# An endogenous growth model with concave consumption functions

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**Abstract**

In this paper, we combine the assumption that the consumption function is concave with an AK production function.

We show that the set of equilibrium steady-state growth rates is an interval. Then we note that when they exists, unegalitarian equilibria are characterized by higher rates of growth than egalitarian ones and, moreover, higher equilibrium growth rates correspond to higher levels of inequality. Also we prove that each path converges either to an egalitarian or to one of unegalitarian equilibria. To what equilibrium a path converges depends on the initial distribution of wealth.

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# 1 Introduction

Considerable recent attention has been focussed on the relationship between inequality and economic growth, in particular, on the dependence of the dynamics of growth on distribution of wealth (for a survey, see, e.g., Aghion et al.(1999)). In this paper, we study this dependence in a framework of an endogenous growth model with the concave consumption function (or, equivalently, the convex saving function).

The assumption that the consumption function is concave dates back to Keynes who wrote that "...with the growth in wealth [comes] the diminishing marginal propensity to consume..." (Keynes, 1936, p.349). Empirical evidence (see, e.g., Lusardi (1996)) shows that the marginal propensity to consume is substantially higher for consumers with low wealth or low income than for consumers with high wealth or income.

As was noticed by Stiglitz (1969), in the case of exogenous growth, if the saving function is convex, the distribution of income and wealth might tend toward a "two-class" equilibrium but his analysis was not detailed. A more detailed analysis of the convex saving function was proposed by Schlicht (1975) who showed that unegalitarian as well as egalitarian equilibria might be locally stable. In Bourguignon (1981) the welfare implications of the coexistence of egalitarian and unegalitarian stable equilibria were considered; it was proved that, when they exist, unegalitarian locally stable equilibria are Pareto superior to egalitarian ones.

In this paper, we combine the assumption that the consumption function is concave with an AK production function. Following Frankel (1962) and Romer (1986) we assume that technological knowledge grows automatically with capital.

We show that the set of equilibrium steady-state growth rates is an interval. Then we note that when they exists, unegalitarian equilibria are characterized by higher rates of growth than egalitarian ones and, moreover, higher equilibrium growth rates correspond to higher levels of inequality. Also we prove that each path converges either to an egalitarian or to one of unegalitarian equilibria. To what equilibrium a path converges depends on the initial distribution of wealth.

## 2 The Model

*Production sector*

The production side of the economy is described as follows. Two factors of production, capital $K$ and effective labour $L$, are used to produce a single good according to a neoclassical production function $F(K, L)$. Capital does not depreciate; the production function gives net output not including the nondepreciating capital. We assume Harrod-neutral technological progress: $L = A\overline{L}$, where $\overline{L}$ is the population and $A$ is the state of technology. Population is constant over time. As for the state of technology, we assume following Frankel (1962) that it is proportional to the current economy-wide average of physical capital per worker:

$$A = \frac{1}{k}\frac{K}{\overline{L}}$$

where $k > 0$ is an exogenously given parameter. It follows that $K/L$ is equal to $k$ and hence is constant over time. The interest rate is $r = F_K(K, L)$ and the wage earned by one unit of effective labour is $w = F_L(K, L)$. Clearly, $r$ and $w$ are also constant over time.

*Consumers*

There is a continuum of families in the economy. They are identical in their exogenous parameters. Suppose that at some time $t$ the amount of effective labour of a family is $l_t > 0$ and that the gross savings of this family are $Z_t > 0$. We assume that each unit of effective labour in the family earns $w$ during the period. Then the gross wealth of the family at time $t+1$ is $(1+r)Z_t + wl_t$. It is divided into consumption $C_t \geq 0$ of the family and savings $Z_{t+1} \geq 0$ of the family. By definition,

$$Z_{t+1} = (1 + r_t)Z_t + w_t l_t - C_t \qquad (1)$$

The consumption decision is made by means of a consumption function $c(\cdot)\colon R_+ \to R_+$. which establishes the relationship between the wealth of a family per unit of effective labour and consumption per unit of effective labour. By assumption it is twice continuously differentiable, strictly increasing, strictly concave and satisfies the following condition: $0 \leq c(y) \leq y$. The family's consumption $C_t$ is given by

