On the Performance of Efficient Portfolios

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Abstract

This paper investigates the performance of efficient portfolios in a financial market with heterogeneous investors including rational traders, noise traders, and chartists. A generalization of the security market line result states that, regardless of the diversity of beliefs, the portfolios of rational investors with mean-variance preferences are mean-variance efficient in the sense of classical CAPM. We show that, depending on the noise traders' behavior, the performance of efficient portfolios when measured by empirical Sharpe ratios can be dominated. Empirical Sharpe ratios may thus be inappropriate indicators for efficient portfolios.

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1 Introduction

The concept of an efficient portfolio and its empirical performance constitute one of the central issues in financial market theory. Starting with the work of Markowitz (1952) and Tobin (1958), economists have investigated portfolios which, given a certain expected return, minimize the risk of future wealth fluctuations. Based on this portfolio theory, Sharpe (1964), Lintner (1965) and Mossin (1966) developed the famous *Capital Asset Pricing Model (CAPM).* One of the fundamental conclusions from the model is that the structure of a *mean-variance efficient* portfolio is exactly that of the so-called market portfolio and that the relation between the expected returns of an asset and the return of this market portfolio is linear.

The notion of efficiency used in the CAPM is an *ex-ante* concept of an optimal trade-off between *expected return* and *expected variance* conditional on available information. It is assumed that the perceived probability distribution of agents is correct and stationary implying that repeated investment decisions that are based on the same beliefs will follow the same principles. Transposed to a dynamic context, this concept of efficiency suggests that any sequence of *actual returns* associated to efficient portfolios will statistically prove to be superior to any other non-efficient portfolio.

A major caveat of the CAPM, however, is its central assumption that all investors have homogeneous self-fulfilling expectations on future returns. This rules out heterogeneous expectations and the possibility of false beliefs including their effects on portfolio decisions. In fact, its static nature leaves unexplained why different agents may hold portfolios that differ considerably from the proposed market portfolio. For a meaningful investigation of efficiency, the static CAPM has therefore to be extended to a fully dynamic setting.

Based on Böhm, Deutscher & Wenzelburger (2000), Böhm & Chiarella (2000) and Brock & Hommes (1998), Wenzelburger (2001) introduced the concept of a *reference portfolio* in a dynamic version of the CAPM with heterogeneous beliefs. The reference portfolio may be seen as a *'modified'* market portfolio that accounts for discrepancies due to incorrect beliefs. A generalization of the famous security market line result is then established stating that regardless of the diversity of beliefs, the portfolios of investors that are collinear to the reference portfolio are mean-variance efficient in the classical sense. It is shown that portfolios of investors with linear mean-variance preferences *and* rational expectations are mean-variance efficient. The prerequisite for this result is the notion of a perfect forecasting rule for first and second moments in the presence of non-rational beliefs of other market participants. These forecasting rules provide correct first and second moments of the price process conditional on the available information, and in this sense they generate orbits with rational expectations.

Contrary to these findings, in many models like the Santa Fe artifical stock market (LeBaron, Arthur & Palmer 1999) or in those of Brock & Hommes (1997a, 1998) and

Chiarella & He (2002) various forms of chartism are applied according to their profitability. It is argued that professional traders use technical trading rules (Taylor & Allen 1992), because they are profitable, e.g., see Brock, Lakonishok & LeBaron (1992) or Lo, Mamaysky & Wang (2000). However, in the context of a dynamic CAPM with rational investors and heterogeneous beliefs matters may change. Economic folklore suggests that only mean-variance efficient portfolios of rational investors will survive in the long run.

Since in a dynamic context trading of assets takes place *before* observing actual return, it is clear that the the *empirical* performance of a portfolio over time has to rely on estimators for the statistical features of the returns associated to that portfolio. In particular, the superiority of efficient portfolios in the sense of CAPM will only show, if the involved estimators for the (conditional) moments are consistent. For inconsistent estimators, portfolios other than the efficient portfolio may appear to perform better. On the other hand, taking into account how subjective expectations determine market clearing asset prices in a sequential asset market (e.g., see Böhm, Deutscher & Wenzelburger 2000), a sequence of returns induced by beliefs has a priori no direct link to the perceived subjective efficiency of a portfolio.

This paper now investigates properties of efficient portfolios under heterogeneous beliefs. We analyze the question to what extent boundedly rational consumers are able to identify professional brokers (called mediators) who hold mean-variance efficient portfolios. It is shown that there is no direct relationship between the ex-ante efficiency and the empirical performance of such portfolios. Taking empirical Sharpe ratios as a performance measure, noise traders or chartists may outperform investors that hold mean-variance efficient portfolios. These findings reveal that based on empirical observations, mediators with mean-variance efficient portfolios may not be identified. In particular, empirical Sharpe ratios might be inaccurate estimators for the slope of the efficiency frontier.

2 The model

Following Böhm & Chiarella (2000) and Wenzelburger (2001), consider an overlapping generations model with a finite number of types $h = 1, \dots, H$ of young households. Each young household of type h lives for two periods, receives an initial endowment $e^{(h)} > 0$ of a non-storable commodity only when young. In order to transfer wealth to the second period of his life, such a consumer will choose a portfolio of $K + 1 \in \mathbb{N}$ retradeable assets whose proceeds he will consume. There is one risk-free real asset which has an exogenously given constant real rate of return $R = 1 + r > 0$. The other K assets correspond to shares of firms whose production activities induce a stochastic process of dividends which are distributed to the shareholders.

Young consumers (households) are characterized by linear mean-variance preferences as in Markowitz (1952) and Tobin (1958). They have no direct access to a stock market and instead select a financial mediator who manages their portfolios. There is a finite number $i = 0, \dots, I$ of fund managers/financial mediators characterized by subjective probability distributions regarding the future cum-dividend price of the assets. We assume that each mediator $i = 0, \dots, I$ selects a probability distribution from a fixed family of subjective distributions parameterized by the first two moments conditional on the available information, i.e., the subjective conditional mean values and subjective conditional (variance-) covariance matrices for future cum-dividend prices.

Let t be an arbitrary trading period and $q_t^{(i)}$ and $V_t^{(i)}$ denote mediator i's subjective mean value and subjective covariance matrix of the future *cum-dividend* price vector $q_{t+1} = p_{t+1} + d_{t+1} \in \mathbb{R}_+^K$. Assume that the dividend payments $d_t \in \mathcal{D}$ are randomly drawn from some subset $\mathcal{D} \subset \mathbb{R}_+^K$ and let p_t denote the corresponding *ex-dividend* price vector. Let $\eta_t^{(hi)} \in [0, 1]$ denote the fraction of households of type h employing mediator i in period t, where $\sum_{i=0}^I \eta_t^{(hi)} = 1$. Then $W_t^{(i)} = \sum_{h=1}^H \eta_t^{(hi)} e^{(h)}$ is the amount of resources of mediator i received from young households in that period. Her earnings from dividend and interest payments from the portfolio of risky assets $x_{t-1}^{(i)} \in \mathbb{R}^K$ and risk-free assets $y_{t-1}^{(i)} \in \mathbb{R}$ obtained after trading in period $t-1$ are $d_t^\top x_{t-1}^{(i)} + r y_{t-1}^{(i)}$. Since aggregate repayment obligations to old households are $(p + d_t)^\top x_{t-1}^{(i)} + R y_{t-1}^{(i)}$, her budget constraint in period t reads $W_t^{(i)} = p^\top x^{(i)} + y^{(i)}$. Then based on the belief $(q_t^{(i)}, V_t^{(i)})$, the aggregate demand function for risky assets of all households which employ i is

$$x^{(i)} = a_t^{(i)} V_t^{(i)-1} [q_t^{(i)} - Rp], \quad p \in \mathbb{R}_+^K, \tag{1}$$

where $a_t^{(i)} := \sum_{h=1}^H \frac{\eta_t^{(hi)}}{\alpha^{(h)}}$ denotes the risk-adjusted fractions of households employing mediator i and $\alpha^{(h)}$ measures risk aversion of household h.

