

Debt stabilizing fiscal rules

Philippe Michel*, Leopold von Thadden[†], Jean-Pierre Vidal[‡]

May 27, 2005

Abstract

Unstable government debt dynamics can typically be corrected by various fiscal instruments, like appropriate adjustments in government spending, public transfers, or taxes. This paper investigates properties of state-contingent debt targeting rules which link stabilizing budgetary adjustments around a target level of long-run debt to the state of the economy. The paper establishes that the size of steady-state debt is a key determinant of whether it is possible to find a rule of this type which can be implemented under all available fiscal instruments. Specifically, considering linear feedback rules, the paper demonstrates that there may well exist a critical level of debt beyond which this is no longer possible. From an applied perspective, this finding is of particular relevance in the context of a monetary union with decentralized fiscal policies. Depending on the level of long-run debt, there might be a conflict between a common fiscal framework which tracks deficit developments as a function of the state of the economy and the unrestricted choice of fiscal policy instruments at the national level.

Keywords: Fiscal regimes, Overlapping generations

JEL classification numbers: E63, H62.

*This paper was started during Philippe Michel's visit at the ECB in June 2004. To our deep regret, he passed away before the paper was completed.

[†]*Corresponding author:* European Central Bank, Kaiserstrasse 29, D-60311 Frankfurt/Main, Germany. e-mail: leopold.von_thadden@ecb.int.

[‡]European Central Bank, Kaiserstrasse 29, D-60311 Frankfurt/Main, Germany. e-mail: jean-pierre.vidal@ecb.int. The views expressed in this paper do not necessarily reflect those of the European Central Bank.

1 Introduction

Unstable government debt dynamics can typically be corrected by appropriate budgetary adjustments. To achieve the needed corrections a government can normally adjust a broad range of fiscal instruments, like government spending, public transfers, or various taxes. Given the multiplicity of fiscal instruments, this paper builds on the idea that there are two different ways to conceptualize state-contingent fiscal rules which stabilize government debt dynamics around a certain long-run level of debt. First, for a particular instrument one can think of rules which link the use of this instrument to the state of the economy. Keeping the other instruments constant, any such rule then implies a certain sequence of budgetary adjustments. Alternatively, one can think of rules which, following the reverse logic, leave the choice of the instrument a priori open and specify directly a state-contingent path of budgetary adjustments. This second reasoning leads then to the question under which of the available instruments a particular specification of such a rule can be implemented.

These two approaches, while algebraically being closely related, offer different insights. This paper argues that the second approach is particularly relevant for the design of fiscal rules in a monetary union with decentralized fiscal policies. Specifically, the logic of the second approach can be used to see that, depending on the target level of government debt, it may not be possible to find a state-contingent prescription of stabilizing budgetary adjustments that can be implemented under all available instruments. In other words, the paper argues that, depending on the target level of government debt, conflicts may arise between a common fiscal framework which tracks deficit developments as a function of the state of the economy and the unrestricted choice of fiscal policy instruments at the national level.

To make this reasoning precise, this paper considers a small and fully tractable model which distinguishes between three distinct fiscal instruments in the government's flow budget constraint, namely government consumption, transfer payments and a wage income tax. The analysis is based on a deterministic overlapping generations economy with government debt dynamics in the spirit of Diamond (1965). To operationalize the notion of unstable government debt dynamics, the paper identifies steady states which are characterized by non-negative levels of government debt and which are locally unstable under the assumption of a permanently balanced primary budget. However, the economy can be stabilized at the corresponding steady-state levels if one allows for appropriate budgetary adjustments. Such adjustments can be brought about by any of the three instruments. For the sake of simple tractability, the paper considers for each instrument a rule which sets the instrument as a linear function of the two state variables of the model, physical capital and real government bonds. For given feedback coefficients associated with the two states of the economy, any such rule, when combined with the linearized flow budget constraint

of the government, generates a ‘debt targeting rule’ which specifies a particular linear reaction of the primary balance to the two states of the economy. In sum, this experiment leads to a broad class of debt targeting rules, defined over the set of admissible feedback coefficients.