$$C_t = l_t c(\frac{(1+r)Z_t + wl_t}{l_t})$$

It should be noted that the relationship between the wealth of a family and the consumption of this family changes over time in such a way that if the family's wealth grows with the same rate as technological knowledge, the average propensity to consume does not change. At the same time, if the family's wealth grows slowly (faster) than technological knowledge, the average propensity to consume increases (decreases). Note that we assume that consumption is a function of wealth but it is not difficult to show that the results will be the same if consumption is a function of income.

It is convenient to introduce the saving function $s(\cdot)$:

$$s(y) = y - c(y)$$

Clearly, the saving function $s\colon R_+ \to R_+$ is twice differentiable, strictly increasing, strictly convex and satisfies the following condition: $0 \leq s(y) \leq y$. Rewriting (1) we get

$$Z_{t+1} = l_t s(\frac{(1+r)Z_t + wl_t}{l_t}) \tag{2}$$

We will denote by $z_t$ the savings per unit of effective labour: $z_t = Z_t/l_t$

Let us introduce the economy's growth rate, $n_t$, by:

$$1 + n_t = \frac{K_{t+1} + F(K_{t+1}, L_{t+1})}{K_t + F(K_t, L_t)} = \frac{K_{t+1}}{K_t} = \frac{L_{t+1}}{L_t}$$

where $L_t$ is the amount of effective labour in the economy at time $t$. Rewriting (2) in intensive form, we get:

$$(1 + n_t)z_{t+1} = s((1+r)z_t + w) \tag{3}$$

Now introduce the function $\phi(\cdot)\colon R_+ \to R_+$:

$$\phi(z) = s((1+r)z + w)$$

Clearly, $\phi(\cdot)$ is differentiable, strictly increasing and strictly convex. Moreover, $\phi(0) = s(w) > 0$ and $0 < \phi(z) \leq (1+r)z + w$.

For the family under consideration we have:

$$(1 + n_t)z_{t+1} = \phi(z_t). \tag{4}$$

*Dynamics of the model*

Suppose that at time $t = 0$ the population is divided into a finite number of different groups of families in such a way that the savings per unit of effective labour of the families that belong to the same group are equal. Let there be $N$ groups and $0 < \alpha_t^j < 1$ be the fraction of group $j \in \{1, \ldots, N\}$ in the population. The amount of effective labour in every group is proportional to the number of individuals in the group, that is the ratio of amount of effective labour in one group to another remains constant over time.

Taking this into consideration, we get $L_t^j = \alpha_t^j L_t$, where $L_t^j$ is the amount of effective labour of group $j$ at time $t$. It is clear that $\sum_{j=1}^N \alpha_t^j = 1$. We assume that the fraction of each group is constant over time, that is $\alpha_t^j = \alpha_0^j$ $(t = 0, 1, 2, \ldots)$.

The stock of capital at each time $t$, $K_t$, is equal to the gross savings in the economy:

$$K_t = \sum_{j=1}^N Z_t^j.$$

Dividing both sides of this equation by $L_t$ we get

$$k = \sum_{j=1}^N \frac{Z_t^j}{L_t^j} \frac{L_t^j}{L_t} = \sum_{j=1}^N \alpha_t^j z_t^j. \tag{5}$$

At the end of the period $[t, t+1]$, the gross wealth of the economy equals:

$$K_t + F(K_t, L_t) = K_t + K_t F_K'(K_t, L_t) + L_t F_L'(K_t, L_t) =$$
$$= (1+r)K_t + wL_t = \sum_{j=1}^{N}((1+r)Z_t^j + wL_t^j) \qquad (6)$$

and the wealth of group $j$ is $(1+r)Z_t^j + wL_t^j$. Gross savings of this group at time $t+1$ are given by equation (2) which determines the stock of capital at time $t+1$.

