An Evolutionary Game Theory Explanation of ARCH Effects

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Abstract

ARCH/GARCH models have been widely used to examine financial markets data, but formal explanations for the sources of conditional volatility are lacking. This paper demonstrates the existence of GARCH effects similar to those found in asset returns, in a standard asset pricing model with heterogeneous agents. Evolutionary game theory describes how agents endogenously switch between different forecasting strategies based on past forecast errors. We show conditions under which agents agree on the fundamental forecast and those where agents adopt strategies that use extraneous information. Divergence from agreement on fundamentals could lead to periods of high volatility in prices and returns. Econometric tests of simulated data show the existence of GARCH effects, we examine the impact of changes in the model parameters.

The goal of volatility analysis must ultimately be to explain the causes of volatility. While time series structure is valuable for forecasting, it does not satisfy our need to explain volatility. .... Thus far, attempts to find the ultimate cause of volatility are not very satisfactory.

- Robert Engle [2001]

Few models are capable of generating the type of ARCH one sees in the data. .... Most of these studies are best summarized with the adage that "to get GARCH you need to begin with GARCH."

- Adrian Pagan [1996]

1 Introduction

ARCH/GARCH models have been widely used for estimating and forecasting time series with conditional volatility. These models have been used to describe the behavior of inflation, interest rates and exchange rates[1], and they have become the standard tool for analyzing returns in financial markets (Engle [2001]). As the above quotes indicate, however, formal explanations of the sources of conditional volatility have been elusive.

This paper demonstrates ARCH effects using a game theoretic approach where agents endogenously choose forecasting strategies. An evolutionary game theory mechanism describes how fractions $x_t = (x_{1,t}, ..., x_{k,t})$ of the population using forecasting strategies $(e_{1,t}, ..., e_{k,t})$ evolve according to the performance of the strategies. The asset price is a function such that

$$y_t = f(x_t, Z_t, \Theta_t)$$

where Z_t is the information set of the agents and Θ_t is a vector of stochastic elements that includes dividends and other, possibly extraneous, shocks. A key observation is that the asset price y_t depends on the evolution of agents beliefs, represented by changes in x_t as follows.

$$x_{t+1} = g(x_t, y_t, Z_t)$$

[1] Bollerslev (2001) examined inflation dynamics with a GARCH model. Engle, Lilien and Robins [1987] use the ARCH in mean model to study yield curve issues. Diebold and Nerlove [1989] use a multivariate ARCH model to analyze exchange rates.

The conditional variance of the asset price can be written as

$$V\left(y_t|Z_t\right) = h\left(x_t, V\left(\Theta_t\right)\right)$$

that is, volatility depends on the state of agents' beliefs and the magnitude of the shocks. Conditional volatility arises when agents adopt strategies that propagate the variance of Θ_t to the variance of y_t for a number of periods.

ARCH effects would not arise if agents settled on a single forecast, as many representative agent models assume. Recent observations of the stock market suggest that changing beliefs is a driving force behind the dramatic movements in asset prices, such as those seen in technology stocks.

We provide a particular example of such a process, focusing on three forecasting strategies that satisfy rational expectations. The *fundamentalist* only uses expected future dividends. The *mystic* also uses fundamentals but may also experiment with other extraneous information. The *reflectivist* forms the true mathematical expectation by using all the information about the other forecasts and the state of the population. Agents switch to strategies that have shown lower forecast errors, and an evolutionary dynamic of Hofbauer and Weibull [1996] allows us to parameterize how aggressively they do so.

Standard econometric tests of the simulated data confirm the existence of ARCH and GARCH effects for certain standard deviations of the stochastic elements. For small shocks to dividends and the martingale innovation, mysticism never increases its following and conditional heteroscedasticity does not appear. However, as the magnitude of the shocks rise, the asset price shows occasional bubble behavior and the returns data show ARCH and GARCH effects for many of the simulations. Similarly, if agents are slow to switch to better performing strategies, ARCH and GARCH effects are not present, but as agents become more aggressive, these effects are increasingly common. We also note that these effects diminish when shocks to the dividends are very large and/or agents are extremely aggressive. In this situation, there is so much noise in the returns process that detecting ARCH and GARCH effects is less likely.

Brock, Hommes and Wagener [2001] extend a standard mean-variance optimization model of asset prices to an environment with heterogeneous agents.[2] We adopt a similar approach, but

[2] Other studies with heterogeneous expectations include Chiarella [2002], Degrauwe [1993], Lux [1998] and DeLong, Shleifer, Summers and Waldmann [1990]. Our study differs in our focus on forecasts satisfying rational expectations.

assume that agents know all the parameters of the model and stick to forecasts based on rational expectations. LeBaron, Arthur and Palmer [1999] study the time series features of a simulated asset market and show the existence of ARCH effects and many other features of financial markets data. They use a computational approach with many trader types introduced throughout the simulation according to a genetic algorithm. The dividend process in their model has serial dependence, which may be a factor in the existence of conditional volatility, while dividends in the simulated model of this paper are serially independent. Another advantage to our approach is our ability to compare the merits of a small number of strategies that can be interpreted in standard economic models.

The focus on the possibility of heterogeneous forecasts in this paper stands in contrast to those who argue that martingale solutions should be ruled out, according to criteria such as transversality (see Cochrane [2001] p. 27), minimum state variable (McCallum [1983, 1997]), and expectational stability (Evans and Honkapohja [1994, 2001]). Parke and Waters [2002] analyze conditions under which the population is robust to the introduction of mysticism and the agents stick to forecasts based on fundamentals and conditions where heterogeneity in the forecasts could persist. The present paper presents a particular example of the class of models whose stability characteristics are extensively analyzed in Parke and Waters [2002].

The convergence to rational expectations literature, such as the papers on least squares learning of Marcet and Sargent [1989a, 1989b], is also related. Woodford [1990] and Howitt and McAfee [1992] show the possibility of learning sunspot equilibria, which depends on accidental correlations between sunspots and fundamentals, similar to the present work. These papers focus on agents leaning model parameters over time, while our agents know the parameters and choose forecasts that are based on multiple solutions to the model.

A paper that does have a formal model explaining conditional volatility in a different environment is the study of real interest rate fluctuations in den Haan and Spear [1998]. They construct on optimizing model where agents hold saving in the form of bonds. Agents are heterogeneous, as in the present paper, and receive idiosyncratic shocks. Volatility clustering arises due to borrowing constraints that vary across the business cycle.

The organization of the paper is as follows. Section 2 develops the asset pricing model with heterogeneous agents. Section 3 describes the different forecasts, shows how the asset price and forecast errors are determined. Section 4 shows how agents' choices of strategies evolve over time.

Section 5 reports the simulations results showing ARCH and GARCH effects. Section 6 concludes.

2 Asset Pricing with Heterogeneous Agents

Brock, Hommes and Wagener [2001][3] extend a standard asset pricing model to the situation where agents can have heterogeneous beliefs about future prices. Agents are myopic, mean-variance optimizers who can choose between a risky asset and a riskless asset with gross rate of return R. An agent's wealth W_t evolves according to

$$W_{t+1} = RW_t + (y_{t+1} + u_{t+1} - Ry_t)\, z_t$$

where the price of the risky asset is y_t, the dividend payment is u_t and z_t is the number of shares purchased by the agent in time t. The asset price and dividend process are stochastic so agents does not have precise knowledge of y_{t+1} or u_{t+1} when making decisions about z_t.

