Value-at-Risk and Expected Shortfall for Quadratic Portfolio of Securities with Mixture of Elliptic Distributed Risk Factors

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Abstract

Generally, in the financial literature, the notion of quadratic VaR is implicitly confused with the Delta-Gamma VaR, because more authors dealt with portfolios that contained derivatives instruments. In this paper, we postpone to estimate both the expected shortfall and Value-at-Risk of a quadratic portfolio of securities (i.e equities) without the Delta and Gamma Greeks, when the joint log-returns changes with multivariate elliptic distribution. To illustrate our method, we give special attention to mixture of normal distributions, and mixture of Student t-distributions.

Key Words: Classical analysis, Computational Finance, Elliptic distributions, Risk Management .

1 Introduction

Value-at-Risk is a market risk management tool that permit to measure the maximum loss of the portfolio with certain confidence probability $1-\alpha$, over a certain time horizon such as one day. Formally , if the price of portfolio's $P(t,S(t))$ at time t is a random variable where $S(t)$ represents a vector of risk factors at time t, then VaR be implicitly given by the formula

$$Prob\{-P(t,S(t))+P(0,S(0))>VaR_{\alpha}\}=\alpha.$$

Generally, to estimate the VaR_{α} for portfolios depending non-linearly on the return , or portfolios of non-normally distributed assets, one turns to Monte Carlo methods. Monte Carlo methodology has the obvious advantage of being almost universally applicable, but has the disadvantage of being much slower than comparable parametric methods, when the latter are available.

In this paper we are concerned with the numerical estimation of the losses that the portfolio of equities faces due to the market, as a function of the future values of S. Following the quadratic Delta-Gamma Portfolio, we introduce the notion of quadratic portfolio of equities due to the analytic approximation of Taylor in 2^{nd} order of log-returns for very small variations of time. Quadratic approximations have also been the subject of a number of papers dedicated to numerical computations for VaR (but these have been done for portfolio that contains derivatives instruments). We refer the reader to Cardenas and al.(1997) [2] for a numerical method to compute quadratic VaR using fast Fourier transform . Note that in [1], Brummelhuis,

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Cordoba, Quintanilla and Seco have dealt with the similar problem, but their work have been done analytically for Delta-Gamma Portfolio VaR when the joint underlying risk factors follow a normal distribution which is a particular case of elliptic distribution . All our calculus will be done according to the assumption that the joint securities (i.e equities) log-returns follow an elliptic distribution. To illustrate our method, we will take some examples of elliptic distributions as mixture of *multivariate t-student* and mixture of normal distributions. Note also that, Following RiskMetrics, Sadefo-Kamdem [11](2003) have generalized the notion of Δ-normal VaR by introducing the notion of Δ-Elliptic VaR , with special attention to Δ-Student VaR , but this concerned the linear portfolio. In this paper, we will do the same for nonlinear quadratic portfolios without derivatives instruments.

The rest of the paper is organized as follows: In section 2, we introduced the notion of quadratic portfolio of securities(i.e equities) due to the 2^{nd} order Taylor approximation of log returns . Our calculus is made with the more generalized assumptions that the joint underlying log-returns follow an elliptic distribution. That is why in section 3, following [4], we recall the definition of elliptic distribution and we show that under the hypothesis of elliptic distribution the VaR estimation of such portfolios is reduced to a multiple integral equation. Next following the paper of Alan Genz [6] , we recall the notions of symmetric interpolar rules for multiple integrals over hypersphere and we use this method to reduce our problem to one dimensional integral equation. In the same section, we illustrate our method by giving an explicit equation with solution VaR (Value-at-Risk) when the joint log-returns follow some particular mixture of elliptic distributions named mixture of multivariate Student t-distributions or the mixture of normal distributions, in these cases the VaR estimation is reduced to finding the zero's of a certain specials functions. In section 5 we treats the expected shortfall for general elliptic quadratic portfolios of securities without derivatives instruments and we illustrate with the special case of normal distribution. Finally, in section 6 we give a conclusion.

2 Quadratic Portfolio of Securities(i.e Equities)

A portfolio of n securities is a vector $\theta \in \mathbb{R}^n$; the component θ_i represents the number of holdings of the i^{th} instruments, which in practise does not need to be an integer. So at time t the price of the portfolio of n securities is given by:

$$P(t) = \sum_{i=1}^{n} \theta_i S_i(t) \tag{1}$$

where $S(t) = (S_1(t), \ldots, S_n(t))$ such that

$$P(t) - P(0) = \sum_{i=1}^{n} \theta_i (S_i(t) - S_i(0)) = \sum_{i=1}^{n} S_i(0).\theta_i.(\frac{S_i(t)}{S_i(0)} - 1)$$

For small fluctuations of time and market, we assume that log-return is given by :

$$log(S_i(t)/S_i(0)) = \eta_i(t) \tag{2}$$

therefore

$$S_i(t) - S_i(0) = S_i(0)(\frac{S_i(t)}{S_i(0)} - 1) = S_i(0)(exp(\eta_i(t)) - 1)$$

Then we have that

$$S(t) = (S_1(0)exp(\eta_1), \ldots, S_n(0)exp(\eta_n)).$$

By using Taylor's expansion of the exponential's due to small fluctuations of returns with a time, we have that :

$$exp(\eta_i(t)) - 1 \approx \eta_i(t) + \frac{\eta_i(t)^2}{2}. \tag{3}$$

If we assume that $\eta = (\eta_1, \ldots, \eta_n)$ is an elliptic distribution note by $N_n(\mu, \Sigma, \phi)$ then

$$P(t) - P(0) = \sum_{i=1}^{n} S_i(0) \cdot \theta_i \cdot (exp(\eta_i(t)) - 1) \approx \sum_{i=1}^{n} S_i(0).\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2}). \quad (4)$$

By following the usual convention of recording portfolio losses by negative numbers, but stating the Value-at-Risk as a positive quantity of money, The VaR_α at confidence level of $1 - \alpha$ is given by solution of the following equation:

$$Prob\{|P(t) - P(0)| \geq VaR_\alpha\} = \alpha$$

In the probability space of losses $|P(t) - P(0)| = -P(t) + P(0)$, therefore recalling (4), we have

$$Prob\{\sum_{i=1}^{n} S_i(0).\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2}) \leq -VaR_\alpha\} = \alpha, \quad (5)$$

then an elementary mathematical tool give that :

$$\eta_i(t) + \frac{\eta_i(t)^2}{2} = \frac{1}{2}(\eta_i(t) + 1)^2 - 1 \quad (6)$$

therefore

$$Prob\{\sum_{i=1}^{n} S_i(0).\frac{\theta_i}{2}(\eta_i(t) + 1)^2) \leq -VaR_\alpha + \sum_{i=1}^{n} \frac{\theta_i}{2} \cdot S_i(0)\} = \alpha. \quad (7)$$