In line with the motivation of the opening paragraph, this class of debt targeting rules can be investigated in two directions. First, we derive for each instrument the range of *instrument-specific* feedback coefficients which stabilize government debt at the target value. When considered in isolation such ranges are shown to exist for all instruments. Second, we follow the reverse logic and establish whether there exists a debt targeting rule which can be implemented under all instruments with *common* feedback coefficients. This amounts to check whether the ranges of stabilizing feedback coefficients induced by the three instruments do overlap. The paper shows that the answer to this question is not trivial and depends on the level of long-run debt around which the economy is stabilized. Specifically, the key theoretical results of the paper state that at a zero level of steady-state debt there always exists a debt targeting rule which can be implemented under all three instruments with common feedback coefficients. As the level of steady-state debt rises, however, the instrument-specific adjustment paths become increasingly diverse. As we show, this feature implies that there may well exist a critical long-run level of debt beyond which there exists no longer a debt targeting rule that is implementable under all three instruments with common feedback coefficients.

The novelty of these results can be assessed from different angles. First, the results relate to the literature on fiscal closure rules, as typically used in large scale macroeconomic models. In this literature it is widely understood that different instruments, when residually used to enforce the intertemporal budget constraint of the government, lead to different dynamic outcomes which preclude simple comparisons across simulations, as discussed in Bryant and Zhang (1996), Mitchell et al. (2000), and Pérez and Hiebert (2002). Yet, by construction, this literature offers few explicit analytical findings and we are not aware of a systematic discussion of the role of government debt in this context.

Turning to tractable small scale models, stability features of Diamond-models have been discussed in a number of studies, but typically not with the intention to compare between the stabilization properties of different fiscal instruments.¹ Closer to

¹For a detailed discussion of dynamic equilibria in Diamond-models with production, but without government debt, see Galor and Ryder (1989). For comprehensive surveys of the dynamics with debt and with zero (and, more generally, constant) primary deficits, see Azariadis (1993), and de la Croix and Michel (2002). Special features of constant deficit rules are discussed by Farmer (1986), with a focus on cyclical adjustment patterns, and Chalk (2000), with a focus on sustainability issues. For a comparison of adjustment dynamics under a balanced primary budget and a time-varying unbalanced budget, stressing labour market aspects, see Kaas and von Thadden (2003). For further discussions of fiscal rules in overlapping generations models, see Marin (2002),

the spirit of this paper, Schmitt-Grohé and Uribe (1997), Guo and Harrison (2004), and Giannitsarou (2004), all considering Ramsey economies with infinitely lived agents, show that for a given fiscal rule (in their context: a balanced-budget rule) equilibria can be locally unique or indeterminate, depending on whether budget balance is achieved by distortionary income taxes, consumption taxes or government spending adjustments. Our paper shares with these papers the descriptive nature of the fiscal rule, but our focus is not on balanced budget dynamics and the role of debt is substantially different.²

Implementation issues of fiscal policy have also been addressed in a large number of papers which explicitly solve for optimal fiscal (and monetary) policies from the perspective of Ramsey economies.³ In such modelling environments, however, there is little conceptual agreement about the optimal target level of long-run debt itself.⁴ Reflecting this feature, recent studies by Kollmann (2004), Lambertini (2004) and Schmitt-Grohé and Uribe (2004), for example, consider only a restricted set of optimal policies, in the sense that the long-run level of debt around which the optimization takes place is pre-specified. By contrast, the long-run target levels of government debt analyzed in our overlapping generations structure have a simple normative foundation because of dynamic efficiency considerations.⁵

Why may it not always be possible in our set-up to find a state-contingent prescription of stabilizing budgetary adjustments that can be implemented under all available instruments? Intuitively, this finding reflects that the model economy consists of two parts: the budget constraint of the government and a block which summarizes all the remaining private sector activities in the economy. By construction, the source of instability is confined to the first part, while the instruments which can be used to achieve the required budgetary adjustments affect the second part

Annicchiarico and Giammarioli (2004) and Fernandez Huertas-Moraga and Vidal (2004).

²Related to this literature, see also the dynamic analysis of tax changes in Judd (1987), Turnovsky (1990) and Mankiw and Weinzierl (2004). However, these papers do not explicitly focus on the stabilization properties of different fiscal instruments, but rather compare between short- and long-run features of equilibria which are characterized by different tax structures.

³This literature goes back to Lucas and Stokey (1983). For recent authoritative treatments, see, in particular, Chari and Kehoe (1999) and Benigno and Woodford (2003).