Let $\bar{x} \in \mathbb{R}_+^K$ denote the total amount of retradeable risky assets in the economy which must be equal to the sum of previous positions $\sum_{i=0}^I x_{t-1}^{(i)}$. Given a list of arbitrary beliefs

$(q_t^{(i)}, V_t^{(i)})_{i=0}^I$ one immediately obtains an explicit functional form of the ex-dividend price p_t from the market-clearing condition, given by

$$(2) \qquad p_t = S\big((a_t^{(i)}, q_t^{(i)}, V_t^{(i)})_{i=0}^I\big) := \frac{1}{R}\left(\sum_{i=0}^I A_t^{(i)}\, q_t^{(i)} \; - A_t\bar{x}\right)$$

with

$$(3) \qquad A_t := \left(\sum_{i=0}^I a_t^{(i)} V_t^{(i)-1}\right)^{-1} \quad \text{and} \quad A_t^{(i)} := a_t^{(i)} A_t V^{(i)-1}, \quad i = 0, \ldots, I.$$

Assuming that all covariance matrices are positive definite, A_t is well defined, symmetric, and itself positive definite. The map (2) is an economic law with an expectations feedback in the sense of Böhm & Wenzelburger (1999, 2002, 2003).

The decision of a household is based on the performance of a mediator. Having invested the amount $W_t^{(i)} = \sum_{h=1}^H \eta_t^{(hi)} e^{(h)}$, mediator i's return from selling the portfolio $x_t^{(i)} := a_t^{(i)} V_t^{(i)-1}[q_t^{(i)} - Rp_t]$ in period $t+1$ is

$$(4) \qquad R_{t+1}^{(i)} \;=\; r + \tfrac{a_t^{(i)}}{W_t^{(i)}}\,[q_{t+1} - Rp_t]^\top V_t^{(i)-1}[q_t^{(i)} - Rp_t].$$

The behavior of households is modeled using a LOGIT model, e.g., see Anderson, de Palma & Thisse (1992). For each i, the sample means $\hat{\mu}_t^{(i)}$ and the sample standard deviations $\hat{\sigma}_t^{(i)}$ of the time series $\{R_s^{(i)}\}_{s=0}^t$ are recursively given by

$$(5) \qquad \begin{aligned} \hat{\mu}_t^{(i)} &:= \tfrac{1}{t+1}\textstyle\sum_{s=0}^t R_s^{(i)} \;=\; \tfrac{1}{t+1}[R_t^{(i)} + t\hat{\mu}_{t-1}^{(i)}], \\ \hat{\sigma}_t^{(i)} &:= \Big[\tfrac{1}{t+1}\textstyle\sum_{s=0}^t (R_s^{(i)} - \hat{\mu}_s^{(i)})^2\Big]^{\frac{1}{2}} \;=\; \Big[\tfrac{(R_t^{(i)} - \hat{\mu}_t^{(i)})^2}{t+1} + \tfrac{t}{t+1}\hat{\sigma}_{t-1}^{(i)\,2}\Big]^{\frac{1}{2}}, \end{aligned}$$

where $\hat{\mu}_{-1}^{(i)} \geq 0$ and $\hat{\sigma}_{-1}^{(i)} \geq 0$. Using (5), an estimator for the *Sharpe ratio* associated with the realized returns (4) of mediator i is given by $(\hat{\mu}_t^{(i)} - r)/\hat{\sigma}_t^{(i)}$. The fraction $\eta_t^{(hj)}$ of households of type h which employs a particular mediator j in period t is now assumed to be determined by the *discrete-choice probability*

$$(6) \qquad \eta_t^{(hj)} := \frac{\exp\Big(\beta^{(h)}\,(\hat{\mu}_{t-1}^{(j)} - r)/\hat{\sigma}_{t-1}^{(j)}\Big)}{\sum_{i=0}^I \exp\Big(\beta^{(h)}\,(\hat{\mu}_{t-1}^{(i)} - r)/\hat{\sigma}_{t-1}^{(i)}\Big)}, \quad t > 0,$$

with arbitrary $\eta_0^{(hj)} \geq 0$, $\sum_{i=0}^I \eta_0^{(hi)} = 1$. The parameter $\beta^{(h)}$ appearing in the *discrete choice model* (6) describes the *intensity of choice* of a household of type h, that is, how fast a typical consumer of type h will switch to a different mediator.

To obtain the evolution of the asset prices, we need to specify the probabilistic assumptions on the the exogenous dividend process.

Assumption 2.1 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ *an increasing family of sub-*σ*-algebras of* $\mathcal{F}$*. The dividend payments are described by a* $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$*-adapted stochastic process* $\{d_t\}_{t\in\mathbb{N}}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^K$*. Moreover, each covariance-variance matrix* $\mathbb{V}_t[d_{t+1}] \equiv \mathbb{V}_d$*,* $t \in \mathbb{N}$ *is constant over time, with* $\mathbb{V}_d$ *positive definite.*

Given the price law (2) for ex-dividend prices and Assumption 2.1 for the dividend process, the cum-dividend price in period t is

$$q_t = S\big((a_t^{(i)}, q_t^{(i)}, V_t^{(i)})_{i=0}^I\big) + d_t, \quad t \in \mathbb{N}. \tag{7}$$

Since p_t defined by (2) is $\mathcal{F}_{t-1}$ measurable, the conditional mean values and the conditional covariance matrices of the cum-dividend price, i.e., its first two conditional moments, are

$$\begin{aligned} \mathbb{E}_{t-1}[q_t] &= \mathbb{E}_{t-1}\Big[S\big(a_t, q_t^{(i)}, V_t^{(i)})_{i=0}^I\big)\Big] + \mathbb{E}_{t-1}[d_t] \\ &= \tfrac{1}{R}\Big[\textstyle\sum_{i=1}^I A_t^{(i)} q_t^{(i)} + A_t^{(0)} q_t^{(0)} - A_t \bar{x}\Big] + \mathbb{E}_{t-1}[d_t], \end{aligned} \tag{8}$$

and

$$\mathbb{V}_{t-1}[q_t] = \mathbb{V}_{t-1}[d_t] = \mathbb{V}_d, \tag{9}$$

respectively. This reveals that the uncertainty in cum-dividend prices and in the traded quantities of assets rests solely on the dividend process.[1]

3 Perfect forecasting rules for moments

Suppose that a boundedly rational mediator 0 (also referred to as fundamentalist) seeks efficient reference portfolios in the sense of Sec. 4. Given the context of mean-variance preferences, it then suffices to analyze the case in which mediator 0 is able to correctly predict the first two moments of the price process conditional on all available information, whereas other market participants may have non-rational beliefs. As a short hand, we will use the term *rational expectations* to describe the situation in which the first two moments of mediator 0's subjective distributions, i.e., the conditional mean values and the conditional covariance matrices, coincide with the respective moments of the true distributions.