By posing $X = (\eta_1 + 1, ..., \eta_n + 1)$, it is straightforward that , X is an elliptic distribution due to the fact that it is a linear combination of elliptic distribution η. We note $X \sim N(\mu + 1, \Sigma, \phi^{'})$ with a continuous density function $h_1(x)$. Remark that

$$\sum_{i=1}^{n} \alpha_i(\eta_i(t) + 1)^2 = (x, \Lambda.x)$$

with $\Lambda = (\alpha_{ii})_{i=1..n}$ is a diagonal matrix with diagonal values $\alpha_{ii} = \frac{S_i(0).\theta_i}{2} \geq 0$ and $\mu^{'} = (\mu_1{+}1, \ldots, \mu_n{+}1) = \mu + \mathbb{I}$ is the mean vector of X , therefore (7) becomes

$$Prob\{(X, \Lambda.X) \leq k\} = \alpha \quad (8)$$

with $k = -VaR_\alpha + \sum_{i=1}^{n} \alpha_{ii} = \frac{P(0)}{2} - VaR_\alpha$. We will suppose that $k > 0$, this means that the Value-at-Risk of our portfolio's is not greater than P(0)/2 .

Remark 2.1 *We remark that to estimate the Value-at-Risk of a portfolio of securities (i.e equities), the computation of our model need as inputs the quantity θ_i and the initial security price $S_i(0)$ for each $i = 1..n$ as given in (1). Recall that in literature, the computation of Quadratic Delta-Gamma (Δ-Γ) VaR need as inputs the sensitivity vector Δ and the sensitivity matrix Γ, because of the presence of derivatives products in the portfolio.*

3 Reduction to an Integral Equation

In this section, we will reduce the problem of computation of the Value at Risk for quadratic portfolio of equities to the study of the asymptotic behavior of the density function distribution over the hyper-sphere.

3.1 Notions of Elliptic Distributions

The following definitions will be given as in [4](2002) .

3.1.1 Spherical Distribution

Definition 3.1 *A random vector $X = (X_1, X_2, ..., X_n)^t$ has a spherical distribution if for every orthogonal map $U \in R^{n\times n}$ (i.e. maps satisfying $UU^t = U^tU = I_{n\times n}$)*

$$UX =_d X.^1$$

we note that: $X \sim S_n(\phi)$.If X has a density f(x) then this is equivalent to $f(x) = g(x^tx) = g(\|x\|^2)$ for some function $g : R_+ \longrightarrow R_+$, so that the spherical distributions are best interpreted as those distributions whose density is constant on spheres.

Elliptical distributions extend the multivariate normal $N_n(\mu, \Sigma)$, for which μ is mean and Σ is the covariance matrix. Mathematically, they are the affine maps of spherical distributions in $\mathbb{R}^n$.

3.1.2 Elliptic Distribution

Definition 3.2 *Let $T : R^n \longrightarrow R^n$, $y \longmapsto Ay + \mu$, $A \in R^{n\times n}$, $\mu \in R^n$. X has an elliptical distribution if X = T (Y) and $Y \sim S_n(\phi)$. If Y has a density $f(y) = g(y^ty)$ and if A is regular ($det(A) \neq 0$ so that $\Sigma = A^tA$ is strictly positive),then $X = AY + \mu$ has a density*

$$h(x) = g((x-\mu)^t\Sigma^{-1}(x-\mu))/\sqrt{det(\Sigma)} \tag{9}$$

and the contours of equal density are now ellipsoids. An elliptical distribution is fully described by its mean, its covariance matrix and its characteristic generator.

- Any linear combination of an elliptically distributed random vector is also elliptical with the same characteristic generator ϕ . If $Y \sim N_n(\mu, \Sigma, \phi)$, $b \in \mathbf{R}^m$ and $B \in \mathbf{R}^{m\times n}$ then $B.Y + b \sim N_m(B\mu + b, B\Sigma B^t, \phi)$.

3.2 Integral Equation with solution VaR

Since X is an elliptic distribution, its density take the following form:

$$h_1(x) = h((x - \mathbb{1}))$$

$\mathbb{1}$ is the vector of unities and h is the density function of η which take the form :

$$h(x) = g((x-\mu)\Sigma^{-1}(x-\mu)^t)/\sqrt{det(\Sigma)},$$

therefore we have the following equation

$$Prob\{(X, \Lambda X) \geq -VaR_\alpha + \sum_1^n \alpha_{ii}\} = 1 - \alpha = I(k)$$

with VaR_α, as solution such that $k = -VaR_\alpha + \sum_1^n \alpha_{ii}$.

In terms of our elliptic distribution parameters we have to solve the following equation:

$$I(k) = \int_{\{(x,\Lambda.x)\geq k\}} h_1(x)dx = 1 - \alpha \tag{10}$$

with $X \sim E_n(\mu + 1, \Sigma, \phi)$, $AA^t = \Sigma$ and $I(k)$ given as follow:

$$I(k) = \int_{\{(x,\Lambda.x)\geq k\}} g((y-\mu-\mathbb{1})^{\mathrm{t}}\Sigma^{-1}(\mathrm{y}-\mu-\mathbb{1}))\frac{\mathrm{dy}}{\sqrt{\det(\Sigma)}}$$

[1] $=_d$ denote equality in distribution

since Λ is a diagonal matrix with all positive diagonal values, we decompose $\Lambda = \Lambda^{1/2} \cdot \Lambda^{1/2}$ therefore the equation (10) becomes

$$I(k) = \int_{\{<\Lambda^{1/2}(Az+\mu+\mathbb{1}),\Lambda^{1/2}(\mathrm{Az}+\mu+\mathbb{1})>\geq \mathrm{k}\}} g(\|z\|^2)dz = \int_{\|\Lambda^{1/2}(Az+\mu+\mathbb{1})\|_2^2 \leq k\}} g(\|z\|^2)dz.$$

Cholesky decomposition states that $\Sigma = AA^t$ when Σ is suppose to be positive, therefore if we changing the variable $z = A^{-1}(y - \mu - \mathbb{1})$, the precedent integral becomes :

$$I(k) = \int_{(Az+\mu+\mathbb{1})^{\mathrm{t}}\Lambda(\mathrm{Az}+\mu+\mathbb{1})\geq \mathrm{k}\}} g(\|z\|^2)dz.$$