⁴In the framework of Aiyagari et al. (2002) the optimal long-run level of government debt is shown to be negative because of the non-distortionary nature of the interest income that a government receives in such a constellation. Benefits of positive government debt, like the loosening of private sector borrowing constraints because of an enhanced liquidity position, are discussed in Aiyagari and McGrattan (1998). Costs and benefits of long-run debt levels are also discussed in Martin (2004), with a focus on time consistency issues in the presence of non-indexed government debt. Similarly, see Díaz-Giménez et al. (2004).

⁵We do not optimize over the feedback coefficients in the debt targeting rule. The studies by Schmitt-Grohé and Uribe as well as by Kollmann show that simple feedback rules may well have welfare properties similar to those one obtains from fully optimizing programs. Whether a similar claim can be made in this context for overlapping generations economies needs to be investigated.

through different margins. The level of steady-state debt determines the relative importance of these margins within the set of intertemporal equilibrium conditions. At a zero level of steady-state debt, these margins carry zero weights, ensuring thereby that there always exists a debt targeting rule which can be implemented under all three instruments. For positive and rising levels of steady-state debt, however, these margins gain importance, implying that for any particular debt targeting rule the instrument-specific stabilization profiles become increasingly distinct. Exploiting this feature, we show that there may well exist a critical debt level beyond which it is no longer possible to find a debt targeting rule that can be implemented under all three instruments.

We think that one particularly interesting application of our results is given by monetary unions in which member states remain responsible for their national budgetary policies, subject to the provisions of a common fiscal framework that are needed to keep free-riding incentives at the national level in check. The European Monetary Union is a good example for this since the Treaty and the Stability and Growth Pact constitute a rule-based fiscal framework that sets certain limits to deficits and debt levels and strengthens multilateral budgetary surveillance.⁶ Moreover, whenever corrective fiscal policy measures are needed the framework respects, in principle, national preferences with respect to the implementation of such measures, in line with the subsidiarity principle.⁷ Evidently, the broad modelling assumptions maintained in this paper cannot fully capture further institutional details which characterize this particular arrangement. Yet, the analysis of this paper clearly indicates that a sufficiently low level of average debt facilitates the smooth functioning of any carefully balanced arrangement of this type.

The paper is organized as follows. As a particularly tractable starting point, Section 2 presents a Diamond-type overlapping generations model with an exogenous labour supply, enriched with a government sector and public debt. The model allows for three fiscal instruments, namely government consumption, lump-sum taxes levied on young agents and lump-sum transfers to old agents. Section 3 introduces the notion of the debt targeting rule and derives the main results of the paper. Section 4 establishes the robustness of the main results of Section 3 along two dimensions. First, we show that our results remain unaffected if the debt targeting rule is no longer expressed in terms of state-contingent adjustments of the primary balance, but imposes instead state-contingent restrictions on the path of the overall deficit or, alternatively, on the path of newly emitted debt. Second, at the expense of

⁶For arguments in favour of a rule-based framework in a monetary union and for further references on this widely studied topic, see Chari and Kehoe (1998), Fatás et al. (2003), and Uhlig (2002).

⁷According to the subsidiarity principle, as laid down in Article 5 of the Treaty (and Article I-9 of the draft Constitution), the Union shall act only if and insofar as the objectives of the intended action cannot be sufficiently achieved by the Member States.

more tedious algebra, we allow for an endogenous labour supply and distortionary income taxation. This modification, while mitigating some of the effects, does not change qualitatively any of our results. Section 5 offers conclusions. Proofs and some technical issues are delegated to two Appendices at the end of the paper.

2 The model with exogenous labour supply

For simple tractability the first part of the paper is based on a version of a Diamond-type overlapping generations economy with exogenous labour supply and lump-sum taxes and transfers.

Problem of the representative agent

In period t , the economy is populated by a large number N_t of young agents and N_{t-1} of old agents. Each agent lives for two period and has a time-invariant, fixed labour supply $l = 1$ when being young and a zero labour supply when being old. The population grows at the constant rate $n > 0$, i.e. $N_t = (1 + n) \cdot N_{t-1}$. Let preferences of the representative agent born in period t be given by

$$U(c_t, d_{t+1}),$$

where c_t and d_{t+1} denote first-period and second-period consumption, respectively.

(A 1) The function $U(c, d)$ is twice continuously differentiable, strictly increasing, strictly quasi-concave and satisfies for all $\bar{c}, \bar{d} > 0$, $\lim_{c \rightarrow 0} U_c(c, \bar{d}) \rightarrow \infty$ and $\lim_{d \rightarrow 0} U_d(\bar{c}, d) \rightarrow \infty$.

