope. In the case of American puts, d'Halluin, Forsyth and Vetzal (2003) apply a fixed-point iteration method.

In the pure-diffusion case, Meyer and van der Hoek (1997) use the method of lines to find both the price and free boundary for American call and put options. They demonstrate that the method is highly efficient, and produces accurate results that converge to the true solution as the level of discretisation is increased. Meyer (1998) subsequently extends this idea to Merton's jump-diffusion model, in the case where the density for the jump size is discrete. For a small number of potential jump sizes, Meyer demonstrates that the method of lines can be applied iteratively to find both the price and free boundary for American calls and puts. Again, the method is proven to be convergent, and it displays a substantial level of accuracy.

Chiarella, El-Hassan and Kucera (1999) demonstrate how Fourier-Hermite series expansions can be used to price both European and American options under pure-diffusion dynamics. The method is extremely fast to compute, and yields highly accurate prices, at the cost of some loss of accuracy in the free boundary estimate near expiry. An additional benefit is that unlike any of the approaches cited previously, Fourier-Hermite series require only that the time dimension be discretised, since our estimate of the price will be given in terms of continuous basis functions of the underlying asset price. Furthermore, the option price sensitivities, such as delta and gamma, can be readily calculated from the polynomial price estimate using direct differentiation.

In this paper we explore another alternative numerical method for the evaluation of American call options under Merton's jump-diffusion model. We propose to extend the path-integral approach of Chiarella, El-Hassan and Kucera (1999) to the jump-diffusion case by considering an American call option where the density for the jump sizes is log-normal. This corresponds to one of the examples considered by Merton (1976) for European options. It is anticipated that the Fourier-Hermite method will be well-suited to this problem, since the log-normal density is naturally related to the orthogonality-weighting function for Hermite polynomials.

The remainder of this paper shall be as follows. Section 2 establishes the pricing problem in the case of a European call option with log-normally distributed jump sizes. Section 3 details how Fourier-Hermite series can be used to approximate the solution for a European call. The method is expanded to the American call case in Section 4, with a discussion of numerical implementation issues given in Section 5. Some numerical results are presented in Section 6, with price and free boundary comparisons between the series-expansion, numerical integration and method of lines solutions. Conclusions are provided in Section 7, with most of the details for mathematical proofs provided in appendices.

2 Problem Statement

Let $C(S_t, t)$ denote the price of an option contract written on the underlying asset S_t at present time t. $C(S_t, t)$ has strike price K, and matures at time $T > t$. Following Merton (1976), we assume that S_t follows a jump-diffusion process, whose risk-neutral dynamics are given by

$$dS_t = (r - q - \lambda k)S_t dt + \sigma S_t dW + (Y - 1)S_t d\bar{q}, \tag{1}$$

where r is the risk-free rate, q is the continuously compounded dividend yield of S_t, σ is the instantaneous volatility per unit time and W is a standard Wiener process. For the jump component, $\bar{q}$ is a Poisson process whose increments satisfy

$$d\bar{q} = \begin{cases} 1, & \text{with probability } \lambda dt, \\ 0, & \text{with probability } (1 - \lambda dt). \end{cases}$$

We allow the proportional jump size, Y, to be a random variable with probability measure Q_Y, and corresponding density function $G(Y)$. Thus the expected jump size, k, is given by

$$k = \mathbb{E}^{Q_Y}[Y - 1] = \int_0^\infty (Y - 1)G(Y)dY.$$

For the purpose of this paper we shall assume that $G(Y)$ is a log-normal density. Specifically, as in Merton (1976) let

$$G(Y) = \frac{1}{Y\delta\sqrt{2\pi}} e^{\frac{\ln Y - (\gamma - \delta^2/2)^2}{2\delta^2}}, \tag{2}$$

where we set $\gamma \equiv \ln(1 + k)$, and δ^2 is the variance of $\ln Y$.

Given the stochastic differential equation (SDE) in (1), we can solve the corresponding Kolmogorov backward equation to find the transition density for S_t. Let $p(S_T, T|S_t, t)$ denote the probability of observing price S_T at future time T, given that we observe price S_t at the current time t, where S_t follows the risk-neutral dynamics in (1). The transition density is therefore

$$p(S_T, T|S_t, t) = \frac{e^{-\lambda(T-t)}}{S_T\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n}{n! v_n \sqrt{T-t}} \exp\left\{ \frac{-[\ln(S_T/S_t) - (r_n - q - v_n^2/2)(T-t)]^2}{2v_n^2(T-t)} \right\} \quad (3)$$

where $r_n = r - \lambda k + n\gamma/(T-t)$ and $v_n^2 = \sigma^2 + n\delta^2/(T-t)$. Thus $p(S_T, T|S_t, t)$ is a Poisson-weighted sum of log-normal density functions, where each density in the sum is considered on the condition that n jumps have been observed in the time interval $(T-t)$.

As in Merton (1976), we assume that the jump-risk can be fully diversified by the option holder. Applying the Feynman-Kac formula, the price of $C(S_t, t)$ is given by

$$C(S_t, t) = \mathbb{E}_t[g(S_T)] = e^{-r(T-t)} \int_0^{\infty} g(S_T) p(S_T, T|S_t, t) dS_T, \quad (4)$$

where $g(S_T) \equiv C(S_T, T)$ is the payoff function for $C(S_t, t)$.

In order to apply the Fourier-Hermite expansion technique, we will need to transform equation (4) to one where the domain of integration spans the interval $(-\infty, \infty)$. This is achieved by the change of variable $\xi_T = \ln(S_T/K)/\theta$, where θ is a "volatility scaling" constant[1] whose value depends upon the relative values of σ, λ, γ and δ. Furthermore, let $Kf(\xi_t, t) = C(S_t, t)$. Under this transformation, equation (4) becomes

$$f(\xi_t, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{g(Ke^{\theta\xi_T})}{K} \Pi(\xi_T, T|\xi_t, t) d\xi_T, \quad (5)$$

where

$$\Pi(\xi_T, T|\xi_t, t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n e^{-\lambda(T-t)} \theta}{n! v_n \sqrt{2\pi(T-t)}} \exp\left\{ \frac{-[\xi_T - v_n \theta^{-1} \sqrt{2(T-t)} \mu_n(\xi_t, T-t)]^2}{2v_n^2 (T-t)\theta^{-2}} \right\}, \quad (6)$$

[1]In the pure-diffusion case, Chiarella, El-Hassan and Kucera (1999) set $\theta \equiv \sigma$.

with

$$\mu_n(\xi_t, T-t) = \frac{\theta}{v_n\sqrt{2(T-t)}}\left[\xi_t + \left(r_n - q - \frac{v_n^2}{2}\right)\frac{(T-t)}{\theta}\right]. \tag{7}$$

Using the Chapman-Kolmogorov equation, it is possible to form a backward recursion for the transformed price $f(\xi_t, t)$. Firstly, discretise the time domain into J sub-intervals, each of length Δt. Introducing the notation $f^j(\xi_j) \equiv f(\xi_{j\Delta t}, j\Delta t)$, with $f^J(\xi_J) = g(Ke^{\theta\xi_T})/K$, we can apply the same methods as used in Chiarella, El-Hassan and Kucera (1999) to express $f^{j-1}(\xi_{j-1})$ as

$$f^{j-1}(\xi_{j-1}) = e^{-r\Delta t}\int_{-\infty}^{\infty} f^j(\xi_j)\Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1})d\xi_j, \quad (j = J, J-1, \cdots, 1). \tag{8}$$

Note that $f^0(\xi_0)$ represents the transformed option price at the current time t.

To evaluate the integral term in equation (8), we will estimate $f^j(\xi_j)$ using a Fourier-Hermite series expansion. Chiarella, El-Hassan and Kucera (1999) recommend the use of Hermite polynomials because their weighting function is closely related to the functional form of $\Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1})$. Furthermore, series expansions have the advantage that they result in a price estimate which is a continuous function of the underlying, eliminating the need to extrapolate prices for various values of ξ_t.

3 Evaluation of European Call Options

We begin our application of the Fourier-Hermite series expansion method by firstly considering the case of a European call option. This example will allow us to provide a clear explanation of how the Hermite series method works before considering the added complexity that results from having an early exercise feature. In addition, there are several key results that arise from the European case which are required for the American option, making the European problem an efficient starting point for the American call.

In the case of the European call, the payoff function $g(S_T)$ becomes

$$g(S_T) = \max(S_T - K, 0),$$

and therefore

$$f^J(\xi_J) = \max(e^{\theta \xi_J} - 1, 0).$$

Substituting for $\Pi(\xi_j, t_j | \xi_{j-1}, t_{j-1})$ in equation (8), we have

$$f^{j-1}(\xi_{j-1}) = e^{-r\Delta t} \int_{-\infty}^{\infty} f^j(\xi_j) \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n e^{-\lambda \Delta t} \theta}{n! v_n \sqrt{2\pi \Delta t}} \exp\left\{ \frac{-[\xi_j - v_n \theta^{-1} \sqrt{2\Delta t} \mu_n(\xi_{j-1}, \Delta t)]^2}{2 v_n^2 \Delta t \theta^{-2}} \right\} d\xi_j$$

where we note that $r_n = r - \lambda k + n\gamma/\Delta t$, and set $\hat{v}_n^2 \equiv (\sigma^2 + n\delta^2/\Delta t)/\theta^2 = v_n^2/\theta^2$. Changing the variable of integration from ξ_j to $\hat{v}_n \sqrt{2\Delta t} \xi_j$ gives

$$f^{j-1}(\xi_{j-1}) = \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{-\infty}^{\infty} f^j(\hat{v}_n \sqrt{2\Delta t} \xi_j) e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} d\xi_j. \quad (9)$$

Next we expand $f^j(\xi_j)$ in a Fourier-Hermite series according to

$$f^j(\xi_j) = \sum_{m=0}^{\infty} \alpha_m^j H_m(\xi_j), \quad (10)$$

where the α_m^j coefficients are given by[2]

$$\alpha_m^j = \frac{1}{2^m m!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_j^2} f^j(\xi_j) H_m(\xi_j) d\xi_j. \quad (11)$$

For practical purposes, we must truncate the summation in (10) at some finite number of basis functions, N. Thus our goal is to determine the coefficients α_m^j.

Proposition 1:

[2]Refer to Abramowitz and Stegun (1970) for standard results regarding Hermite polynomials.