Agents may have heterogeneous expectations about an endogenous variable X_t. Let an agent of type j have the expectation $e_{j,t}(X_{t+1})$ formed at time t for the next period value X_{t+1}. Agents of type j maximize the following objective over z_t, where $V_{j,t}$ denotes the conditional variance at time t for agent j.

$$e_{j,t}(W_{t+1}) - \frac{a}{2} V_{j,t}(W_{t+1})$$

The parameter a denotes the level of risk aversion. Assuming that the conditional variance $V_{j,t}(W_{t+1}) = \sigma_W^2$ is the same constant for all types and denoting the demand for shares of type j as $z_{j,t}$, the optimization yields the condition

$$z_{j,t} = \left(\frac{1}{a\sigma_W^2}\right) e_{j,t}(y_{t+1} + u_{t+1} - Ry_t)\,.$$

There is a constant supply of shares z_s per investor and the fraction of the population of type j is $x_{j,t}$. Summing the demand over the n types and equating supply and demand yields the

[3] This model, which they refer to as the *Adaptive Belief Sytem*, was introduced in the technical literature in Brock and Hommes [1999].

following condition for the price of the risky asset.

$$Ry_t = \sum_{j=1}^{n} x_{j,t} e_{j,t} \left(y_{t+1} + u_{t+1} \right) - C \tag{1}$$

where $C = a\sigma_W^2 z_s$.

If agents are homogeneous and have rational expectations (1) becomes

$$y_t = \alpha E_t \left(y_{t+1} + u_{t+1} \right) - C \tag{2}$$

for $\alpha = R^{-1}$. The fundamental, perfect foresight, bubble-free solution is

$$y_t^* = \sum_{s=1}^{\infty} \alpha^s u_{t+s} - \frac{C}{1-\alpha}, \tag{3}$$

which corresponds to the present value model of asset pricing where α is the discount factor. However, the self-fulfilling nature of expectations admits the class of solutions

$$y_t = E_t y_t^* + \alpha^{-t} m_t \tag{4}$$

to equation (2) where m_t is any martingale such that $m_t = m_{t-1} + \eta_t$ for some *iid*, mean zero stochastic process η_t. Some argue for the exclusion of such solutions on the basis of a transversality condition, but in a study of short term dynamics, we find such criteria unpersuasive.

In the present approach agents use forecasting strategies based on the general solutions (4) to (2). Agents agree on the expectations of future dividends so $E_t y_t^*$ is common knowledge, but they may disagree on the use of a martingale in making forecasts.

The key underlying assumptions of Brock, Hommes and Wagener [2001] used in the present approach are the common beliefs about the dividend process and the constant conditional variance of the asset prices. However, while Brock, Hommes and Wagener [2001] postulate a number of different forecasting strategies, such as trend chasing or perfect foresight, we restrict ourselves to forecasts based on the general solutions (4).

3 Strategies

This section postulates three possible strategies for forecasting the asset price y_t based on the general solutions (4). The realization of y_t is determined by the forecasts and the fractions of the population using the different strategies. Determining y_t allows the construction of forecast errors for each strategy, which act as payoff in the evolutionary game theory dynamics.

A fraction of the population γ_t uses the *fundamentalist forecast* based on (3).

$$e_{\gamma,t} = E_t y^*_{t+1}$$

Since the one period ahead expectations determine the current asset price, these forecasts of $y_{t+1}$are formed without knowledge of y_t. Another fraction λ_t of the population uses a *mystic forecast* based on a martingale solution of the form (4).

$$e_{\lambda,t} = E_t y^*_{t+1} + \alpha^{-t-1} m_t$$

One potential reasons agents might consider the mystic forecast is the existence of a spurious (sunspot) variable that some agents believe affects asset prices.

While both of the above forecasting strategies may be attractive to some, they do not use all the information in this environment with heterogeneous forecasts. We postulate a a fraction of the population β_t using a third strategy, the *reflective forecast* $e_{\beta,t}$, which takes into account the other forecasts and the fractions of the population using each. The asset pricing formula for heterogeneous agents (1) with the above three forecasts yields

$$y_t = \alpha \left(\gamma_t e_{\gamma,t} + \lambda_t e_{\lambda,t} + \beta_t e_{\beta,t} + E_t u_{t+1}\right) - \alpha a \sigma^2 z_s \tag{5}$$

using the fact that agents have a common expectation about future dividends. The reflectivist forecast uses all available information to satisfy the asset pricing equation under rational expectations (2) so it must satisfy

$$y_t = \alpha \left(e_{\beta,t} + E_t u_{t+1}\right) - \alpha a \sigma^2 z_s. \tag{6}$$

Combining (5) and (6), the reflectivist forecast may be represented as

$$e_{\beta,t} = n_t e_{\lambda,t} + (1 - n_t)\, e_{\gamma,t}$$

where

$$n_t = \frac{\lambda_t}{\lambda_t + \gamma_t}.$$

The reflectivist forecast is an average of the other two forecasts, weighted according to their relative popularity. The reflectivist strategy is analogous to Keynes' [1935] "beauty contest" interpretation of predicting stock prices. Agents give primary attention to anticipating the choices of others as opposed to the intrinsic value of the asset.

Agents choose forecasts based on past performance of the different strategies. As with many evolutionary models[4], we must specify the payoff to a strategy. Following LeBaron et. al. [1999], agents judge strategies according to squared forecast error[5]. The forecasts described above determine the realization of the asset price, which in turn determine the forecast errors for each strategy. To determine forecast errors, note that agents' forecasts of y_t in the previous period are

$$\begin{aligned} e_{\gamma,t-1} &= E_{t-1}y_t^* \\ e_{\lambda,t-1} &= E_{t-1}y_t^* + \alpha^{-t} m_{t-1} \\ e_{\beta,t-1} &= E_{t-1}y_t^* + \alpha^{-t} n_{t-1} m_{t-1}. \end{aligned}$$

Note that the reflectivist forecast includes the martingale, weighted according to the popularity of the mystic forecast, represented by n_t. The fraction $\gamma_t, \lambda_t, \beta_t$ and n_t are determined once y_{t-1} is realized and held constant until y_t is determined. At time t, the updated fundamental solution $E_t y_{t+1}^*$ and martingale m_t become available. Agents still do not know the contemporaneous price y_t, but can use the new information to form the following forecasts of y_{t+1}.

[4] Blume and Easley [1992] discuss this point in the context of an evolutionary study of asset pricing. They are concerned with long run survival of strategies.

[5] In constrast, Brock, Hommes and Wagener [2001] use realized profits. While this is reasonable, accuracy of forecast is another good an indicator of future performance of a strategy.

$$
\begin{aligned}
e_{\gamma,t} &= E_t y^*_{t+1} \\
e_{\lambda,t} &= E_t y^*_{t+1} + \alpha^{-t-1} m_t \\
e_{\beta,t} &= E_t y^*_{t+1} + \alpha^{-t-1} n_t m_t
\end{aligned}
$$

These forecasts with the asset price equation (5) yield the realization

$$y_t = E_t y^*_t + \alpha^{-t} n_t m_t \tag{7}$$

The negatives of the three period t squared forecast errors can be written as

$$\pi_{\beta,t} = -U_t^2, \tag{8}$$

$$\pi_{\gamma,t} = -U_t^2 - 2n_t U_t A_t - n_t^2 A_t^2, \tag{9}$$

$$\pi_{\lambda,t} = -U_t^2 + 2\left(1 - n_t\right) U_t A_t - \left(1 - n_t\right)^2 A_t^2, \tag{10}$$

where $A_t = \alpha^{-t} m_{t-1}$ is the level of the martingale term. The reflective forecast error $U_t = F_t + G_t$ includes the innovation in the fundamentals for y_t

$$F_t = E(y^*_t | \Omega_t) - E(y^*_t | \Omega_{t-1})$$

and a term involving the innovation in the martingale multiplied by the weight n_t measuring the importance of mysticism

$$G_t = \alpha^{-t} n_t (m_t - m_{t-1}).$$

Intuitively, U_t depends on serially independent shocks while A_t depends on the martingale, which has autoregressive behavior.