In any period t , agents take the wage rate (w_t) and the return factor R_{t+1} on savings (s_t) as given. There exists a tax-transfer-system such that young agents pay lump-sum taxes $\eta_t > 0$, while they receive lump-sum transfers θ_{t+1} when being old.⁸ This leads to the pair of budget constraints

$$\begin{aligned} w_t - \eta_t &= c_t + s_t \\ d_{t+1} &= R_{t+1}s_t + \theta_{t+1}, \end{aligned}$$

which can be used to rewrite the objective as:

$$U(w_t - \eta_t - s_t, R_{t+1}s_t + \theta_{t+1}).$$

The optimal choice of savings is characterized by the first-order condition

$$U_1 = R_{t+1}U_2.$$

⁸We do not make any explicit sign restrictions regarding the second-period lump-sum payment θ_{t+1} . Strictly speaking, the term ‘tax-transfer’-system would refer only to a scenario with $\theta_{t+1} > 0$.

To characterize the savings decision of agents we refer to the well-investigated Diamond model without second-period transfers (i.e. $\theta_{t+1} = 0$) and assume $w - \eta > 0$. Then, according to (A 1), there exists the savings function

$$s(w - \eta, R) = \arg \max U(w - \eta - s, Rs), \quad (1)$$

with $s(w - \eta, R) : R_{++} \times R_{++} \rightarrow R_{++}$ being continuously differentiable. In order to extend (1) to a situation with $\theta_{t+1} \neq 0$ it is assumed that the present value of the income of agents is positive, i.e. $w - \eta + \frac{\theta}{R} > 0$. Then, savings will be given by

$$s_t = s\left(w_t - \eta_t + \frac{\theta_{t+1}}{R_{t+1}}, R_{t+1}\right) - \frac{\theta_{t+1}}{R_{t+1}},$$

and s_t satisfies $-\frac{\theta_{t+1}}{R_{t+1}} < s_t < w_t - \eta_t$, ensuring non-negative consumption in both periods.⁹ Finally, to impose further structure on the function $s(w, R)$, we make the customary assumption:

(A 2) Consider $U(c, d)$ and assume that consumption goods are normal and gross substitutes. Then, $0 < s_w < 1$ and $s_R \geq 0$.

Production

It is assumed that there exists a larger number of competitive firms with access to a standard neoclassical technology $F(K_t, L_t)$, where K and L denote the aggregate levels of physical capital and labour, respectively.

(A 3) The function $F(K, L) : R_{++} \times R_{++} \rightarrow R_{++}$ is positive valued, twice continuously differentiable, homogenous of degree 1, increasing and satisfies $F_{KK}(K, L) < 0$.

Firms are price takers in input and output markets. In a competitive equilibrium, labour market clearing requires $L_t = N_t$. Let $k_t = K_t/N_t$ denote the capital stock per young agent, giving rise to the familiar pair of first-order conditions

$$R_t = 1 - \delta + F_K(k_t, 1) = R(k_t) \quad (2)$$

$$w_t = F_L(k_t, 1) = w(k_t), \quad (3)$$

with δ denoting the depreciation rate on capital. According to (2) and (3), the equilibrium return rates of the two production factors depend only on the equilibrium capital intensity and change along the factor price frontier with $R'(k_t) = F_{KK}(k_t, 1) < 0$ and $w'(k_t) = F_{LK}(k_t, 1) > 0$.

⁹For a more detailed discussion of the savings problem under second-period lump-sum payments, see de la Croix and Michel p. 130 f.

Government

In the representative period t , the government consumes an amount G_t of aggregate output which does not affect the utility of consumers.¹⁰ Let

$$\Pi_t = N_t \eta_t - N_{t-1} \theta_t - G_t$$

denote the aggregate primary surplus. In intensive form this reads as

$$\pi_t = \eta_t - \frac{\theta_t}{1+n} - g_t$$

where g_t and π_t denote government consumption and the primary surplus per young individual, respectively. It is assumed that agents perceive investments in physical capital and government bonds (in real terms) as perfect substitutes with identical return factor R_t . Then, the flow budget constraint of the government, expressed per young agent, reads as

$$(1+n)b_{t+1} = R(k_t)b_t - \pi_t.$$

Intertemporal equilibrium conditions

In sum, we obtain a version of the intertemporal equilibrium conditions of the Diamond-model, modified for the existence of a simple tax-transfer system and the possibility of a primary balance that does not have to be balanced in every period