The coefficients α_m^j *can be generated recursively using the relationship*

$$\alpha_m^{j-1} = e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{N} A_{m,i}^{(n)} \alpha_i^j, \quad (j = J-1, J-2, \cdots, 2, 1), \qquad (12)$$

where the $A_{m,i}^{(n)}$ *terms are given by*

$$A_{m,i}^{(n)} = \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + z w_n) dz, \qquad (13)$$

with

$$b_n = \left(r_n - q - \frac{v_n^2}{2} \right) \frac{\Delta t}{\theta},$$

and

$$w_n = \sqrt{1 + 2\Delta t \hat{v}_n^2}.$$

Proof: Refer to Appendix A.1.

□

In order to implement the recursion (12) for the coefficients of the Hermite expansions, we must first evaluate (13). By using the recurrence relations for Hermite polynomials (Abramowitz and Stegun, 1970) we can also generate recursions for the $A_{m,i}^{(n)}$ terms.

Proposition 2:

The terms $A_{m,i}^{(n)}$*, defined by equation (13) can be found using the recurrence relation*

$$A_{m,i}^{(n)} = \frac{i}{m} A_{m-1,i-1}^{(n)}, \quad (m, i = 1, 2, \cdots, N), \qquad (14)$$

where

$$A_{0,i}^{(n)} = 2b_n A_{0,i-1}^{(n)} + 2(i-1)(w_n^2 - 1) A_{0,i-2}^{(n)}, \quad (i = 2, 3, \cdots, N), \qquad (15)$$

$$A_{0,0}^{(n)} = 1, \quad A_{0,1}^{(n)} = 2b_n,$$

and

$$A_{m,i}^{(n)} = 0, \; for\; m > i.$$

Proof: Refer to Appendix A.2.

□

Combining the results of Propositions 1 and 2, we now have all that is required to determine the α_m^j coefficients, with the exception of those at time step $(J-1)$. As recommended by Chiarella, El-Hassan and Kucera (1999) we avoid expanding the piecewise linear payoff function $f^J(\xi_J)$ in a Fourier-Hermite series, and instead evaluate the initial α_m^{J-1} coefficients directly.

Proposition 3:

The coefficients at the first time step prior to expiry, α_m^{J-1}, are given by the recurrence relation

$$\begin{aligned}\alpha_m^{J-1} &= \frac{\theta}{2m}\left[\alpha_{m-1}^{J-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^{m-1}(m-1)!}\right.\\ &\qquad\left.\times\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^{m-1}}\frac{1}{\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-(b_n/w_n)^2}\right],\\ &(m = 2, 3, \cdots, N),\end{aligned} \tag{16}$$

with

$$\begin{aligned}\alpha_0^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2}\\ &\quad\times\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}erfc\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right) - erfc\left(-\frac{b_n}{w_n}\right)\right\},\end{aligned} \tag{17}$$

and

$$\alpha_1^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{2}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}erfc\left(-\frac{b_n}{w_n}-\frac{\theta w_n}{2}\right)\right\}. \tag{18}$$

Proof: Refer to Appendix A.3.

□

At this point we now have all that is required to find the European call price using Hermite series expansions, with the exception of the value of the scaling parameter θ. The issue of selecting appropriate θ values is discussed at length in Section 5, but we note at this point that Merton (1976) provides a closed-form solution for the European call price under the dynamics given by (1)-(2). It is thereby possible for us to choose θ such that the Hermite-series result accurately reproduces the closed-form solution.

4 Evaluation of American Call options

With the European call solution using Fourier-Hermite series established, we now address the task of pricing an American call option. Given the same underlying dynamics from (1)-(2), the American call price is given by

$$C_A(S_t, t) = \max_{t \leq \tau \leq T} \{\mathbb{E}_t[e^{-r(\tau - t)} \max(S_\tau - K, 0)]\}. \tag{19}$$

The expectation is taken over the range of possible stopping times, τ. The optimal stopping time τ^* is the smallest time for which it is optimal to exercise early, and is defined according to

$$\tau^* = \inf\{s \in [t, T] : F(S_s, s) = S_s - K\}.$$

Applying the same time discretisation as was used for the European call, we can evaluate the American call price using the backward recursion

$$C_A(S_t, t) = \max\{\max(S_\tau - K, 0), e^{-r\Delta t}\mathbb{E}_t[C_A(S_{t+\Delta t}, t + \Delta t)]\}, \quad (0 \leq t \leq T).$$

This is equivalent to finding the discounted expected call value at time step t, given the value at time $t + \Delta t$, and then applying the external max[ ] operator to the price profile for all relevant values of S to to determine at which underlying asset values early exercise has become optimal. This is the same method commonly applied when pricing American options using binomial trees and finite difference methods.

Using the same change of variable for the underlying from Section 2, and defining $KF^j(\xi_j) \equiv C_A(S_{j\Delta t}, j\Delta t)$, the value of the American call becomes

$$F^{j-1}(\xi_{j-1}) = \max\{\max(e^{\theta\xi_{j-1}} - 1, 0), e^{-r\Delta t}\mathbb{E}_{t_{j-1}}[F^j(\xi_j)]\}, \quad (j = J, J-1, \cdots, 1).$$

As demonstrated by Chiarella, El-Hassan and Kucera (1999), we can account for the early exercise feature within the Fourier-Hermite series expansion method by way of a three-step procedure, implemented for $j = J, J-1, \cdots, 1$:

Step 1. Determine $V^{j-1}(\xi_{j-1})$, which is given by

$$\begin{aligned} V^{j-1}(\xi_{j-1}) &= e^{-r\Delta t}\mathbb{E}_{t_{j-1}}[F^j(\xi_j)] \\ &= e^{-r\Delta t}\int_{-\infty}^{\infty} \Pi(\xi_j, t_j|\xi_{j-1}, t_{j-1})F^j(\xi_j)d\xi_j. \end{aligned} \tag{20}$$

This is the value at t_{j-1} of the American call option unexercised.

Step 2. Solve for the early exercise value of the state variable at time t_{j-1}, denoted by ξ^*_{j-1}. This is the value of ξ which solves

$$V^{j-1}(\xi) = e^{\theta\xi} - 1. \tag{21}$$

Step 3. The value of the American call at time t_{j-1} is determined by

$$F^{j-1}(\xi_{j-1}) = \begin{cases} V^{j-1}(\xi_{j-1}) & \text{for } -\infty < \xi_{j-1} < \xi^*_{j-1}, \\ e^{\theta\xi_{j-1}} - 1 & \text{for } \xi^*_{j-1} < \xi_{j-1} < \infty. \end{cases} \tag{22}$$

The most complicated component in this three-step procedure is the calculation of $V^{j-1}(\xi_{j-1})$

in Step 1. This calculation is achieved by first expanding $V^{j-1}(\xi_{j-1})$ in a Fourier-Hermite series according to

$$V^{j-1}(\xi_{j-1}) = \sum_{m=0}^{N} \alpha_m^{j-1} H_m(\xi_{j-1}), \tag{23}$$

whose coefficients are given by the Proposition 4.

Proposition 4:

The coefficients α_m^{j-1} are generated recursively using the relationship

$$\alpha_m^{j-1} = \gamma_m^{j-1} + e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{N} A_{m,i}^{j,n} \alpha_i^j, \quad (j = J-1, J-2, \cdots, 2, 1), \tag{24}$$

where the $A_{m,i}^{j,n}$ terms are given by

$$\begin{aligned} A_{m,i}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_k^{(n)}} e^{-z^2} H_m(z) H_i(b_n + w_n z) dz, \\ & \quad (m, i = 0, 1, 2, \cdots, N). \end{aligned} \tag{25}$$

The γ_m^{j-1} terms are found recursively using

$$\begin{aligned} \gamma_m^{j-1} &= \frac{\theta \gamma_{m-1}^{j-1}}{2m} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{-(z_j^{(n)})^2}}{w_n^m \sqrt{\pi}} \\ & \quad \times \left\{ H_{m-1}(z_j^{(n)})[e^{\theta(b_n + w_n z_j^{(n)})} - 1] + w_n \theta H_{m-2}(z_j^{(n)}) \right\}, \quad (m = 2, \cdots, N), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \gamma_0^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \left\{ e^{b_n \theta} e^{\frac{w_n^2 \theta^2}{4}} erfc\left(z_j^{(n)} - \frac{w_n \theta}{2} \right) - erfc(z_j^{(n)}) \right\}, \\ \gamma_1^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n} \Big\{ \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} e^{\theta(w_m z_j^{(n)} + b_n)} \\ & \quad + \frac{\theta w_n}{2} e^{b_n \theta} e^{\frac{w_n^2 \theta^2}{4}} erfc\left(z_j^{(n)} - \frac{w_n \theta}{2} \right) - \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} \Big\}, \end{aligned}$$

with

$$z_j^{(n)} = \frac{\xi_j^* - b_n}{w_n},$$

and b_n, w_n are as defined in Proposition 1.

Proof: Refer to Appendix B.1.

□

At this point, we are again required to evaluate an integral equation, in this case (25), in order to implement the recurrence for α_m^{j-1}. By use of the recurrence relations for Hermite polynomials, we can develop a recurrence to find the $A_{m,i}^{j,n}$ terms for the American call.

Proposition 5:

The terms $A_{m,i}^{j,n}$, defined by equation (25) can be generated by use of the recurrence relation

$$\begin{aligned} A_{m,i}^{j,n} &= \frac{i}{m}A_{m-1,i-1}^{j,n} - \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}H_{m-1}(z_j^{(n)})H_i(b_n + w_n z_j^{(n)})e^{-(z_j^{(n)})^2}, \quad (27) \\ & (m, i = 1, 2, \cdots, N), \end{aligned}$$

where

$$A_{m,0}^{j,n} = -\frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}H_{m-1}(z_j^{(n)}), \quad (m = 1, 2, \cdots, N), \quad (28)$$

$$\begin{aligned} A_{0,i}^{j,n} &= 2(w_n^2 - 1)(i-1)A_{0,i-2}^{j,n} + 2b_n A_{0,i-1}^{j,n} \\ &\quad - \frac{w_n}{\sqrt{\pi}}H_{i-1}(b_n + w_n z_j^{(n)})e^{-(z_j^{(n)})^2}, \quad (i = 2, 3, \cdots, N), \quad (29) \end{aligned}$$

with

$$A_{0,0}^{j,n} = \frac{1}{2}erfc(-z_j^{(n)}),$$

and

$$A_{0,1}^{j,n} = b_n erfc(-z_j^{(n)}) - \frac{w_n}{\sqrt{\pi}}e^{-(z_j^{(n)})^2}.$$

Proof: Refer to Appendix B.2.

□

All that remains is to initiate the algorithm with respect to time. As for the European call, this requires us to calculate α_m^{J-1}. Since the American call has the same payoff as the European call, and the early exercise condition is simply given by the value of the underlying asset relative to the strike price[3], the α_m^{J-1} coefficients for the American call are the same as those for the corresponding European option. Thus for the first time step, α_m^{J-1} are given by equations (16)-(18) from Proposition 3.