To investigate the existence of forecasting rules that correctly predict the first moments of cum-dividend prices, consider the case with two mediators, say mediator 0 and mediator 1, and assume without loss of generality that $a_t^{(0)} > 0$ for all times $t \in \mathbb{N}$ throughout

[1] The underlying OLG structure is not essential for this paper. Hillebrand (2003) shows that the price process (7) is structurally the same when consumers have a multi-period planning horizon. This holds true also for consumers with infinite lives who maximize wealth myopically as in Brock & Hommes (1997a, 1998), Kirman (1998) and Chiarella & He (2002).

this section.[2] Suppose that mediator 1 is a *noise trader* in the sense of De Long, Shleifer, Summers & Waldmann (1990, p. 709) whose cum dividend forecast $q_t^{(1)}$ deviates from the cum-dividend forecast $q_t^{(0)}$ by some randomly drawn perturbation ϵ_t, such that

$$q_t^{(1)} = q_t^{(0)} + \epsilon_t.$$

Using (8), the condition that the conditional forecast errors of the cum-dividend prices for mediator 0 vanishes is

$$\mathbb{E}_{t-1}[q_t] - q_{t-1}^{(0)} = \mathbb{E}_{t-1}\left[S\big((a_t^{(i)}, q_t^{(i)}, V_t^{(i)})_{i=0}^I\big)\right] + \mathbb{E}_{t-1}[d_t] - q_{t-1}^{(0)} = 0 \tag{10}$$

for all times t. Inserting the second expression in (8) into (10) and solving for $q_t^{(0)}$, yields an explicit expression for the new forecast $q_t^{(0)}$, given by

$$\begin{aligned} q_t^{(0)} &= \psi^{(0)}\big(\epsilon_t, \mathbb{E}_{t-1}[d_t], (a_t^{(i)}, q_t^{(i)}, V_t^{(i)})_{i=0}^I, q_{t-1}^{(0)}\big) \\ &:= (A_t^{(0)} + A_t^{(1)})^{-1}\left[R\big(q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t]\big) - A_t^{(1)}\epsilon_t - \textstyle\sum_{i=2}^I A_t^{(i)} q_t^{(i)} + A_t\bar{x}\right]. \end{aligned} \tag{11}$$

Notice that the inverse $(A_t^{(0)} + A_t^{(1)})^{-1}$ is well defined, because $A_t^{(0)}$ is positive definite and $A_t^{(1)}$ is positive semi-definite. The function (11) is therefore well-defined as well and may now be interpreted as a *forecasting rule* for cum-dividend prices of mediator 0, because it defines a functional relationship between the actual forecast $q_t^{(0)}$ and historical data. The forecasting rule (11) provides unbiased forecasts for cum-dividend prices for mediator 0 in the sense that (10) is satisfied and will henceforth be referred to as *unbiased forecasting rule* or, simultaneously, as *perfect forecasting rule for first moments*.

By (9), the second moments of the price process are independent of any expectations and equal to the covariance matrix of the dividend process. Thus, *perfect forecasting rules for second moments* which correctly predict second moments of the prices process always exist. For mediator 0, such a forecasting rule takes the form

$$V_t^{(0)} = \mathbb{V}_d \quad \text{for all} \quad t \in \mathbb{N}. \tag{12}$$

It is shown in Wenzelburger (2001) that mediator 0 has to know the excess demand function of all market participants and thus their investment behavior in order to apply the unbiased forecasting rule (11). The main informational constraint for applying the forecasting rule (11) is the fact that neither the fraction of households joining a particular mediator nor the beliefs of the mediators are observable quantities. Wenzelburger (2001) shows how this missing information can successfully be retrieved from a suitable estimation of the excess demand function.

[2] This can always be guaranteed by setting $\eta_0^{(h'0)} > 0$ and $\beta^{h'} = 0$ for some household h' in the discrete choice model (6).

4 Risk premia

Recall that the households' investment decisions are based on the performance of mediators. Following Wenzelburger (2001), we introduce the *reference portfolio* of period $t \in \mathbb{N}$ by

$$x_t^{ref} = \mathbb{V}_t[q_{t+1}]^{-1}[\mathbb{E}_t[q_{t+1}] - Rp_t], \tag{13}$$

noting that by Assumption 2.1 (i) and (9), the conditional covariance matrix $\mathbb{V}_t[q_{t+1}]$ is positive definite and hence invertible. The reference portfolio x_t^{ref} is fictitious in the sense that it is not necessarily traded in the market. If $r = R - 1$ is the risk-free interest rate, then the return of the portfolio (13) in period $t+1$ after investing one unit of the consumption good in period t is

$$R_{t+1}^{ref} = r + [q_{t+1} - Rp_t]^\top x_t^{ref}. \tag{14}$$

The conditional variance of R_{t+1}^{ref} is $\mathbb{V}_t[R_{t+1}^{ref}] = x_t^{ref\top}\mathbb{V}_t[q_{t+1}]x_t^{ref}$. Combined with (13), this shows that the *conditional risk premium* of the reference portfolio satisfies

$$\mathbb{E}_t[R_{t+1}^{ref}] - r = [\mathbb{E}_t[q_{t+1}] - Rp_t]^\top \mathbb{V}_t[q_{t+1}]^{-1}[\mathbb{E}_t[q_{t+1}] - Rp_t] = \mathbb{V}_t[R_{t+1}^{ref}] \tag{15}$$

and for this reason is always non-negative. On the contrary, the conditional risk premium of a mediator may, in general, well be negative.

The security market line result Theorem 3.1 in Wenzelburger (2001) states that the risk premium $\mathbb{E}_t[R_{t+1}^{(i)}] - r$ of any mediator i can only be higher than the risk premium of the reference portfolio (13) at the expense of higher risk, i.e. $\mathbb{V}_t[R_{t+1}^{(i)}] \geq \mathbb{V}_t[R_{t+1}^{ref}]$. Moreover, taking *Sharpe ratios* conditional on information at date t, it is shown that

$$\frac{\mathbb{E}_t[R_{t+1}^{(i)}] - r}{\sqrt{\mathbb{V}_t[R_{t+1}^{(i)}]}} \leq \frac{\mathbb{E}_t[R_{t+1}^{ref}] - r}{\sqrt{\mathbb{V}_t[R_{t+1}^{ref}]}} = \sqrt{\mathbb{V}_t[R_{t+1}^{ref}]} \quad \text{for all times } t. \tag{16}$$

Using (4) and (14), the Sharpe ratios (16) of mediator i's returns take the form

$$\frac{\mathbb{E}_t[R_{t+1}^{(i)}] - r}{\sqrt{\mathbb{V}_t[R_{t+1}^{(i)}]}} = \frac{[\mathbb{E}_t[q_{t+1}] - Rp_t]^\top x_t^{(i)}}{\sqrt{x_t^{(i)\top}\mathbb{V}_t[q_{t+1}]x_t^{(i)}}}. \tag{17}$$

In view of (1), any mediator i who is able to correctly predict the first two conditional moments of the price process at date t will hold the scalar fraction $a_t^{(i)}$ of the reference portfolio, i.e., $x_t^{(i)} = a_t^{(i)} x_t^{ref}$. As a consequence, (16) implies that this mediator will always obtain the highest Sharpe ratio and hence hold mean-variance efficient portfolios in the sense of classical CAPM. Portfolios of non-rational investors will always be inefficient.