Putting equations (5) and (4) together, we obtain the system of equations, describing the dynamics of our model. For $t = 0, 1, \ldots,$

$$\left\{ \begin{array}{rcll} \alpha_t^j & \equiv & \alpha_0^j & j = 1, \ldots, N \\ k & = & \sum_{j=1}^N \alpha_j z_t^j & \\ z_{t+1}^j & = & k \frac{\phi(z_t^j)}{\sum_{i=1}^N \alpha_i \phi(z_t^i)} & j = 1, \ldots, N \end{array} \right. \qquad (7)$$

In this system $z_0^j$ should be considered as predetermined values of endogeneous variables ($j = 1, \ldots, N$). In our opinion the numbers $\alpha_t^j$ also should not be interpreted as exogeneous parameters, it is better to consider them as predetermined values of endogeneous variables. Nevertheless, as we consider $\alpha_t^j$ as constant over time we will futher denote $\alpha_t^j$ by $\alpha_j$ for conveniency. Note that

$$1 + n_t = \frac{\sum_{i=1}^N \alpha_i \phi(z_t^i)}{k}.$$

Without loss of generality, we will futher assume that if $N > 1$, then

$$z_0^1 < z_0^2 < \ldots < z_0^N \qquad (8)$$

## 3 Steady-state equilibria

As usually, we define a steady-state equilibrium as a state where the level of savings per unit of effective labour in each group and the rate of growth do not change over time.

Prior to defining steady-state equilibria formally, several points should be clarified. Suppose that we are given an equilibrium growth rate $n^*$. Then for each family savings per unit of effective labour, $z$, in a steady-state equilibrium corresponding to this growth rate must satisfy the following equality:

$$\phi(z) = (1 + n^*)z \tag{9}$$

Since $\phi(\cdot)$ is a strictly convex function, this equation has at most two solutions. Denote the smaller solution by $z_l^*$ and the larger one by $z_h^*$. Thus each family can find itself at one of two possible steady-state positions and the population can split at most into two groups. Those whose savings per unit of effective labour are $z_l^*$ will be called *spenders* and those whose savings per unit of effective labour are $z_h^*$ — *savers*.

Let $\sigma$ equal the proportion of savers in the population and $1 - \sigma$ equal the proportion of spenders. These proportions are determined endogenously in the following definition.

**Definition 1** *An array $(n^*, z_l^*, z_h^*, \sigma^*)$, where $\sigma^* \in [0, 1]$, is called a steady-state equilibrium if:*

1. *$z_l^*$ is the smaller solution to (9)*
2. *$z_h^*$ is the larger solution to (9)*

*3.* $\sigma^* z_h^* + (1-\sigma^*) z_l^* = k.$

A steady-state equilibrium $(n^*, z_l^*, z_h^*, \sigma^*)$ will be called dividing if $z_l^* < z_h^*$ and $0 < \sigma^* < 1$; otherwise it will be called non-dividing. Note that in the non-dividing equilibrium either $z_l^* = z_h^* = k$ or $\sigma^* \in \{0, 1\}$. Without loss of generality, we suppose for a non-dividing equilibrium $(n^*, z_l^*, z_h^*, \sigma^*)$ that $z_l^* = z_h^* = k$.

Let us start our analysis with the study of the function $\phi(z)/z$. Define

$$g := \lim_{z\to\infty} \phi(z)/z$$

**Lemma 1** *The function $\phi(z)/z$ behaves in one of the following two ways:*

*either it monotonically decreases on* $[0; \infty)$

*or there exists $z_{min} \in R$ such that the function $\phi(z)/z$ reaches its minimum at $z_{min}$; $\phi(z)/z$ monotonically decreases on $[0; z_{min}]$ and it monotonically increases on $[z_{min}; \infty)$.*