5 Numerical Implementation: American Call

In order to numerically implement the three-step backwards recursion for the American call, we must address two further issues. The first is the matter of solving for the optimal exercise boundary, ξ_j^*, at each time step. This is achieved by applying a root-finding method to equation (21) in Step 2. Here we use the same iterative method supplied by Chiarella, El-Hassan and Kucera (1999) for the pure-diffusion case. Specifically, ξ_{j-1}^* is given by

$$\xi_{j-1}^{i+1} = \frac{1}{\theta}\ln(1 + V^{j-1}(\xi_{j-1}^i)) \text{ for } i = 0, 1, 2, \cdots$$

which is iterated until $|\xi_{j-1}^{i+1} - \xi_{j-1}^i| < \varepsilon$ for some arbitrarily small ε, where $\xi_{j-1}^0 = \xi_j^*$. We also assume that $\xi_J^* = 0$, since it is known that for an American call, $\xi_J^* \geq 0$. This method typically displays fast convergence, but in the cases where it does not, it can be replaced with any appropriate alternative, such as the bisection method.

The second unresolved issue at this point is the form of the scaling parameter θ. In the pure-diffusion case (i.e. when $\lambda = 0$), Chiarella, El-Hassan and Kucera (1999) set $\theta = \sigma$. This has the effect of transforming the problem to one with a unit coefficient for the diffusion term. While the authors present no details on the purpose of this transformation, practical experiments demonstrate that the results of the Hermite series expansion method are far more

[3]Strictly speaking, the free boundary at expiry time, J, is equivalent to the strike price, K. We also know the limit of the free boundary as time to expiry tends to 0^+, as demonstrated in by Chiarella and Ziogas (2004). This limit has no impact on the value of the payoff function at expiry, thus making the value of $z_J^{(n)}$ irrelevant for the purpose of calculating α_m^{J-1}.

accurate when this volatility scaling transformation is applied.

In the jump-diffusion case, it is not as simple to perform an equivalent volatility scaling to the jump-diffusion SDE (1). The theoretical equivalent to the pure-diffusion case would be to define θ as

$$\theta^2 = \sigma^2 + \lambda(e^{2\gamma+\delta^2} - 2e^{\gamma} + 1),$$

however in practice this does not consistently produce sufficiently accurate prices. In particular, when the jump component is significantly volatile, such a definition appears to consistently underestimate θ. Furthermore, there is evidence that when the diffusion volatility is significantly large in relation to the volatility contributed by the jump term, then $\theta = \sigma$ can often prove sufficient, and the more complex definition leads to an overestimation of θ.

While there is no closed-form solution for the American call price under the dynamics in (1), there is a formula for the corresponding European call, derived by Merton (1976). By comparing the Fourier-Hermite series solution for the European call to the exact solution, we are able to numerically explore the values of θ that maximise the accuracy of the method. Such analysis demonstrates that θ is clearly a function of four SDE parameters, such that

$$\theta \equiv \theta(\sigma, \lambda, \gamma, \delta).$$

Determining the exact functional form of θ, however, is not as straightforward, due to the most natural starting point proving ineffective, and the complex four-dimensional form required.

Without a specific function for θ, we instead propose a simple optimisation method based on European options. Given Merton's closed-form solution for the European call, we first select a value of θ such that the Hermite series solution is sufficiently accurate in a neighbourhood around the strike. This accuracy can be assessed using an arbitrary error measure (such as the root mean square error) for a range of spot prices centred at K. When generating our results in this paper, we estimate θ by trial and error to around 2-3 significant figures. Our aim is not to develop an efficient optimisation technique for selecting θ, but rather to demonstrate that a sufficiently optimal value of θ exists, and that this can be confirmed by use of the pricing formula for the European call.

6 Results

We now demonstrate the accuracy and efficiency of the Fourier-Hermite series expansion method by generating prices for the American call under a range of parameter values. As a basis for comparison, we also calculate the call prices using two alternative methods. The first method is direct numerical integration of the integral equations for the price and free boundary of the American call. A derivation of these equations, using McKean's incomplete Fourier transform method, is provided by Chiarella and Ziogas (2004), along with a corresponding numerical integration scheme. In using this method, we initially discretise the time-domain into 50 steps. The process is then repeated using 100 time steps, and the two results are combined into a final solution using Richardson extrapolation. We also apply a fine grid for the initial 4 time steps, consisting of 40 sub-steps, to help improve the free boundary estimate near expiry.

Since the existing literature offers no specific numerical method as the "true" solution for the problem at hand, call prices are also generated by a second method for comparison purposes. Given that Meyer (1998) proves the method of lines is convergent for American calls and puts with discrete jumps, we shall use it as an additional benchmark for the Fourier-Hermite method. We implement the method of lines for the American call as outlined by Meyer, with a few minor modifications. For all necessary interpolations, we use cubic splines rather than the cubic Lagrangian suggested by Meyer. 50 time steps are used to maintain consistency with the numerical integration results. We apply 10,000 space steps in the region $0 \leq S \leq 4$, in the case where the strike is equal to 1. The large number of space steps was necessary to ensure that the resulting free boundary was sufficiently smooth. Since the method demands that the distribution for the jump sizes be discrete, an approximation was used for the log-normal density, $G(Y)$, consisting of 200 evenly-spaced values in the region $-10 \leq \ln Y \leq 10$.

When implementing the Fourier-Hermite series, we again set the number of time steps to be 50, and use $N = 40$ basis functions for the series expansion of the price. We consider a 6-month American call option with a strike of 100 for a range of parameter values. In all cases we first find the exact price of the corresponding European call option, and then apply the Fourier-Hermite method to the European case for several values of θ, until the relative errors in the prices at $S = 80, 90, 100, 110$, and 120 are sufficiently small (usually less than 1%, and always less than 0.1% at the strike). The required values of θ were found to vary as σ, γ, δ and λ were varied, but remained unaffected by changes in r and q. In all cases $\theta > \sigma$. The final value was determined using simple trial and error, but could readily be computed via a suitable optimisation algorithm.

The code for all three methods was implemented using LAHEY$^{\text{TM}}$FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system. The typical computation time for each of the numerical methods is reported in Table 1. Numerical integration is by far the slowest method, taking over 29 minutes to compute, and this value increases exponentially as the number of time steps increases. The method of lines provides a significant saving, with only 93.578 seconds required to solve the problem. The main contributions to this runtime are the large number of space steps required to achieve a monotonic early exercise boundary, and the large number of discrete jump sizes used to approximate the log-normal distribution in equation (2). When compared to the method of lines, the Fourier-Hermite series is exceptionally fast, requiring only 1.359 seconds to calculate the call price and free boundary. This does not include the time spent determining the optimal value of θ, but since the method requires even less computation for the European call, a good optimisation method should add very little to this runtime, which we anticipate to be no more than 10-15 seconds in total. This fast computation is attributable to the method's heavy reliance on recurrence relations, both for the Hermite polynomial evaluations and the various coefficient calculations.

****Insert Table 1 here****

A range of American call prices are presented in Tables 2-5. In all of these tables we report the value of the American call at spot values of $S = 80, 90, 100, 110$ and 120. The relative difference between the numerical integration and Fourier-Hermite series methods are also included, as they are devoid of any discretisation error that may be introduced when approximation $G(Y)$ for the method of lines solution. Tables 2-4 focus on the prices as the mean jump size e^{γ} is changed for various values of r and q, and with $\sigma = 0.40$. Table 5 considers two additional cases with smaller diffusion coefficients of $\sigma = 0.20$.

Table 2 presents the 6-month American call price for $e^{\gamma} = 1$, representing jumps centered around the current underlying asset price. In Table 3 we have $e^{\gamma} = 1.05$, indicating upwards jumps on average, while Table 4 has $e^{\gamma} = 0.95$, implying that downward jumps are expected. The value of δ was adjusted in each case to ensure that the volatility of $\ln Y$ was fixed at 20% and the Poisson intensity is set at $\lambda = 1$ throughout. In all cases the relative difference between the numerical integration and Fourier-Hermite methods is less than 1%. This appears insensitive to the relative values of r and q. In most cases the three methods are found to be equivalent to the first 2-3 significant figures.

****Insert Table 2 here****

****Insert Table 3 here****

****Insert Table 4 here****

****Insert Table 5 here****

Given that the diffusion coefficient of $\sigma = 0.40$ is quite large, Table 5 considers two cases where this has been reduced to 0.20. The first example in Table 5 reduces σ while maintaining $\lambda = 1$. The Fourier-Hermite series continues to yield prices of suitable magnitude, with the largest relative difference being around 2.1%. In the second part of Table 5, we increase the Poisson intensity to $\lambda = 5$, and observe the impact of more frequent jumps on the results. It is interesting to note that this leads to relative differences that are again consistently less than

1%.

To complete the analysis, we provide some free boundary profiles for the three methods under consideration. In Figures 1-2, we present the early exercise boundary for two different 6-month American call options with strike price $K = 1.00$. For Figure 1 we have set $r = 3\%$, $q = 5\%$, $\lambda = 1$, $\gamma = 0$ and $\delta = 0.1988$. Given that the numerical integration and method of lines results are extremely close together, we shall assume that these best represent the true free boundary. The Fourier-Hermite result deviates from the other methods in two critical ways. Firstly, the free boundary near expiry, $\tau = 0$, is quite poor. The Fourier-Hermite estimate is significantly less than the true solution. The second discrepancy arises near the current time, $\tau = 0.50$. While the Fourier-Hermite is now quite close to the true solution, it appears to have converged to a function that is parallel to the desired result. Figure 2 repeats the results of Figure 1, but this time we take $\sigma = 0.20$, $\gamma = 0.0488$ and $\delta = 0.1888$. Once again, it is clear that the numerical integration and method of lines results are extremely close, while the Fourier-Hermite solution deviates greatly near expiry, and runs parallel close to the true solution near the current time.

****Insert Figure 1 here****

****Insert Figure 2 here****

Given the nature of the Fourier-Hermite solution, it is possible to offer some justification for the observed free boundary estimates, as well as their anticipated impact on the American call price. Near expiry, it is clear that the result is tending to some definite function that is parallel to the true solution. The difference between the Fourier-Hermite solution and the exact free boundary is most likely due to the fact that the series approximation for the American call price is centred about the strike. Since the observed free boundaries in Figures 1-2 are quite far from the strike for any significant amount of time prior to expiry, it is not unsurprising to find that the series expansion contains some small margin of error when approximating the free boundary for τ values greater than 0.15.