In the payoffs above, the third terms in the payoffs to fundamentalism and mysticism (9) and (10), referred to as martingale terms, is detrimental to each payoff. However, the second term in each involves $U_t A_t$, referred to as covariance terms, are potentially beneficial to one of the payoffs depending on the sign of $U_t A_t$.

Evolution of β_t, γ_t and λ_t depends on the *fitness* of each strategy, which is the difference between each payoff and the population average payoff $\bar{\pi}_t = \beta_t \pi_{\beta,t} + \gamma_t \pi_{\gamma,t} + \lambda_t \pi_{\lambda,t}$. The reflectivist strategy has an intrinsic advantage in that is will always have non-negative fitness, but it is possible for the mystic or fundamentalist payoff to be the best. Using the payoffs (8), (9) and (10) above, the fitness for reflectivism is

$$\pi_{\beta,t} - \bar{\pi}_t = \frac{\gamma_t \lambda_t}{\gamma_t + \lambda_t} A_t^2 \geq 0. \tag{11}$$

The covariance terms in (9) and (10) cancel in the population average payoff, but the martingale terms do not so the population average will always be worse than the payoff to reflectivism.

Despite this property of reflectivism, if the covariance term $2(1-n_t) U_t A_t$ in the payoff to mysticism is positive and sufficiently large then $\pi_{\lambda,t} > \pi_{\beta,t} > \pi_{\gamma,t}$. If there are few followers of mysticism, the sign of the covariance term depends on whether the martingale and dividend process are positively correlated, so $U_t A_t > 0$ means that the mystic has conjured a fortuitously accurate forecast.

These observations give some intuition about the interaction between the strategies. The realization of the asset price (7) gives some insight into the potential effect on the time series data. If mysticism is eliminated, so that $n_t = 0$, then the reflective and fundamental forecasts are identical, y_t follows fundamentals and returns will be determined by dividends alone. However, if $n_t > 0$ the martingale could influence the asset price and lead it away the fundamental y_t^*. Further, if n_t changes over time, both behaviors could be observed for different stretches of time. The possibility of the price switching between fundamental and bubble behavior creates the potential for the detection of ARCH effects.

4 Evolution

This section describes a particular evolutionary framework based on the idea that agents imitate the strategies of other successful agents[6]. Agents will switch to other strategies which have better payoffs, meaning lower forecast errors. We use a discrete time version of evolutionary dynamics.

[6] DeLong, Schleifer, Summer, and Waldman [1990] use imitation in their noise trader model.

Let $r_{i,t}$ be the fraction of agents using forecast i who review their choice of strategy at time t, and let $p^i_{j,t}$ be the probability that an agent using forecast j in period t who reviews switches to forecast i in the next period. If there are k available forecasts, then the change in $x_{i,t}$ is given by

$$x_{i,t+1} - x_{i,t} = \sum_{j=1}^{k} r_{j,t} x_{j,t} p^i_{j,t} - r_{i,t} x_{i,t}. \tag{12}$$

This is the discrete time version of equation (4.24) in Weibull (1997).[7]

All agents review every period regardless of the payoff so $r_{i,t} \equiv 1$, but that the transition probabilities $p^i_{j,t}$ depend on the performance of the strategies. Agent will tend to switch to other strategies with better payoffs, meaning lower forecast errors. Agents use payoff weighting functions $w(\pi_{i,t})$ to arrive at the transition probabilities

$$p^i_{j,t} = \frac{w(\pi_{i,t})\, x_{i,t}}{\overline{w}_t}, \tag{13}$$

where $\overline{w}_t = \sum_{h=1}^{n} w(\pi_{h,t})\, x_{h,t}$. When the weighting function $w(\cdot)$ is linear $\overline{w}_t$ corresponds to the population average payoff. The transition probability $p^i_{j,t}$ into strategy i depends on its current weighted payoff $w(\pi_{i,t})$ relative to the population average $\overline{w}_t$ and its current popularity $x_{i,t}$. Substituting (13) into (12) with $r_{i,t} \equiv 1$ yields

$$x_{i,t+1} = x_{i,t}\, \frac{w(\pi_{i,t})}{\overline{w}_t}. \tag{14}$$

The fraction using strategy i will obviously increase if its payoff $\pi_{i,t}$ is large relative to the payoffs to the other strategies, but the dynamics of the system depend on the exact specification for $w(\cdot)$.

The above evolution equation by itself shows that $x_{i,t} = 0$ is steady state so if a strategy has no followers it cannot acquire any. Avocates of minimum state variables require that all agents use the fundamentalist strategy, while those persuaded by rational bubbles would think that all agents use mysticism. The present paper takes a middle course, since each strategy has attractive features. Pricing of assets based on fundamentals focuses on real events. Mysticism recognizes the potential for large short term deviations in asset prices. Reflectivism uses all the information

[7] Hofbauer and Weibull (1996, pp. 564-6) consider two specific behavioral models contained within this general framework. Parke and Waters [2001] investigate both in depth.

available to form a rational expectation. Therefore, we study the evolution of the system under the following.

Condition 1 *The fractions γ_t,λ_t and β_t do not fall below certain minimums $\gamma_{\min}$, $\lambda_{\min}$ and $\beta_{\min}$.*

This condition is similar to the evolutionary game theory concept of drift, studied by Binmore, Gale and Samuelson [1995], who analyze which strategies are robust to the introduction of small fraction of the population using other strategies.

In practice, we choose a $\lambda_{\min}$ to be much lower than the other minimums so that if mysticism is near its minimum it has very little affect on the asset price and y_t essentially follows the fundamental forecast[8]. Our goal is to find out whether mysticism can gain sufficient following to impact the system, causing bubble-like behavior and inducing ARCH effects in the time series data.

Another element determinig the dynamics is the specification of the weighting function $w(\pi)$. Parke and Waters [2002] shows that for a linear[9] $w(\pi)$ and bounded errors, reflectivism is the dominant strategy. For a convex weighting function, the conditions for agents to adopt the mystic forecast improve. Convexity of $\omega(\pi)$ implies that agents are seeking out the best performing strategy more aggressively. Compared to the replicator dynamic, convexity of the weighting function means the population shares change *overproportionally* with the fitness of the strategies[10]. Let the weighting function be the following exponential form $w(\pi) = e^{\sigma\pi}$ where σ parameterizes the convexity of the function. For higher σ, agents are switching to the best forecast more aggressively.

The evolution equation (14) and the payoffs (8), (9) and (10) determine equations of motion for the fractions of the population following each strategy. Using the exponential weighting function above, the second degree Taylor approximation in A_t of the equation describing the motion of β_t becomes

$$\frac{\beta_t}{\beta_{t+1}} \cong 1 - \sigma n_t (1 - n_t) A_t^2 + 2\sigma^2 n_t (1 - n_t) A_t^2 U_t^2.$$

Note that $\dfrac{\beta_t}{\beta_{t+1}} < 1$ implies that reflectivism's share is increasing. This approximation demonstrates the conditions under which β_t might decrease, giving mysticism a chance to succeed. If

[8] Parke and Waters [2001] relaxes this assumption on the minimum fraction following mysticism and provides a more nuanced analysis of the stability of the system.