**Proof.**

First note that $\lim_{z\to 0} \phi(z)/z = +\infty$ and $g < +\infty$

Now we compute the derivative of $\phi(z)/z$:

$$(\phi(z)/z)' = \frac{\phi'(z)z - \phi(z)}{z^2}$$

To determine the sign of the derivative it is sufficient to determine the sign of the numerator. Put $g(z) = \phi'(z)z - \phi(z)$. Since $\phi'(0) < +\infty$, it is clear that $g(0) = -\phi(0) < 0$. Moreover

$$g'(z) = \phi''(z)z \geq 0.$$

To conclude the proof it remains to check if there exists $z_{min}$ such that $g(z) < 0$ for $z < z_{min}$ and $g(z) > 0$ for $z > z_{min}$. If the answer is "yes" this concludes the proof. If such $z_{min}$ does not exist then, clearly, $g(z) < 0$ for all $z > 0$ and $\phi(z)/z$ is a monotonically decreasing function.

*Examples.*

1. The simplest example is represented by a linear function: $s(y) = hy, \quad 0 < h < 1$. In this case

$$\phi(z) = h(1+r)z + hw$$

It is clear that

$$\frac{\phi(z)}{z} = h(1+r) + \frac{hw}{z}$$

is a monotonically decreasing function. Therefore for a linear saving function and a function close to linear $\phi(z)/z$ monotonically decreases.

2. Let $s(y) = y - \ln(1+y)$. Then

$$\phi(z) = w + (1+r)z - \ln(1 + w + (1+r)z)$$

and

$$g(z) = z\phi'(z) - \phi(z) = -\frac{(1+w)(w+(1+r)z)}{1+w+(1+r)z} - \ln(1+w+(1+r)z)$$

Here, it is clear that $g(0) = -w + \ln(1+w) < 0$ and $g(z) \to_{z\to\infty} +\infty$. Therefore, $\phi(z)/z$ reaches its minimum at some $z_{min} < \infty$.

3. Let $s(y) = y - 1 + \exp(-y)$. Clearly, $s'(y) > 0$ and $s''(y) > 0$. We have:

$$\phi(z) = w + (1+r)z - 1 + \exp(-w - (1+r)z)$$

$$g(z) = -\exp(-w - (1+r)z)(1 + (1+r)z) + 1 - w$$

$$g(0) = -\exp(-w) + 1 - w < 0.$$

Thus $\lim_{z\to\infty} g(z) = 1 - w$, that is the sign of $\lim_{z\to\infty} g(z)$ depends on $w$. If $w < 1$ then $z_{min}$ exists. Otherwise, that $z_{min}$ does not exist.

**Theorem 1** *1. If $\phi(k)/k \geq g$ then there exists only a non-dividing equilibrium $(n^*, z_l^*, z_h^*, \sigma^*)$. For this equilibrium $z_l^* = z_h^* = k$ and $1 + n^* = \phi(k)/k$.*

*2. If $\phi(k)/k < g$ then for each $n^*$ satisfying $g > 1 + n^* \geq \phi(k)/k$, there exist $z_l^*$, $z_h^*$, $\sigma^*$ such that $(n^*, z_l^*, z_h^*, \sigma^*)$ is an equilibrium. If $1 + n^* = \phi(k)/k$ then this equilibrium is non-dividing; if $1 + n^* > \phi(k)/k$, then it is dividing.*

**Proof.** The proof follows from the previous proposition.

We illustrate the two opportunities by Fig.1 and Fig.2.

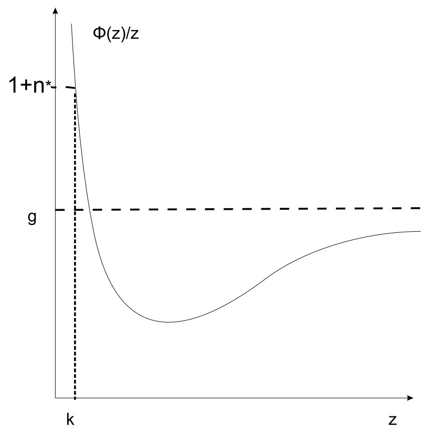


Figure 1:

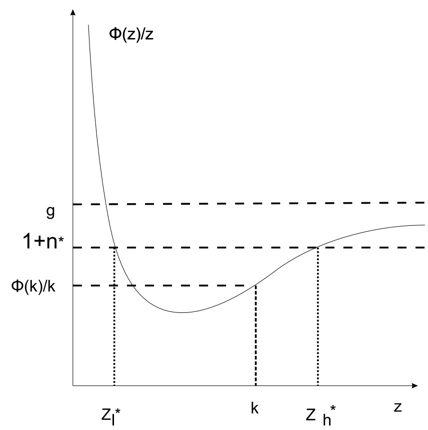


Figure 2:

For $z_l^*$ and $z_h^*$ it holds

$$\frac{\phi(z_l^*)}{z_l^*} = \frac{\phi(z_h^*)}{z_h^*} \tag{10}$$

and

$$z_l^* \leq k \leq z_h^* \tag{11}$$

Therefore statement 1 is trivial because there does not exist more than one $z$ such that equation (10 ) and equation (11) hold together. Statement 2 is trivial also because for any pair $(z_l^*, z_h^*)$ that satisfies (10) we can find $\sigma^*$ such that inequalities (11) are true.

Since it should be noted that the value of $z_h^*/z_l^*$ may be considered as an index of inequality in the economy, it is readily seen, that the growth rate of economy is related to inequality in the economy: the higher is the growth rate, the higher is inequality.

**Proposition 1** *An equilibrium rate of growth corresponding to any dividing equilibrium is higher than an equilibrium rate of growth corresponding to any non-dividing equilibrium.*

*If the economy is in a dividing equilibrium, then the higher is $n^*$, the higher is $z_h^*/z_l^*$.*

**Proof.** The proof is straightforward.

# 4 Asymptotics

Now let us procede with the dynamics of the model. Our aim now is to prove that each path converges to a dividing or non-dividing equilibrium. At first, we prove this result under the assumption that $\phi(z)/z$ monotonically decreases. In this case each path converges to a non-dividing equilibrium.

**Proposition 2** *Suppose $\phi(z)/z$ monotonically decreases for $z \geq 0$. Then for any path $\{z_t^j\}_{j=1}^{N}\,{}_{t=0}^{\infty}$:*

$$z_t^j \to_{t\to\infty} k \qquad \forall j = 1, \ldots, N$$

**Proof.** Since we assumed (8) it holds

$$\phi(z_0^1)/z_0^1 > \phi(z_0^2)/z_0^2 > \ldots > \phi(z_0^N)/z_0^N$$

Therefore for all $t$,

$$\phi(z_t^N)/z_t^N = \min_j \phi(z_t^j)/z_t^j < 1 + n_t < \max_j \phi(z_t^j)/z_t^j = \phi(z_t^1)/z_t^1$$

Using (7), we get $z_{t+1}^N < z_t^N$, $t = 0, \ldots, \infty$. Consequently $\{z_t^N\}_{t=0}^{\infty}$ is a monotonically decreasing sequence. Thus, it has a limit :

$$z_t^N \to_{t\to\infty} z^N$$

Similarly, $z_t^1$ is a monotonically increasing sequence and

$$z_t^1 \to_{t\to\infty} z^1$$

Consider any $i$ и $j$, such that for some $t$, $z_t^i < z_t^j$. The system of equations (7) gives us:

$$z_{t+1}^i = \frac{\phi(z_t^i)}{1 + n_t}$$
$$z_{t+1}^j = \frac{\phi(z_t^j)}{1 + n_t}$$

Combining this with the monotonicity of $\phi(z)/z$ we get

$$\ldots > \frac{z_{t+1}^i}{z_{t+1}^j} > \frac{z_t^i}{z_t^j}$$

Thus the sequence $\{\frac{z_t^i}{z_t^j}\}_{t=0}^{\infty}$ is monotonically increasing and is bounded by 1. Hence it converges.