Near expiry, however, the differences are far more dramatic. This is because the option price, for small values of τ, is very close in shape to the piecewise-linear payoff function for the call. In this time-region, the option price will not be well approximated by a Fourier-Hermite series, since we are fitting an N-degree polynomial to a function that is almost piecewise-linear. It is interesting to note, however, that despite the poor approximation near expiry, the prices for the 6-month call options produced by the Fourier-Hermite method are still very accurate. In particular, the minor error in the free boundary for $\tau > 0.2$ seems to have had no significant impact on the prices produced by the method. This is in keeping with the well known result that the prices of American options are highly insensitive to small changes in the free boundary[4]. Hence one major shortcoming of polynomial series expansions is that they cannot easily handle piecewise-linear functions. In particular, to ensure that the method remains stable for all time steps after the first, we must use $b(0) = K$ at the start of the time-stepping procedure, and cannot take advantage of our knowledge of the limit $b(0^+)$ from Chiarella and Ziogas (2004). Thus there appears no robust way to extract a more accurate free boundary approximation for small values of τ. Should one require a precise estimate of the free boundary near expiry, this could be quickly achieved using an alternative method, such as the method of lines, applied to the interval $0 \leq \tau \leq 0.15$. Another possible method would be to form a small-time expansion for the free boundary near expiry, and use this to approximate $b(\tau)$ when τ is near zero.

7 Conclusion

In this paper we have presented a generalisation of the Fourier-Hermite series expansion method of Chiarella, El-Hassan and Kucera (1999) for the pricing of European and American call options. This extension applies the Fourier-Hermite series method to Merton's (1976) jump-diffusion model, where the jump sizes are log-normally distributed. We derive the recurrence relations for both the European and American call option under jump-diffusion, and present the special time-stepping algorithm to account for early exercise in the American

[4]See for example AitSahlia and Lai (2001), and Chiarella and Ziogas (2003).

case. When implementing the method for the jump-diffusion model, an unspecified scaling parameter is required to be known. Using Merton's closed-form solution for the European call price, we provide a means for estimating this scaling parameter's value for a given global volatility level.

The series expansion method was used to generate a range of American call prices, and the results compared with those generated using the numerical integration method of Chiarella and Ziogas (2004), as well as the method of lines approach of Meyer (1998). We find that all three methods produce relatively consistent prices, and in particular that the Fourier-Hermite prices are always within 1% of the numerical integration results, with only two reported exceptions. The results indicate that for a sufficiently large global volatility, the Fourier-Hermite method yields excellent levels of accuracy when compared with the standards displayed in the existing literature on the subject. Furthermore, the Fourier-Hermite method has proven to be extremely efficient, requiring significantly less computation time than either of the alternatives presented.

The most notable short-coming for the Fourier-Hermite approach was in estimating the early exercise boundary. The method was incapable of reproducing the correct free boundary near expiry, and was only able to achieve a parallel solution near the current time. The expiry issue we contribute to the poor performance of polynomial approximations when estimating functions that are close to piecewise linear in form, such as the value of an American call or put near expiry . For the current-time discrepancy, we suggest that the centralisation of the series expansions around the strike are a likely cause. This cannot be easily remedied without foregoing price accuracy in the critical region around the strike. It has been of interest to note that even with these small inaccuracies in the free boundary estimate, the resulting prices have been highly accurate. This demonstrates that the series expansion technique has a potential trade-off in the form of increased computation speed at the cost of accuracy in estimating the early exercise boundary, most predominantly near expiry. This does not diminish the value of the method as an efficient means of pricing American options under jump-diffusion processes where the jump sizes follow a specified continuous distribution. Further computation time is saved in that there is never any need to interpolate option prices for various values of the spot, since the price estimate is a continuous function of the underlying asset. It is also trivial to estimate the delta and gamma for the American call once the Fourier-Hermite series approximation has been found.

There are several avenues that could be pursued for future research. Given that the free boundary estimate near expiry is suboptimal, some alternative estimate would be of significant value. A small-time expansion of the free boundary near expiry remains unaddressed for American calls under jump-diffusion. In presenting multiple benchmark prices for the American call option, there is still no clear consensus as to what the exact price is for the American call under consideration. This continues to cast some doubt regarding the accuracy of any numerical method being considered. While the presented method has the advantage of being well-suited to the case where jump sizes follow a log-normal distribution, it is not yet known how the method would perform for jump sizes with discrete distributions. Finally, we have not offered an explicit optimisation routine for selecting the scaling parameter prior to finding the American call price. Determining and verifying an explicit optimisation routine, or an explicit form for the scaling parameter in terms of the global volatility of the jump-diffusion process, would further increase the robustness of the method.

Appendix A. Hermite Coefficients for the European Call

A.1 Proof of Proposition 1

From equation (11), α_m^{j-1} is given by

$$\alpha_m^{j-1} = \frac{1}{2^m m!}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} f^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1}.$$

Substituting for $f^{j-1}(\xi_{j-1})$ from (9), we obtain

$$\begin{aligned}
\alpha_m^{j-1} &= \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) \\
&\quad \times \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2}d\xi_j d\xi_{j-1} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)I_m^{(n)}(\xi_j)d\xi_j
\end{aligned}$$

where

$$I_m^{(n)}(\xi_j) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-[\xi_j-\mu_n(\xi_{j-1},\Delta t)]^2-\xi_{j-1}^2} H_m(\xi_{j-1})d\xi_{j-1}. \tag{30}$$

To evaluate $I_m^{(n)}(\xi_j)$ we complete the square in the exponent. Recalling the definition of μ_n from (7), it is simple to show that

$$[x-\mu_n(\xi,\Delta t)]^2+\xi^2 = \left[\frac{w_n\xi}{\hat{v}_n\sqrt{2\Delta t}} - \frac{(x\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2 + \left[\frac{\hat{v}_n\sqrt{2\Delta t}x-b_n}{w_n}\right]^2, \tag{31}$$

where we set $b_n \equiv (r_n - q - \frac{v_n^2}{2})\Delta t/\theta$ and $w_n \equiv \sqrt{1+2\Delta t\hat{v}_n^2}$. Thus $I_m^{(n)}(\xi_j)$ can be expressed as

$$\begin{aligned}
I_m^{(n)}(\xi_j) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{w_n\xi_{j-1}}{\hat{v}_n\sqrt{2\Delta t}} - \frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\} \\
&\quad \times\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\} H_m(\xi_{j-1})d\xi_{j-1}.
\end{aligned}$$

If we make the change of variable $y = w_n\xi_{j-1}/\hat{v}_n\sqrt{2\Delta t}$, $I_m^{(n)}(\xi_j)$ becomes

$$\begin{aligned}
I_m^{(n)}(\xi_j) &= \frac{1}{\sqrt{\pi}}\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n} \\
&\quad \times\int_{-\infty}^{\infty}\exp\left\{-\left[y-\frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n\hat{v}_n\sqrt{2\Delta t}}\right]^2\right\}H_m\left(\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}y\right)dy.
\end{aligned} \tag{32}$$

To evaluate this integral, we refer to a result form Erdéyli et al (1953b, p195, eq'n. (30)), which states that

$$\frac{1}{\sqrt{2\pi u}}\int_{-\infty}^{\infty} H_m(z)\exp\left\{-\frac{(z-v)^2}{2u}\right\}dz = (1-2u)^{\frac{m}{2}}H_m\left(\frac{v}{\sqrt{1-2u}}\right).$$

Letting $y = z/\sqrt{2u}$, this becomes

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_m(\sqrt{2u}y)\exp\left\{-\left[y-\frac{v}{\sqrt{2u}}\right]^2\right\}dy = (1-2u)^{\frac{m}{2}}H_m\left(\frac{v}{\sqrt{1-2u}}\right).$$

Thus if we equate $u = \hat{v}_n^2\Delta t/w_n^2$ and $v = (\xi_k\hat{v}_n\sqrt{2\Delta t}-b_n)/w_n^2$, we can now evaluate $I_m^{(n)}(\xi_j)$ as

$$\begin{aligned} I_m^{(n)}(\xi_j) &= \exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n}\left(1-\frac{\hat{v}_n^2 2\Delta t}{w_n^2}\right)^{\frac{m}{2}} \\ &\quad\times H_m\left(\frac{(\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)}{w_n^2}\sqrt{\frac{w_n^2}{w_n^2-2\hat{v}_n^2\Delta t}}\right) \\ &= \frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}}H_m\left(\frac{\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n}{w_n}\right)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}. \end{aligned} \tag{33}$$

Using equation (33), the expression for α_m^{j-1} becomes

$$\begin{aligned} \alpha_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_{-\infty}^{\infty} f^j(\hat{v}_n\sqrt{2\Delta t}\xi_j)\frac{\hat{v}_n\sqrt{2\Delta t}}{w_n^{m+1}} \\ &\quad\times H_m\left(\frac{\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n}{w_n}\right)\exp\left\{-\left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j-b_n}{w_n}\right]^2\right\}d\xi_j. \end{aligned}$$

If we define $z = (\xi_j\hat{v}_n\sqrt{2\Delta t}-b_n)/w_n$, we now have

$$\alpha_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n! w_n^m}\int_{-\infty}^{\infty} f^j(b_n+zw_n)H_m(z)e^{-z^2}dz.$$

Expanding $f^j(b_n + zw_n)$ in a Fourier-hermite series as defined in (10) we obtain

$$\begin{aligned}
\alpha_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \sum_{i=0}^{\infty} \alpha_i^k \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz \\
&= e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{\infty} \alpha_i^k A_{m,i}^{(n)},
\end{aligned}$$

where

$$A_{m,i}^{(n)} \equiv \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz.$$

Truncating the number of basis functions at order N, we obtain equations (12)-(13) of Proposition 1. Note that while we must truncate the order of the Hermite-series expansion, the same is not true for the summation over the number of observed jumps, n. This must be computed for increasing values of n until convergence is obtained, according to some pre-specified accuracy level.