To compare the performance of a portfolio of fundamentalist, who is able to correctly predict the first two moments of the price process, with the performance of that of a noise trader, notice first that their realized returns (4) depend crucially on differences in beliefs. This is seen as follows. Let I_K denote the $K \times K$ identity matrix. Using $I_K = \sum_{i=0}^{I} A_t^{(i)}$ for all times t and (1) , the portfolio $x_t^{(j)}$ of risky assets of mediator j after trading in period t takes the form

$$x_t^{(j)} = a_t^{(j)} V_t^{(j)-1} A_t \bar{x} + \sum_{i=0}^{I} a_t^{(j)} V_t^{(j)-1} A_t^{(i)} [q_t^{(j)} - q_t^{(i)}]. \tag{18}$$

In the special case of homogeneous beliefs on first and second moments, we have

$$x_t^{(j)} = \frac{a_t^{(j)}}{a} \bar{x} \quad \text{with} \quad a = \sum_{h=1}^{H} \tfrac{1}{\alpha^{(h)}}. \tag{19}$$

Then all mediators hold fractions of the market portfolio $\bar{x}$, where the fractions are determined by the risk-adjusted group sizes, i.e., the (risk-adjusted) market shares. In view of (4), the differences in the respective returns are then exclusively determined by the market shares.

To facilitate further investigations, assume from now on that both mediators are able to correctly predict the conditional second moments of the cum-dividend price process. This implies that the subjective covariance matrices of the fundamentalist (mediator 0) and the noise trader (mediator 1) coincide and are correct such that for each $t \in \mathbb{N}$,

$$V_t^{(0)} = V_t^{(1)} = \mathbb{V}_d. \tag{20}$$

Using (4) and the reference portfolio (13), the excess return of the fundamentalist becomes

$$R_{t+1}^{(0)} - r = \tfrac{a_t^{(0)}}{W_t^{(0)}} [q_{t+1} - R p_t]^\top x_t^{ref}. \tag{21}$$

It follows from (15) that the *risk premium* of the fundamentalist is always positive, that is, $\mathbb{E}[R_{t+1}^{(0)}] - r > 0$. Analogously, since $q_t^{(1)} = q_t^{(0)} + \epsilon_t$, the excess return of a noise trader becomes

$$R_{t+1}^{(1)} - r = \tfrac{a_t^{(1)}}{W_t^{(1)}} [q_{t+1} - R p_t]^\top [x_t^{ref} + \mathbb{V}_d^{-1} \epsilon_t]. \tag{22}$$

Taking conditional expectations, it follows from (13) and (15) that

$$\mathbb{E}_t[R_{t+1}^{(1)} - R_{t+1}^{(0)}] = \left(\tfrac{a_t^{(1)}}{W_t^{(1)}} - \tfrac{a_t^{(0)}}{W_t^{(0)}} \right) \mathbb{V}_t[R_{t+1}^{ref}] + \tfrac{a_t^{(1)}}{W_t^{(1)}} \epsilon_t^\top x_t^{ref}. \tag{23}$$

This shows that the risk-adjusted group sizes weighted by the invested amount of resources, i.e., $\frac{a_t^{(1)}}{W_t^{(1)}}$ and $\frac{a_t^{(0)}}{W_t^{(0)}}$ together with the correlation between the reference portfolio

x_t^{ref} and the error term ϵ_t have a key influence on whether or not the risk premia of a noise trader are higher than the risk premia of a fundamentalist.

In the special case of only one type of household, i.e. $H = 1$, the first effect cancels out. It is straightforward to see that the ratios $\frac{a_t^{(i)}}{W_t^{(i)}}$, $i = 0, \ldots, I$ are constant over time, such that

$$\frac{a_t^{(i)}}{W_t^{(i)}} = \frac{1}{\alpha^{(1)} e^{(1)}}, \quad i = 0, \ldots, I$$

for all times t. The following proposition follows directly from the law of iterated expectations.

Proposition 4.1 *Assume that there exists only one type of household, i.e., $H = 1$. Then*

$$\mathbb{E}[R_{t+1}^{(1)}] - r = \mathbb{E}[R_{t+1}^{(0)}] - r + \frac{1}{\alpha^{(1)} e^{(1)}} \mathbb{E}[\epsilon_t^\top x_t^{ref}].$$

Moreover,

$$\mathbb{E}[R_{t+1}^{(1)}] \geq \mathbb{E}[R_{t+1}^{(0)}] \quad \textit{if and only if} \quad \mathbb{E}[\epsilon_t^\top x_t^{ref}] \geq 0.$$

Unfortunately, an analytical comparison of the respective (unconditional) variances $\mathbb{V}[R_{t+1}^{(1)}]$ and $\mathbb{V}[R_{t+1}^{(0)}]$ does not give as much insight as for the first moments. Nevertheless we collected some results in Appendix A.1.

5 The performance of efficient portfolios

In many models like the Santa Fe artifical stock market (LeBaron, Arthur & Palmer 1999) or in those of Brock & Hommes (1997a, 1998) and Chiarella & He (2002) various forms of chartism are investigated. This is justified by the observation that professional traders use technical trading rules (Taylor & Allen 1992), because they are profitable, e.g., see Brock, Lakonishok & LeBaron (1992) or Lo, Mamaysky & Wang (2000). However, given the availability of perfect forecasting rules for first and second moments, matters change. Since with these forecasting rules efficient portfolios can actually be traded, economic folklore suggests that these portfolios will eventually attract all capital and outperform all other investment strategies. Hence, only portfolios of rational mediators will survive in the long run. In a scenario in which households are allowed to select mediators, one is tempted to hypothesize that eventually all or at least the vast majority of agents will hold efficient portfolios.

However, as the examples of the next two sections show, there exist robust situations in which non-rational mediators remain in the market while their portfolios perform better than the efficient one. Thus, it is by no means guaranteed that the principle *'choose the best performer'* achieves its intended goal. From a CAPM perspective such a finding appears to be counter intuitive and in fact is opposite to the results obtained in Böhm & Chiarella (2000) for homogeneous groups.

The present section reinforces results obtained by Tonn (2001) who showed that in the CAPM case with two fixed groups of noise traders and fundamentalists empirical Sharpe ratios of noise traders may be higher than those of fundamentalists and, in fact, that the returns of mean-variance efficient portfolios may on average be lower than those of noise traders, while at the same time, the empirical standard deviation is higher.