Recall that we have got the convergence of $\{z_t^N\}_{t=0}^{\infty}$. Combining with the convergence of $\{z_t^i/z_t^j\}_{t=0}^{\infty}$ we obtain that $z_t^j$ converges to some $z^j$ for all $j$. Therefore, $1+n_t$ converges to some $1+n^*$. Taking into account (7), we get:

$$z^i = \frac{\phi(z^i)}{1+n^*}$$

It follows that $\frac{\phi(z^i)}{z^i}$ is uniform for all $i$. Using the monotonicity of $\phi(z)/z$, we get $z^i = k, \quad i = 1,\dots,N$ and hence $z_t^i \to_{t\to\infty} k$.

Now we shall study the case where $\phi(z)/z$ reaches its minimum at some $z_{min} < \infty$. The following proposition claims that any path converges either to a non- dividing or to a dividing equilibrium. In the latter case we state that for all $j$ the sequence $\{z_t^j\}_{t=0}^{\infty}$ converges. Moreover, for all $j$ but $N$, the sequences $\{z_t^j\}_{t=0}^{\infty}$ converge to a common limit, different from the one the sequence $\{z_t^N\}_{t=0}^{\infty}$ converges to.

**Theorem 2** *Any path* $\{z_t^j\}_{j=1}^{N}\,{}_{t=0}^{\infty}$

*either converges to a non-dividing equilibrium:* $z_t^j \to k$, $j = 1,\dots,N$,

*or converges to a dividing equilibrium* $(n^*, z_l^*, z_h^*, \sigma^*)$ *such that* $\sigma^* = \alpha_N$, *that is:*

$$z_t^N \to z_h^* \quad z_t^j \to z_l^* \quad j = 1,\dots,N-1$$

If $\phi(k)/k < g$ and $k \neq z_{min}$ then it is possible to introduce $k_-$ and $k_+$ such that $\frac{\phi(k_-)}{k_-} = \frac{\phi(k_+)}{k_+} = \frac{\phi(k)}{k}$ and either $k_-$ or $k_+$ equals to $k$ with $k_- < k_+$.

If $\phi(k)/k \geq g$ or $k = z_{min}$, put $k_- = k_+ = k$.

First let us prove the following simple lemma.

**Lemma 2** *For any path* $\{z_t^j\}_{t=0}^{\infty}\,{}_{j=1}^{N}$*:*

1. $z_t^i < k_-$ *implies* $z_{t+s}^i < k_-$ $\quad (s = 1, 2, \ldots)$

2. $z_t^i < z_{min}$ *implies* $z_{t+s}^i < z_{min}$ $\quad (s = 1, 2, \ldots)$

3. $z_t^i < k_+$ *implies* $z_{t+s}^i < k_+$ $\quad (s = 1, 2, \ldots)$

**Proof.**

1. Let us suppose that for some $t$, $z_t^i < k_-$. Clearly,

$$\sum_j \alpha_j \phi(z_t^j) > \phi(k)$$

because $\phi$ is a convex function. Thus

$$z_{t+1}^i = \frac{k\phi(z_t^i)}{\sum_j \alpha_j \phi(z_t^j)} < k\frac{\phi(z_t^i)}{\phi(k)} = k_- \frac{\phi(z_t^i)}{\phi(k_-)} < k_-$$

2. Similarly, if for some $t$, $z_t^i < z_{min}$, then

$$z_{t+1}^i = \frac{k\phi(z_t^i)}{\sum_j \alpha_j \phi(z_t^j)} < k\frac{\phi(z_t^i)}{\phi(k)} < z_{min} \frac{\phi(z_t^i)}{\phi(z_{min})} < z_{min}$$

3. By the same argument as in statement 1, if for some $t$, $z_t^i < k_+$, then

$$z_{t+1}^i = \frac{k\phi(z_t^i)}{\sum_j \alpha_j \phi(z_t^j)} < k\frac{\phi(z_t^i)}{\phi(k)} = k_+ \frac{\phi(z_t^i)}{\phi(k_+)} < k_+$$

Continuing in the same way, we complete the proof of the lemma.