$$(1+n)(k_{t+1} + b_{t+1}) = s_t = s(w(k_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta_{t+1}}{R(k_{t+1})} \quad (4)$$

$$(1+n)b_{t+1} = R(k_t)b_t - \pi_t \quad (5)$$

$$\pi_t = \eta_t - \frac{\theta_t}{1+n} - g_t. \quad (6)$$

Initial conditions

In each period t , the state of the economy is summarized by the pair (b_t, k_t) , denoting the beginning-of-period per capita values of the capital stock and of real government bond holdings which are predetermined by past investment decisions undertaken in period $t-1$. Hence, when we subsequently classify the dynamic behaviour of the system (4)-(6) under various fiscal closures, it is natural to assume that dynamics are characterized by two initial conditions, b_0 and k_0 .¹¹

¹⁰If publicly provided public goods enter the utility of the representative consumer in an additively separable manner, they do not affect the consumer's saving decision. Our analysis would still hold under this assumption.

¹¹Note, however, that there is also a branch of the literature which stresses the role of bubbles in closely related models and treats real government debt as a jumping variable, see Tirole (1985). More recently, the treatment of real government debt as a jumping variable plays also a key role in the logic of the fiscal theory of the price level, as summarized, for example, in Woodford (2001).

Dynamics under a permanently balanced primary budget

The equations (4)-(6), without further restrictions, allow for a rich set of dynamic equilibria. In the remainder of this paper, however, we focus on the local stability behaviour of the system around steady states which have the particular feature of a balanced primary budget (i.e. $\pi = 0$). Moreover, to set the stage for a meaningful discussion of stabilizing off-steady-state adjustments in the primary balance in Section 3, we establish benchmark steady states of (4)-(6) which, assuming $b_0 \neq b$ and $k_0 \neq k$, are locally unstable if the primary budget is *permanently* balanced, i.e. if $\pi_t \equiv 0$ for all t . Specifically, consider a stationary tax-transfer system with $g \equiv \eta - \frac{\theta}{1+n} > 0$, leading to the two-dimensional dynamic system in k_t and b_t

$$(1+n)(k_{t+1} + b_{t+1}) = s_t = s(w(k_t) - \eta + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})} \quad (7)$$

$$(1+n)b_{t+1} = R(k_t)b_t. \quad (8)$$

Using a first-order approximation, dynamics around steady states of (7) and (8) evolve according to

$$A_1 \cdot dk_{t+1} + (1+n) \cdot db_{t+1} = A_2 \cdot dk_t \quad (9)$$

$$(1+n) \cdot db_{t+1} = R'(k)b \cdot dk_t + R(k) \cdot db_t, \quad \text{with:} \quad (10)$$

$$A_1 = 1+n - R'(k) \cdot [s_R + (1-s_w)\frac{\theta}{R(k)^2}] \quad (11)$$

$$A_2 = s_w w'(k) > 0 \quad (12)$$

Existence and stability conditions of steady states of (7) and (8) have been widely discussed in the literature. In particular, under mild assumptions the system is associated with two distinct types of steady states which are unstable under a permanently balanced primary budget: i) *steady states with zero debt and underaccumulation* ($k > 0, b = 0, R(k) > 1+n$) and ii) *golden rule steady states with positive debt* ($k > 0, b_{gr} > 0, R(k) = 1+n$). To this end, we make the assumption

(A 4) There exist steady states of (7) and (8) with $k > 0$ and $b \geq 0$, satisfying $A_2 < A_1$.

Remark: Assumption (A 4) is not very restrictive. For illustration, assume first $g = \eta = \theta = 0$. Then, if assumptions (A 1)-(A 3) are satisfied and, for example, the aggregate production function is of Cobb-Douglas type, there exists a unique steady state with $k > 0$ and $b = 0$, satisfying $A_2 < A_1$.¹² Moreover, if at this steady

¹²If the production function is of the more general CES-type this reasoning extends to the case of an elasticity of substitution larger than one. If the elasticity is less than one, there are zero or two steady states with $k > 0$ and $b = 0$. In the latter case, the high activity steady state satisfies $A_2 < A_1$.