[9] This case corresponds to the *replicator dynamic*, the original evolutionary game theory model from biology. Economists have developed a broader class of evolutionary models from foundations involving social interaction.

[10] Hofbauer and Weibull [1996] examine the specification of the weighting function in detail.

the second order term $2\sigma^2 n_t (1 - n_t) A_t^2 U_t^2$ is zero, then the equation corresponds to the linear weighting case and β_t is monotone increasing. However, if the second order term is large enough then β_t will fall. If the correlation between U_t and A_t favors mysticism as well, the share using mysticism λ_t will increase. Writing the approximation as

$$\frac{\beta_t}{\beta_{t+1}} \cong 1 - \sigma n_t (1 - n_t) A_t^2 [1 - 2\sigma U_t^2],$$

β_t will fall if σ and the magnitude of U_t is sufficiently large. So if agents are being aggressive and the variance of the stochastic terms is high mysticism should have a chance to gain a following, potentially drawing the price of the asset away from the fundamental forecast.

Reflectivism has a natural advantage and will have a superior payoff to the population average at all times. However, the mystic payoff can be superior if the covariance term in its payoff is sufficiently large and has the right sign. If, in addition, agents are actively seeking the best forecasting strategy, many of them may switch to mysticism when its payoff is the best.

5 Simulations

Simulations of model with exponential weighting show conditions under which mysticism can gain a following, potentially producing bubble behavior in the asset price. Econometric tests of the simulated data demonstrate that the present model with agents switching between forecasts can generate ARCH and GARCH effects in the data for returns on the asset. If the variance of the stochastic processes is sufficiently large and agents are aggressively seeking the best performing strategy, then the asset price may deviate from the fundamental forecast and returns could display ARCH properties.

Figures [1] and [2] show simulated realization for the model given by the evolution equation (14) with exponential weighting $w(\pi) = e^{\sigma\pi}$ and the payoffs (8), (10) and (9) with different variances[11] of the dividend process u_t. The graphs also show the shares of the population using each strategy across time. The fraction following mysticism λ_t is the vertical distance from zero, while the fraction following fundamentalism γ_t is the distance from one. The gap between the two represents β_t, the fraction using reflectivism.

[11] The exponential weighting function parameter is set to $\sigma = 1$ unless otherwise noted.

All simulation begin at the potentially stable point where reflectivism has its maximum number of followers. A small fraction $\varepsilon_\lambda = 0.001$ of the population using the mystic forecast is introduced but at a much lower level than the minimums for the other strategies $\varepsilon_\gamma = \varepsilon_\beta = 0.05$. The purpose of setting a low ε_λ is so that effect on the asset price realization, given by (7), is small when the fraction following mysticism is near its minimum, since n_t will also be low. Simulations in the computational finance literature often introduce new strategies regularly during the simulation. The model in this paper could be extended to include many different mystic forecasts operating simultaneously, but we focus on a single mystic forecast for clarity. If the following for one mystic forecast drops below the minimum, we allow λ_t to remain at ε_λ and set the martingale such that $m_{t-1} = 0$, representing a new mystic forecast.

Both simulations in Figures [1] and [2] have the same martingale innovation process[12], but the standard deviation of the dividend process σ_u is set to the relatively low value $\sigma_u = 0.25$ for the simulation shown in Figure 1 while $\sigma_u = 1.25$ in Figure 2. The two simulations show dramatically different behavior. In Figure 1, when the standard deviation of the dividend process is small, the population share for mysticism remains at its minimum and the asset price remains close to the fundamental forecast. However, in Figure 2, when the dividend shocks are larger, there are periods when mysticism succeeds in attracting adherents and becoming the dominant strategy at times. Stretches of time when mysticism dominates often show bubble behavior in the asset price, represented by the large deviations from zero in the fundamentalist forecast error. Such bubbles never last indefinitely. The existence of a minimum fraction following fundamentalism ensures that the reflective and mystic forecasts do not coincide and, as the martingale becomes large, the reflective forecast eventually outperforms the mystic forecast.[13]

Econometric tests confirm that the model produces time series features often associated with financial markets. ARCH and GARCH models are often used to analyze returns on assets. For

[12] Both the dividend process u_t and the martingale innovation process η_t are distributed normally for these simulations. The mean of the dividend process is 0.2 throughout. In Figures 1-2, the standard deviation of η_t, $\sigma_\eta = 0.5$. The evolution parameter $\sigma = 1$ for both simulations as well.

[13] See Parke and Water [1999] for a full discussion of this issue. In some models of rational bubbles, such as Hall, Psaradakis and Sola [1999] have, the bubble collapses with some exogenously given probability. The collapse in our model occurs endogenously, given the assumption about the minimum fraction using fundamentalism.

the present model returns are simply

$$RET_t = \frac{u_t + y_{t+1} - y_t}{y_t}. \tag{15}$$

Assuming markets are efficient implies that returns should be serially independent and fluctuate about the mean. The dividend process is serially uncorrelated and distributed normally $u_t \sim N(5, \sigma_u^2)$[14]. As explained in section 2, agents are assumed to all have the same constant conditional variance of future wealth σ_W. We take $\sigma_W = E_t\left(y_{t+1}^{*2}\right) + \sigma_u^2$ which is the highest conditional variance wealth could take if asset prices follow the fundamental solution. Of course if agents expect bubbles, σ_W could be higher, but as long as it is constant, there would be no qualitative difference in the results. More sophisticated agent could try to estimate an ARCH model every period and try to anticipate changes in volatility, but this would only increase the serial correlation in volatility. In our approach, ARCH effects arise solely from heterogeneity in the forecasting strategies.

5.1 ARCH

Observing the returns formula above (15) shows the potential for serial correlation in the squared errors. Using the expression for the realized price (7), the difference $y_{t+1} - y_t$ will contain the term $\alpha^{-t-1} m_t (n_{t+1} - n_t)$. Since the evolution equation (14) implies that the fractions of the population following different strategies are serially correlated, n_t will be as well. This correlation will be particularly pronounced when the mystic is gaining a following. The square of this term will appear in the squared errors for the return so the presence of ARCH/GARCH effect seems to be quite possible.

We test sample runs of 1000 periods for the presence of heteroscedasticity in the returns. Table 1 shows results from applying Engle's [1989] test, constructing the deviations ε_t of RET_t from its mean as follows.

$$RET_t = \overline{RET} + \varepsilon_t \tag{16}$$

The next step is to test for serial correlation of the squared deviations ε_t^2 using least square on the lags $\varepsilon_{t-1}^2, \varepsilon_{t-2}^2, \varepsilon_{t-3}^2, \ldots$. Each entry in the table shows the percentage of the sample runs, out of

[14] Other parameter values are $a = 0.25, \alpha = 0.99$ and $z_s = 1.0$.

10,000 trials, that showed significant (at the 99% level) serial correlation of the squared errors for a given lag length and for various standard deviations of the dividends and martingale innovation. The individual charts demonstrate the results for a given σ_η, and the last row gives the percent of the runs which showed significance in each of the first five lags.

A Monte Carlo analysis shows the percentage, above which the detected ARCH effects are not spurious. Performing the same test on white noise returns showed significant ARCH effect for the first lag in 0.0107 of the runs with a standard deviation of 0.0116. Invoking Chebyshev's inequality, if ARCH effects appear in more than the threshold of 0.0687 of the sample runs, they are not spurious with a 99% probability. ARCH effects at longer lags were even less frequent in the case of white noise returns.