**Proof of theorem 2.**

Firstly we shall concentrate our attention on elements of sequences $\{z_t^j\}_{t=0}^{\infty}$, that can be found on $[0; z_{min}]$ . Consider $i$ and $j$ such that $z_t^i < z_t^j < z_{min}$ for some $t$. Combining with lemma (2) and (7), we obtain

$$\frac{z_{t+1}^i / z_t^i}{z_{t+1}^j / z_t^j} = \frac{\phi(z_t^i)/z_t^i}{\phi(z_t^j)/z_t^j} > 1$$

Consequently,

$$\frac{z^i_{t+1}}{z^j_{t+1}} > \frac{z^i_t}{z^j_t}$$

and $z^i_s/z^j_s \to_{s\to\infty} q \leq 1$. Let us show that $q = 1$. First let us introduce some notation:

$$\{a_s\} \approx \{b_s\} \Leftrightarrow |a_s - b_s| \to_{s\to\infty} 0$$

In this notation $\{z^i_s\} \approx \{qz^j_s\}$. Since $z^i_{t+1} = \phi(z^i_t)/1 + n_t$, we get

$$\{qz^j_{s+1}\} \approx \{\frac{\phi(qz^j_s)}{1+n_s}\}$$

and

$$\{\phi(qz^j_s)\} \approx \{q\phi(z^j_s)\}$$

Therefore

$$\{\frac{\phi(qz^j_s)}{qz^j_s}\} \approx \{\frac{\phi(z^j_s)}{z^j_s}\}$$

and combining this with the monotonicity of $\phi(z)/z$ we obtain

$$|z^i_s - z^j_s| \to_{s\to\infty} 0$$

Let us study the behaviour of the sequence $\{z^j_t\}^{\infty}_{t=0}$ for $j$ such that $z^j_t \in [z_{min}; A]$, $t = 0, 1, \ldots, \infty$, where $A = max_j(k/\alpha_j)$. (Clearly, $z^j_t < A$ for all $j = 1, \ldots, N, \quad t = 0, 1, \ldots$) We prove that $\{z^{N-1}_t\}^{\infty}_{t=0}$ cannot be such a sequence, that is there exists $T$ such that $z^{N-1}_T < z_{min}$. Assume the converse, that is $z_{min} < z^{N-1}_t < z^N_t$ for all $t$. By the same argument as before,

$$\frac{z^{N-1}_{t+1}/z^{N-1}_t}{z^N_{t+1}/z^N_t} = \frac{\phi(z^{N-1}_t)/z^{N-1}_t}{\phi(z^N_t)/z^N_t} < 1$$

Consequently,

$$\frac{z^{N-1}_{t+1}}{z^N_{t+1}} < \frac{z^{N-1}_t}{z^N_t} < \ldots < \frac{z^{N-1}_0}{z^N_0} < 1$$

and $z_s^{N-1}/z_s^N \to_{s\to\infty} q < 1$. As above

$$\{\frac{\phi(qz_s^N)}{qz_s^N}\} \approx \{\frac{\phi(z_s^N)}{z_s^N}\}$$

Hence

$$|z_s^N - z_s^{N-1}| \to_{s\to\infty} 0$$

This contradicts to the fact that $q < 1$.

Summing up, we have proved that there exists at most one $j$ such that $z_t^j > z_{min}$ for all $t$ and if such $j$ exists then this $j$ must be equal to $N$.

Also we have proved that $|z_t^i - z_t^j| \to 0$ for $j = 1, \ldots, N-1$. It remains to check that there exists $j$ such that the sequence $\{z_t^j\}_{t+0}^{\infty}$ has a limit.

Using the previous statement, $z_t^j < z_{min}$ for all $j = 1, \ldots N-1$ and some $t$. Because of monotonicity of $\phi(z)/z$ on $[0; z_{min}]$ we have

$$\frac{\phi(z_t^{N-1})}{z_t^{N-1}} < \ldots < \frac{\phi(z_t^1)}{z_t^1}$$

Concerning $\phi(z_t^N)/z_t^N$ three opportunities are possible:

i1) $\phi(z_t^N)/z_t^N < \phi(z_t^{N-1})/z_t^{N-1}$

i2) $\phi(z_t^N)/z_t^N > \phi(z_t^1)/z_t^1$

i3) $\phi(z_t^{N-1})/z_t^{N-1} \leq \phi(z_t^N)/z_t^N \leq \phi(z_t^1)/z_t^1$

Consider them all.

