A DYNAMIC ANALYSIS OF MOVING AVERAGE RULES

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ABSTRACT. The methods of various moving average rules remain popular with financial market practitioners. These rules have recently become the focus of empirical studies. However there seem to have been very few studies on the analysis of the type of financial market dynamics resulting from the fact that some agents engage in such strategies. In this paper we seek to fill this gap in the literature by proposing a market of financial market dynamics in which demand for traded assets has both a fundamentalist and a chartist component. The chartist demand is governed by the difference between a long run and a short run moving average. Both types of traders are bounded rational in the sense that, based on a certain fitness measure, traders switch from strategy with low fitness to the one with high fitness. We characterise first the stability and bifurcation properties of the underlying deterministic model via the reaction coefficient of the fundamentalists, the extrapolation rate of the chartists and the lengths used for the moving averages. By increasing the switching intensity, we then examine various rational routes to randomness for different, but fixed, long run moving average. The price dynamics of moving average is also examined and it is found that an increase of the window length of the long moving average can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour.

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1. INTRODUCTION

Technical analysts, also known as "chartists" (referring here to analysts who use the various moving average rules), attempt to forecast prices by the study of patterns of past prices and a few other related summary statistics about security trading. Basically, they believe that shifts in supply and demand can be detected in charts of market movements. In an environment of efficient markets, technical trading rules should not be useful for generating excess returns. However, despite all the evidence presented in academic journals that security prices follow random walks, and consequently that these security markets are at least weak-form efficient, as defined by Fama (1970), the use of technical trading rules still seems to be widespread amongst financial markets traders.

There have been various studies of the use and profitability of technical analysis in recent years. Taylor and Allen (1992) document the enduring popularity of the trading rules in their survey of currency traders in London. Of the respondents, 90% replied that technical trading rules are an important component of short-term investment strategies. Allen and Taylor (1990, 1992) suggest that this is an important finding given the apparent ability of exchange rates to move far from fundamentals over protracted periods of time, as documented by Frankel and Froot (1986, 1990). Earlier empirical literature on stock returns finds evidence that daily, weekly and monthly returns are predictable from past returns. Pesaran and Timmermann (1994, 1995) present further evidence on the predictability of excess returns on common stocks for the S&P 500 and Dow Jones Industrial portfolios and examine the roubstness of the evidence on the predictability of U.S. stock returns. To investigate the sources of the predictability, in Brock *et al* (1992), two of the simplest and most popular trading rules, moving average and the trading range break rules, are tested through the use of bootstrap techniques. They find that returns obtained from buy (sell) signals are not likely to be generated by four popular null models, which are the random walk, the AR(1), the GARCH-M and the EGARCH models. They document that buy signals generate higher returns than sell signals and the returns following buy signals are less volatile than returns on sell signals. This asymmetric nature of the returns and the volatility of the Dow series over the periods of buy and sell signals suggest the existence of nonlinearities in the data generation mechanism. Recent efforts, such as Lo *et al* (2000) and Goldbaum (2003), have also examined explicitly the profitability of technical trading rules and the implications for market efficiency. The profit generating potential of trading rules has also been scrutinised within the genetic programming framework by Neely at al (1997) and by the use of artificial neural networks by Gencay (1998) and Fernandez-Rodriguez et al (2000).

Most of the cited research has focused on empirical studies. Furthermore, in terms of agents' actual demand (that are based on the various signals) and tests involving real world data, the hypothesis of profitability of trading rules is highly and ideally simplified. To apply the results in practice, the question as to how to determine the amount to buy/sell and how the market prices are affected following these buy/sell actions is not clear. There seem to have been very few studies on the analysis of the type of financial market dynamics resulting from the fact that some agents engage in technical trading strategies. This paper seeks to fill this gap in the literature by proposing a market of financial market dynamics in which demand for traded assets has both a fundamentalist and a chartist component. Chartists are assumed to chase the trend generated by long moving average. Both types of traders are boundedly rational in the sense that, based on a certain fitness measure, traders switch from strategies with low fitness to the ones with high fitness.

The paper develops and analyses a model in which individual boundedly rational behaviour leads to inefficiencies in an asset market which can be exploited through use of various moving average rules. The main objectives of this paper are to analyze the stability properties of the model, particularly in relation to the moving average trading strategies, and the potential for the model to generate complex dynamics, and to examine the impact of the moving average trading rules on the market dynamics.

The plan of the paper is as follows. In the following section, we focus on one of the simplest cases when the fundamentalists' demand is determined by mean reversion to the fundamental price, while the chartists' demand is based on the sign of the difference of short and long moving averages, as in Chiarella (1992) and Brock and Hommes (1997, 1998). Based on certain fitness measures, such as observed differences in payoffs, the traders can make an endogeneous selection of which trading strategies to use, such as in Blume et al (1994), Brock and Hommes (1997), Brock and LeBaron (1996) and Brown and Jennings (1989). Consequently, an adaptive heterogeneous asset pricing model with a market maker scenario is developed. In Section 3, the existence, local stability and bifurcation of the fundamental price, in terms of the reaction coefficient of the fundamentalists, the extrapolation rate of the chartists, the lengths used for the moving averages, and switching intensity, is analyzed when the lengths of the long moving average is small. The analysis, combined with some results on general window length for some special cases, gives us some important insights into the effect of increasing the length of the long moving average. In Section 4, by increasing the switching intensity among the two strategies, we examine numerically various rational routes to randomness for different, but fixed, long-run moving averages. The price dynamics induced by the moving average rule is then examined numerically in Section 5 and it is found that an increase of the window length of the long moving average can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour. Section 6 concludes the paper.

2. An Asset Pricing Model under a Market Maker

Following the framework of Brock and Hommes (1998), this section sets up an asset pricing model with different types of heterogeneous traders who trade according to different technical trading rules, such as fundamental analysis and charting. The market clearing price is arrived via a market maker scenario in line with Chiarella and He (2003*b*) rather than the Walrasian scenario used in Brock and Hommes (1998) and

Chiarella and He (2002). Whilst the market maker and Walrasian auctioneer mechanisms are highly stylized accounts of how the market claering price is arrived at, we feel that the former is closer to what is going on in real markets. To focus on the pricing dynamics of technical trading rules, we motivate the excess demand functions of different types of traders by their trading rules directly, rather that the demand functions driven from utility maximization of their portfolio investment with both risky and risk-free assets (as for example in Brock and Hommes (1998) and Chiarella and He (2002, 2003*b*)).