At lower levels of σ_η, ARCH effects do not appear in more than this threshold fraction. The small innovations lead to a small A_t term in the payoffs, (8), (9) and (10), making the payoffs similar so there is little incentive to switch strategies. Furthermore, a small martingale will have relatively little impact on the asset price and returns. At higher standard deviations of the dividends, mysticism can gain a following even with small σ_η, but here the noise created by the dividends tends to drown out the impact of mysticism on returns.

At higher levels of the standard deviation of the martingale innovation, conditions for the success of mysticism and the appearance of ARCH effects improves. As the magnitude of the shocks rise, ARCH effects first occur when $\sigma_\eta = 0.5$ and $\sigma_u = 1.0$, where $0.1180 > 0.0687$ of the runs had significant ARCH effects at the first lag. In addition, ARCH effects in this case appeared at all tested lags.

As σ_η rises further, ARCH effects become increasingly common. Larger shocks to the martingale innovation increase the magnitude of the martingale in A_t as well as U_t opening the possibility that the covariance terms in the payoffs to mysticism and fundamentalism, (9) and (10), will play an important role. The magnitude of the dividends remains important, however. At low σ_u, such as $\sigma_u = 0.5$, significant ARCH effects do not appear for any σ_η, but for $\sigma_u = 1.0$ they are present at all $\sigma_\eta \geq 0.5$. Larger shocks to u_t raises the magnitude of U_t in the payoffs helping to make the covariance term bigger, which can help mysticism, bigger, in comparison to the martingale term, which worsens the mystic payoff.

However, for very large dividend shocks, ARCH effects are diminished. For example, when

$\sigma_\eta = 1.0$, there are significant levels of ARCH for $\sigma_u = 1.0$ and 2.0 but not for $\sigma_u = 4.0$, a pattern which persists at higher levels of σ_η. Again, when dividend shocks are large, they can be the dominant factor determining returns, and, since dividends are uncorrelated, returns are less likely to exhibit serial correlation of any kind.

These tests clearly show that heterogeneity of forecasting strategies with agents choosing these strategies endogenously can produce ARCH effects in asset returns, as is commonly seen in the financial markets data. The combination of sufficiently large shocks to the martingale innovation and moderate dividend shocks creates condition where mysticism can gain a following, bubbles can form and serial correlation can appear in the squared errors of asset returns.

5.2 GARCH

A natural next step is to examine the simulated data that showed ARCH effects to see whether it is well represented by a GARCH model. GARCH (Generalized ARCH) models, introduced by Bollerslev [1986], are commonly used to examine financial markets data and offers a useful extension of the ARCH approach. GARCH models reduce the number of parameters estimated and allow for serial correlation in both the squared errors and the conditional variance of the endogenous variable, which separates short and long term variability in volatility.

We examine the simulated data using a GARCH(1,1) model that is often used with financial markets data[15]. This models the conditional variance of the errors $E_{t-1}\left(\varepsilon_t^2\right) = \sigma_{R,t}^2$ such that

$$\sigma_{R,t}^2 = \kappa + \varphi\sigma_{R,t-1}^2 + \psi\varepsilon_{t-1}^2$$

so it depends on the previous periods conditional variance and squared errors. The advantage of this specification is its parsimony, multiple lags of ε_t^2 are not required as in the ARCH test, and the separation of the effects of the long term conditional variance $\sigma_{R,t-1}^2$ and the short term squared errors ε_{t-1}^2. Of course it is possible to include further lags of either variable, but, as Engle [2001] notes, the GARCH(1,1) with one lag of each, has proved sufficient for most financial markets data.

Table 2 reports the results of estimates of the GARCH(1,1) model of simulated data from the model described in the previous subsection. Each chart in the table gives results for a given

[15] Bollerslev [1992] surveys ARCH modeling in finance.

standard deviation of the martingale innovation while each row summarizes the results from 1000 sample runs of 1000 periods for a given standard deviation of the dividend process. The first column give σ_u and second column (*garch signif*) reports the percent of the runs where φ is significantly greater than zero[16]. The following column (*arch signif*) shows the percent of the runs where ψ is significantly positive, among the sample runs where φ is significant. The next two columns (*garch est* and *std garch est*) show the average estimate of ϕ, among those that are significantly positive, and the standard deviation of the estimates, to give a sense of the variability of the parameter. Similarly, the next two columns (*arch est* and *std arch est*) report the mean and standard deviation of the estimated ψ's, from the sample runs where both ψ and ϕ are significantly positive. Finally, we conduct a Ljung-Box Q test to see if the remaining residuals lacked serial correlation. The final column (*diag*) reports the percentage of the sample runs which showed no serial correlation of the samples where ϕ was significantly positive, passing the diagnostic test.

The table shows that many of the sample runs that showed ARCH effects are well represented econometrically by a GARCH(1,1) model. In all cases, the percentage showing significant GARCH effects (the estimate of $\varphi > 0$) is similar to the percentage that showed significant ARCH effects at a single lag. Furthermore, the percent passing the diagnostic check for a lack of serial correlation in the residuals is very high, always over 70% and over 90% in more than half the cases. Examination of individual samples shows that some runs are better modeled by including a moving average or autoregressive term or both in the returns equation (16), which is common with high frequency data (see Enders [2004], p. 145). Combining the information in these columns (*garch signif* and *diag*) gives a good estimate of the percentage of runs well represented by the GARCH(1,1) specification. For example, for $\sigma_\eta = \sigma_u = 1.0$, the fraction showing significant GARCH effects is 0.459, and the fraction of these passing the diagnostic test is 0.8998 so the product 0.413 is the faction with a good fit to the model. The estimates of φ tend to be large, over 0.5, for most of the sample runs but estimates of the ARCH parameter ψ are often very small and/or insignificant. Such features are also common to financial markets data (Engle [2001]).

The results of the GARCH estimations show similar pattern to the table concerning the ARCH tests. Higher levels of the standard deviation of the martingale innovation tend to show a greater

[16] To be counted, the returns also are required to show significant ARCH effects for one lag and the estimate of ψ must be positive.

frequency of significant GARCH effects. This observation is true for the standard deviation of the dividend process, but as the level of σ_u rises, GARCH effects are less likely to appear. At high levels of σ_η, when GARCH effect are common the adoption and subsequent rejection of mysticism happens quite quickly. Mysticism can gain a large following but it usually last for less than 10 periods before it is rejected. Outbreaks of mysticism that last longer, as shown in Figure 2, which tend to occur for more moderate σ_η may or may not show GARCH effects. A bubble that forms slowly may show a small change in the volatility of the returns. These observations suggest that ARCH and GARCH effects are particularly prevalent when agents adopt a forecast that is extremely divergent from fundamentals, but subsequently reject it quickly.

The result in Table 2 demonstrate that the present model, where agents have the option of adopting a successful bubble forecast, can produce ARCH and GARCH effects that are very similar to those found in financial markets. For large shocks in the martingale innovation and moderate dividend shocks, a very significant fraction of the sample simulations are well modeled by a GARCH(1,1) process with parameter values that closely resemble those found in studies of financial markets data.

For some choices of parameters, GARCH effects are quite common. For $\sigma_\eta = 2.0$ and $\sigma_u = 1.0$, over two-thirds of the runs showed significant GARCH effects and passed the diagnostic test. An example of one of these runs is shown in Figure 3. Volatility clustering in the returns is quite evident and occurs around short outbursts of mysticism. Engle [2001] suggests that clusters of large shocks must be the result of news, and we can interpret our simulations as agents temporarily responding to a new variable but quickly discarding it as irrelevant. Any news, whether is matters to future dividends or not, can have an impact if enough agents think it is important, even for only a short period of time.