5.1 Fundamentalists versus noise traders

Consider the case of two mediators $i = 0, 1$ in K financial markets. Assume that mediator 0 has rational expectations on cum-dividend prices and that mediator 1 is a noise trader as discussed above (De Long, Shleifer, Summers & Waldmann 1990, p. 709). In this case $I = 1$ and $A_t^{(0)} + A_t^{(1)} = I_K$. Suppose furthermore that the subjective covariance matrices of both mediator 0 and 1 coincide and are correct, such that (20) holds for all $t \in \mathbb{N}$. Setting $a = \sum_{h=1}^{H} \frac{1}{\alpha^{(h)}}$ for the aggregate risk tolerance, this implies

$$A_t = \frac{1}{a}\mathbb{V}_d \quad \text{with } a = \sum_{i=0}^{I} a_t^{(i)} \tag{24}$$

for all times $t \in \mathbb{N}$. The resulting cum-dividend price process is then given by a set of random difference equations

$$\begin{cases} q_t &= q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t] + d_t, \\ q_t^{(1)} &= q_t^{(0)} + \epsilon_t, \\ q_t^{(0)} &= R\big(q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t]\big) - \frac{a_t^{(1)}}{a}\,\epsilon_t + \frac{1}{a}\mathbb{V}_d\,\bar{x}, \end{cases} \tag{25}$$

where the last equation is the unbiased forecasting rule corresponding to (11). The system of equations (25) is a nonlinear random difference equation in expected prices with additive noise which has nonlinearity induced by the risk-adjusted market share $a_t^{(1)}$. Since the ex-dividend price is $p_t = q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t]$, it is straightforward to verify that the reference portfolio (13) takes the form

$$x_t^{ref} = \frac{1}{a}\left[\bar{x} - a_t^{(1)}\mathbb{V}_d^{-1}\epsilon_t\right].$$

Under the hypotheses of Proposition 4.1, we obtain the following Corollary.

Corollary 5.1 *Assume that there exists only one type of household, i.e., $H = 1$. Then*

$$\mathbb{E}[R_{t+1}^{(1)}] \geq \mathbb{E}[R_{t+1}^{(0)}] \quad \textit{if and only if} \quad \mathbb{E}[\epsilon_t]^\top \bar{x} \geq \mathbb{E}\left[a_t^{(1)}\epsilon_t^\top \mathbb{V}_d^{-1}\epsilon_t\right].$$

Let us call a noise trader optimistic if $\mathbb{E}[\epsilon_t] >> 0$ and pessimistic if $-\mathbb{E}[\epsilon_t] >> 0$. Then Corollary 5.1 implies that the average return of a sufficiently optimistic noise trader is higher than the average return of a fundamentalist, whereas the average return of a pessimistic noise trader will never be higher than the average return of the fundamentalist.

The long-run behavior of the random dynamical system, i.e., equations (25) together with the discrete choice model (6) is described by random attractors which is the random analogue of an attractor of a deterministic system (see Arnold 1998, p. 483). Each orbit starting from the corresponding domain of attraction will then eventually end up on such an attractor. Typical candidates for these special orbits are generated by asymptotically stable *random fixed points*. A random fixed point may be seen as a special solution to the difference equation (25) that induces a stationary and ergodic process. Loosely speaking, asymptotic stability of a random fixed point means that for almost all perturbations $\omega \in \Omega$, all orbits starting from sufficiently close initial conditions eventually converge to orbits of the random fixed point.

The ergodicity property of an asymptotically stable random fixed point implies in particular that the sample means and the sample standard deviations given in (5) converge almost surely to the mean value and the standard deviation of the stationary distribution associated to the random fixed point. In case of such a random fixed point we therefore expect the empirical Sharpe ratios to converge almost surely to constant values. This implies that the market shares given by the group sizes (6) as well will almost surely converge to constant values. As a consequence, the long-run behavior of the system (25) is then governed by a stochastic system which is asymptotically linear.

To gain a first insight into the behavior of (25), consider the case in which the switching process has settled down to a situation with constant market shares, that is, $a_t^{(1)} \equiv a^{(1)}$ for all times t. In this case the system (25) is an affine (linear) random difference equation for which a unique random fixed point exists. In classical terminology, a random fixed point of (25) then corresponds to a *'steady state solution'* of a linear stochastic system as given in Hannan & Deistler (1988, Chap. 1). For (25) this process takes the form

$$(26) \qquad q_{\star t}^{(0)} := \sum_{s=0}^{\infty} R^s \left[\frac{1}{a} \mathbb{V}_d \, \bar{x} - \mathbb{E}_{t-1-(s+1)}[d_{t-(s+1)}]\big) - \frac{a^{(1)}}{a} \, \epsilon_{t-(s+1)} \right].$$

If $R < 1$, the random fixed point is globally asymptotically stable implying that all orbits will eventually converge to orbits of the stationary process (26), see Böhm & Chiarella (2000). If $R > 1$, then the random fixed point is unstable and the resulting process is explosive.

While in the linear case, it is analytically tractable to verify the existence of an asymptotically stable fixed point, the situation is considerably more complicated in the nonlinear case. Since the general existence results (Schmalfuß 1998, 1996) are difficult to apply, we rely on numerical experiments in order to obtain some qualitative properties.

Consider the situation with two risky assets ($K = 2$) such that the system (25) has a 2-dimensional state space. The forecast errors $\{\epsilon_t = (\epsilon_t^{(1)}, \epsilon_t^{(2)})\}_{t\in\mathbb{N}}$ are governed by iid processes with symmetric triangular density functions, parameterized by a list

$$(\epsilon_{min}^{(k)}, \epsilon_{max}^{(k)}, c^{(k)}), \quad k = 1, 2,$$

where $-\epsilon_{min}^{(k)} = \epsilon_{max}^{(k)} = 0.325$ and the $c^{(k)}$ describe the modes of the densities, respectively (see Appendix A.2).

For all numerical experiments, the safe rate of return is kept fixed and set to $R = 0.99$ while the dividend payments in both markets follow uncorrelated AR(1) processes (see Appendix A.2). In addition, the switching parameters for $H = 3$ households will be set to the same level, i.e. $\beta^{(h)} = \beta$, $h = 1, 2, 3$, and noise parameters will be set to the same level, i.e. $c^{(k)} = c$, $k = 1, 2$. This assumption facilitates the numerical investigation of the role of the noise parameter c and the switching parameter β, where by construction no household will switch between mediators for $\beta = 0$. Finally, initial market shares are equal, i.e., $\eta_t^{(0)} = \eta_t^{(1)} = 1/2$

Figure 1 about here.

Figure 1 shows for each asset a relatively fast convergence of three different cum-dividend price paths to a joint price path. Here, we have set $c = 0.1625$ and $\beta = 2.5$ for a relatively low switching intensity. All three price paths are simulated using the *same* path of realizations $\omega_1 \in \Omega$ of the exogenous noise process. Numerical experiments indicate that this convergence seems to be independent of initial forecasts (and other initial conditions such as price levels). Moreover, a convergence to a joint path seems to take place for every other randomly drawn path $\omega \in \Omega$ of the noise process. For this reason, these observations are interpreted as numerical evidence for an asymptotically stable random fixed point.

Figure 2 about here.

Figure 2 (a) exhibits a time series of asymptotically constant market shares, confirming that the system (25) is asymptotically linear, whereas Figure 2 (b) show the realized returns of the fundamentalist (mediator 0) and the noise trader (mediator 1), respectively. These findings suggest that the limiting behavior of the system (25) is governed by a stationary and ergodic process generated by a random fixed point. Moreover, notice that Figure 2 (a) displays a situation in which the noise trader outperforms the fundamentalist, because $\eta_t^{(0)} < 0.5$ for large t, even though none of the mediators will leave the market, because the market share $\eta_t^{(0)}$ stays well between 0 and 0.5. Figure 2 (b) provides evidence for the intuition that the additional random fluctuations of noise traders are absorbed in the return process of the fundamentalist.