Consider an asset pricing model with only one risky asset. Let P_t be the price (cum dividend) per share of the risky asset at time t. Let N be the total number of traders (assumed to be a constant) in the market, among which there are $N_{h,t}$ of type h traders at time t with $h = 1, 2, \cdots, H$ and $\sum_{h=1}^{H} N_{h,t} = N$. Then the market fractions of different types of traders at time t can be defined as

$$n_{h,t} = N_{h,t}/N, \qquad h = 1, 2, \cdots, H. \tag{1}$$

Let the excess demand for the risky asset of trader i at time t be $D_{i,t}$. Then the aggregate excess demand at time t is given by

$$D_t = \sum_{i=1}^{N} D_{i,t} = N \sum_{i=1}^{h} n_{h,t} D_{h,t}, \tag{2}$$

where $D_{h,t}$ corresponds to the representative excess demand function of type h traders. We assume that prices are set period by period via a market maker mechanism and adjusted according to the aggregate excess demand. Thus

$$P_{t+1} = P_t + \frac{\mu}{N} D_t + \tilde{\epsilon}_t = P_t + \mu \sum_{i=1}^{h} n_{h,t} D_{h,t} + \sigma_\epsilon \tilde{\epsilon}_t, \tag{3}$$

where $\tilde{\epsilon}_t$ is an *i.i.d.* normally distributed random variable that captures a random excess demand process either driven by unexpected news about fundamentals, or representing noise created by *noise traders* with $\tilde{\epsilon}_t \sim N(0, 1)$ and $\sigma_\epsilon \geq 0$. In equation (3) the parameter $\mu > 0$ measures the speed of price adjustment of the market maker to the excess demand.

For simplicity, we consider throughout this paper that there are only two types of traders: fundamentalists and chartists, who in fact are the most widespread types of traders in financial markets and whose trading strategies and excess demand functions are specified in the following discussion. We assume that there are $N_{f,t}$ fundamentalists and $N_{c,t}$ chartists at time t. Then the market fraction of fundamentalists and chartists at time t are given by, respectively

$$n_{f,t} = \frac{N_{f,t}}{N}, \qquad n_{c,t} = \frac{N_{c,t}}{N}. \tag{4}$$

The aggregate excess demand D_t at time t in (2) is then given by

$$D_t = N[n_{f,t} D_t^f + n_{c,t} D_t^c], \tag{5}$$

where D_t^f and D_t^c are the excess demands of the fundamentalist and chartist, respectively. Set

$$m_t = n_{f,t} - n_{c,t},$$

so that $n_{f,t} = (1 + m_t)/2$ and $n_{c,t} = (1 - m_t)/2$. Following from (3)-(5), the market price of the risky asset is then determined by

$$P_{t+1} = P_t + \frac{\mu}{2}[(1 + m_t)D_{f,t} + (1 - m_t)D_{c,t}] + \sigma_\epsilon \tilde{\epsilon}_t. \tag{6}$$

Fundamentalists—The fundamentalists believe that the market price should be given by the fundamental price that they have estimated based on various types of fundamental information, such as earnings, exports, general economic forecasts and so forth. They buy/sell the stock when the current price is below/above the fundamental price. For simplicity, we assume that the fundamental price is a positive constant P^* and the excess demand of the fundamentalists is given by[1]

$$D_t^f = \alpha(P^* - P_t), \tag{7}$$

where the parameter $\alpha > 0$ is a combined measure of the aggregate risk tolerance of the fundamentalists and their reaction to the *mis-pricing*.

Chartists—Unlike the fundamentalists, the chartists trade based on charting signals generated from the costless information contained in the history of the price, such as moving averages among various technical trading rules used in financial markets. The chartists' excess demand is assumed to be based on signals generated by moving averages[2]. More precisely, a moving average of length k at time t is defined as

$$ma_t^k = \frac{1}{k}\sum_{i=0}^{k-1} P_{t-i}, \quad (k \geq 1).$$

A trading signal is defined as difference between a short-run moving average ma_t^S and a long-run moving average ma_t^L, namely

$$\psi_t^{S,L} = ma_t^S - ma_t^L, \tag{8}$$

where $L \geq S$ are positive integers. For the chartists, their excess demands are assumed to be governed by

$$D_t^c = h(\psi_t^{S,L}), \tag{9}$$

where the function h has the general properties

$$h(0) = 0, \qquad h'(x) > 0, \qquad xh''(x) < 0.$$

[1] Given an annual risk free rate r, the excess demand function in (7) should be formed by $D_t^f = \alpha[P^* - (1 + r/K)P_t]$, where K corresponds to the trading frequency per year. To characterize asset price dynamics at a high-frequency (such as $K = 250$ for daily) the risk-free rate per trading period r/K is very small, so here we simply treat it as zero.

[2] As we stated in the introduction a variety of moving average trading rules are used by market traders.

This corresponds to the very popular technical trading rule based on the crossing of the long run and short run moving averages. By setting $S = 1$ we obtain the moving average rule whereby chartists wish to be long (short) when the current price is above (below) the moving average. For $S > 1$ we obtain the double moving average rule according to which the chartists go long (short) when the short run moving average moves above (below) the long run moving average. In this paper, we select

$$h(x) = \tanh(ax), \qquad a = h'(0) > 0.$$

We note that this form of chartist excess demand function allows us to capture some elements of the filtered moving average rules. This is so since, when a is small, the chartists initially react cautiously to the long/short signals, in a sense waiting to confirm the maintenance of the change in sign of the signal. In this way they minimize the costs incurred if the signal changes frequently in a short time period.

Fitness Measure and Population Evolution—In order to introduce the adaptiveness of agents, we follow the mechanism of Brock and Hommes (1998) and define the fitness functions $\pi_{f,t}, \pi_{c,t}$ as their realized capital gains[3]:

$$\pi_{f,t} = D^f_{t-1}(P_t - P_{t-1}) - C_f, \qquad \pi_{c,t} = D^c_{t-1}(P_t - P_{t-1}) - C_c, \tag{10}$$

where $C_f, C_c \geq 0$ are the costs of their strategies. Then the population fractions are assumed to be updated by the well known *discrete choice model* or 'Gibbs' probabilities (e.g. Manski and McFadden (1981))

$$n_{f,t} = \frac{e^{\beta U_{f,t}}}{e^{\beta U_{f,t}} + e^{\beta U_{c,t}}}, \qquad n_{c,t} = \frac{e^{\beta U_{c,t}}}{e^{\beta U_{f,t}} + e^{\beta U_{c,t}}}, \tag{11}$$

where

$$U_{f,t} = \pi_{f,t} + \eta U_{f,t-1}, \qquad U_{c,t} = \pi_{c,t} + \eta U_{c,t-1}, \tag{12}$$

and $\eta \in [0, 1]$ measures the memory of the cumulated fitness function and $\beta \geq 0$ measures the switching intensity among the two strategies. In particular, if $\beta = 0$, there is no switching between strategies among agents. See Brock and Hommes (1998) for a more extensive discussion of this switching mechanism.