5.3 Agent Aggression

For mysticism to have the opportunity to gain a following, agents must be sufficiently aggressive in switching to the best performing forecast. Such behavior is represented by the parameter σ in the exponential weighting function. Higher σ has the effect of placing greater weight on better forecasts[17], meaning that a greater fraction will change their forecasting strategy to the better forecasts, increasing the possibility that mysticism could draw so adherents from reflectivism. Therefore, an increase in σ raises the probability that ARCH and GARCH effects will be detected in the simulated data, up to a point.

Table 3 shows very few sample runs[18] with significant GARCH effects for $\sigma = 0.5$, but for $\sigma = 2$ and 4 over one third of the runs showed significant GARCH effects and passed the Q-test on the residuals. Starting from low levels of σ, when agents are more aggressive, GARCH effects are more prevalent. However, for very high level of $\sigma = 8$, for example, the frequency of GARCH effects falls. Examination of some samples[19] shows that mysticism has not been driven out. On the contrary, the fractions following mysticism and fundamentalism fluctuate wildly, creating noisy realizations for the asset price and returns, making conditional volatility effects more difficult to detect.

6 Conclusion

ARCH / GARCH models have proved to be extremely successful econometric techniques for detecting conditional volatility, particularly for returns in financial markets. This paper shows a formal model explaining how such effects arise endogenously through agents choices of heterogeneous forecasting strategies. Agent switching to forecasts based on martingale solutions can produce bubble behavior in the asset price and GARCH effects in the returns.

Agents know all the parameters of the model and only use strategies satisfying rational expectations. Their choice of forecast depends on the previous performance of the different strategies based on forecast errors. An evolutionary game theory mechanism describes how the fractions of the population using the different strategies change over time and allows comparisons of the dynamics depending on how quickly agents change strategies.

The reflectivist forecast, which uses the mathematical expectation of the asset price, has an intrinsic advantage over the other strategies, but mysticism, which uses a forecast based on a

[17] The role of σ here is similar to the *search intensity* parameter in the multinomial logit approach used by Brock, Hommes and Wagener [2001]. See Parke and Waters [1999] for a discussion of the relative merits of the two approaches.

[18] For these runs $\sigma_\eta = 0.5$ and $\sigma_u = 1.0$

[19] See Waters [2002] for numerous examples.

martingale solution, can gain a temporary following under certain conditions. If the magnitude of the shocks to dividends and the martingale innovation are large enough and agents are sufficiently aggressive about switching to the best strategy, then mysticism can gain a following, the asset price may be led away from the fundamental forecast and ARCH and GARCH effect could appear in the returns. The simulation results reported here conclusively demonstrate the presence of these effects in the model with endogenous switching between forecasting strategies.

As Engle [2001] notes, the source of conditional volatility must be news of some kind. Our results suggest that the new information need not be relevant in any fundamental sense. If enough investor beliive that a piece of information is important, it will be. The process of experimenting with and rejecting sources of information is a key factor in the appearance of ARCH effects. The evolution of heterogeneous beliefs is key to understanding the behavior of financial markets.

References

Binmore, K., J. Gale, and L. Samuelson (1995): "Learning to be Imperfect: The Ultimatum Game," *Games and Economic Behavior*, 8, 56–90.

Blume, L., and D. Easley (1992): "Evolution and Market Behavior," *Journal of Economic Theory*, 58, 9–40.

Bollerslev, T. (1986): "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31(3), 307–27.

——— (2001): "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 100(1), 131–51.

Brock, W. A., and C. H. Hommes (1997): "A Rational Route to Randomness," *Econometrica*, 65, 1059–1095.

——— (1998): "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model," *Journal of Economic Dynamics and Control*, 22, 1235–1274.

Brock, W. A., C. H. Hommes, and F. O. Wagener (2001): "Evolutionary Dynamics in Financial Markets with Many Trader Types," *working paper, Center for Nonlinear Dynamics in Economics and Finance, University of Amsterdam.*

Chiarella, C., and X.-Z. He (2002): "Heterogeneous Beliefs, Risk and Learning in a Simple Asset Pricing Model," *Computational Economics*, 19, 95–132.

Cochrane, J. H. (2001): *Asset Pricing*. Princeton University Press, Princeton, NJ.

De-Grauwe, P. (1993): *Exchange Rate Theory: Chaotic Models of Foreign Exchange Markets.* Blackwell, Cambridge, MA.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann (1990): "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98, 703–738.

den Haan, W. J., and S. A. Spear (1998): "Volatility Clustering in Real Interest Rates: Theory and Evidence," *Journal of Monetary Economics*, 41(3), 431–53.

Diebold, F. X., and M. L. Nerlove (1989): "The Dynamics of Exchange Rate Volatility: A Multivariate Latent Factor ARCH Model," *Journal of Applied Econometrics*, 4(1), 1–21.

Engle, R. (2001): "The Use of ARCH/GARCH Models in Applied Econometrics," *Journal of Economic Perspectives*, 15(4), 157–168.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, N.J.

Hall, S. G., Z. Psaradakis, and M. Sola (1999): "Detecting Periodically Collapsing Bubbles: A Markov-Switching Unit Root Test," *Journal of Applied Econometrics*, 14(2), 143–54.

Hofbauer, J., and J. W. Weibull (1996): "Evolutionary Selection against Dominated Strategies," *Journal of Economic Theory*, 71, 558–573.

Howitt, P., and R. P. McAfee (1992): "Animal Spirits," *American Economic Review*, 82, 493–507.

Keynes, J. M. (1935): *The General Theory of Employment, Interest and Money*. Harcourt Brace, New York.

LeBaron, B., W. B. Arthur, and R. Palmer (1999): "Time Series Properties of an Artificial Stock Market," *Journal of Economic Dynamics and Control*, 23, 1487–1516.

Marcet, A., and T. J. Sargent (1989a): "Convergence of Least-Squares Learning in Environmen with Hidden State Variables and Private Information," *Journal of Political Economy*, 97, 1306–1322.

——— (1989b): "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

McCallum, B. T. (1983): "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, 11, 139–168.

——— (1997): "The Role of the Minimum State Variables Criterion in Rational Expectations Models," *International Journal of Tax and Finance*, 6(4), 621–639.

Pagan, A. (1996): "The Econometrics of Financial Markets," *Journal of Empirical Finance*, 3, 15–102.

Parke, W. R., and G. A. Waters (1999): "Mysticism, Fundamentalism and Rational Expectations," *manuscript, University of North Carolina - Chapel Hill*.

——— (2002): "An Evolutionary Game Theory Approach to Rational Expectations," *manuscript, University of North Carolina - Chapel Hill*.

Robert Engle, D. L., and R. Robins (1987): "Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Model," *Econometrica*, 55(2), 391–47.

Samuelson, L. (1998): *Evolutionary Games and Equilibrium Selection*. MIT Press, Cambridge, MA.

Weibull, J. W. (1997): *Evolutionary Game Theory*. MIT Press, Cambridge, MA.

Woodford, M. (1990): "Learning to Believe in Sunspots," *Econometrica*, 58, 277–307.