state $R(k) < 1 + n$, there exists a golden rule steady state with $k > 0$, $b > 0$ and $R(k) = 1 + n$, also satisfying $A_2 < A_1$. If $g = \eta > 0$ and $\theta = 0$, this reasoning can be extended as long as g is smaller than some positive bound \bar{g} . Finally, by continuity, if $g \equiv \eta - \frac{\theta}{1+n} > 0$ and $\theta \neq 0$, such steady states continue to exist as long as θ is sufficiently small.¹³

From a normative perspective, the two mentioned steady-state types are relevant benchmarks because of dynamic efficiency considerations. For further reference, we conclude this section with a brief discussion of why these steady states are locally unstable. In general, according to the linearized system (9)-(10), dynamics without government debt dynamics are stable if $A_2 < A_1$, as ensured by (A 4). In the presence of government debt dynamics, however, instability can occur because of two partial effects. First, assuming a constant interest rate ($R'(k) = 0$), interest payments induce a *snowball effect on debt*, and this effect is unstable whenever the interest rate is higher than the (population) growth rate of the economy, i.e. whenever $R(k) > 1 + n$. Second, out of steady state the interest rate does not stay constant in an economy with capital stock dynamics, implying that, for any initial level of debt, there is an additional effect on debt according to $R'(k)b \cdot dk_t$. Any crowding out of capital leads over time to a higher interest rate which reinforces debt dynamics. We call this second channel the *interest rate effect on debt*.

Benchmark 1: Underaccumulation steady state ($k > 0$, $b = 0$, $R(k) > 1 + n$)
 Since $b = 0$, the interest rate effect on debt in the linearized dynamics is zero and the instability is entirely caused by the snowball effect. Because of the absence of the interest rate effect, government debt dynamics are independent of (9) and it is easy to verify that the two eigenvalues of (9) and (10) are given by $\lambda_1 = A_2/A_1 \in (0, 1)$, and $\lambda_2 = R(k)/1 + n > 1$. This pattern of eigenvalues implies that, for initial values $k_0 \neq k$ and $b_0 \neq 0$ close to the steady state, dynamics are locally unstable under a balanced primary budget rule.

Benchmark 2: Golden rule steady state ($k > 0$, $b_{gr} > 0$, $R(k) = 1 + n$)
 At the golden rule steady state with positive debt, the snowball effect is associated with a unit root, and strict instability is ensured by the additionally operating interest rate effect on debt. This can be verified from the characteristic polynomial associated with (9) and (10), evaluated at the golden rule steady state:

$$p(\lambda) = \lambda^2 - \left[1 + \frac{A_2}{A_1} - \frac{R'(k) \cdot b_{gr}}{A_1}\right] \cdot \lambda + \frac{A_2}{A_1}.$$

¹³For a detailed discussion of the existence and stability of dynamic equilibria in Diamond-models with zero primary deficits, see the references quoted in footnote 1. Specifically, de la Croix and Michel (2002) focus in detail on aspects of lump-sum tax and transfer systems, see p. 195 ff.

Then, $p(0) = A_2/A_1 \in (0, 1)$ and $p(1) = R'(k)b_{gr}/A_1 < 0$, implying $0 < \lambda_1 < 1 < \lambda_2$. Hence, at the golden rule steady state with $b_{gr} > 0$ dynamics are locally unstable under a balanced primary budget rule.

3 Stability under a debt targeting rule: a common framework with three instruments

Compared with the analysis of the previous section, we now allow for adjustments in the primary balance with the idea to stabilize the benchmark steady states, i.e. $\pi_t \neq 0$ is admitted for the off-steady-state dynamics. For a generic discussion of such stabilizing adjustments it seems natural to think of state-contingent fiscal rules which link the variations in π_t to deviations of the two state variables b_t and k_t from their steady-state values, as given by

$$\pi_t = \pi(k_t - k, b_t - b), \quad \text{with: } \pi(0, 0) = 0.$$

In combination with the budget constraint of the government, appropriate rules of this type give rise to the expression

$$(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi(k_t - k, b_t - b), \quad (13)$$

which, in contrast to the budget *identity* (5), describes a generic *debt targeting rule* which aims to stabilize the economy at the benchmark steady states. To operationalize (13), the use of at least one of the three instruments needs to be linked to the states of the economy. In the following, we distinguish between three scenarios in which adjustments are achieved by variations of one of the three available fiscal instruments g_t , η_t or θ_{t+1} , while holding the other two instruments constant at their steady-state values.