A Complete Asset Pricing Model—It follows from (5)-(6) that the market price of the risky asset is determined according to

$$P_{t+1} = P_t + \frac{\mu}{2}\big[(1 + m_t)\alpha(P^* - P_t) + (1 - m_t)h(\psi_t^{S,L})\big] + \tilde{\epsilon}_t \tag{13}$$

and, from (10)-(11), that the difference of population fractions m_t evolves according to

$$m_t = \tanh\big[\frac{\beta}{2}(U_t - C)\big], \tag{14}$$

[3]As indicated in footnote 1, we assume the risk free rate for the trading period is zero.

where $C = C_f - C_c \geq 0$, $\mu \geq 0$ measures the speed of price adjustment of the market maker based on the excess demand, and

$$U_t = [D_{t-1}^f - D_{t-1}^c][P_t - P_{t-1}] + \eta U_{t-1}, \tag{15}$$

By setting $\sigma_\epsilon = 0$, the nonlinear stochastic dynamical system (13)-(15) becomes a nonlinear deterministic system where the price follows

$$P_{t+1} = P_t + \frac{\mu}{2}\Big[(1+m_t)\alpha(P^* - P_t) + (1-m_t)h(\psi_t^{S,L})\Big]. \tag{16}$$

In general system (14)-(16) is an $L+2$ dimensional non-linear difference system. We seek principally to understand how its dynamic behaviour is affected by the reaction coefficient of the fundamentalists, the excess demand function of the chartists, the lengths used for the moving averages, and the switching intensity.

3. Stability and Bifurcation Analysis

In this section, we consider price dynamics of the deterministic system (14)-(16). We first state the following result on the existence of the unique steady state and the corresponding characteristic equation.

Proposition 3.1. *For the deterministic system (14)-(16), assume $\eta \in [0,1)$. Then there exists a unique steady state $(P_t, m_t, U_t) = (P^*, m^*, 0)$, where P^* is the constant fundamental price and $m^* = \tanh(-\beta C/2)$. In addition, the characteristic equation of this steady state is given by*

$$\Gamma(\lambda) = \lambda(\lambda - \eta)\Gamma_{S,L}(\lambda), \tag{17}$$

where

$$\begin{aligned}\Gamma_{S,L}(\lambda) \equiv \lambda^L - (1-\bar{\alpha})\lambda^{L-1} - \bar{a}(\frac{1}{S} - \frac{1}{L})(\lambda^{L-1} + ... + \lambda^{L-S}) \\ + \frac{\bar{a}}{L}(\lambda^{L-S-1} + \cdots + \lambda + 1) = 0\end{aligned} \tag{18}$$

and

$$\bar{\alpha} = \alpha\mu(1+m^*)/2, \quad \bar{a} = a\mu(1-m^*)/2. \tag{19}$$

Proof. See Appendix A.1 □

The parameter $\bar{\alpha}$ measures the combined effect of, the speed of price adjustment of the market maker μ toward the aggregate excess demand, the speed of current price adjustment of the fundamentalists towards their expected fundamental price α, and the market equilibrium fraction m^*. The parameter $\bar{a}$ measures the combined effect of the speed of, price adjustment of the market maker μ, the extrapolation rate of the chartists to the difference of short and long run moving averages of the past prices a, and the equilibrium market fraction m^*. Obviously, $m^* = 0$ when $C = 0$.

We now analyse the local stability of the unique steady state and its bifurcation properties. Given the structure of equation (17), the local stability and bifurcations are determined by the eigenvalues of $\Gamma_{S,L}(\lambda) = 0$. For the simplicity, we take $S = 1$. In the following discussion, we concentrate on the case $S = 1$ and $L \geq 1$.

For general $L \geq 1$, we first obtain the following result.

Lemma 3.2. *Let $S = 1$ and $L > 1$.*

(i). *If $\bar{\alpha} = 1 + \bar{a}$, then the eigenvalues λ_i of $\Gamma_{1,L}$ satisfy $|\lambda_i| < 1$ if and only if $0 < \bar{a} < L$. In addition, for $\bar{a} = L$, the λ_i satisfy $\lambda_i \neq 1$ and $(1-\lambda_i^L)/(1-\lambda_i) = 0$.*

(ii) *A necessary condition for $|\lambda_i| < 1$ for all i is*

$$0 < \bar{a} < L, \qquad 0 < \bar{\alpha} < \begin{cases} 2 + \bar{a} & \text{for } L = 2l; \\ 2 + \frac{L-1}{L}\bar{a} & \text{for } L = 2l + 1. \end{cases} \tag{20}$$

Proof. See Appendix A.2. □

The above Lemma 3.2 leads to the following corollary.

Corollary 3.3. *For $S = 1$ and $L \geq 1$,*

- *if $\bar{\alpha} = 1 + \bar{a}$, then the steady state price P^* is locally stable for $0 < \bar{a} < L$. In addition, at $\bar{a} = L$, there occurs a $1 : L + 1$ resonance bifurcation*[4].
- *a necessary condition for the steady state price to be stable is given by (20). In addition, $\bar{\alpha} = 2 + \bar{a}$ for even L and $\bar{\alpha} = 2 + \bar{a}(L-1)/L$ for odd L leads to a flip bifurcation.*

The first result in Corollary 3.3 indicates the stability of the fundamental price along the line $\bar{\alpha} = 1 + \bar{a}$ only, as indicated in Fig. 1. However, it has two implications. First, along the line, the stability region is proportionally enlarged as lags for the long moving average L increase. Secondly, for fixed lag L, the stability line segment $\bar{\alpha} = 1 + \bar{a}$ for $0 < \bar{a} < L$ is part of the stability region on the parameter $(\bar{\alpha}, \bar{a})$ parameter plane. Consequently, we may conjuncture that the stability region is enlarged as lag L increases. However this conjuncture is in general not true and this will become clear from the following theoretical results for cases of $L = 1, 2, 3$ and 4 in this section and numerical results for higher lags L in Sections 4 and 5. The second result in Corollary 3.3 gives us necessary stability boundaries for $\bar{a}$ and $\bar{\alpha}$, indicated by Fig. 1. For general lag L, we have the following result that gives more precisely common stability region for any lags.