A.2 Proof of Proposition 2

To develop a recurrence relation for $A_{m,i}^{(n)}$, we note from Abramowitz and Stegun (1970) that the recurrence relation for Hermite polynomials is

$$H_m(z) = 2zH_{m-1}(z) - 2(m-1)H_{m-2}(z),$$

and furthermore, the derivative of a Hermite polynomial can be defined recursively as

$$H_m'(z) = 2mH_{m-1}(z).$$

Applying the recurrence relation to equation (13), we have

$$\begin{aligned}
A_{m,i}^{(n)} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} H_m(z) H_i(b_n + zw_n) dz \\
&= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2z e^{-z^2} H_{m-1}(z) H_i(b_n + zw_n) dz
\end{aligned}$$

$$-\frac{2(m-1)}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2} H_{m-2}(z)H_i(b_n+zw_n)dz.$$

Using integration by parts on the first integral, we have

$$\begin{aligned}
A_{m,i}^{(n)} &= \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\Big\{ \Big[-e^{-z^2}H_{m-1}(z)H_i(b_n+zw_n)\Big]_{-\infty}^{\infty} \\
&\quad + \int_{-\infty}^{\infty} e^{-z^2}[2iw_n H_i(b_n+zw_n)H_{m-1}(z) + 2(m-1)H_{m-2}(z)H_i(b_n+zw_n)]dz\Big\} \\
&\quad -\frac{2(m-1)}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{m-2}(z)H_i(b_n+zw_n)dz \\
&= \frac{2iw_n}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_m+zw_n)H_{m-1}(z)dz \\
&= \frac{i}{m}\frac{1}{2^{m-1}(m-1)! w_n^{m-1}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_m+zw_n)H_{m-1}(z)dz \\
&= \frac{i}{m}A_{m-1,i-1}^{(n)}, \quad (m,i=1,2,\cdots,N),
\end{aligned}$$

which is equation (14) of the main text.

To implement the recurrence for $A_{m,i}^{(n)}$, we require expressions for $A_{m,0}^{(n)}$, $A_{0,i}^{(n)}$ and $A_{0,0}^{(n)}$. Firstly, $A_{m,0}^{(n)}$ is given by

$$\begin{aligned}
A_{m,0}^{(n)} &= \frac{1}{2^m m! w_n^m}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_m(z)dz \\
&= 0 \text{ for } m \neq 0,
\end{aligned}$$

where the last equality follows from the orthogonality result for Hermite polynomials. This subsequently implies that $A_{m,i}^{(n)} = 0$ for all $m > i$.

Through use of the Hermite polynomial recurrence relation, $A_{0,i}^{(n)}$ is given by

$$\begin{aligned}
A_{0,i}^{(n)} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_i(b_n+zw_n)dz \\
&= \frac{w_n}{\sqrt{\pi}}\int_{-\infty}^{\infty} 2ze^{-z^2}H_{i-1}(b_n+zw_n)dz + \frac{2b_n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-1}(b_n+zw_n)dz \\
&\quad \frac{-2(i-1)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}H_{i-2}(b_n+zw_n)dz.
\end{aligned}$$

Applying integration by parts to the first integral term, $A_{0,i}^{(n)}$ becomes

$$\begin{aligned} A_{0,i}^{(n)} &= \frac{w_n}{\sqrt{\pi}}\left\{\left[-e^{-z^2}H_{i-1}(b_n+zw_n)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-z^2}2(i-1)H_{i-2}(b_n+zw_n)w_n dz\right\} \\ &\quad +2b_nA_{0,i-1}^{(n)}-2(i-1)A_{0,i-2}^{(n)} \\ &= 2b_nA_{0,i-1}^{(n)}+2(i-1)(w_n^2-1)A_{0,i-2}^{(n)}, \quad (i=2,3,\cdots,N), \end{aligned}$$

which is an additional recurrence relation for $A_{0,i}^{(n)}$ as given in equation (15).

It is straightforward to show that

$$A_{0,0}^{(n)}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}dz=1,$$

and to use the recurrence for $A_{0,i}^{(n)}$, we also require $A_{0,1}^{(n)}$, which can be evaluated as

$$\begin{aligned} A_{0,1}^{(n)} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}H_1(b_n+zw_n)dz \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}2(b_n+zw_n)dz \\ &= 2b_n+\frac{w_n}{\sqrt{\pi}}[-e^{-z^2}]_{-\infty}^{\infty} \\ &= 2b_n. \end{aligned}$$

A.3 Proof of Proposition 3

To generate α_m^{J-1}, recall that at time step $j=J$

$$f^J(\xi_J)=\max(e^{\theta\xi_J}-1,0),$$

and the transition density is given by

$$f^{J-1}(\xi_{J-1})=\frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\int_0^{\infty}(e^{v_n\sqrt{2\Delta t}\xi_J}-1)e^{-[\xi_J-\mu_n(\xi_{J-1},\Delta t)]^2}d\xi_J.$$

Expanding the solution at time step $j = J - 1$ in a Fourier-Hermite series according to

$$f^{J-1}(\xi_{J-1}) = \sum_{m=0}^{\infty} \alpha_m^{J-1} H_m(\xi_{J-1}),$$

the expression for α_m^{J-1} becomes

$$\alpha_m^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_0^{\infty} (e^{v_n \sqrt{2\Delta t}\xi_J} - 1) I_m^{(n)}(\xi_J) d\xi_J,$$

where $I_m^{(n)}(\xi_J)$ is given by equation (33). Substituting $I_m^{(n)}(\xi_J)$ into α_m^{J-1} we have

$$\begin{aligned} \alpha_m^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{\hat{v}_n \sqrt{2\Delta t}}{w_n^{m+1}} \\ & \quad \times \int_0^{\infty} (e^{v_n \sqrt{2\Delta t}\xi_J} - 1) H_m \left(\frac{\xi_J \hat{v}_n \sqrt{2\Delta t} - b_n}{w_n} \right) \exp \left\{ - \left[\frac{\hat{v}_n \sqrt{2\Delta t}\xi_J - b_n}{w_n} \right]^2 \right\} d\xi_J. \end{aligned}$$

Making the change of variable $z = (\xi_J \hat{v}_n \sqrt{2\Delta t} - b_n)/w_n$, α_m^{J-1} becomes

$$\begin{aligned} \alpha_m^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{1}{w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} (e^{\theta w_n z} e^{\theta b_n} - 1) e^{-z^2} H_m(z) dz \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{1}{w_n^m} \left\{ e^{\theta b_n} \Psi_m^{(n)} \left(-\frac{b_n}{w_n} \right) - \Omega_m^{(n)} \left(-\frac{b_n}{w_n} \right) \right\}, \end{aligned}$$

where

$$\Omega_m^{(n)} \left(-\frac{b_n}{w_n} \right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2} H_m(z) dz, \tag{34}$$

and

$$\Psi_m^{(n)} \left(-\frac{b_n}{w_n} \right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2} e^{\theta w_n z} H_m(z) dz. \tag{35}$$

Firstly consider the integral $\Omega_m^{(n)}$. Using the three-term recurrence relation for $H_m(z)$, we have

$$\Omega_m^{(n)} \left(-\frac{b_n}{w_n} \right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} 2z e^{-z^2} H_{m-1}(z) dz - 2(m-1) \Omega_{m-2}^{(n)} \left(-\frac{b_n}{w_n} \right).$$

Applying integration by parts, we find that

$$\Omega_m^{(n)}\left(-\frac{b_n}{w_n}\right) = \frac{1}{\sqrt{\pi}} H_{m-1}\left(-\frac{b_n}{w_n}\right) e^{-\left(\frac{b_n}{w_n}\right)^2}. \tag{36}$$

Note that when $m = 0$ we have

$$\Omega_0^{(n)}\left(-\frac{b_n}{w_n}\right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2} dz = \frac{1}{2}\text{erfc}\left(-\frac{b_n}{w_n}\right),$$

and when $m = 1$,

$$\Omega_1^{(n)}\left(-\frac{b_n}{w_n}\right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2} 2z dz = \frac{1}{\sqrt{\pi}} e^{-\left(\frac{b_n}{w_n}\right)^2}.$$

Next we consider $\Psi_m^{(n)}$. Again using the three-term recurrence we find that

$$\begin{aligned}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} 2z e^{-z^2} e^{\theta w_n z} H_{m-1}(z) dz - 2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right) \\ &= \Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) - 2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right)\end{aligned}$$

where

$$\Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) = \frac{1}{\sqrt{\pi}} \int_{-\frac{b_n}{w_n}}^{\infty} 2z e^{-z^2} e^{\theta w_n z} H_{m-1}(z) dz.$$

Through the use of integration by parts, $\Phi_m^{(n)}$ becomes

$$\begin{aligned}\Phi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\Big\{ \left[-e^{-z^2} H_{m-1}(z) e^{\theta w_n z}\right]_{-\frac{b_n}{w_n}}^{\infty} \\ &\quad + \int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2}[e^{\theta w_n z} 2(m-1) H_{m-2}(z) + H_{m-1}(z) w_n e^{w_n z}\theta] dz\Big\} \\ &= \frac{1}{\sqrt{\pi}} H_{m-1}\left(-\frac{b_n}{w_n}\right) e^{-b_n\theta} e^{-\left(\frac{b_n}{w_n}\right)^2} + \theta w_n \Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right) \\ &\quad + 2(m-1)\Psi_{m-2}^{(n)}\left(-\frac{b_n}{w_n}\right).\end{aligned}$$

Thus a recurrence for $\Psi_m^{(n)}$ is given by

$$\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) = \frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-b_n\theta}e^{-\left(\frac{b_n}{w_n}\right)^2} + w_n\theta\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right), \quad (m = 1, 2, \cdots, N). \tag{37}$$

For $m = 0$ we have

$$\begin{aligned}\Psi_0^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} e^{-z^2}e^{\theta w_n z}dz \\ &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} e^{-(z-\frac{\theta w_n}{2})^2}e^{\frac{\theta^2 w_n^2}{4}}dz \\ &= \frac{1}{2}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n} - \frac{\theta w_n}{2}\right),\end{aligned}$$

and when $m = 1$,

$$\begin{aligned}\Psi_1^{(n)}\left(-\frac{b_n}{w_n}\right) &= \frac{1}{\sqrt{\pi}}\int_{-\frac{b_n}{w_n}}^{\infty} 2ze^{-z^2}e^{\theta w_n z}dz \\ &= \frac{1}{\sqrt{\pi}}\left[-e^{-z^2}e^{\theta w_n z}\right]_{-\frac{b_n}{w_n}}^{\infty} + \int_{-\frac{b_n}{w_n}}^{\infty}\theta e^{-z^2}w_n e^{w_n z}dz \\ &= \frac{1}{\sqrt{\pi}}\exp\left\{-\left(\frac{b_n}{w_n}\right)^2 - \theta b_n\right\} + \frac{\theta w_n}{2}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n} - \frac{\theta w_n}{2}\right).\end{aligned}$$

Hence the coefficients α_m^{J-1} are given by

$$\alpha_m^{J-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m}\left\{e^{\theta b_n}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) - \frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}\right\}.$$

We can now use equation (37) to derive a recurrence for α_m^{J-1}, independent of $\Psi_m^{(n)}$. Firstly, rearrange the expression for α_m^{J-1} to give

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) &= e^{(r+\lambda)\Delta t}2^m m!\alpha_m^{J-1} \\ &\quad + \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m}\frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}.\end{aligned} \tag{38}$$