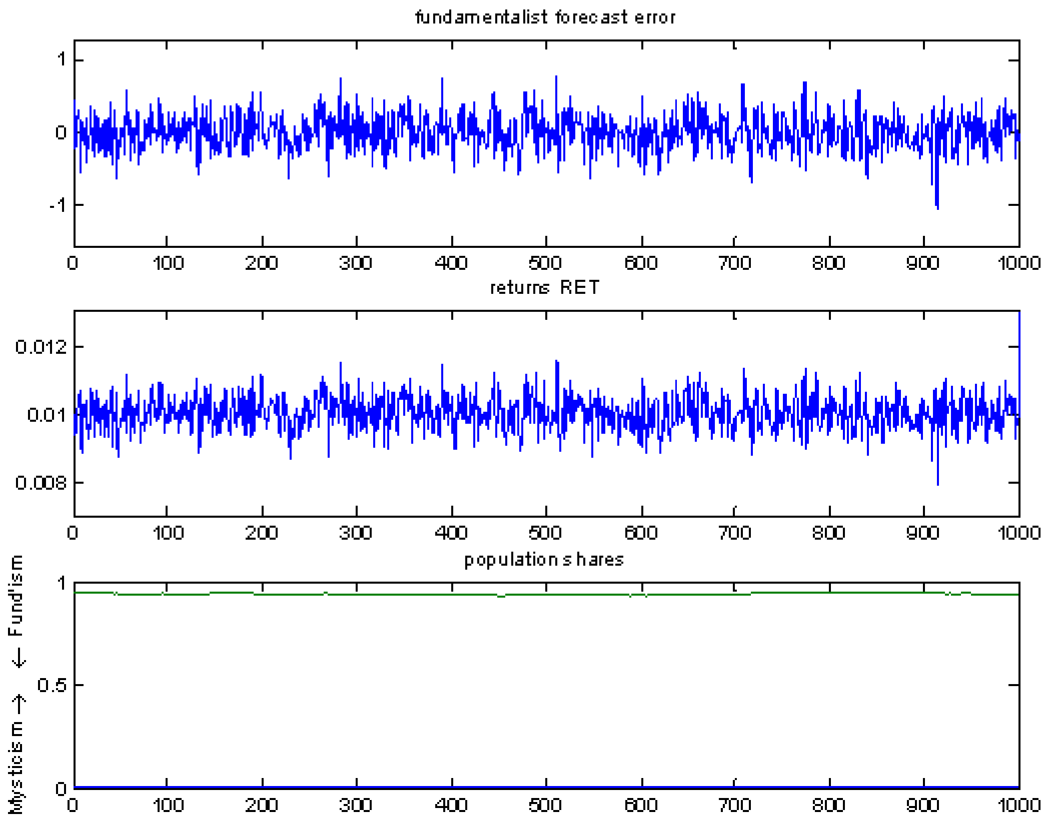

Figure 1: $\sigma_u = 0.25$, $\sigma_\eta = 0.5$

	Lag / stdevu	0.25	0.5	1	2	4	8
	1	0.0106	0.0112	0.0416	0.0115	0.0098	0.0084
	2	0.0103	0.0098	0.0437	0.0136	0.0094	0.0085
stdeveta = 0.25	3	0.0118	0.0103	0.0450	0.0115	0.0104	0.0090
	4	0.0107	0.0110	0.0432	0.0125	0.0092	0.0081
	5	0.0102	0.0118	0.0429	0.0141	0.0091	0.0099
	10	0.0106	0.0109	0.0424	0.0143	0.0088	0.0094
	15	0.0104	0.0090	0.0395	0.0138	0.0087	0.0099
	20	0.0086	0.0084	0.0376	0.0122	0.0087	0.0082
	sig 1-5	0.0014	0.0019	0.0199	0.0028	0.0017	0.0010

	Lag / stdevu	0.25	0.5	1	2	4	8
	1	0.0092	0.0089	0.1180	0.0457	0.0081	0.0096
	2	0.0118	0.0107	0.1310	0.0461	0.0117	0.0088
stdeveta = 0.5	3	0.0114	0.0104	0.1356	0.0445	0.0112	0.0079
	4	0.0113	0.0103	0.1388	0.0441	0.0092	0.0092
	5	0.0110	0.0099	0.1388	0.0444	0.0091	0.0110
	10	0.0107	0.0098	0.1353	0.0373	0.0101	0.0131
	15	0.0102	0.0098	0.1312	0.0326	0.0099	0.0110
	20	0.0098	0.0091	0.1243	0.0282	0.0088	0.0097
	sig 1-5	0.0016	0.0018	0.0756	0.0181	0.0012	0.0011

	Lag / stdevu	0.25	0.5	1	2	4	8
	1	0.0078	0.0124	0.2776	0.1112	0.0122	0.0093
	2	0.0138	0.0143	0.3151	0.1113	0.0110	0.0114
stdeveta = 1	3	0.0140	0.0135	0.3325	0.1100	0.0097	0.0109
	4	0.0136	0.0144	0.3426	0.1094	0.0103	0.0110
	5	0.0131	0.0166	0.3482	0.1049	0.0104	0.0120
	10	0.0132	0.0161	0.3589	0.0952	0.0110	0.0112
	15	0.0120	0.0137	0.3533	0.0827	0.0092	0.0094
	20	0.0114	0.0132	0.3452	0.0743	0.0083	0.0087
	sig 1-5	0.0013	0.0034	0.2108	0.0518	0.0016	0.0019

	lag / stdevu	0.25	0.5	1	2	4	8
	1	0.0044	0.0071	0.4729	0.1933	0.0554	0.0095
	2	0.0250	0.0138	0.5445	0.2147	0.0615	0.0105