[4]Resonance bifurcations occur when the eigenvalues lie on the unit circle. When $\bar{a} = L$, the eigenvalues are given by $\lambda_k = e^{2k\mu\pi i}$ with $\mu = 1/(L+1)$. Geometrically, the L eigenvalues correspond to the $L+1$ unit roots distributed evenly on the unit circle, excluding $\lambda = 1$. When $L = 1$, a flip or period-doubling bifurcation occurs. When $L = 2$, according to Kuznetsov (1995), the bifurcation is known as a 1:3 strong resonance, which may lead to two sets of period three cycles with one set stable and other set unstable (see Chiarella and He (2000) for more details). For $L \geq 2$, according to Sonis (2000), the bifurcation is given by $1 : L + 1$ periodic resonances. For $L_1 = L_2 = L = 3, 4$, instability of the steady state leads to 1:4 and 1:5 periodic resonance bifurcations, respectively, and similar dynamics to 1:3 resonance bifurcation are also found. Theoretical analysis for such types of bifurcation of higher dimensional discrete systems can be exceedingly complicated and not yet completely understood, (see Example 15.34 in Hale and Kocak (pp. 481-482, (1991)))

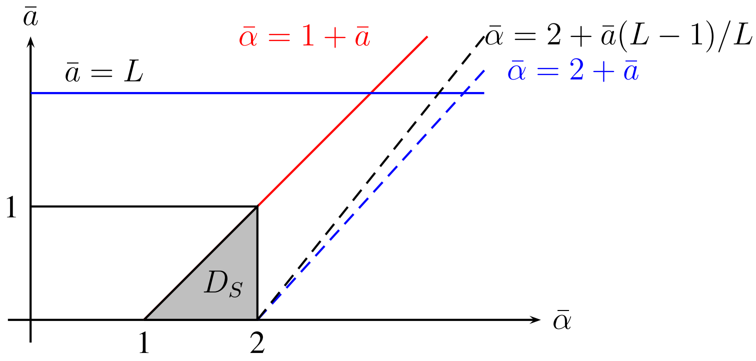

FIGURE 1. Common stability region D_S and necessary stability boundaries $\bar{a} = L$ and $\bar{\alpha} = 2+\bar{a}$ for even lag L and $\bar{\alpha} = 2+\bar{a}(L-1)/L$ for odd lag L for $S = 1$ and general lag $L > 1$.

Proposition 3.4. *For $S = 1$ and $L > 1$. If $1 + \bar{a} < \bar{\alpha} < 2$ then P^* is locally asymptotically stable (LAS).*

Proof. See Appendix A.3 □

The stability region and necessary stability boundaries in terms of parameters $(\bar{\alpha}, \bar{a})$ given by Proposition 3.4 are plotted in Fig. 1. The region D_S corresponds to the stability region defined by Proposition 3.4. Note that the stability condition holds for all L, indicating that the region D_S is the common stability region for all L. This becomes clear from the following results where stability and bifurcation are analysed for $L = 1, 2, 3$ and 4.

For $L = 1, 2, 3$ and 4, the following proposition describes explicitly the regions of local asymptotic stability (LAS) in the $(\bar{\alpha}, \bar{a})$ plane and the bifurcation behaviour at the boundaries of those regions where local asymptotic stability turns to instability.

Proposition 3.5. *For $S = 1$ and $L = 1, 2, 3, 4$, the local stability and bifurcation of the fixed point P^* can be described as follows.*

(i) *For $L = 1$, P^* is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{11}(\bar{\alpha}, \bar{a}) = \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < 2, 0 < \bar{a}\}.$$

In addition

- *a flip bifurcation occurs when $\bar{\alpha} = 2$, and*
- *a saddle-node bifurcation occurs when $\bar{\alpha} = 0$.*

(ii) *For $L = 2$, P^* is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{12}(\bar{\alpha}, \bar{a}) = \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \bar{a} + 2, 0 < \bar{a} < 2\}.$$

Furthermore,

- *a saddle-node bifurcation occurs when $\bar{\alpha} = 0$,*
- *a Hopf bifurcation occurs when $\bar{a} = 2$, and*

– *a flip bifurcation occurs when* $\bar{\alpha} = \bar{a} + 2$.

(iii) *For* $L = 3$, P^* *is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{13}(\bar{\alpha}, \bar{a}) = \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \frac{2}{3}\bar{a} + 2, \bar{a}(2 - \bar{\alpha} + \bar{a}) < 3\}.$$

Furthermore,

– *a saddle-node bifurcation occurs when* $\bar{\alpha} = 0$,

– *a Hopf bifurcation occurs when* $\bar{a}(2 - \bar{\alpha} + \bar{a}) = 3$, *and*

– *a flip bifurcation occurs when* $\bar{\alpha} = \frac{2}{3}\bar{a} + 2$.

(iv) *For* $L = 4$, P^* *is LAS if*

$$\begin{aligned}(\bar{\alpha}, \bar{a}) \in D_{14}(\bar{\alpha}, \bar{a}) =&\{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \frac{3}{4}\bar{a} + 2, 0 < \bar{a} < 4, \\ &(5\bar{a} - 4\bar{\alpha})(4 + \bar{a})^2 < \bar{a}(8 + 3\bar{a} - 4\bar{\alpha})^2\}.\end{aligned}$$

In addition, a flip bifurcation occurs when $\bar{\alpha} = \frac{3}{4}\bar{a} + 2$.

Proof. See Appendix A.4. □

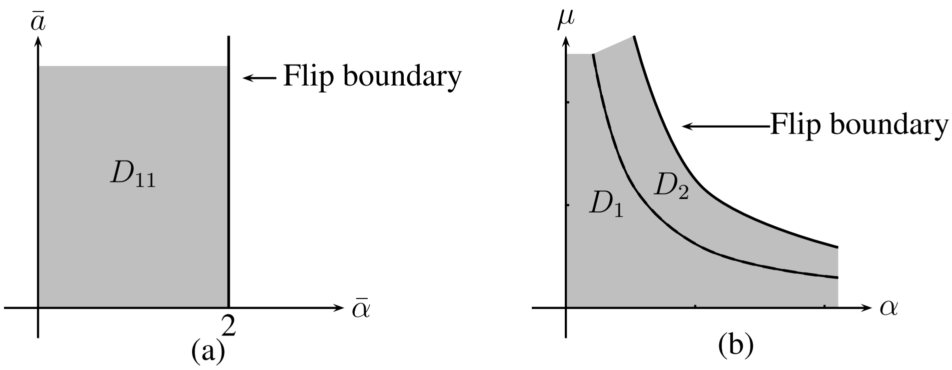

FIGURE 2. Stability region D_{11} and bifurcation boundary in the $(\bar{\alpha}, \bar{a})$ plane (a), and the (α, μ) plane (b), where $D_{11} = D_1 \cup D_2$ and $\lambda \in (0, 1)$ in D_1, $\lambda \in (-1, 0)$ in D_2.

Consider the case $S = L = 1$ for which the stability region is D_{11}. Obviously, the stability condition is independent of $\bar{a}$, as shown in Fig. 2(a). By assuming $m_t = 1$ in this case, the price equation is simplified to $P_{t+1} - P^* = [1 - \alpha\mu](P_t - P^*)$. Hence the stability condition $0 < \bar{\alpha} = \alpha\mu < 2$ can be expressed in terms of the speed of the price adjustment of the fundamentalists towards the fundamental price (α) and the speed of price adjustment of the market maker (μ). The stability region in terms of the parameters α and μ is plotted in Fig. 2(b), indicating that the stability of the steady state price P^* is maintained only when the reaction speeds from both the fundamentalists and the market maker are balanced in a certain way. Note that, when $\alpha\mu = 1$, the prices stay at the constant steady state price P^*. The stability region D_{11} is then divided into two regions $D_1(\alpha\mu < 1)$ and $D_2(1 < \alpha\mu < 2)$. On the one hand, the eigenvalue $\lambda = 1 - \alpha\mu$ is positive for $(\alpha, \mu) \in D_1$ and negative for $(\alpha, \mu) \in D_2$.