Observe that $c \geq 0$ ($c < 0$) implies that the noise trader chooses a more optimistic (more pessimistic) price forecast than the (unbiased) one of the fundamentalist. Figure 3 provides numerical evidence that a noise trader may outperform the fundamentalist, if the noise trader is sufficiently optimistic. Leaving all other parameters at their levels fixed above, we vary $c \in [-0.325, 0.325]$ and $\beta \in [0, 10]$ as in a two-parameter bifurcation diagram. Monte Carlo simulations suggest that the system (25) becomes asymptotically linear for each parameter constellation, because the market shares settle down to constant values. We conclude from this observation that limiting behavior is stationary and ergodic.

Figure 3 about here.

Figure 3 portrays the market share of the fundamentalist $\eta_T^{(0)}$ after $T = 1000$ periods. Using two different randomly chosen noise paths ω_1 and ω_2, which are kept fixed for each panel, several features may be observed. First, the blue and purple regions in panels (a) and (b) show parameter combinations for which, depending on the noise path, the noise trader will outperform the fundamentalist. On the contrary, the green, yellow and light red regions indicate parameter values for which the fundamentalist outperforms the noise trader. In this region the two mediators have non-zero market shares and remain in the market.

Second, high switching intensities β may induce market shares $\eta_T^{(0)}$ close to zero as indicated by the deep red purple coloring in Figure 3. In these parameter regions the fundamentalist die out and only noise traders survive. On the other hand, noise traders have zero market shares in the regions of vermillion red.

Third, the sharp vertical contour breaks indicate that there is a strong sensitivity of the limiting behavior on the mode c and therefore on the noise process. The contour lines suggest the presence of coexisting attractors with associated basins of attraction for these parameter values.

These findings show that the portfolio of a noise trader may outperform those of a fundamentalist, in spite of the fact that the fundamentalist maintains the efficient portfolio. Since the market shares are a monotone transformation of the empirical Sharpe ratios, the empirical Sharpe ratios of noise traders may be higher than the empirical Sharpe ratios of fundamentalists. This demonstrates that a particular choice behavior may not be able to identify an efficient mediator.

The bifurcation diagrams have to be interpreted with care. The underlying presumption is that the long-run behavior is generated by a random fixed point such that the system is asymptotically stationary and ergodic. This presumption was verified by numerical experiments, however, no analytical result is yet available. Apart from the patience of households and the optimism of the noise traders, the long-run behavior of the economy seems to depend strongly on the chosen noise path. See our website *www.wiwi.uni-bielefeld.de/~boehm* for more simulation results.

5.2 Fundamentalists, noise traders, and chartists

In the second example, a so-called chartist is added to the other two mediators: mediator 0 with rational expectations on cum-dividend prices and mediator 1 who is a noise trader. Although we will not treat this case in the same full length of Section 5.1, similar phenomena will be observed with the advantage that there exist an asymptotically stable random fixed point for $R > 1$.

Let mediator 2 be a chartist who uses the simple *technical trading rule*

$$q_t^{(2)} = \psi^{(1)}(q_{t-1}, \dots, q_{t-L}) := \sum_{l=1}^{L} D^{(l)} q_{t-l} \tag{27}$$

as a forecasting rule, where $D^{(l)} = \mathrm{diag}(\delta_1^{(l)}, \dots, \delta_K^{(l)})$, $l = 1, \dots, L$ are diagonal matrices whose non-zero entries denote the expected weighted trends of the asset prices, respectively. Thus, mediator 2 behaves like a trend trader in $K \geq 1$ risky assets. As before, assume that all subjective covariance matrices coincide and are correct, such that for each $t \in \mathbb{N}$, $V_t^{(i)} = \mathbb{V}_d$, $i = 1, 2, 3$, and hence, as before,

$$A_t = \left(\sum_{h=1}^{H} \frac{1}{\alpha^{(h)}} \right)^{-1} \mathbb{V}_d, \quad t \in \mathbb{N}. \tag{28}$$

The resulting cum-dividend price process is given by a set of random difference equations

$$\begin{cases} q_t & = & q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t] + d_t, \\ q_t^{(2)} & = & \sum_{l=1}^{L} D^{(l)} q_{t-l}, \\ q_t^{(1)} & = & q_t^{(0)} + \epsilon_t, \\ q_t^{(0)} & = & R\left(1 + \frac{a_t^{(2)}}{a_t^{(0)} + a_t^{(1)}}\right)\left(q_{t-1}^{(0)} - \mathbb{E}_{t-1}[d_t]\right) - \frac{a_t^{(1)}}{a_t^{(0)} + a_t^{(1)}} \epsilon_t \\ & & \qquad - \frac{a_t^{(2)}}{a_t^{(0)} + a_t^{(1)}} \sum_{l=1}^{L} D^{(l)} q_{t-l} + \frac{1}{a_t^{(0)} + a_t^{(1)}} \mathbb{V}_d\, \bar{x}, \end{cases} \tag{29}$$

where the last equation corresponds to the unbiased forecasting rule (11). Together with the discrete choice model (6), these equations describe the evolution of cum-dividend prices. To gain some insight into the behavior of (29), let us first discuss the case in which the switching process has settled down to a situation with constant market shares. In this case Proposition 5.2 establishes conditions under which an asymptotically stable random fixed point obtains.

Proposition 5.2 *Assume that the following hypotheses are satisfied.*

(i) The group sizes $a_t^{(i)} \equiv a^{(i)} > 0$, $i = 0, 1, 2$ *are constant over time.*

(ii) Let $1 \leq k \leq K$ be arbitrary but fixed. Assume that all zeros of the characteristic polynomial χ_k, $k = 1, \ldots, K$ associated with (29), given by

$$\chi_k(\lambda) = \lambda^{L+1} - R\left(1 + \tfrac{a^{(2)}}{a^{(0)}+a^{(1)}}\right)\lambda^L + \tfrac{a^{(2)}}{a^{(0)}+a^{(1)}} \sum_{l=1}^{L} \delta_k^{(l)} \lambda^{L-l},$$

lie inside the unit circle.

Then the price process for the k-th asset admits a globally asymptotically stable random fixed point.

The proof of Proposition 5.2 is analogous to the proof of Proposition 5.2 in Wenzelburger (2001). Under the hypotheses of Proposition 5.2, the existence of globally asymptotically stable random fixed point implies that with probability one all solutions to the equation (29) will eventually behave like the stationary stochastic process induced by that random fixed point.

For all numerical experiments, we consider two risky assets ($K = 2$) and a risk-free rate of return set to $R = 1.01$, while as before, the dividend payments in both markets follow uncorrelated AR(1) processes. The forecast errors $\{\epsilon_t = (\epsilon_t^{(1)}, \epsilon_t^{(2)})\}_{t\in\mathbb{N}}$ are governed by iid processes with a triangular distribution with $-\epsilon_{min}^{(k)} = \epsilon_{max}^{(k)} = 0.325$ and $c^{(k)} = c$, $k = 1, 2$. To facilitate numerical investigations, the switching parameters for $H = 3$ households are again set to the same level, i.e. $\beta^{(h)} = \beta$, $h = 1, 2, 3$. Moreover the length of the chartist's memory is $L = 2$ with $\delta^{(1)} = \delta^{(2)} = 0.65$ and the initial market shares are $\eta_0^{(0)} = \eta_0^{(1)} = \eta_0^{(2)} = 1/3$.