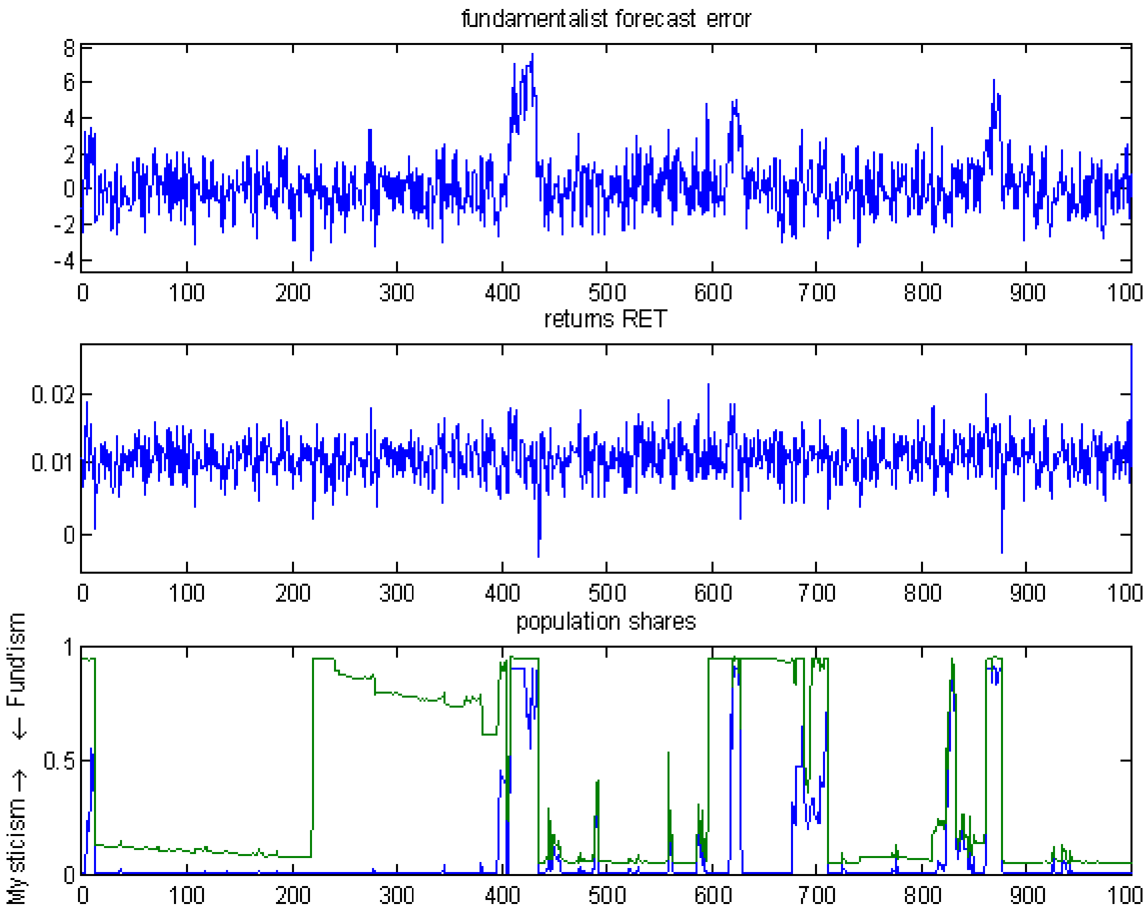

Figure 2: $\sigma_u = 1.25$, $\sigma_\eta = 1.25$

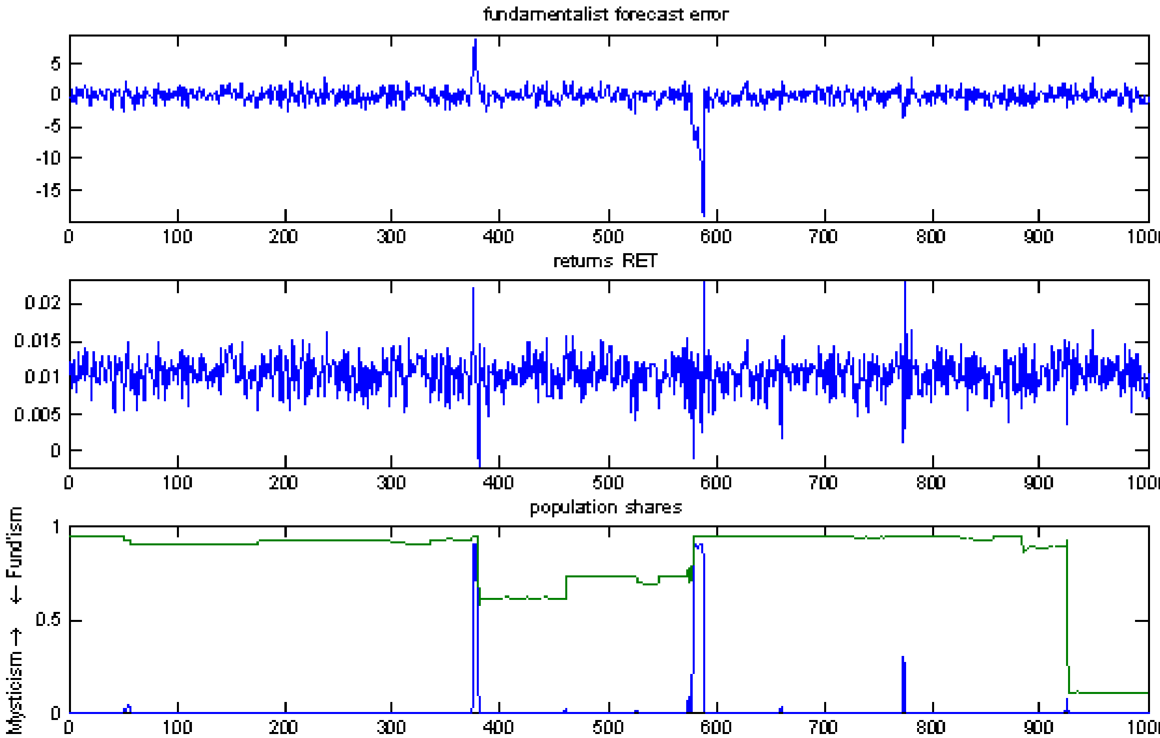

Figure 3: $\sigma_u = 1.0$, $\sigma_\eta = 2.0$

Table 1: Fractions With Significant ARCH Effects

stdev eta = 0.25

stdev u	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.25	0.0300	0.3667	0.8838	0.0074	0.0112	0.0085	0.9333
0.5	0.0240	0.6667	0.8759	0.0114	0.0201	0.0131	0.9167
1	0.0610	0.7377	0.8605	0.0581	0.0294	0.0226	0.9016
2	0.0100	0.6000	0.7921	0.1999	0.0338	0.0319	1.0000
4	0.0110	1.0000	0.7202	0.1708	0.0573	0.0289	0.9091

stdev eta = 0.5

stdev u	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.25	0.0360	0.5556	0.8786	0.0106	0.0170	0.0120	0.9722
0.5	0.0250	0.5600	0.8786	0.0134	0.0172	0.0141	0.8000
1	0.2100	0.8714	0.8543	0.0543	0.0369	0.0234	0.9286
2	0.0450	0.5556	0.8133	0.1248	0.0303	0.0253	0.9556
4	0.0070	1.0000	0.7171	0.1696	0.0435	0.0162	1.0000

stdev eta = 1.0

stdev u	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.25	0.1140	0.8246	0.8748	0.0150	0.0225	0.0147	0.9211
0.5	0.0360	0.6667	0.8791	0.0103	0.0178	0.0112	0.9444
1	0.4790	0.9708	0.7938	0.1424	0.0670	0.0515	0.9144
2	0.2680	0.7239	0.7931	0.1653	0.0419	0.0406	0.8022
4	0.0050	1.0000	0.6627	0.1879	0.0673	0.0286	1.0000

stdev eta = 2.0

stdev u	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.25	0.3990	0.9424	0.8650	0.0251	0.0316	0.0224	0.9699
0.5	0.1290	0.7829	0.8746	0.0193	0.0209	0.0144	0.9845
1	0.7270	0.9904	0.6363	0.1951	0.1337	0.0796	0.9271
2	0.2410	0.9959	0.2862	0.1725	0.2863	0.0863	0.7137
4	0.0410	1.0000	0.6129	0.2056	0.0793	0.0384	0.8537

stdev eta = 4.0

stdev u	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.25	0.5990	0.9716	0.8280	0.0966	0.0504	0.0507	0.9683
0.5	0.3640	0.8764	0.8319	0.1224	0.0320	0.0379	0.9698
1	0.6840	0.9927	0.4593	0.2089	0.2087	0.1059	0.9444
2	0.6400	1.0000	0.1478	0.0856	0.6524	0.1167	0.8453
4	0.1980	1.0000	0.2450	0.1609	0.2888	0.0903	0.6768

Table 2: GARCH estimation results

sigma	garch signif	arch signif	avg garch est	stdev garch est	avg arch est	stdev arch est	diag
0.5	0.0290	0.3793	0.8782	0.0156	0.0176	0.0132	0.9655
1	0.1830	0.8251	0.8295	0.1152	0.0432	0.0366	0.9180
2	0.4390	0.9134	0.8370	0.1080	0.0417	0.0301	0.8360
4	0.4370	0.5973	0.8419	0.1055	0.0391	0.0315	0.8032
8	0.2200	0.4136	0.8443	0.1162	0.0319	0.0256	0.8864
16	0.1528	0.4318	0.8458	0.1170	0.0297	0.0216	0.9091
32	0.1340	0.3134	0.8602	0.0636	0.0249	0.0141	0.8806
64	0.0970	0.2784	0.8646	0.0651	0.0232	0.0145	0.9072

Table 3: GARCH estimation for varying σ's