Consequently, relative to the steady state price, the returns of the market price P_t are positively (negatively) correlated in the region D_1 (D_2). On the other hand, in the region D_1 the market price is under-adjusted (or under-reacted) by either the market maker or the fundamentalists, while in the region D_2 the market price is over-adjusted (or over-reacted) by both the market maker and the fundamentalists. We thus call D_1 (D_2) a region of under-reaction (over-reaction) from the point of view of either the market maker or the fundamentalists. In addition, $\bar{\alpha} = 2$ leads to a flip bifurcation, resulting from overreaction of either the market maker or the fundamentalists.

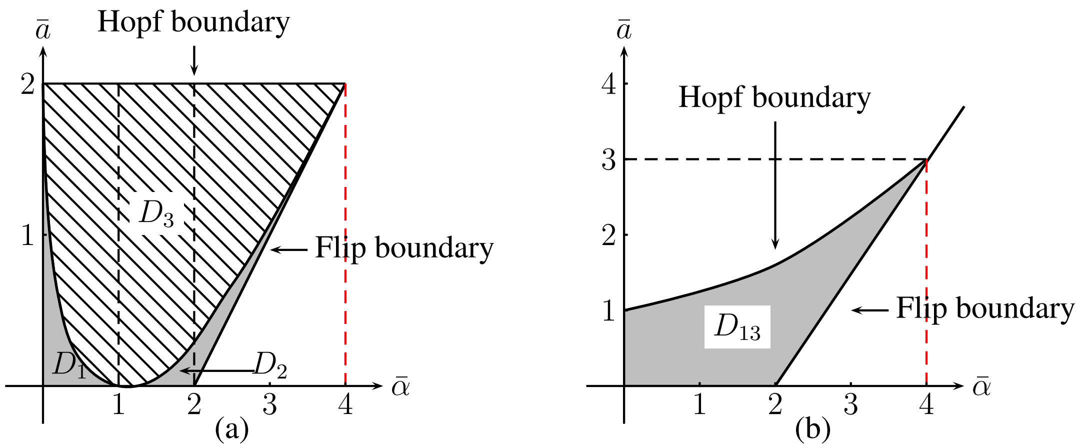

FIGURE 3. (a) Stability region $D_{12} = D_1 \cup D_2 \cup D_3$ and bifurcation boundaries for $S = 1$ and $L = 2$; (b) Stability region D_{13} and bifurcation boundaries for $S = 1$ and $L = 3$.

Consider next the case $L = 2$. The stability region D_{12} and bifurcation boundaries are plotted in Fig. 3(a) in the $(\bar{\alpha}, \bar{a})$ parameter plane. The stability region D_{12} can be divided into three regions $D_{12} = D_1 \cup D_2 \cup D_3$ with both the eigenvalues $\lambda_{1,2}$ are positive in D_1, negative in D_2, and complex in D_3. Along the boundary between D_1, D_2 and D_3, we have real double eigenvalues. The Hopf bifurcation boundary is defined by $\bar{a} = 2$ and $\bar{\alpha} \in (0, 4)$. The nature of the Hopf bifurcation is determined by the value ω of the complex eigenvalues $\lambda_{1,2} = e^{\pm 2\pi i\omega}$, and hence the value of $\rho = 2\cos(2\pi\omega)$ (see Chiarella and He (2003a) for detailed discussion on the nature of the bifurcation). It can be verified that, along the Hopf bifurcation boundary,

$$\rho = 2 - \bar{\alpha} \in [-2, 2] \qquad \text{for} \qquad \bar{\alpha} \in [0, 4].$$

In the case $L = 3$. The stability region D_{13} and the bifurcation boundaries are plotted in Fig. 3(b) on the $(\bar{\alpha}, \bar{a})$ parameter plane. Different from the cases $S = 1$ and $S = 2$, the Hopf bifurcation now depends on both parameters $\bar{\alpha}$ and $\bar{a}$. The nature of the Hopf bifurcation is determined by

$$\rho \equiv 2\cos(2\pi\omega) = \frac{3}{\bar{a}} - 1 \in [0, 2] \qquad \text{for} \qquad \bar{a} \in [1, 3].$$

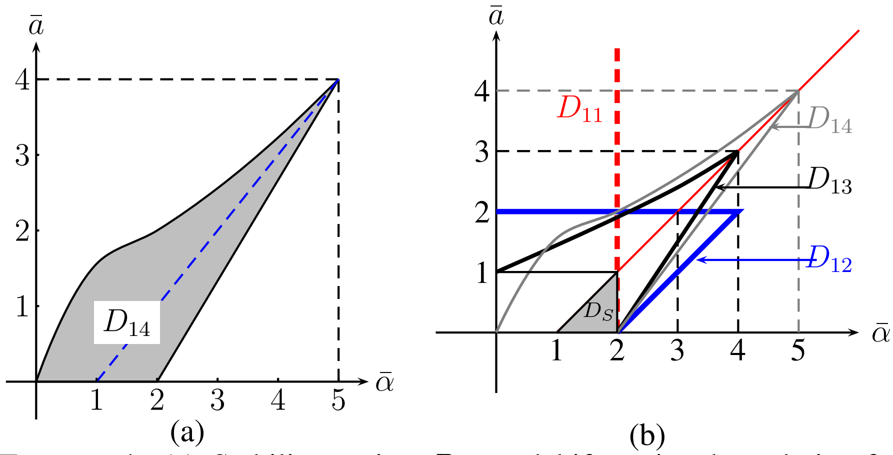

FIGURE 4. (a) Stability region D_{14} and bifurcation boundaries for $S = 1$ and $L = 4$; (b) Comparison of stability regions and bifurcation boundaries D_{1L} for $L = 1, 2, 3, 4$.

In the case $L = 4$. With the help of numerical calculation, the stability region D_{14} can be plotted in Fig. 4(a) on the $(\bar{\alpha}, \bar{a})$ parameter plane. One difference from the previous cases $L = 1, 2$ and 3 is that the stability region of the parameter $\bar{a}$ becomes smaller for small values of $\bar{\alpha}$ (i.e. $0 < \bar{\alpha} < 2$).