Figure 4 displays two parameter bifurcation diagrams with $c \in [-0.325, 0.325]$ and $\beta \in [0, 2]$ corresponding to the same two randomly drawn noise paths of Section 5.1. As before, these are kept fixed for each panel, respectively. Monto Carlo simulations indicate that for each parameter combination, a random fixed point exists such that the long-run behavior is stable. The panels of each of the figures portray the market shares of fundamentalists and noise traders after $T = 1000$ periods. The panels display a rich variety of possible outcomes for which all mediators coexist in the long run while the chartist may achieve a surprisingly high market share. Moreover, the contour lines are predominantly a horizontally oriented indicating that the parameter c has less influence than the switching intensity β.

6 Conclusions

The paper provided a first simulation analysis of the empirical performance of portfolios in a competitive financial market with heterogeneous investors. For such markets, the generalization of the security market line result (Wenzelburger 2001) states that the portfolios of investors with mean-variance preferences *and* rational expectations are efficient in the ex-ante sense of classical CAPM theory. It was shown that these mean-variance efficient portfolios may fail to empirically outperform other (inefficient) portfolios. Taking empirical Sharpe ratios as a performance measure, noise traders or chartists may appear to perform better than investors holding mean-variance efficient portfolios. These findings reveal that mean-variance efficient mediators may not be identified. They provide some evidence that empirical performance measures of portfolios in an environment with heterogeneous beliefs may be misleading.

A Appendix

A.1 Comparison of variances

Let us now turn to the variances of the respective returns. Since

$$\mathbb{V}[R_{t+1}^{(1)}] = \mathbb{V}[R_{t+1}^{(0)}] + \mathbb{C}\mathrm{ov}[R_{t+1}^{(1)} + R_{t+1}^{(0)}\,, R_{t+1}^{(1)} - R_{t+1}^{(0)}],$$

we have

$$\mathbb{V}[R_{t+1}^{(1)}] \leq \mathbb{V}[R_{t+1}^{(0)}], \tag{30}$$

if and only if

$$\mathbb{C}\mathrm{ov}[R_{t+1}^{(1)} + R_{t+1}^{(0)}\,, R_{t+1}^{(1)} - R_{t+1}^{(0)}] = \mathbb{C}\mathrm{ov}[R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r\,, R_{t+1}^{(1)} - R_{t+1}^{(0)}] \leq 0.$$

Using the definition of the covariance, (30) holds, if and only if

$$\mathbb{E}\big[(R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r)(R_{t+1}^{(1)} - R_{t+1}^{(0)})\big] \leq \mathbb{E}\big[R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r\big]\,\mathbb{E}\big[R_{t+1}^{(1)} - R_{t+1}^{(0)}\big]. \tag{31}$$

To see whether or not (31) can be satisfied, notice that (21) and (22) imply

$$R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r = \Big(\tfrac{a_t^{(1)}}{W_t^{(1)}} + \tfrac{a_t^{(0)}}{W_t^{(0)}}\Big)\,[q_{t+1} - Rp_t]^\top x_t^{ref} + \tfrac{a_t^{(1)}}{W_t^{(1)}}[q_{t+1} - Rp_t]^\top \mathbb{V}_d^{-1}\epsilon_t.$$

Taking conditional expectations, it follows from (13) and (15) that

$$\mathbb{E}_t[R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r] = \Big(\tfrac{a_t^{(1)}}{W_t^{(1)}} + \tfrac{a_t^{(0)}}{W_t^{(0)}}\Big)\,\mathbb{V}_t[R_{t+1}^{ref}] + \tfrac{a_t^{(1)}}{W_t^{(1)}}\,\epsilon_t^\top x_t^{ref}. \tag{32}$$

Clearly, (32) is positive whenever (23) is positive. In this case, taking (unconditional) expectations, the r.h.s. of (31) is positive. Under the hypotheses of Proposition 4.1, this observation holds true, whenever $\mathbb{E}[\epsilon_t^\top x_t^{ref}] \geq 0$.

The positivity of the r.h.s. of (31) is an important prerequisite for (31) to be satisfied. Considering the l.h.s. of (31), the following observation can be made. First, we calculate

$$
\begin{aligned}
&(R_{t+1}^{(1)} + R_{t+1}^{(0)} - 2r)\,(R_{t+1}^{(1)} - R_{t+1}^{(0)}) \\
&\quad = \left(\left[\frac{a_t^{(1)}}{W_t^{(1)}}\right]^2 - \left[\frac{a_t^{(0)}}{W_t^{(0)}}\right]^2\right)\left([q_{t+1} - Rp_t]^\top x_t^{ref}\right)^2 \\
&\qquad +2\left[\frac{a_t^{(1)}}{W_t^{(1)}}\right]^2\left([q_{t+1} - Rp_t]^\top x_t^{ref}\right)\left([q_{t+1} - Rp_t]^\top \mathbb{V}_d^{-1}\epsilon_t\right) \\
&\qquad + \left[\frac{a_t^{(1)}}{W_t^{(1)}}\right]^2\left([q_{t+1} - Rp_t]^\top \mathbb{V}_d^{-1}\epsilon_t\right)^2 .
\end{aligned}
$$

Taking expectations, we see that the l.h.s. of (31) may indeed be small enough such that equation (31) may be satisfied.

A.2 Simulation of the exogenous noise

This appendix collects most of the information relevant for the simulation results.

Assumption A.1 *The dividend payments for the $K = 2$ assets are described by an AR(1) process $d_t = Bd_{t-1} + \xi_t$, where $B = diag(0.76, 0.76)$ and $\{\xi_t = (\xi_t^{(1)}, \xi_t^{(2)})\}_{t\in\mathbb{N}}$ is an iid process with each $\xi_t^{(k)}$, $k = 1, 2$ uniformly distributed on the compact interval $[0, 0.024]$. Each covariance-variance matrix satisfies*

$$\mathbb{V}_t[d_{t+1}] = \mathbb{V}[\xi_t] = diag(0.000048, 0.000048), \quad t \in \mathbb{N}.$$

Assumption A.2 *Forecast errors of noise traders $\{\epsilon_t = (\epsilon_t^{(1)}, \epsilon_t^{(2)})\}_{t\in\mathbb{N}}$ follow an iid process. For each asset $k = 1, 2$, the respective distribution function is a triangular distribution (Evans, Hastings & Peacock 1993, S. 149f), given by the density function*

$$
f_k(z^{(k)}) = \begin{cases} \frac{2(z^{(k)} - \epsilon_{min}^{(k)})}{(\epsilon_{max}^{(k)} - \epsilon_{min}^{(k)})(c^{(k)} - \epsilon_{min}^{(k)})} & \forall \quad \epsilon_{min}^{(k)} \leq z^{(k)} \leq c^{(k)} \\ \frac{2(\epsilon_{max}^{(k)} - z^{(k)})}{(\epsilon_{max}^{(k)} - \epsilon_{min}^{(k)})(\epsilon_{max}^{(k)} - c^{(k)})} & \forall \quad c^{(k)} \leq z^{(k)} \leq \epsilon_{max}^{(k)}, \end{cases}
$$