If we do the following decomposition $(Az + \mu + \mathbb{1})^{\mathrm{t}}\Lambda(\mathrm{Az} + \mu + \mathbb{1}) = (\mathrm{z} + \mathrm{v})^{\mathrm{t}}\mathrm{D}(\mathrm{z} + \mathrm{v}) + \delta$, with $D = A^t \cdot \Lambda \cdot A$, $v = A^{-1}(\mu + \mathbb{1})$ and $\delta = 0$, after some elementary calculus

$$I(k) = \int_{\{(z+v)^t D(z+v)\geq k-\delta\}} g(\|z\|^2)dz.$$

If we suggesting $k_1 = k - \delta = R^2$, $z + v = u$, $dz = du$, we find that

$$I(k) = \int_{\{u^t.D.u\geq R^2\}} g(\|u - v\|^2)dz = 1 - \alpha.$$

By introducing the variable $z = D^{1/2}u/R$, we have that:

$$I(R) = R^n \int_{\{\|z\|\geq 1\}} g(\|RD^{\frac{-1}{2}}z - v\|^2)\frac{dz}{\sqrt{det(D)}}$$

next, by using spherical variables $z = r.\xi$ with $\xi \in S_{n-1}$ and $dz = r^{n-1}drd\sigma(\xi)$, where $d\sigma(z)$ is a elementary surface of z on $S_{n-1} = \{\xi | \xi \in \mathbb{R}^n, \xi_1^2 + \xi_2^2 + ... + \xi_n^2 = 1\}$ and by introducing the function $J(r, R)$ such that

$$J(r, R) = \int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}}\xi - v\|^2)d\sigma(\xi) \tag{11}$$

we obtain

$$R^{-n} \cdot I(R) = \int_1^\infty r^{n-1}\Big[\int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}}\xi - v\|^2)d\sigma(\xi)\Big]\frac{dr}{\sqrt{det(D)}} = \int_1^\infty r^{n-1}J(r, R)\frac{dr}{\sqrt{det(D)}}. \tag{12}$$

Next By introducing the function

$$H(s) = s^n \int_1^\infty r^{n-1}J(r, s)dr, \tag{13}$$

our goal will be to solve the following equation

$$H(s) = (1 - \alpha)\sqrt{det(D)}. \tag{14}$$

In the following section, we propose to approximate J(r,R) by applied the numerical methods giving in the paper of Alan Genz (2003), (see [6] for more details).

3.3 Numerical approximation of J(r,R)

In this section, we estimate the integral J(r,R) by a numerical methods given by Alan Genz in [6] .

3.4 Some interpolation rules on S_{n-1}

The paper [6] of Alan Genz, give the following method. Suppose that we need to estimate the following integral

$$J(f) = \int_{S_{n-1}} f(z)d\sigma(z)$$

where $d\sigma(z)$ is an element of surface on $S_{n-1} = \{z|z \in \mathbb{R}^n, z_1^2 + z_2^2 + ... + z_n^2 = 1\}$.

In effect, let be the n-1 simplex by $T_{n-1} = \{x|x \in \mathbb{R}^{n-1}, 0 \leq x_1 + x_2 + ... + x_{n-1} \leq 1\}$ and for any $x \in T_{n-1}$, define $x_n = 1 - \sum_{i=1}^{n-1} x_i$. Also $t_p = (t_{p_1}, ..., t_{p_{n-1}})$ if points $t_0, t_1, \ldots, t_m$ are given, satisfying the condition : $|t_p| = \sum_{i=1}^{n} t_{p_i} = 1$ whenever $\sum_{i=1}^{n} p_i = m$, for non-negative integers $p_1, \ldots, p_n$, then the Lagrange interpolation formula (sylvester [12] for a function g(x) on T_{n-1} is given by

$$L^{(m,n-1)}(g,x) = \sum_{|p|=m} \prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{x_i^2 - t_j^2}{t_{p_i}^2 - t_j^2} g(t_p)$$

$L^{(m,n-1)}(g,x)$ is the unique polynomial of degree m which interpolates g(x) at all of the C_{m+n-1}^m points in the set $\{x|x = (t_{p_1}, ..., t_{p_{n-1}}), |p| = m\}$. Silvester provided families of points, satisfying the condition $|t_p| = 1$ when $|p| = m$, in the form $t_i = \frac{i+\mu}{m+\theta n}$ for i=0,1,...,m, and μ real. If $0 \leq \theta \leq 1$, all interpolation points for $L^{(m,n-1)}(g,x)$ are in T_{n-1}. Sylvester derived families of interpolators rules for integration over T_{n-1} by integrating $L^{(m,n-1)}(g,x)$. Fully symmetric interpolar integration rules can be obtained by substitute $x_i = z_i^2$, and $t_i = u_i^2$ in $L^{(m,n-1)}(g,x)$, and define

$$M^{(m,n)}(f,z) = \sum_{|p|=m} \prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{z_i^2 - u_j^2}{u_{p_i}^2 - u_j^2} f\{u_p\}$$

where $f\{u\}$ is a symmetric sum defined by

$$f\{u\} = 2^{-c(u)} \sum_s f(s_1u_1, s_2u_2, \ldots, s_nu_n)$$

with c(u) the number of nonzero entries in $(u_1, \ldots, u_n)$, and the $\sum_s$ taken over all of the signs combinations that occur when $s_i = \pm 1$ for those i with u_i different to zero.

Lemma 3.3 *If*

$$w_p = J(\prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{z_i^2 - u_j^2}{u_{p_i}^2 - u_j^2})$$

then

$$J(f) = R^{(m,n)}(f) = \sum_{|p|=m} w_p f\{u_p\}$$

$$f\{u\} = 2^{-c(u)} \sum_s f(s_1u_1, s_2u_2, ..., s_nu_n)$$

, with c(u) the number of nonzero entries in $(u_1, ..., u_n)$, and the $\sum_s$ taken over all of the signs combinations that occur when $s_i = \pm 1$ for those i with u_i different to zero.

The proof is given in [6] by (Alan Genz (2003)) as follow:
Let $z^k = z_1^{k_1} z_2^{k_2} \cdot z_n^{k_n}$. J and R are both linear functionals, so it is sufficient to show that $R^{(m,n)}(z^k) = J(z^k)$ whenever $|k| \leq 2m+1$. If k has any component k_i that is odd, then $J(z^k) = 0$, and $R^{(m,n)}(z^k) = 0$ because ever term u_q^k in each of the symmetry sums $f\{u_p\}$ has a cancelling term - u_q^k . Therefore, the only monomials that need to be considered are of the form z^{2k}, with $|k| \leq m$. The uniqueness of $L^{(m,n-1)}(g,x)$ implies

$L^{(m,n-1)}(x^k, x) = x^k$ whenever $|k| \leq m$, so $M^{(m,n)}(z^{2k}, z) = z^{2k}$, whenever $|k| \leq m$. Combining these results:

$$J(f) = M^{(m,n)}(f, z)) = \sum_{|p|=m} w_p f\{u_p\} = R^{(m,n)}(z^k)$$

whenever $f(z) = z^k$, with $|k| \leq 2m+1$, so $R^{(m,n)}(f)$ has polynomial degree 2m+1. For more details (cf. Genz [6]).