A comparison of the stability regions D_{1L} for $L = 1, 2, 3$ and 4 is plotted in Fig. 4(b), which leads to the following observations:

- As L increases, the stability region for the parameter $\bar{a}$ becomes smaller for smaller values of $\bar{\alpha}$ (say $\bar{\alpha} < 2$), and is enlarged for larger values of $\bar{\alpha}$.
- The steady state can only be locally stable when either the fundamentalists reduce their speed of price adjustment towards their expected fundamental price and the chartists extrapolate the difference of the moving averages weakly, or the reactions of the chartist and fundamentalist are balanced in a certain way (that is, the parameters $\bar{\alpha}$ and $\bar{a}$ are near the line $\bar{\alpha} = 1 + \bar{a}$, as indicated in Lemma 3.2 and Fig. 4(b)).
- Based on the analytical results for $L = 1, 2, 3, 4$ and Corollary 3.3 and Proposition 3.4 for general L, we conjecture that: *as lag L increases, the stability region tends to shrink towards, but stretch along, the line $\alpha = 1 + \bar{a}$ with common stability region D_S.*

Given a large of variety on moving averages used in financial markets and difficulty of eigenvalue analysis for high-order characteristic equations, it is not clear how different moving averages influence the stability of the steady state price and types of bifurcation differently. However the analysis has given some important insights into the fact that local asymptotic stability depends on some subtle balance between the reactions of fundamentalists and chartists. We also able to conjecture the general effect of the lag length of the long moving average. This conjecture is partial verified by the numerical simulations in the following sections. In the following section, we examine numerically some rational routes to randomness when agents' switching intensity increases for different moving average rules.

4. Rational Routes to Randomness

In this section, we consider the effect of the switching intensity on the price dynamics of the deterministic system (14)-(16). In order to see the effect under different long-run moving average, we choose $S = 1$ and consider two extreme cases on the long-run moving average $L = 4$ and $L = 100$, respectively. In both cases, we select a fixed set of parameters as follows:

$$\alpha = 1, \mu = 2, \eta = 0.2, a = 1, C = 1. \tag{21}$$

For $\beta = 0$, it follows from (19) that $\bar{\alpha} = 1$ and $\bar{a} = 1$.

4.1. **Case $L = 4$.** For $\beta = 0$, the fundamental price P^* is locally stable. As β increases, $\bar{a}$ increases and $\bar{\alpha}$ decreases. It then follows from Proposition 3.5(iv) that the fundamental price becomes unstable as the switching intensity increases. This is verified by numerical simulations. To illustrate the effect of the switching intensity β on the price and population dynamics, we include phase plots, in terms of (P_t, m_t), for different values of $\beta = 0.2, 0.3, 0.49, 0.52, 0.555$ and 0.57 in Fig. 4.1. It is found that, once the fundamental price P^* becomes unstable, the solutions converge to *figure-eight shaped* attractors for low switching intensity(e.g. the case $\beta = 0.2$ and 0.3). As the switching intensity increases, the figure-eight shaped attractor has grown initially (for $\beta = 0.3, 0.4$) and then stretched to a *scissors-shaped* attractor (for $\beta = 0.49$). As the intensity increases further, the simple attractor becomes more complicated (for $\beta = 0.52$) and eventually leads to strange attractors (for $\beta = 0.555$ and 0.57). The bifurcation diagram of the price and the corresponding Lyapunov exponent with respect to the switching intensity parameter β are plotted in Fig. 4.3. One can see that the market price variation increases as the switching intensity increases. These patterns are similar to the rational routes to randomness studied extensively in Brock and Hommes (1997, 1998). A common interesting feature displayed in Fig. 4.1 is that all the attractors are symmetric about the constant fundamental price. This feature is also shared in most of cases for general lag L. A much more extensive analysis would be required to determine the nature of the mechanism generating such a feature, it may be caused by either the Hopf bifurcations or special structure of the model.

Insert Figures 4.1 and 4.2 here

To illustrate the time series behind the interesting phase plots in Fig. 4.1, the price time series for $\beta = 0.2, 0.49, 0.52$ and 0.57 are plotted in Fig. 4.2 over the first 500 trading periods. It is found that, as the switching intensity increases, the prices oscillate first around the fundamental price periodically or quasi-periodically and then irregularly. Also, the price becomes more volatile (e.g. the case $\beta = 0.52$ and 0.57).

Insert Figure 4.3 here

4.2. **Case $L = 100$.** For $\beta = 0$, we conjectured earlier that the fundamental price P^* is unstable for large L with the selected parameters and this is confirmed by numerical simulations. To illustrate the effect of the switching intensity β on the price and population dynamics when a long moving average of $L = 100$, instead of the case $L = 4$ in the previous discussion, we include phase plots, in terms of (P_t, m_t), for different values of $\beta = 0.05, 0.1, 0.2, 0.3, 0.35, 0.42, 0.45, 0.46$ and 0.4652 in Fig. 4.4. As β increases, the attractor starts with narrow *figure-eight shaped* ones (for $\beta = 0.05$ and 0.1) and is then stretched (or extrapolated) by the chartists towards the extreme high/low price levels (for $\beta = 0.2$). The closed attractors are then broken down to *Lorenz-like attractors* of the 3-dimensional continuous Lorenz system, see Peitgen *et al* (1992) (for β between 0.3 and 0.35), which is not observed for the case $L = 4$. As the switching intensity increases further, the *Lorenz-like* attractors merge into one connected strange attractor (for $\beta = 0.42$) and then to strange attractors (for $\beta = 0.45, 0.46$ and 0.4652). The bifurcation diagram and the corresponding Lyapunov exponent with respect to the switching intensity parameter β are plotted in Fig. 4.6. Similar to the case of $L = 4$, as the switching intensity increases, the volatility of both price and population increases.

Insert Figures 4.4 and 4.5 here

The corresponding price time series are illustrated for $\beta = 0.1, 0.3, 0.35, 0.42$ and 0.46 in Fig. 4.5. One can see that an increase of the switching intensity can generate very interesting price patterns when $L = 100$, comparing with $L = 4$. With lower switch intensity ($\beta = 0.1$), the fundamental price is unstable and extrapolation of the price trend by the chartists push the price away from the fundamental price. Because of their limited long/short position, their fitness or utility become smaller when they reach their limited position. This leads traders to switch back to the fundamentalists, bringing price towards the fundamental price. Because of the increase of the fitness of the chartists, the price is pushed further to different extreme level. As the switching intensity increases (for $\beta = 0.3, 0.35$), such switching from high/low extreme to low/high extreme happens very quickly. At the same time, the price becomes more volatile. As the intensity increase further, the regular switching pattern of the price between two extreme levels is destroyed, leading to highly volatile price pattern (for $\beta = 0.46$).

Insert Figure 4.6 here

It is interesting to see different rational routes to randomness for $L = 4$ and $L = 100$. For $L = 4$, the strange attractors (in terms of the phase plots) become more dense as the switching intensity increases and the frequency of oscillation of the price series are very high (see Fig. 4.2). However, for $L = 100$, the time periods of price staying at either high or low levels are prolonged and prices become even volatile at the extreme levels (see Fig. 4.5). Correspondingly, the strange attractors concentrate more on the extreme levels and become less dense within the attractors (see Fig. 4.4).