Let $\phi(z_t^N)/z_t^N < \phi(z_t^{N-1})/z_t^{N-1}$. Therefore

$$\phi(z_t^N)/z_t^N < 1 + n_t < \phi(z_t^1)/z_t^1$$

Hence $z^1_{t+1} > z^1_t$ and $z^N_{t+1} < z^N_t$. These inequalities imply

$$\frac{\phi(z^1_{t+1})}{z^1_{t+1}} > 1 + n_t > \frac{\phi(z^N_{t+1})}{z^N_{t+1}}$$

Suppose that $z^{N-1}_{t+1} > z^{N-1}_t$. Then $\phi(z^{N-1}_t)/z^{N-1}_t > 1 + n_t$ and therefore $\phi(z^{N-1}_{t+1})/z^{N-1}_{t+1} > 1 + n_t$. Thus,

$$\phi(z^{N-1}_{t+1})/z^{N-1}_{t+1} > \phi(z^N_{t+1})/z^N_{t+1}$$

Or else suppose that $z^{N-1}_{t+1} < z^{N-1}_t$. Then

$$\phi(z^{N-1}_{t+1})/z^{N-1}_{t+1} > \phi(z^{N-1}_{t+1})/z^{N-1}_{t+1} > \phi(z^N_{t+1})/z^N_{t+1}$$

Thus, in both cases,

$$\phi(z^N_{t+1})/z^N_{t+1} < \phi(z^{N-1}_{t+1})/z^{N-1}_{t+1}$$

Therefore

$$z^N_t > z^N_{t+1} > z^N_{t+2} > \dots$$

This yields that the sequence $\{z^N_s\}$ converges as $s$ tends to infinity because it is monotonically decreasing.

Let $\phi(z^N_t)/z^N_t > \phi(z^1_t)/z^1_t$. By the same argument the sequence $\{z^N_s\}^{\infty}_{s=t}$ converges.

Let

$$\phi(z^{N-1}_t)/z^{N-1}_t \leq \phi(z^N_t)/z^N_t \leq \phi(z^1_t)/z^1_t$$

If

$$\phi(z^{N-1}_{t'})/z^{N-1}_{t'} \leq \phi(z^N_{t'})/z^N_{t'} \leq \phi(z^1_{t'})/z^1_{t'}$$

holds for all $t' > t$ then

$$\ldots < z_{t+1}^{N-1} < z_t^{N-1}$$

and

$$\ldots > z_{t+1}^1 > z_t^1$$

Or else there exist $t' > t$, such that one of two inequalities does not hold. In this case we can use the results concerning previous cases. This concludes the proof.

Now we have proved that each path, satisying (7) converges to a steady-state equilibrium. In the following proposition we give some sufficient conditions for the stability of some dividing and some non-dividing equilibria.

**Proposition 3** *Suppose that $N > 1$ and $z_{min}$ exists.*

*In this case*

*1. if $k > z_{min}$ then any path $\{z_t^j\}_{j=1}^N{}_{t=0}^{\infty}$ converges to a dividing equilibrium $(n^*, z_l^*, z_h^*, \sigma^*)$ such that $\sigma^* = \alpha_N$:*

$$z_t^j \to z_l^*, \qquad j = 1, \ldots N-1$$

$$z_t^N \to z_h^*$$

*2. if $\phi(k)/k \geq g$, then any path $\{z_t^j\}_{j=1}^N{}_{t=0}^{\infty}$ converges to a non-dividing equilibrium $(n^*, z_l^*, z_h^*, \sigma^*)$ such that $1 + n^* = \phi(k)/k$:*

$$z_t^j \to k, \qquad j = 1, \ldots N$$

**Proof.** The proof is straightforward.

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