3.5 Application to Numerical approximation of $J(r, R)$

Since our goal is to estimate the integral (10), it is straightforward that the theorem (4.2.1) is applicable to the function f such that

$$f(z) = g(\|rRD^{\frac{-1}{2}}z - v\|^2).$$

Next, since we have that

$$f\{u_p\} = g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2),$$

by introducing the approximate function J_{u_p} that depend to the choice of u_p, (10) becomes

$$J(r, R) \approx \sum_{|p|=m} \sum_{s} w_p \quad g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2) = J_{u_p}(r, R) \tag{15}$$

we note $s.u_p = (s_1 u_1, ..., s_n u_n)^t$.

Remark 3.4 *$J_{u_p}(r, R)$ is the numerical approximation of $J(r, R)$ as given in (15), is depend to the choice of interpolation points u_p on hypersphere. Recall that $J(r, R)$ was a fixed function that depend to R and the density function of our elliptic distribution .*

By introducing H_{u_p}, the approximate function of H as define in (13) that depend of J_{u_p}, such that

$$H_{u_p}(s) = s^n \int_1^{\infty} r^{n-1} J_{u_p}(r, s) dr \approx H(s). \tag{16}$$

By replace $H(s)$ in (14) by $H_{u_p}(s)$ we then prove the following result:

Theorem 3.5 *If we have a quadratic portfolio of securities (i.e equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by $\Delta\Pi \approx \sum_{i=1}^n S_i(0) \cdot \theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, with portfolio weights θ_i. Suppose moreover that the joint log-returns is a random vector $(\eta_1, \ldots, \eta_n)$ that follows a continuous elliptic distribution, with probability density as in (9),where μ is the vector mean and Σ is the variance-covariance matrix, and where we suppose that $g(s^2)$ is integrable over $\mathbb{R}$, continuous and nowhere 0. Then the approximate portfolio's quadratic elliptic VaR^g_{α,u_p} at confidence $(1-\alpha)$ is given by*

$$VaR^g_{\alpha,u_p} = \frac{P(0)}{2} - R^2_{g,u_p} \tag{17}$$

where R_{g,u_p} is the unique solution of the equation

$$H_{u_p}(s) = (1-\alpha) \cdot \sqrt{det(D)} = \frac{(1-\alpha)}{2^{n/2}} \sqrt{det(\Sigma) \prod_{i=1}^{n} \theta_i \cdot S_i(0)}. \tag{18}$$

In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.

Remark 3.6 *The precedent theorem give to us an approximate Quadratic Portfolio Value-at-Risk (VaR^g_{α,u_p}) that depend to our choice of interpolation points on hypersphere ,α and the function g. Therefore it is clear that the best choice of interpolation point will depend to the g function in (9).*

With some simple calculus we have the following remark

Remark 3.7

$$J_{u_p}(r,R) = \sum_{|p|=m} w_p \sum_s g\Big(a(s,u_p,R)\cdot r^2 - 2\cdot b(s,u_p,R,D,v)\cdot r + c(v)\Big) \tag{19}$$

for which $a(s,u_p,R,D) = \|RD^{\frac{-1}{2}}(s.u_p)\|^2$, $b(s,u_p,R,D,v) = R < D^{\frac{-1}{2}}(s.u_p), v >$,$c = \|v\|^2$. *Sometimes, for more simplification we will note a,b,c.*

Since inequality of Schwartz give that $b^2 - ac < 0$, we use the change of variable by posing $b1 = \frac{b^2-ac}{a} < 0$, $u = r - \frac{b}{a}$, by using using the binom of Newton, and by introducing the function $G^{j,g}$ for $j = 0,..,n-1$, such that we have the following remark

Remark 3.8

$$J(r,R) = \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} G^{j,g}_{u_p,s}(R) \tag{20}$$

with

$$G^{j,g}_{u_p,s}(R) = \int_{1-\frac{b}{a}}^{\infty} z^j \cdot g(az^2 - b_1)dz \tag{21}$$

for which a,b and c are defined in (3.7).

By replace $d = b/a = \frac{<D^{\frac{-1}{2}}(s.u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$ by its value in (3.8),we obtain

$$G^{j,g}_{u_p,s}(R) = \int_{1-\frac{<D^{\frac{-1}{2}}(s.u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}}^{\infty} z^j \cdot g\Big(R^2\cdot\|D^{\frac{-1}{2}}(s.u_p)\|^2 z^2 - \frac{< D^{\frac{-1}{2}}(s.u_p), v >^2}{\|D^{\frac{-1}{2}}(s.u_p)\|^2} + \|v\|^2\Big)dz \tag{22}$$

we then have the following theorem

Theorem 3.9 *If we have a quadratic portfolio of securities (i.e equities) for which the joint securities log-returns changes with continuous elliptic distribution with pdf distribution as in* (9)*, then the approximate portfolio's quadratic elliptic* VaR^g_{α,u_p} *at confidence* $(1-\alpha)$ *is given by*

$$VaR^g_{\alpha,u_p} = \frac{P(0)}{2} - R^2_{g,u_p} \tag{23}$$

where R_{g,u_p} *is the unique solution of the equation*

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} R^{j+1}\cdot\Big(\frac{< D^{\frac{-1}{2}}(s.u_p), v >}{\|D^{\frac{-1}{2}}(s.u_p)\|^2}\Big)^{n-1-j} \cdot G^{j,g}_{u_p,s}(R) = \frac{(1-\alpha)}{2^{n/2}}\sqrt{det(\Sigma)\prod_{i=1}^{n}\theta_i\cdot S_i(0)}. \tag{24}$$

In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.

Remark 3.10 *We have reduced our problem to one dimensional integral equation. Therefore, to get an explicit equation to solve, we need to estimate* $G^{j,g}_{u_p,s}(R)$ *that depend to R with parameters g,*u_p*,v and D.*

Therefore, in the case of normal distribution or *t*-distribution, it will suffices to replace g in the expression of (21), an to estimate the one dimensional integral (21).

3.5.1 The case of normal distribution

In the case of normal distribution, the pdf is given by:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} exp(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^t) \tag{25}$$

and specific is given as follow

$$g(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{x}{2}} = C(n).e^{-\frac{x}{2}}$$

therefore it suffices to replace g in (21) then

$$G^j_{u_p,s}(R) = (2\pi)^{-\frac{n}{2}} e^{\frac{b_1}{2}} \int_{1-\frac{b}{a}}^{\infty} u^j e^{-\frac{au^2}{2}} du. \tag{26}$$

If $1 - \frac{b}{a} > 0$ (it is the case when R is sufficiently big such that $|v| < R\|D^{\frac{-1}{2}}(s.u_p)\|$).

$$\frac{G^j_{u_p,s}(R)}{\exp(-\frac{\|v\|^2}{2})(2\pi)^{-\frac{n}{2}}} = \exp\left(\frac{< D^{\frac{-1}{2}}(s.u_p), v >^2}{R\|D^{\frac{-1}{2}}(s^t u_p)\|^2}\right)(2/a)^{\frac{1+j}{2}} \Gamma(\frac{j+1}{2}, \frac{(R\|D^{\frac{-1}{2}}(s^t u_p)\|)^2}{2}(1 - \frac{< D^{\frac{-1}{2}}(s.u_p), v >}{R\|D^{\frac{-1}{2}}(s^t u_p)\|^2})^2) \tag{27}$$

therefore, since $a = (R\|D^{\frac{-1}{2}}(s.u_p)\|)^2$ we have the following theorem