5. Dynamics of Long-Run Moving Average

In this section, we consider the effect of the long-run moving average on the price dynamics of the deterministic system (14)-(16). For comparison, we select a fixed set of parameters as follows:

$$\alpha = 1, \mu = 2, \beta = 0.4, \eta = 0.2, a = 1, C = 0. \tag{22}$$

It follows from (19) that $\bar{\alpha} = 1$ and $\bar{a} = 1$. Hence the fundamental price is locally stable for $L = 2, 3, 4$ and unstable for $L \geq 5$. Fig. 4.7 illustrates how the phase plot (in terms of (P_t, m_t)) changes as the lag L increases.

Insert Figures 4.7 and 4.8 here

For $L = 5$, the attractor is given by an *figure-eight shaped* closed orbit with small price variation (about 1% of the fundamental price level) and there is a trend among the traders to switch from the fundamentalists to the chartists. For $L = 8$, the size of the attractor is enlarged, implying that the deviations of both price and population from the fundamental value, which is $P_t = 100, m_t = 0$, is enlarged. Hence an increase in the moving average window L destabilizes the price dynamics. This destabilizing effect becomes more significant when L is increased further to $L = 9, 10, 50$ and price dynamics even become more complicated for $L = 90$ and 100, as indicated by the phase plots in Fig. 4.7.

In order to understand such destablizing effect of the moving average, let us examine the time series in Fig. 4.8. It is found that, following the cross over of the long moving average and the market price, both the chartists and fundamentalists take the same long/short position initially, but soon after they take opposite positions. This helps to build up either up or down trend, pushing price to either higher or lower level initially, but soon after, their different positions slow down the trend built up initially and bring the price back towards the fundamental price level. The time period for changing position to bring the price towards the fundamental price is proportional to the lag L. When the lag L for the moving average is small, position changes happen quickly. As L increases, it takes longer time to change the price trend and bring the price towards the fundamental price level.

The above destablizing effect of the lag L holds in general for the parameters located within regions in which the fundamental price is locally stable for lower lags and unstable for higher lags, as discussed in the above. However, it can be verified that for the parameters located within regions in which the fundamental price is unstable for lower lags, but stable for higher lags, an increase of lag L can stabilizing an otherwise unstable system. In particular, for the parameters satisfying $\alpha = 1 + \bar{a}$, Corollary 3.3 implies that the fundamental price is locally stable for $\bar{a} < L$. Numerical simulations (not reported here) indicate that, in this case, an increase of L can bring an explosive system to a (locally) stable system.

6. Conclusions

Within Brock and Hommes' (1998) asset pricing model with heterogeneous and adaptive beliefs, price fluctuations are driven by an evolutionary dynamics between different expectation schemes. Consequently various rational routes to randomness are observed when the intensity of choice to switch prediction strategies is high. This analytic oriented framework relies on its mathematical tractability of lower dimension system and it is in general difficult to have a clear picture when the prediction strategies involve a long history of price, such as the long moving average rules. Given the popularity of moving average strategies in both the real market and empirical studies, this paper analyzes the impact of the moving average on the market dynamics and potentially rational routes to randomness. Within the confines of a model of the fundamentalists and trend followers (who trade on the trend generated by the cross of the latest price over the long moving average) we are able to obtain some important qualitative insights into the impact of the moving average in general. Intuitively a long moving average smooths price dynamics and hence an increase of the length of the moving average is expected to stabilize the market price dynamics. However our results show that, within a market maker scenario, this is in general not true (except both the reaction coefficient α of the fundamentalists and the extrapolation rate a of the trend followers are balanced in certain way satisfying $\bar{\alpha} = 1 + \bar{a}$). In fact, the length of the moving average is destabilizing the market price and, to our knowledge, this is a new result related to the dynamics of the moving average. Another contribution of this paper is that a similar rational routes to randomness when the switching intensity is high is observed across various moving average rules. In subsequent research, a more realistic model with large number of trading rules, in particular with agents using different moving average strategy, should be studied extensively by using various numerical simulation tools, such as genetic algorithms and neural networks.

Appendix A. Proofs of Main Results

A.1. **Proof of Proposition 3.1.** The deterministic system (14)-(16) can be written as follows:

$$\begin{cases} P_{t+1} &= F(X_t) \\ U_{t+1} &= H(X_t) \\ m_{t+1} &= G(X_t). \end{cases} \tag{23}$$

where

$$X_t = (P_t, P_{t-1}, \cdots, P_{t-(L-1)}, U_t, m_t),$$

$$F(X_t) = P_t + \frac{\beta}{2}[-(1-m_t)\alpha(P_t - P^*) + (1-m_t)h(\psi_t^{S,L})], \tag{24}$$

$$H(X_t) = [-\alpha(P_t - P^*] - h(\psi_t^{S,L})][F(X_t) - P_t] + \eta U_t, \tag{25}$$

$$G(X_t) = \tanh[\beta(H(X_t) - C)/2]. \tag{26}$$

One can easily see that, for $\eta \in [0,1)$, $(P_t, U_t, m_t) = (P^*, 0, m^*)$ is the unique steady state of the system (23), where P^* corresponds to the constant fundamental price and $m^* = \tanh(-\beta C/2)$. Also, evaluated at the unique steady state,

$$\begin{aligned}
\frac{\partial F}{\partial P_t} &= 1 + \frac{\mu}{2}[-(1+m^*)\alpha + (1-m^*)a(\frac{1}{S} - \frac{1}{L})],\\
\frac{\partial F}{\partial P_{t-1}} &= \frac{\partial F}{\partial P_{t-2}} = \cdots = \frac{\partial F}{\partial P_{t-(L-S)}} = \frac{\mu}{2}(1-m^*)a(\frac{1}{S} - \frac{1}{L}),\\
\frac{\partial F}{\partial P_{t-(L-S-1)}} &= \cdots = \frac{\partial F}{\partial P_{t-(L-1)}} = \frac{\mu}{2}(1-m^*)a(-\frac{1}{L}),\\
\frac{\partial F}{\partial U_t} &= \frac{\partial F}{\partial m_t} = 0,\\
\frac{\partial H}{\partial P_t} &= \frac{\partial H}{\partial P_{t-1}} = \cdots = \frac{\partial H}{\partial P_{t-(L-1)}} = 0,\\
\frac{\partial H}{\partial U_t} &= \eta, \frac{\partial H}{\partial m_t} = 0,\\
\frac{\partial G}{\partial P_t} &= \frac{\partial G}{\partial P_{t-1}} = \cdots = \frac{\partial G}{\partial P_{t-(L-1)}} = 0,\\
\frac{\partial G}{\partial U_t} &= \eta\beta/2, \frac{\partial G}{\partial m_t} = 0.
\end{aligned}$$

Based on these calculations, one can verify that the characteristic equation of the steady state has the form of (17).