In addition, from equation (37) we can readily show that

$$\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^m}\Psi_m^{(n)}\left(-\frac{b_n}{w_n}\right) = \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^m}\frac{1}{\sqrt{\pi}}H_{m-1}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2} + \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta e^{\theta b_n}}{w_n^{m-1}}\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right). \tag{39}$$

Substituting (38) into (39) we have

$$\begin{aligned} e^{(r+\lambda)\Delta t}2^m m!\alpha_m^{J-1} &= \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{e^{\theta b_n}}{w_n^{m-1}}\Psi_{m-1}^{(n)}\left(-\frac{b_n}{w_n}\right) \\ &= e^{(r+\lambda)\Delta t}2^{m-1}(m-1)!\alpha_{m-1}^{J-1}\theta \\ &\quad + \sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{w_n^{m-1}}\frac{1}{\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}, \end{aligned}$$

and hence the recurrence relation for α_m^{J-1} is

$$\begin{aligned} \alpha_m^{J-1} &= \frac{\theta}{2m}\left[\alpha_{m-1}^{J-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^{m-1}(m-1)!}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n^{m-1}}\frac{1}{\sqrt{\pi}}H_{m-2}\left(-\frac{b_n}{w_n}\right)e^{-\left(\frac{b_n}{w_n}\right)^2}\right], \\ &\quad (m = 2, 3, \cdots, N), \end{aligned}$$

as stated in (16) of the main text. To initiate this recurrence, we note that for $m = 0$,

$$\begin{aligned} \alpha_0^{J-1} &= e^{-(r+\lambda)\Delta t}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}\Psi_0^{(n)}\left(-\frac{b_n}{w_n}\right) - \Omega_0^{(n)}\left(-\frac{b_n}{w_n}\right)\right\} \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n} - \frac{\theta w_n}{2}\right) - \operatorname{erfc}\left(-\frac{b_n}{w_n}\right)\right\}, \end{aligned}$$

and when $m = 1$,

$$\begin{aligned} \alpha_1^{J-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{1}{w_n}\left\{e^{\theta b_n}\Psi_1^{(n)}\left(-\frac{b_n}{w_n}\right) - \Omega_1^{(n)}\left(-\frac{b_n}{w_n}\right)\right\} \\ &= \frac{e^{-(r+\lambda)\Delta t}}{2}\sum_{n=0}^{\infty}\frac{\lambda^n(\Delta t)^n}{n!}\frac{\theta}{2}\left\{e^{\theta b_n}e^{\frac{\theta^2 w_n^2}{4}}\operatorname{erfc}\left(-\frac{b_n}{w_n} - \frac{\theta w_n}{2}\right)\right\}. \end{aligned}$$

Appendix B. Hermite Coefficients for the American Call

B.1 Proof of Proposition 4

Substituting the transition density (6) into equation (20), the expression for $V^{j-1}(\xi_{j-1})$ becomes

$$V^{j-1}(\xi_{j-1}) = \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{-\infty}^{\infty} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} F^j(\hat{v}_n \sqrt{2\Delta t}\xi_j) d\xi_j.$$

Using the value of $F^j(\hat{v}_n \sqrt{2\Delta t}\xi_j)$ from equation (22), we have

$$\begin{aligned} V^{j-1}(\xi_{j-1}) &= \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \\ &\quad \times \Big\{ \int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} V^j(\hat{v}_n \sqrt{2\Delta t}\xi_j) d\xi_n \\ &\quad + \int_{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}}^{\infty} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} (e^{\theta \xi_j \hat{v}_n \sqrt{2\Delta t}} - 1) d\xi_j \Big\} \\ &= \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} V^j(\hat{v}_n \sqrt{2\Delta t}\xi_j) d\xi_j \\ &\quad + h^{j-1}(\xi_{j-1}), \end{aligned}$$

where

$$h^{j-1}(\xi_{j-1}) \equiv \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}}^{\infty} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} (e^{v_n \sqrt{2\Delta t}\xi_j} - 1) d\xi_j.$$

Next, expand the functions V^j, V^{j-1} and h^{j-1} in Fourier-Hermite series, such that

$$V^{j-1}(\xi_{j-1}) = \sum_{m=0}^{\infty} \alpha_m^{j-1} H_m(\xi_{j-1}),$$

and

$$h^{j-1}(\xi_j) = \sum_{m=0}^{\infty} \gamma_m^{j-1} H_m(\xi_{j-1}).$$

From the orthogonality conditions for Hermite polynomials, the coefficients for these expansions are given by

$$\alpha_m^{j-1} = \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} V^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1}$$

for V^j, and

$$\gamma_m^{j-1} = \frac{1}{2^m m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} h^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1},$$

for h^{j-1}.

Now we must develop recurrence relations for α and γ. Starting with the γ coefficients, substitute the expression for $h^{j-1}(\xi_{j-1})$ into the γ_m^{j-1} equation to obtain

$$\begin{aligned}
\gamma_m^{j-1} &= \frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) \\
&\quad \times \left\{ \frac{e^{-(r+\lambda)\Delta t}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta}}}^{\infty} e^{-[\xi_j - \mu_n(\xi_{j-1},\Delta t)]^2} (e^{v_n\sqrt{2\Delta}\xi_j} - 1) d\xi_j \right\} d\xi_{j-1} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \Big\{ \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty} (e^{v_n\sqrt{2\Delta t}\xi_j} - 1) \\
&\quad \times \int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} H_m(\xi_{j-1}) e^{-[\xi_j - \mu_n(\xi_{j-1},\Delta t)]^2} d\xi_{j-1} d\xi_j \Big\}.
\end{aligned}$$

Using the result in equation (31) from Appendix A.1, γ_m^{j-1} becomes

$$\begin{aligned}
\gamma_m^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \Big\{ \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty} (e^{v_n\sqrt{2\Delta t}\xi_j} - 1) \exp\left\{ - \left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n} \right]^2 \right\} \\
&\quad \times \int_{-\infty}^{\infty} \exp\left\{ - \left[\frac{w_n \xi_{j-1}}{\hat{v}_n\sqrt{2\Delta t}} - \frac{(\xi_j \hat{v}_n\sqrt{2\Delta t} - b_n)}{w_n \hat{v}_n \sqrt{2\Delta t}} \right]^2 \right\} H_m(\xi_{j-1}) d\xi_{j-1} d\xi_j \Big\}.
\end{aligned}$$

A change of integration variable to $y = w_n \xi_{j-1}/\hat{v}_n\sqrt{2\Delta t}$ yields

$$\gamma_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\pi} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \Big\{ \int_{\frac{\xi_j^*}{\hat{v}_n\sqrt{2\Delta t}}}^{\infty} (e^{v_n\sqrt{2\Delta t}\xi_j} - 1) \exp\left\{ - \left[\frac{\hat{v}_n\sqrt{2\Delta t}\xi_j - b_n}{w_n} \right]^2 \right\}$$

$$\times \int_{-\infty}^{\infty} \exp\left\{-\left[y - \frac{(\xi_j \hat{v}_n \sqrt{2\Delta t} - b_n)}{w_n \hat{v}_n \sqrt{2\Delta t}}\right]^2\right\} H_m\left(\frac{\hat{v}_n \sqrt{2\Delta t}}{w_n} y\right) \frac{\hat{v}_n \sqrt{2\Delta t}}{w_n} dy d\xi_j \Big\}$$

$$= \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}}^{\infty} (e^{v_n \sqrt{2\Delta t} \xi_j} - 1) I_m^{(n)}(\xi_j) d\xi_j$$

where $I_m^{(n)}(\xi_j)$ is given by equation (32). Since $I_m^{(n)}(\xi_j)$ can be evaluated to produce (33), γ_m^{j-1} becomes

$$\gamma_{m-1}^{j} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \Big\{ \int_{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}}^{\infty} (e^{v_n \sqrt{2\Delta t} \xi_j} - 1) \frac{\hat{v}_n \sqrt{2\Delta t}}{w_n^{m+1}} H_m\left(\frac{\xi_j \hat{v}_n \sqrt{2\Delta t} - b_n}{w_n}\right)$$
$$\times \exp\left\{-\left[\frac{\hat{v}_n \sqrt{2\Delta t} \xi_j - b_n}{w_n}\right]^2\right\} d\xi_j \Big\}.$$

If we now let $z = (\xi_j \hat{v}_n \sqrt{2\Delta t} - b_n)/w_n$, and define $z_j^{(n)} \equiv (\xi_j^* - b_n)/w_n$. Thus the integral for γ becomes

$$\gamma_{m-1}^{j} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \int_{z_j^{(n)}}^{\infty} (e^{(w_n z + b_n)\theta} - 1) H_m(z) e^{-z^2} dz.$$

To find a recurrence relation for γ_m^{j-1}, note that

$$\gamma_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \Big\{ \frac{e^{\theta b_n}}{\sqrt{\pi}} \int_{z_j^{(n)}}^{\infty} e^{-z^2} e^{\theta w_n z} H_m(z) dz$$
$$- \frac{1}{\sqrt{\pi}} \int_{z_j^{(n)}}^{\infty} e^{-z^2} H_m(z) dz \Big\}$$
$$= \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \Big\{ e^{\theta b_n} \Psi_m^{(n)}(z_j^{(n)}) - \Omega_m^{(n)}(z_j^{(n)}) \Big\},$$

where $\Omega_m^{(n)}$ and $\Psi_m^{(n)}$ are defined by equations (34) and (35) respectively. Using (36), we can easily show that

$$\Omega_m^{(n)}(z_j^{(n)}) = \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{-(z_j^{(n)})^2},$$

and similarly, equation (37) implies that the recurrence for $\Psi_m^{(n)}(z_j^{(n)})$ is

$$\Psi_m^{(n)}(z_j^{(n)}) = \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{\theta w_n z_j^{(n)} - (z_j^{(n)})^2} + \theta w_n \Psi_{m-1}^{(n)}(z_j^{(n)}).$$

Thus the expression for γ_m^{j-1} becomes

$$\gamma_m^{j-1} = \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \left\{ e^{\theta b_n} \Psi_m^{(n)}(z_j^{(n)}) - \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{-(z_j^{(n)})^2} \right\},$$

which can be rearranged to produce

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{\theta b_n}}{w_n^m} \Psi_m^{(n)}(z_j^{(n)}) &= e^{(r+\lambda)\Delta t} 2^m m! \gamma_m^{j-1} \\ &\quad + \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{-(z_j^{(n)})^2}. \end{aligned}$$