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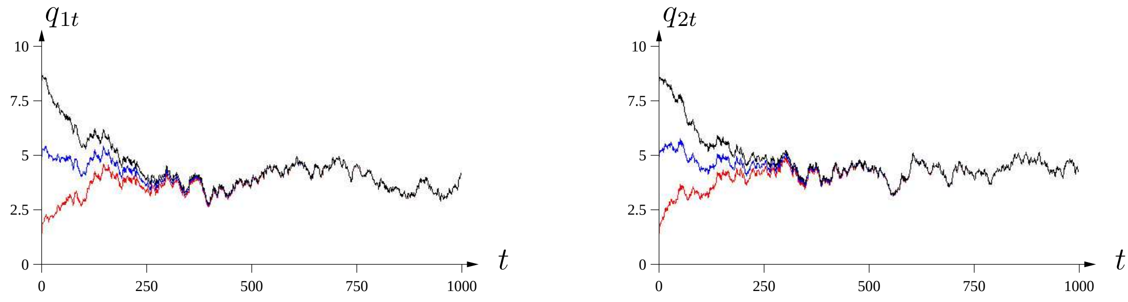
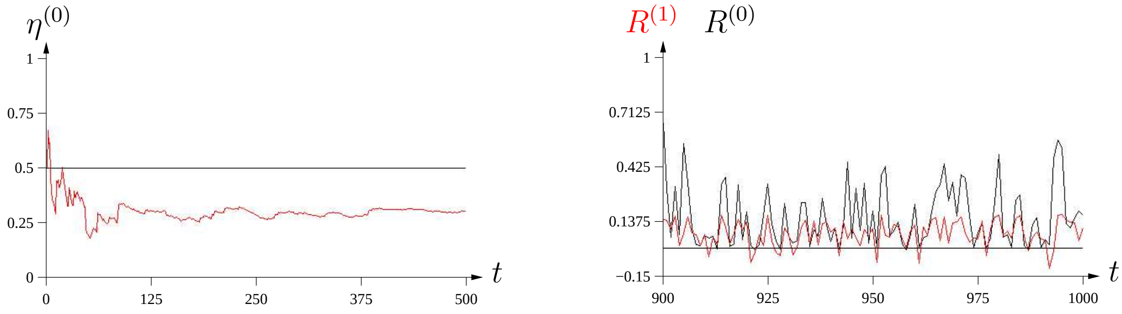


(a) Asset 1

(b) Asset 2

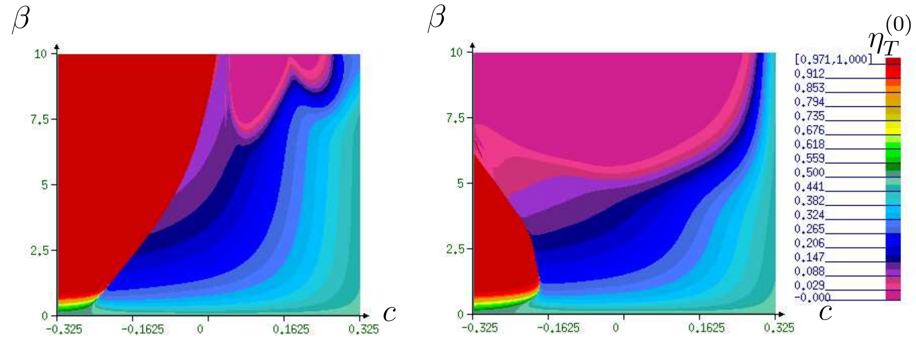
Figure 1: Random fixed point: $\beta = 2.5$, $c = 0.1625$.



(a) Fundamentalist's market share.

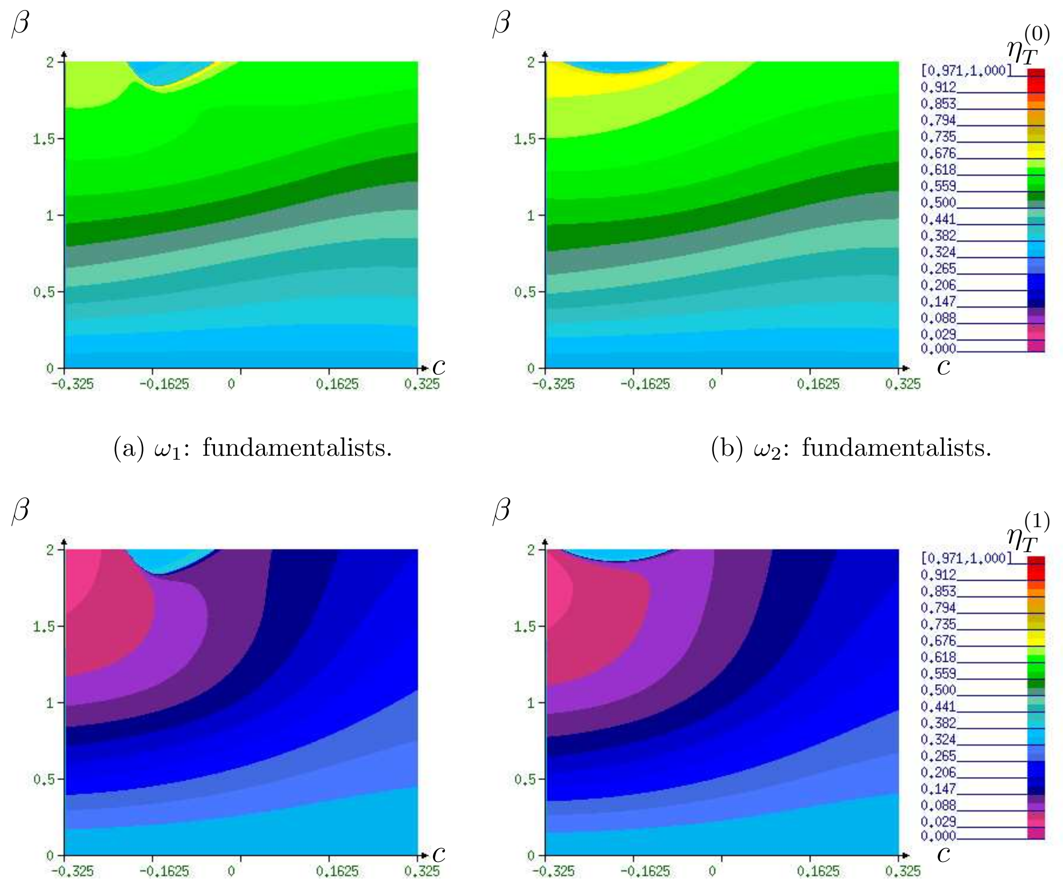
(b) Returns.

Figure 2: Market shares and returns, $c = 0.1625$ and $\beta = 2.5$



(a) ω_1 (b) ω_2

Figure 3: Market shares of fundamentalist $\eta_T^{(0)}$ after $T = 1000$ periods.



(a) ω_1: fundamentalists.

(b) ω_2: fundamentalists.

(c) ω_1: noise traders.

(d) ω_2: noise traders.

Figure 4: Market shares after $T = 1000$ periods.