A.2. **Proof of Lemma 3.2.** For $S = 1$ and $\bar{\alpha} = 1 + \bar{a}$,

$$\Gamma_{1L}(\lambda) \equiv \lambda^L + \frac{\bar{a}}{L}(\lambda^{L-1} + \cdots + \lambda + 1) = 0.$$

It follows from Chiarella and He (2002) that $|\lambda_i| < 1$ iff $-\frac{1}{L} < \frac{\bar{a}}{L} < 1$, i.e., $\bar{a} < L$ (since $\bar{a} > 0$).

In general, following from Jury's test, necessary conditions for $|\lambda_i| < 1$ for all i are $\bar{a}/L < 1$ and

$$\Gamma_{iL}(1) = \bar{\alpha} > 0$$

$$(-1)^L \Gamma_{1L}(-1) = \begin{cases} 2 - \bar{\alpha} + \bar{a} > 0 & \text{for } L = 2l \\ 2 - \bar{\alpha} + \frac{L+1}{L}\bar{a} > 0 & \text{for } L = 2l+1 \end{cases}$$

A.3. **Proof of Proposition 3.4.** Let $f(\lambda) = \lambda^L$ and $g(\lambda) = -(1 - \bar{\alpha} + \bar{a})\lambda^{L-1} + \frac{\bar{a}}{L}[\lambda^{L-1} + \cdots + \lambda + 1]$. Then on $|\lambda| = 1$,

$$|g(\lambda)| < |1 - \bar{\alpha} + \bar{a}| + \bar{a}, \quad |f(\lambda)| = 1.$$

If $1 + \bar{a} < \bar{\alpha} < 2$, then $|g(\lambda)| < |f(\lambda)|$ on $|\lambda| = 1$. Following from Rouche's theorem, $f(\lambda)$ and $\Gamma_{1L}(\lambda) = f(\lambda) + g(\lambda)$ have the same number of zeros inside $|\lambda| = 1$. Therefore $|\lambda_i| < 1$ for $i = 1, 2, \cdots, L$.

A.4. **Proof of Proposition 3.5.** For $S = 1$,

$$\Gamma_{1,L}(\lambda) \equiv \lambda^L - (1 - \bar{\alpha})\lambda^{L-1} - \bar{a}(1 - \frac{1}{L})\lambda^{L-1} + \frac{\bar{a}}{L}(\lambda^{L-2} + \cdots + \lambda + 1) = 0.$$

i.e.

$$\Gamma_{1,L}(\lambda) \equiv \lambda^L - [1 - \bar{\alpha} + \bar{a}(1 - \frac{1}{L})]\lambda^{L-1} + \frac{\bar{a}}{L}(\lambda^{L-2} + \cdots + \lambda + 1) = 0.$$

- For $L = 1$,

  $$\Gamma_{1,L}(\lambda) \equiv \lambda - (1 - \bar{\alpha}) = 0.$$

  Hence $|\lambda| < 1$ iff $0 < \bar{\alpha} < 2$. Also $\lambda = +1$ for $\bar{\alpha} = 0$ and $\lambda = -1$ for $\bar{\alpha} = 2$.
- For $L = 2$,

  $$\Gamma_{2,1}(\lambda) = \lambda^2 + c_1\lambda + c_2 = 0,$$

  where

  $$c_1 = -(1 - \bar{\alpha} + \frac{1}{2}\bar{a}), \quad c_2 = \frac{\bar{a}}{2}.$$

  Following Jury's test, $|\lambda_i| < 1$ iff

  $$\begin{aligned}
  \pi_1 &\equiv 1 + c_1 + c_2 = \bar{\alpha} > 0,\\
  \pi_2 &\equiv 1 - c_1 + c_2 = 2 - \bar{\alpha} + \bar{a} > 0,\\
  \pi_3 &\equiv 1 - c_2 = 1 - \frac{\bar{a}}{2} > 0.
  \end{aligned}$$

Hence P^* is LAS if $(\bar{\alpha}, \bar{a}) \in D_{12}(\bar{\alpha}, \bar{a})$. Also, $\lambda_1 = 1$ and $|\lambda_2| < 1$ when $\pi_1 = 0$, $\lambda_1 = -1$, $|\lambda_2| < 1$ when $\pi_2 = 0$ and $\lambda_{1,2} \in C$, $|\lambda_{1,2}| = 1$ when $\pi_3 = 0$.

- For $L = 3$,

$$\Gamma_{1,3}(\lambda) \equiv \lambda^3 - [1 - \bar{\alpha} + \bar{a}(1 - \frac{1}{3})]\lambda^2 + \frac{\bar{a}}{3}(\lambda + 1) = 0.$$

Denote

$$c_1 = -[1 - \bar{\alpha} + \frac{2}{3}\bar{a}], \quad c_2 = c_3 = \frac{\bar{a}}{3}.$$

Then $|\lambda_i| < 1$ iff

$$\begin{aligned} \pi_1 &\equiv 1 + c_1 + c_2 + c_3 = \bar{\alpha} > 0, \\ \pi_2 &\equiv 1 - c_1 + c_2 - c_3 = 2 - \bar{\alpha} + \frac{2}{3}\bar{a} > 0, \\ \pi_3 &\equiv 1 - c_2 + c_1 c_3 - c_3^2 \\ &= 1 - \frac{\bar{a}}{3}[2 - \bar{\alpha} + \bar{a}] > 0. \end{aligned}$$

Hence P^* is LAS if $(\bar{\alpha}, \bar{a}) \in D_{13}(\bar{\alpha}, \bar{a})$. Furthermore, $\pi_1 = 0$, $\pi_2 = 0$ and $\pi_3 = 0$ give the saddle-node, flip and Hopf bifurcation boundaries, respectively.

- For $L = 4$,

$$\Gamma_{1,4}(\lambda) \equiv \lambda^4 - [1 - \bar{\alpha} + \frac{3}{4}\bar{a}]\lambda^3 + \frac{\bar{a}}{4}(\lambda^2 + \lambda + 1) = 0.$$

Denote

$$p = -[1 - \bar{\alpha} + \frac{3}{4}\bar{a}], \quad q = \frac{\bar{a}}{4}.$$

Then, using Jury's test, $|\lambda_i| < 1$ iff

$$\begin{aligned} \Gamma_{1,4}(1) &= \bar{\alpha} > 0, \\ \Gamma_{1,4}(-1) &= 2 - \bar{\alpha} + \bar{a} > 0, \\ \bar{a} &< 4 \end{aligned}$$

and both the determinants of matrixes

$$A = \begin{pmatrix} 1 & 0 & q \\ p-1 & 1+q & 0 \\ 2q-p & p-1 & 1+p-q \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -q \\ p & 1-q & -q \\ 0 & p-q & 1-p \end{pmatrix}$$

are positive. It can be verified that $|A| > 0, |B| > 0$ iff $(1+q)^2[1+p-2q] + q(p-1)^2 > 0$ and $p < 1$, respectively, which leads to the result.

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