From the recurrence for $\Psi_m^{(n)}$ we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{\theta b_n}}{w_n^m} \Psi_m^{(n)}(z_j^{(n)}) &= \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{1}{w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{\theta b_n + \theta w_n z_j^{(n)} - (z_j^{(n)})^2} \\ &\quad + \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{\theta b_n}}{w_n^{m-1}} \theta \Psi_{m-1}^{(n)}(z_j^{(n)}), \end{aligned}$$

and by substitution we find that

$$\begin{aligned} e^{(r+\lambda)\Delta t} 2^m m! \gamma_m^{j-1} &= \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{1}{w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{-(z_j^{(n)})^2} \left[e^{\theta(b_n + w_n z_j^{(n)})} - 1 \right] \\ &\quad + \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{\theta b_n}}{w_n^{m-1}} \theta \Psi_{m-1}^{(n)}(z_j^{(n)}) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{1}{w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) e^{-(z_j^{(n)})^2} \left[e^{\theta(b_n + w_n z_j^{(n)})} - 1 \right] \\ &\quad + e^{(r+\lambda)\Delta t} 2^{m-1} (m-1)! \gamma_{m-1}^{j-1} \theta \\ &\quad + \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^{m-1}} \frac{1}{\sqrt{\pi}} H_{m-2}(z_j^{(n)}) e^{-(z_j^{(n)})^2} \theta. \end{aligned}$$

Hence the recurrence relation for γ_m^{j-1} is

$$\begin{aligned} \gamma_m^{j-1} &= \frac{\theta \gamma_{m-1}^{j-1}}{2m} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \frac{e^{-(z_j^{(n)})^2}}{w_n^m \sqrt{\pi}} \\ &\quad \times \left\{ H_{m-1}(z_j^{(n)}) [e^{\theta(b_n + w_n z_j^{(n)})} - 1] + w_n \theta H_{m-2}(z_j^{(n)}) \right\}, \end{aligned}$$

with

$$
\begin{aligned}
\gamma_0^{j-1} &= e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \left\{ e^{\theta b_n} \Psi_0^{(n)}(z_j^{(n)}) - \Omega_0^{(n)}(z_j^{(n)}) \right\} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \left\{ e^{b_n \theta} e^{\frac{w_n^2 \theta^2}{4}} \operatorname{erfc}\left(z_j^{(n)} - \frac{w_n \theta}{2} \right) - \operatorname{erfc}(z_j^{(n)}) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_1^{j-1} &= \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n} \left\{ e^{\theta b_n} \Psi_1^{(n)}(z_j^{(n)}) - \Omega_1^{(n)}(z_j^{(n)}) \right\} \\
&= \frac{e^{-(r+\lambda)\Delta t}}{2} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n} \Bigg\{ \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} e^{\theta (w_m z_j^{(n)} + b_n)} \\
&\qquad + \frac{\theta w_n}{2} e^{b_n \theta} e^{\frac{w_n^2 \theta^2}{4}} \operatorname{erfc}\left(z_j^{(n)} - \frac{w_n \theta}{2} \right) - \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} \Bigg\}.
\end{aligned}
$$

Next we consider the α coefficients. Substituting the expression for $V^{j-1}(\xi_{j-1})$ into the equation for α_m^{j-1}, we have

$$
\begin{aligned}
\alpha_m^{j-1} &= \frac{1}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} V^{j-1}(\xi_{j-1}) H_m(\xi_{j-1}) d\xi_{j-1} \\
&= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}} V^j(\hat{v}_n \sqrt{2\Delta t} \xi_j) \\
&\qquad \times \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi_{j-1}^2} e^{-[\xi_j - \mu_n(\xi_{j-1}, \Delta t)]^2} H_m(\xi_{j-1}) d\xi_{j-1} d\xi_j \\
&= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \int_{-\infty}^{\frac{\xi_j^*}{\hat{v}_n \sqrt{2\Delta t}}} V^j(\hat{v}_n \sqrt{2\Delta t} \xi_j) I_m^{(n)}(\xi_j) d\xi_j,
\end{aligned}
$$

where $I_m^{(n)}(\xi_j)$ is given by equation (30), and evaluated to produce (33). With the change of variable $z = (\hat{v}_n \sqrt{2\Delta t} \xi_j - b_n)/w_n$, the expression for α_m^{j-1} becomes

$$
\alpha_m^{j-1} = \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m! \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_m(z) V^j(b_n + w_n z) dz.
$$

Substituting the Fourier-Hermite expansion for $V^j(b_n + w_n z)$ into the expression for α_m^{j-1}, and truncating the series at term N, we obtain

$$\begin{aligned}\alpha_m^{j-1} &= \gamma_m^{j-1} + \frac{e^{-(r+\lambda)\Delta t}}{2^m m!\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n! w_n^m} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_m(z) \sum_{i=0}^{N} \alpha_i^j H_i(b_n + w_n z) dz \\ &= \gamma_m^{j-1} + e^{-(r+\lambda)\Delta t} \sum_{n=0}^{\infty} \frac{\lambda^n (\Delta t)^n}{n!} \sum_{i=0}^{N} A_{m,i}^{j,n} \alpha_i^j, \quad (j = J, J-1, \cdots, 1),\end{aligned}$$

where

$$A_{m,i}^{j,n} = \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_m(z) H_i(b_n + w_n z) dz, \quad (m, i = 0, 1, 2, \cdots, N).$$

B.2 Proof of Proposition 5

If we apply the three-term Hermite polynomial recurrence relation from Appendix A.2 to equation (25) we find that

$$A_{m,i}^{j,n} = \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2z e^{-z^2} H_{m-1}(z) H_i(b_n + w_n z) dz - \frac{1}{2m w_n^2} A_{m-2,i}^{j,n}.$$

By use of integration by parts, this becomes

$$\begin{aligned}A_{m,i}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \Big\{ \Big[-e^{-z^2} H_{m-1}(z) H_i(b_n + w_n z) \Big]_{-\infty}^{z_j^{(n)}} \\ &\quad + \int_{-\infty}^{z_k^{(n)}} e^{-z^2} [2(m-1) H_{m-2}(z) H_i(b_n + w_n z) + 2i w_n H_{i-1}(b_n + w_n z) H_{m-1}(z)] dz \Big\} \\ &\quad - \frac{1}{2m w_n^2} A_{m-2,i}^{j,n}\end{aligned}$$

Thus the recurrence for $A_{m,i}^{j,n}$ is

$$A_{m,i}^{j,n} = \frac{i}{m} A_{m-1,i-1}^{j,n} - \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} H_{m-1}(z_j^{(n)}) H_i(b_n + w_n z_j^{(n)}) e^{-(z_j^{(n)})^2}.$$

To use this recurrence, we require $A_{m,0}^{j,n}$, $A_{0,i}^{j,n}$ and $A_{0,0}^{j,n}$. Beginning with $A_{m,0}^{j,n}$,

$$\begin{aligned} A_{m,0}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_m(z) dz \\ &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2z e^{-z^2} H_{m-1}(z) dz - \frac{1}{2m w_n^2} A_{m-2,0}^{j,n}. \end{aligned}$$

By an application of integration by parts, this simplifies to

$$\begin{aligned} A_{m,0}^{j,n} &= \frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} \left\{ \left[-e^{-z^2} H_{m-1}(z) \right]_{-\infty}^{z_j^{(n)}} + \int_{-\infty}^{z_j^{(n)}} e^{-z^2} 2(m-1) H_{m-2}(z) dz \right\} \\ &\quad - \frac{1}{2m w_n^2} A_{m-2,0}^{j,n} \\ &= -\frac{1}{2^m m! w_n^m} \frac{1}{\sqrt{\pi}} e^{-(z_j^{(n)})^2} H_{m-1}(z_j^{(n)}), \quad (m = 1, 2, 3, \cdots). \end{aligned}$$

Next we consider $A_{0,i}^{j,n}$, which is given by

$$\begin{aligned} A_{0,i}^{j,n} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} H_i(b_n + w_n z) dz \\ &= \frac{w_n}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} 2z e^{-z^2} H_{i-1}(b_n + w_n z) dz + 2b_n A_{0,i-1}^{j,n} - 2(i-1) J_{0,i-2}^{j,n}. \end{aligned}$$

With an application of integration by parts, this becomes

$$\begin{aligned} A_{0,i}^{j,n} &= \frac{w_n}{\sqrt{\pi}} \left\{ \left[-e^{-z^2} H_{i-1}(b_n + w_n z) \right]_{-\infty}^{z_j^{(n)}} + \int_{-\infty}^{z_j^{(n)}} e^{-z^2} 2(i-1) w_n H_{i-2}(b_n + w_n z) dz \right\} \\ &\quad -2(i-1) A_{0,i-2}^{j,n} + 2b_n A_{0,i-1}^{j,n}, \end{aligned}$$

and hence

$$A_{0,i}^{j,n} = 2(w_n^2 - 1)(i-1) A_{0,i-2}^{j,n} + 2b_n A_{0,i-1}^{j,n} - \frac{w_n}{\sqrt{\pi}} H_{i-1}(b_n + w_n z_j^{(n)}) e^{-(z_j^{(n)})^2},$$

is the recurrence for $A_{0,i}^{j,n}$.

Finally, to implement the recurrence for $A_{m,i}^{j,i}$, we must obtain the initial values $A_{0,0}^{j,n}$ and $A_{0,1}^{j,n}$, which are given by

$$A_{0,0}^{j,n} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} dz = \frac{1}{\sqrt{\pi}} \int_{-z_j^{(n)}}^{\infty} e^{-z^2} dz = \frac{1}{2} \text{erfc}(-z_j^{(n)}),$$

and

$$\begin{aligned} A_{0,1}^{j,n} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z_j^{(n)}} e^{-z^2} 2(b_n + w_n z) dz \\ &= b_n \text{erfc}(-z_j^{(n)}) - \frac{w_n}{\sqrt{\pi}} e^{-(z_j^{(n)})^2}. \end{aligned}$$

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Method	Computation Time
McKean (Integration)	29 min 16.578 sec
Method of Lines	1 min 33.578 sec
Fourier-Hermite	1.359 sec

Table 1: Typical computation time for each of the numerical methods. All code was implemented using LAHEY™FORTRAN 95 running on a PC with a Pentium 4 2.40 GHz processer, 512MB of RAM, and running the Windows XP Professional operating system.