Theorem 3.11 *If we have a portfolio of securities (i.e equities), such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^n S_i(0) \cdot \theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, *with portfolio weights* θ_i. *Suppose moreover that the joint log-returns is a random vector* $(\eta_1, \ldots, \eta_n)$ *that follows a continuous multivariate normal distribution with density function in (25), vector mean* μ , *and variance-covariance matrix* Σ, *the Quadratic Value-at-Risk* (VaR_{α,u_p} *at confidence* $1-\alpha$ *is given by the following formula*

$$R^2_{u_p,\alpha} = -VaR_{\alpha,u_p} + \frac{P(0)}{2}$$

for which $R_{u_p,\alpha}$ *is the unique solution of the following transcendental equation.*

$$2(1-\alpha)\frac{\sqrt{det(D)}}{(2\pi)^{\frac{n}{2}}} = \sum_{|p|=m} w_p \sum_s \frac{(< D^{\frac{-1}{2}}(s.u_p), v >)^{(n-j-1)}}{\|D^{\frac{-1}{2}}(s.u_p)\|^{(2n-1-j)}} e^{\frac{b_1}{2}} \sum_{j=0}^{n-1} \binom{n-1}{j} \Gamma(\frac{j+1}{2}, \frac{a}{2}(1-\frac{b}{a})^2) \tag{28}$$

for which $b_1 = \frac{(<D^{\frac{-1}{2}}(s.u_p),v>)^2}{\|D^{\frac{-1}{2}}(s.u_p)\|^2} - \|v\|^2$, $\frac{b}{a} = \frac{<D^{\frac{-1}{2}}(s.u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$, $a = R^2\|D^{\frac{-1}{2}}(s.u_p)\|^2$. *In this case, we implicitly assume* $VaR_{\alpha;u_p} \leq P(0)/2$. Γ *is the incomplete gamma function.*

3.5.2 Case of t-student distribution

If our elliptic distribution is in particular chosen as the multivariate t-student distribution, we will have density function given by

$$g(x) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\nu/2).\pi^{n/2}}\Big(1+\frac{x}{\nu}\Big)^{(\frac{-\nu-n}{2})} = C(n,\nu)\Big(1+\frac{x}{\nu}\Big)^{(\frac{-\nu-n}{2})} \tag{29}$$

therefore by replacing g in (24), we obtain the equation

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} \int_{1-\frac{b}{a}}^{\infty} u^j \Big(1 + \frac{au^2 - b_1^2}{\nu}\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\sqrt{det(D)}}{C(n,\nu)R^n}. \tag{30}$$

suggesting $c_1 = \nu - b_1^2$, our equation is reduce to

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} \int_{1-\frac{b}{a}}^{\infty} u^j \Big(au^2 + c_1\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\sqrt{det(D)}}{\nu^{\frac{\nu+n}{2}} C(n,\nu) R^n}. \tag{31}$$

changing variable in this integral according to $v = u^2$ and $\beta = \frac{a}{c_1}$, we find that

$$R^n \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} c_1^{\frac{-n-\nu}{2}} \int_{(1-\frac{b}{a})^2}^{\infty} v^{\frac{j+1}{2}-1} \Big(\beta v + 1\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\pi^{n/2}\Gamma(\nu/2)\sqrt{det(D)}}{\nu^{\frac{\nu+n}{2}}\Gamma(\frac{\nu+n}{2})}. \tag{32}$$

For the latter integral equation, we will use the following formula from [7]:

Lemma 3.12 *(cf. [7], formula 3.194(2)). If $|arg(\frac{u}{\beta})| < \pi$, and $Re(\nu_1) > Re(\mu) > 0$, then*

$$\int_u^{+\infty} x^{\mu-1}(1+\beta x)^{-\nu_1} dx = \frac{u^{\mu-\nu_1}\beta^{-\nu_1}}{\nu_1 - \mu} {}_2F_1(\nu_1, \nu_1 - \mu; \nu_1 - \mu + 1; -\frac{1}{\beta \cdot u}). \tag{33}$$

Here ${}_2F_1(\alpha;\beta,\gamma;w)$ is the hypergeometric function.

In our case,$\nu_1 = \frac{\nu+n}{2}$, $u = (1-b/a)^2$, $\nu_1 - \mu = \frac{n+\nu-j-1}{2}$, $\nu_1 - \mu + 1 = \frac{n+\nu-j+1}{2}$ therefore If we replace in (32), we will obtain the following result.

Theorem 3.13 *If we have a portfolio of securities (i.e equities), such that the Profit & Loss function over the time window of interest is, to good approximation, given by $\Delta\Pi \approx \sum_{i=1}^n S_i(0)\cdot\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, with portfolio weights θ_i. Suppose moreover that the joint log-returns is a random vector $(\eta_1,\ldots,\eta_n)$ that follows a continuous multivariate t-distribution with density function given by (29), vector mean μ , and variance-covariance matrix Σ, the Quadratic Value-at-Risk (VaR_{α,u_p} at confidence $1-\alpha$ is given by the following formula*

$$R_{u_p,\alpha}^2 = -VaR_{\alpha,u_p} + \frac{P(0)}{2}$$

for which $R_{u_p,\alpha}$ is the unique solution of the following transcendental equation.

$$\frac{R^n}{(1-\alpha)} \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n}{j} \frac{(b/a)^{n-1-j}}{(\nu - b_1^2)^{\frac{-n-\nu}{2}}} \frac{{}_2F_1\left[\frac{n+\nu}{2}, \frac{n+\nu-j-1}{2}; \frac{n+\nu-j+1}{2}; \frac{b_1^2-\nu}{a(1-\frac{b}{a})^2}\right]}{(n+\nu-j-1)\sqrt{det(\Sigma)}\prod_{i=1}^n \theta_i\cdot S_i(0)} = \frac{(\pi/2)^{n/2}\Gamma(\nu/2)}{\nu^{\frac{\nu+n}{2}}\Gamma(\frac{\nu+n}{2})} \tag{34}$$

for which $b_1 = \frac{(<D^{\frac{-1}{2}}(s.u_p),v>)^2}{\|D^{\frac{-1}{2}}(s.u_p)\|^2} - \|v\|^2$, $\frac{b}{a} = \frac{<D^{\frac{-1}{2}}(s.u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$, $a = R^2\|D^{\frac{-1}{2}}(s.u_p)\|^2$. In this case, we implicitly assume that $VaR_{\alpha,u_p} \le P(0)/2$

Remark 3.14 Note that, Hypergeometric ${}_2F_1$'s have been extensively studies, and numerical software for their evaluation is available in Maple and in Mathematica.