$\mathbb{E}[Y] = e^{\gamma} = 1.00$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.05	4.09	4.07	0.5097%
	90	7.67	7.69	7.70	0.3875%
	100	12.68	12.67	12.72	0.2633%
	110	18.94	18.91	18.97	0.1670%
	120	26.22	26.19	26.25	0.1048%
$r = 0.03, q = 0.05$					
	80	4.07	4.07	4.10	0.7333%
	90	7.76	7.73	7.80	0.5239%
	100	12.83	12.77	12.88	0.3359%
	110	19.14	19.06	19.18	0.2066%
	120	26.46	26.37	26.49	0.1255%

Table 2: Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.2082$ and $\theta = 0.60$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

$\mathbb{E}[Y] = e^{\gamma} = 1.05$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.12	4.19	4.14	0.5325%
	90	7.71	7.77	7.75	0.4136%
	100	12.68	12.72	12.71	0.2865%
	110	18.89	18.91	18.93	0.1863%
	120	26.14	26.15	26.17	0.1197%
$r = 0.03, q = 0.05$					
	80	3.74	3.81	3.76	0.7495%
	90	7.10	7.16	7.14	0.5555%
	100	11.82	11.86	11.88	0.5037%
	110	17.82	17.84	17.88	0.3800%
	120	24.91	24.92	24.96	0.2225%

Table 3: Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0.0488$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1888$ and $\theta = 0.60$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

$\mathbb{E}[Y] = e^{\gamma} = 0.95$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$r = 0.05, q = 0.03$					
	80	4.07	4.07	4.10	0.7333%
	90	7.76	7.73	7.80	0.5239%
	100	12.83	12.77	12.88	0.3359%
	110	19.14	19.06	19.18	0.2066%
	120	26.46	26.37	26.49	0.1255%
$r = 0.03, q = 0.05$					
	80	3.67	3.67	3.70	0.7495%
	90	7.11	7.08	7.16	0.5555%
	100	11.92	11.86	12.00	0.5037%
	110	18.00	17.93	18.09	0.3800%
	120	25.15	25.07	25.22	0.2225%

Table 4: Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = -0.0513$. Other parameter values are $\sigma = 0.40$, $K = 100$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.2082$ and $\theta = 0.67$ for the Fourier-Hermite scaling parameter. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

$\mathbb{E}[Y] = e^{\gamma} = 1.00$	S	McKean (Integration)	Method of Lines	Fourier-Hermite	Relative Difference
$\lambda = 1, \theta = 0.50$					
	80	1.10	1.20	1.10	0.5630%
	90	3.03	3.13	3.09	2.0669%
	100	6.95	6.98	7.07	1.8161%
	110	13.11	13.09	13.23	0.9130%
	120	21.06	21.01	21.17	0.5091%
$\lambda = 5, \theta = 0.675$					
	80	4.29	4.54	4.29	0.0306%
	90	7.69	7.91	7.72	0.3908%
	100	12.45	12.57	12.52	0.6008%
	110	18.50	18.52	18.58	0.4195%
	120	25.64	25.59	25.68	0.1514%

Table 5: Comparing the Fourier-Hermite American call price with results obtained from numerical integration and the method of lines, in the case where $\gamma = 0$, with the smaller diffusion volatility of $\sigma = 0.20$. Other parameter values are $r = 0.03$, $q = 0.05$, $K = 100$, $T - t = 0.50$, and $\delta = 0.1980$. The relative difference is calculated as $|C_{McKean} - C_{Fourier-Hermite}|/C_{McKean}$.

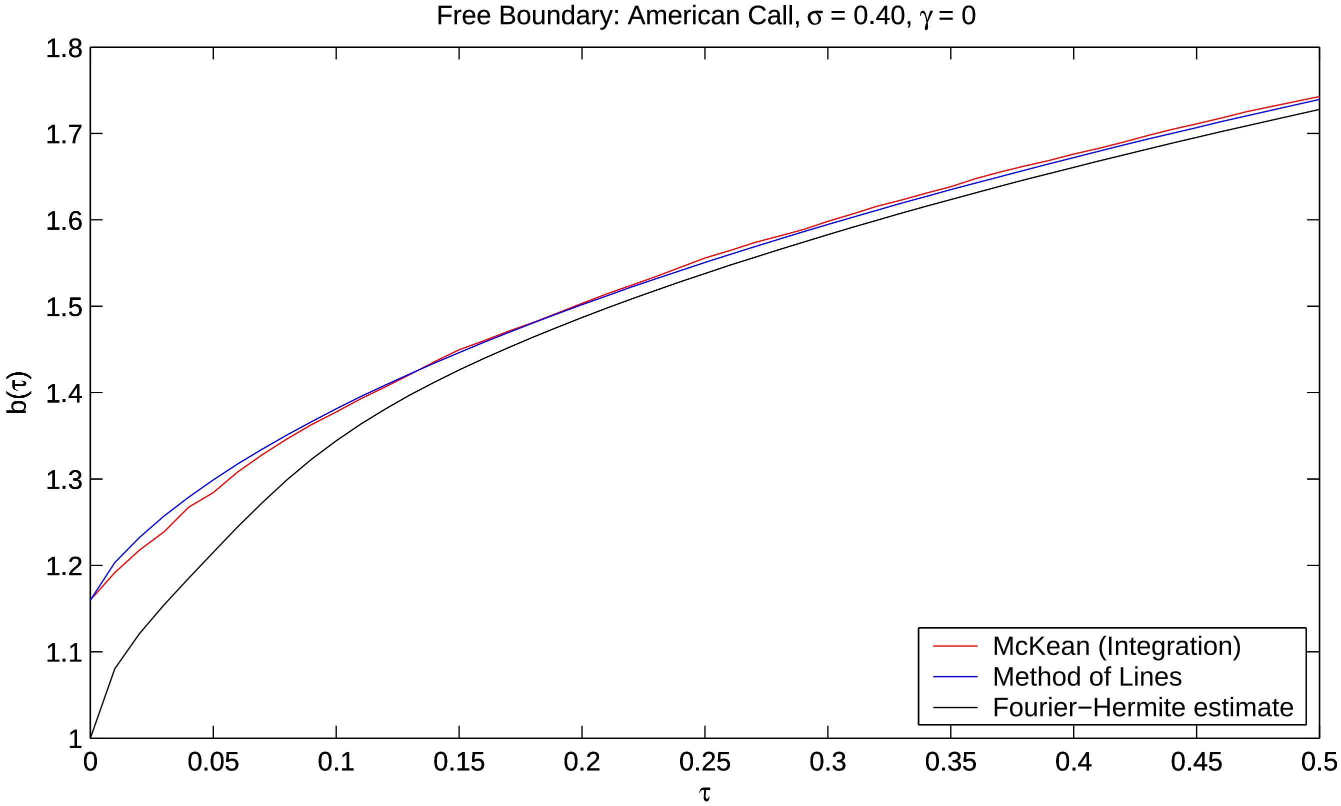

Figure 1: Comparing the early exercise boundary approximation for the American call using numerical integration, method of lines, and Fourier-Hermite series, where the diffusion volatility is $\sigma = 0.40$ and $\gamma = 0$. Other parameters are $K = 1$, $r = 0.03$, $q = 0.05$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1988$ and $\theta = 0.60$ for the Fourier-Hermite method.

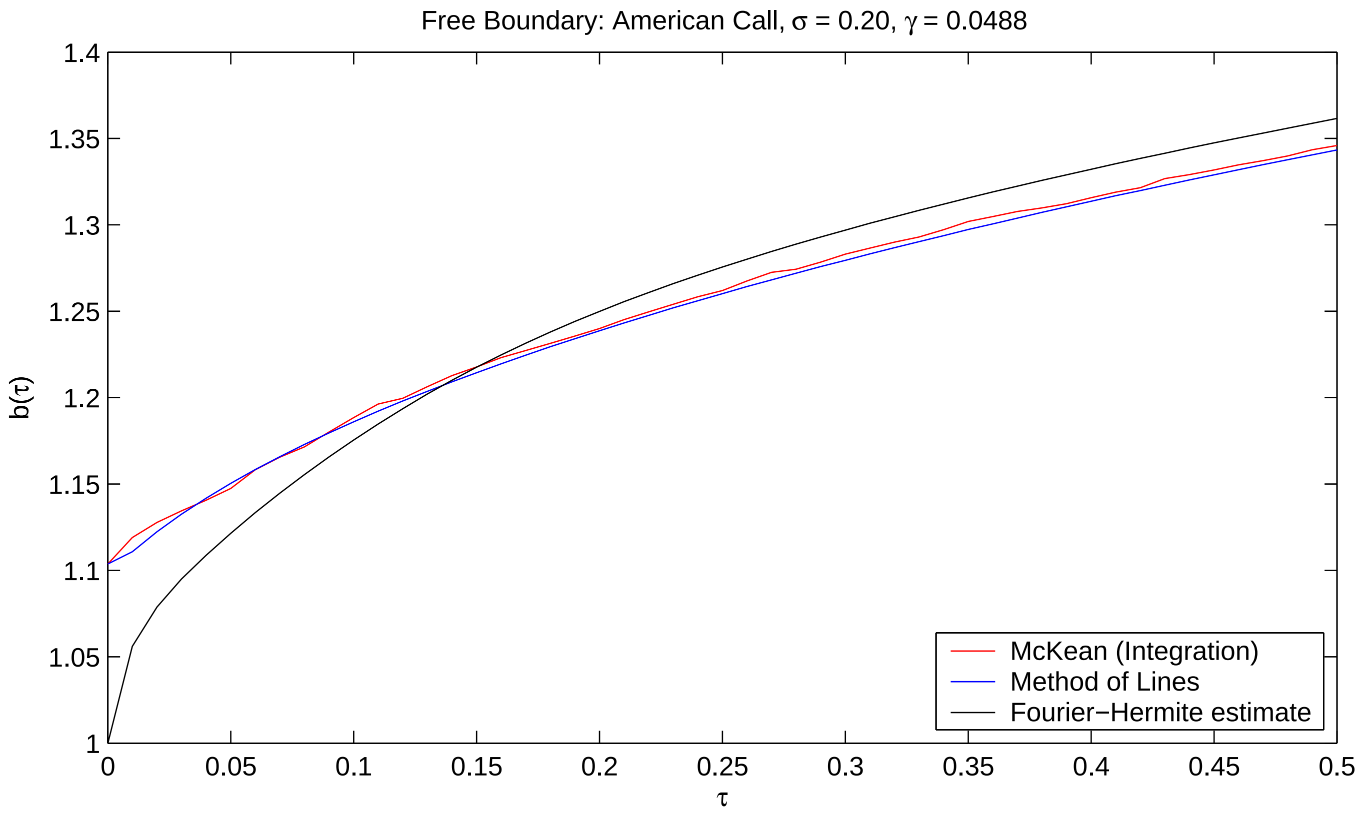

Figure 2: Comparing the early exercise boundary approximation for the American call using numerical integration, method of lines, and Fourier-Hermite series, where the diffusion volatility is $\sigma = 0.20$ and $\gamma = 0.0488$. Other parameters are $K = 1$, $r = 0.03$, $q = 0.05$, $T - t = 0.50$, $\lambda = 1$, $\delta = 0.1888$ and $\theta = 0.485$ for the Fourier-Hermite method.