4 Quadratic VaR with mixture of elliptic Distributions

Mixture distributions can be used to model situations where the data can be viewed as arising from two or more distinct classes of populations; see also [9]. For example, in the context of Risk Management, if we divide trading days into two sets, quiet days and hectic days, a mixture model will be based on the fact that returns are moderate on quiet days, but can be unusually large or small on hectic days. Practical applications of mixture models to compute VaR can be found in Zangari (1996), who uses a mixture normal to incorporate fat tails in VaR estimation. In this section, we sketch how to generalize the preceding section to the situation where the joint log-returns follow a mixture of elliptic distributions, that is, a convex linear combination of elliptic distributions.

Definition 4.1 We say that $(X_1, \dots, X_n)$ has a joint distribution that is the mixture of q elliptic distributions $N(\mu_j, \Sigma_j, \phi_j)^2$, with weights $\{\beta_j\}$ (j=1,..,q ; $\beta_j > 0$; $\sum_{j=1}^q \beta_j = 1$), if its cumulative distribution function can be written as

$$F_{X_1,\dots,X_n}(x_1, \dots, x_n) = \sum_{j=1}^{q} \beta_j F_j(x_1, \dots, x_n)$$

with $F_j(x_1, \dots, x_n)$ the cdf of $N(\mu_j, \Sigma_j, \phi_j)$.

Remark 4.2 In practice, one would usually limit oneself to $q = 2$, due to estimation and identification problems; see [9].

We will suppose that all our elliptic distributions $N(\mu_j, \Sigma_j, \phi_j)$ admit a pdf :

$$f_j(x) = |\Sigma_j|^{-1/2} g_j((x - \mu_j){\Sigma_j}^{-1}(x - \mu_j)^t) \tag{35}$$

for which each g_j is continuous integrable function over $\mathbb{R}$, and that the g_j never vanish jointly in a point of $\mathbb{R}^q$. The pdf of the mixture will then simply be $\sum_{j=1}^q \beta_j f_j(x)$.

Let

$$\Sigma_j = A_j^t \, A_j$$

So, following (12),we introduce $J_k(r, R)$ such that

$$\alpha R^{-n} = \sum_{k=1}^{q} \int_1^\infty r^{n-1} \Big[\int_{S_{n-1}} g_k(\|rRD_k^{\frac{-1}{2}} \xi - v_k\|^2) d\sigma(\xi) \Big] \frac{dr}{\sqrt{det(D_k)}} = \sum_{k=1}^{q} \int_1^\infty r^{n-1} J_k(r, R) dr \tag{36}$$

Next following (21), we introduce the function

$$G^{j,g}_{u_p,s,k}(R) = R^n \int_{1-\frac{b_k}{a_k}}^\infty z^j \cdot g_k(a_k z^2 - b_{1k}) dz \tag{37}$$

with $a_k = \|RD^{\frac{-1}{2}}(s.u_{pk})\|^2$, $b_k = R < D_k^{\frac{-1}{2}}(s.u_{pk}), v >$,$c_k = \|v_k\|^2$,$b_{1k} = \frac{b_k^2 - a_k c_k}{a_k}$, then we have the following corollary

Theorem 4.3 *If we have a portfolio of securities (i.e equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^n S_i(0) \cdot \theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$*, with portfolio weights* θ_i*. Suppose moreover that the joint log-returns is a random vector* $(\eta_1, \dots, \eta_n)$ *is a mixture of q elliptic distributions, with density*

$$h(x) = \sum_{j=1}^{q} \beta_j |\Sigma_j|^{-1/2} g_j((x - \mu_j)\Sigma_j^{-1}(x - \mu_j)^t)$$

where μ_j *is the vector mean, and* Σ_j *the variance-covariance matrix of the* j*-th component of the mixture. We suppose that each* g_j *is integrable function over* $\mathbb{R}$*, and that the* g_j *never vanish jointly in a point of* $\mathbb{R}^m$*. Then the value-at-Risk, or* Quadratic mixture-elliptic VaR*, at confidence* $1 - \alpha$ *is given as the solution of the transcendental equation*

$$\sum_{k=1}^{q} \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b_k/a_k)^{n-1-j} \frac{G^{j,g}_{u_p,s,k}\Big((\frac{P(0)}{2} - VaR^g_{\alpha,u_p})^{1/2} \Big)}{\sqrt{det(\Sigma_k)} \prod_{i=1}^n \theta_i \cdot S_i(0)} = \frac{(1-\alpha)}{2^{n/2}} \tag{38}$$

for which $G^{j,g}_{u_p,s,k}$ *is defined in (37). In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.*

[2] or $N(\mu_j, \Sigma_j, g_j)$ if we parameterize elliptical distributions using g_j instead of ϕ_j

Remark 4.4 One might, in certain situations, try to model with a mixture of elliptic distributions which all have the same variance-covariance and the same mean, and obtain for example a mixture of different tail behaviors by playing with the g_j's.

The preceding can immediately be specialized to a mixture of normal distributions: the details will be left to the reader.

4.1 Application with mixture of Student t-Distributions

We will consider a mixture of q Student t-distributions such that, the k^{th} density function $i = 1, .., q$ will be given by

$$g_k(x) = \frac{\Gamma(\frac{\nu_k+n}{2})}{\Gamma(\nu_k/2).\pi^{n/2}}\Big(1+\frac{x}{\nu_k}\Big)^{(\frac{-\nu_k-n}{2})} = C(n,\nu_k)\Big(1+\frac{x}{\nu_k}\Big)^{(\frac{-\nu_k-n}{2})} \tag{39}$$

and $\Sigma_k = A_k^t A_k$ therefore by replacing g_k by g in (40) and since integration is a linear operation, we obtain the following theorem

Theorem 4.5 *If we have a portfolio of securities (i.e equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^{n} S_i(0)\cdot\theta_i(\eta_i(t)+\frac{\eta_i(t)^2}{2})$*, with portfolio weights* θ_i*. Suppose moreover that the joint log-returns is a random vector* $(\eta_1,\ldots,\eta_n)$ *is a mixture of q t-distributions, with density*

$$h(x) = \sum_{j=1}^{q}\beta_j|\Sigma_j|^{-1/2}\frac{\Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2).\pi^{n/2}}\Big(1+\frac{(x-\mu_j)\Sigma_j^{-1}(x-\mu_j)^t}{\nu_j}\Big)^{-\frac{n+\nu_j}{2}}$$

where μ_j *is the vector mean, and* Σ_j *the variance-covariance matrix of the* j*-th component of the mixture. We suppose that each* g_j *is integrable function over* $\mathbb{R}$*, and that the* g_j *never vanish jointly in a point of* $\mathbb{R}^m$*. Then the value-at-Risk, or* Quadratic mixture-student VaR*, at confidence* $1-\alpha$ *is given by :*

$$R_{u_p}^2 = -VaR_\alpha + \frac{P(0)}{2}$$

for which R_{u_p} *is the unique positive solution of the following equation:*

$$\sum_{k=1}^{q}\sum_{|p|=m}\frac{w_p\Gamma(\frac{\nu_k+n}{2})}{\Gamma(\nu_k/2).}\sum_{s}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{R^n(b_k/a_k)^{n-1-j}}{((\nu_k-b_{1k}^2)/\nu_k)^{\frac{-n-\nu}{2}}}\frac{{}_2F_1\left[\frac{n+\nu_k}{2},\frac{n+\nu_k-j-1}{2};\frac{n+\nu_k-j+1}{2};\frac{b_{1k}^2-\nu}{a_k(1-\frac{b_k}{a_k})^2}\right]}{(n+\nu_k-j-1)\sqrt{|\Sigma_k|\prod_{i=1}^{n}\theta_i\cdot S_i(0)}} = \frac{(1-\alpha)}{(\pi/2)^{\frac{-n}{2}}} \tag{40}$$

for which $b_{1k} = \frac{(<D_k^{\frac{-1}{2}}(s.u_{pk}),v_k>)^2}{\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2} - \|v_k\|^2$, $\frac{b_k}{a_k} = \frac{<D_k^{\frac{-1}{2}}(s.u_{pk}),v_k>}{R\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2}$, $a_k = R^2\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2$ *and* $det(\Sigma_k) = |\Sigma_k|$ *. In this case, we implicitly assume that our losses will not be greater than* $P(0)/2$.

5 Elliptic Quadratic Expected Shortfall for portfolio of securities

Expected shortfall is a sub-additive risk statistic that describes how large losses are on average when they exceed the VaR level. Expected shortfall will therefore give an indication of the size of extreme losses when the VaR threshold is breached. We will evaluate the expected shortfall for a quadratic portfolio of securities under the hypothesis of elliptically distributed risk factors. Mathematically, the expected shortfall associated with a given VaR is defined as:

$$\text{Expected Shortfall } = \mathbb{E}(-\Delta\Pi| -\Delta\Pi > VaR),$$

see for example [9]. Assuming again a multivariate elliptic pdf $f(x) = |\Sigma|^{-1} g((x-\mu)\Sigma^{-1}(x-\mu)^t)$, the Expected Shortfall at confidence level $1-\alpha$ is given by

$$\begin{aligned}
-ES_\alpha &= \mathbb{E}(\Delta\Pi \mid \Delta\Pi \leq -VaR_\alpha) \\
&= \frac{1}{\alpha}\mathbb{E}\left(\Delta\Pi \cdot 1_{\{\Delta\Pi \leq -VaR_\alpha\}}\right) \\
&= \frac{1}{\alpha}\int_{\{(x,\Lambda.x)-P(0)/2\leq -VaR_\alpha\}} ((x,\Lambda.x)-P(0)/2)\ h_1(x)\ dx \\
&= \frac{|\Sigma|^{-1/2}}{\alpha}\int_{\{(x,\Lambda.x)\leq -VaR_\alpha+P(0)/2\}} ((x,\Lambda.x)-P(0)/2)\ g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)dx.
\end{aligned}$$

Using the definition of VaR_α and by replace $\Delta\Pi = (X,\Lambda.X) - \frac{P(0)}{2}$, with random vector X define in section 2,

$$ES_\alpha = \frac{P(0)}{2} - \frac{|\Sigma|^{-1/2}}{\alpha}\int_{\{(x,\Lambda.x)\leq -VaR_\alpha+P(0)/2\}} (x,\Lambda.x)\ g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)dx \tag{41}$$

Let $\Sigma = A^t\ A$, as before.Doing the same linear changes of variables as in section 2 and section 3, we arrive at:

$$\begin{aligned}
ES_\alpha &= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}\Big[\int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}}\xi - v\|^2)d\sigma(\xi)\Big]dr \\
&= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}J(r,R)dr \\
&\approx \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}J_{u_p}(r,R)dr \\
&= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\sum_{|p|=m}\sum_s w_p \quad \int_0^1 r^{n+1} g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2) \quad dr \quad ,
\end{aligned}$$

By introducing the function $Q^g_{u_p,s}$ such that

$$Q^g_{u_p,s}(R) = R^{n+2}\int_0^1 r^{n+1} g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2) \quad dr \tag{42}$$

we have the following theorem

Theorem 5.1 *Suppose that the portfolio is quadratic in the risk-factors $X = (X_1,\cdots,X_n)$: $\Delta\Pi = (X,\Lambda\cdot X) - \frac{P(0)}{2}$ and that $X \sim N(\mu+1,\Sigma,\phi)$, with pdf $f(x) = |\Sigma|^{-1} g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)$. If the VaR_α is given, then the expected Shortfall at level α is given by :*

$$ES_\alpha = \frac{P(0)}{2} - \frac{|D|^{-1/2}}{\alpha}\sum_{|p|=m}\sum_s w_p \quad Q^g_{u_p,s}\Big(\big(\frac{P(0)}{2} - VaR_\alpha\big)^{1/2}\Big). \tag{43}$$

we introduce I^g_1 and I^g_2 such that

$$R^{-n-2}Q^g_{u_p,s}(R) = \int_0^1 r^{n+1} g(ar^2 - 2br + c)dr = \int_0^\infty - \int_1^\infty = I^g_{1,u_p,s}(R) - I^g_{2,u_p,s}(R)$$

Following the Integral (21)

$$I^g_{2,u_p,s}(R) = \sum_{j=0}^{n+1} \binom{n+1}{j} (b/a)^{n+1-j} \cdot G^{j,g}_{u_p,s}(R)$$

for which a,b and c are defined in remark (3.7) and

$$I^g_{1,u_p,s}(R) = \int_0^\infty r^{n+1} g(ar^2 - 2br + c)dr$$

5.1 Expected Shortfall with normal distribution

In the case of normal distribution, the pdf is given by (25) and the specific g is given as follow

$$g(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{x}{2}} = C(n).e^{-\frac{x}{2}}$$

therefore it suffices to replace g in (21) then

$$\begin{aligned} G^j_{u_p,s}(R) &= (2\pi)^{-\frac{n}{2}} \exp\Big(\frac{b^2-ac}{2a}\Big) \int_{1-\frac{b}{a}}^\infty u^j \exp(-\frac{au^2}{2}) du \\ &= (2\pi)^{-\frac{n}{2}} \exp\Big(\frac{b^2-ac}{2a}\Big) (2/a)^{\frac{1+j}{2}} \Gamma\Big(\frac{j+1}{2}, \frac{a}{2}(1-\frac{b}{a})^2\Big). \end{aligned}$$

By using the following lemma

Lemma 5.2 *(cf. [7], formula 3.462(1)). If $Re(\nu) > 0$, and $Re(\beta) > 0$, then*

$$\int_0^{+\infty} x^{\nu-1} \exp(-\beta x^2 - \lambda x) dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\Big(\frac{\lambda^2}{8\beta}\Big) \mathbb{D}_{-\nu}\Big(\frac{\lambda}{\sqrt{2\beta}}\Big) \tag{44}$$

Here $\mathbb{D}_{-\nu}$ is the parabolic cylinder function with

$$\mathbb{D}_{-\nu}(z) = 2^{\frac{-\nu}{2}} e^{\frac{-z^2}{2}} \Big[\frac{\sqrt{\pi}}{\Gamma(\frac{1+\nu}{2})} \Phi\Big(\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\Big) - \frac{\sqrt{2\pi} z}{\Gamma(\frac{\nu}{2})} \Phi\Big(\frac{1+\nu}{2}, \frac{3}{2}; \frac{z^2}{2}\Big) \Big]$$

where Φ is the confluent hypergeometric function (for more details see [7] page 1018).

we next obtain

$$I^g_{2,u_p,s}(R) = (2\pi)^{-\frac{n}{2}} \sum_{j=0}^{n+1} \binom{n+1}{j} (b/a)^{n+1-j} \cdot \exp\Big(\frac{b^2-ac}{4a}\Big) (2/a)^{\frac{1+j}{2}} \Gamma\Big(\frac{j+1}{2}, \frac{a}{2}(1-\frac{b}{a})^2\Big)$$

and

$$\begin{aligned} I^g_{1,u_p,s}(R) &= (2\pi)^{-\frac{n}{2}} \exp(-\frac{\|v\|^2}{2}) \int_0^1 r^{n+1} \exp\Big(-\frac{ar^2-2br}{2}\Big) dr \\ &= (2\pi)^{-\frac{n}{2}} \exp(-\frac{\|v\|^2}{2}) a^{\frac{n+2}{2}} \Gamma(n+2) \exp\Big(\frac{b^2}{4a}\Big) \mathbb{D}_{-n-2}\Big(\frac{-b}{\sqrt{a}}\Big) \end{aligned}$$

for which $\mathbb{D}_{-n-2}$ is the parabolic cylinder function. We have therefore prove the following result:

Theorem 5.3 *Suppose that the portfolio is quadratic in the risk-factors $X = (X_1, \cdots, X_n)$: $\Delta\Pi = (X, \Lambda \cdot X) - \frac{P(0)}{2}$ and that X is a multivariate normal distribution, If the VaR_α is given, then the expected Shortfall at level α is given by :*

$$ES_\alpha = \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha} \sum_{|p|=m} \sum_s w_p \Big[I^g_{1,u_p,s}(R) - I^g_{2,u_p,s}(R) \Big]. \tag{45}$$

for which $R = \sqrt{\frac{P(0)}{2} - VaR_\alpha}$

The preceding can immediately be specialized to a mixture of normal distributions. The details will be left to the reader.

5.2 Student t-distribution Quadratic Expected Shortfall

Following the precedent section 3,4,5, and particularly the lemma (33) , the application can be specialized to a Student t-distribution. The details will be left to the reader.

5.3 How to choose an interpolation points u_p on hypersphere

In order to obtain a good approximation of our integral, one will choose the points of interpolation u_p of our g function such that our approximation is the best as possible. In the case where the g function decreases quickly with all its derivatives of all order, in inspiring of the classical analysis, one will choose the points which render the maximum function $\|rRD^{-1/2}(s.u_p) - v\|$.

6 Conclusion

By following the notion of Delta-Gamma Portfolio that contains derivatives instruments, we have introduced a Quadratic Portfolios of securities (i.e equities) without the use of Delta and Gamma. By using the assumption that the joint securities log-returns follow a mixture of elliptic distributions, we have reduced the estimation of VaR of such quadratic portfolio, to the resolution of a multiple integral equation, that contain a multiple integral over hypersphere. To approximate a multiple integral over hypersphere, we propose to use a numerical approximation method given by Alan Genz in[6]. Therefore, the estimation of VaR is reduced to the resolution of one dimensional integral equation. To illustrate our method, we give special attention to mixture of normal distribution and mixture of multivariate t-student distribution. In the case of t-distribution, we need the hypergeometric special function. For given VaR, we also show how to estimate the expected shortfall of the Quadratic portfolio without derivatives instruments, when the risk factors follow an elliptic distributions and we illustrate our proposition with normal distribution by using the parabolic cylinder function. Note that this method will be applicable to capital allocation, if we could consider an institution as a portfolio of multi-lines businesses. In the sequel paper, we will dealt with this numerical Quadratic method, when the Portfolio contains derivatives instruments (i.e options). A concrete application need the estimation of w_p such that $|p| = m$, therefore we send the reader to Alan Genz [6].

References

[1] R. Brummelhuis, A. Cordoba, M. Quintanilla and L. Seco, Principal Component Value-at-Risk, *Mathematical Finance* **12**, 2002, pp. 23-43.

[2] Cardenas, J., E.Fruchard, E.Koehler, C.Michel, and I.Thomaseau(1997): VaR: One Step beyond, RISK 10(10), 72-75.

[3] Kevin Dowd,(1998) Beyond Value-at-Risk: the new science of risk Management (Wiley series in Frontiers in Finance).

[4] P.Embrechts , A.McNeil and D.Strauman . Correlation and dependance in Risk Management Properties and Pitfalls. In M.A.H. Dempster, editor, Risk Management: Value at Risk and Beyond, pages 176-223. Cambridge University Press, 2002.

[5] D.Duffie and J.Pan. An overview of value-at-risk. *Journal of Derivatives, 4(3):7-49,1997.*

[6] A. Genz, Fully Symmetric Interpolatory Rules for Multiple Integrals over Hyper-Spherical Surfaces, *J. Comp. Appl. Math.* **157** (2003), pp. 187–195.

[7] I.S Gradshteyn, I.M. Ryzbik Table of integrals, series, and products (2000) Editor Alan Jeffrey

[8] J.C. Hull, *Options, Future and Other Derivatives*, 4^{th} Edition, , Prentice Hall, Englewood Cliffs, NJ.

[9] Jorge Mina and Jerry Yi Xiao (2001), Return to Riskmetrics: The Evolution of the standard (www.riskmetrics.com). 2001 RiskMetrics Group Inc.

[10] J.P. Morgan/Reuters, RiskMetrics Technical Document, 4^{th} Edition, 1996, J.P. Morgan.

[11] J. Sadefo-Kamdem, Value-at-Risk and Expected Shortfall for a Linear Portfolios with Elliptically Distributed Risk Factors (2003), submitted.

[12] P.Sylvester, Symmetric quadrature formulae for simplexes, Math. Comp. 24 (1970), pp.95-100.

[13] Zangari, P. (1996). An Improved Methodology for Measuring VaR, RiskMetrics Monitor, 2nd quarter pp.7-25. http://www.riskmetrics.com/research/journals