For simplicity, we assume in all scenarios that the instruments are set as a linear function of the states. According to (6), this implies that the primary surplus π_t will also be linear in the two states.¹⁴ Hence, (13) can be written as

$$(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b), \quad (14)$$

where (14) describes a broad class of debt targeting rules, parametrized by the pair of linear feedback coefficients π_k and π_b . For further reference, we summarize briefly these three scenarios, all of them being consistent with (14).

¹⁴In the generalized version of the model in Section 4.2., tax revenues η_t are no longer lump sum, but the product of a tax rate and the tax base which itself depends on the states of the economy. Then, assuming a linear instrument rule, the coefficients π_k and π_b summarize the induced reaction of the primary surplus in the linearized budget dynamics.

Instrument 1 : Variations in government spending g_t

Assume the government satisfies (14) by adjusting government spending, according to $g_t - g = -\pi_t$.¹⁵ Then, the intertemporal equilibrium conditions can be summarized as

$$(1+n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \eta + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})} \quad (15)$$

$$(1+n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (16)$$

$$g_t = \tilde{g}(k_t, b_t) = g - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (17)$$

Importantly, adjustments in the primary balance via variations in g_t do not affect the accumulation equation, i.e. (15) is identical to (4). In other words, variations in g_t offer a particularly convenient, non-distortionary channel to stabilize the two-dimensional benchmark system (7)-(8). Since (17) does not feed back into the first equation the linearized dynamics read as

$$A_1 \cdot dk_{t+1} + (1+n) \cdot db_{t+1} = A_2 \cdot dk_t \quad (18)$$

$$(1+n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t \quad (19)$$

Instrument 2 : Variations in lump-sum taxes η_t

We assume now, alternatively, that the government satisfies (14) by adjusting first-period lump-sum taxes such that $\eta_t - \eta = \pi_t$. Then, the intertemporal equilibrium evolves according to

$$(1+n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \tilde{\eta}(k_t, b_t) + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})} \quad (20)$$

$$(1+n)b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (21)$$

$$\eta_t = \tilde{\eta}(k_t, b_t) = \eta + \pi_k(k_t - k) + \pi_b(b_t - b)$$

Again, dynamics are two-dimensional in k_t and b_t but adjustments in the primary balance via variations in η_t do affect the disposable income of young agents and, hence, the accumulation equation. Linearization of (20)-(21) yields

$$A_1 \cdot dk_{t+1} + (1+n) \cdot db_{t+1} = (A_2 - s_w \pi_k) \cdot dk_t - s_w \pi_b \cdot db_t \quad (22)$$

$$(1+n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t, \quad (23)$$

where the link between the instrument and the accumulation equation is captured by the use of the partial derivatives $\tilde{\eta}_k = \pi_k$ and $\tilde{\eta}_b = \pi_b$.

¹⁵Primary surpluses require $g_t < g$. Recall the assumption of $g > 0$. In the following, we assume that g is sufficiently positive such that for the local dynamics around the steady state the inequality $g_t > 0$ is always satisfied.

Instrument 3 : Variations in transfers θ_{t+1}

Finally, we consider the case where the government satisfies (14) by adjusting second-period transfers such that $\theta_{t+1} - \theta = -(1+n)\pi_{t+1}$. Then, the intertemporal equilibrium conditions are given by

$$(1+n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \eta + \frac{\tilde{\theta}(k_{t+1}, b_{t+1})}{R(k_{t+1})}, R(k_{t+1})) - \frac{\tilde{\theta}(k_{t+1}, b_{t+1})}{R(k_{t+1})} \quad (24)$$

$$(1+n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (25)$$

$$\theta_{t+1} = \tilde{\theta}(k_{t+1}, b_{t+1}) = \theta - (1+n) \cdot [\pi_k(k_{t+1} - k) + \pi_b(b_{t+1} - b)]$$

Variations in θ_{t+1} affect the second-period disposable income of agents and, hence, the accumulation equation. This is reflected in the linearized versions of (24) and (25)

$$[A_1 - (1-s_w)\frac{(1+n)}{R(k)}\pi_k] \cdot dk_{t+1} + [1+n - (1-s_w)\frac{(1+n)}{R(k)}\pi_b] \cdot db_{t+1} = A_2 \cdot dk_t \quad (26)$$

$$(1+n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t, \quad (27)$$

where we have used $\tilde{\theta}_k = -(1+n)\pi_k$ and $\tilde{\theta}_b = -(1+n)\pi_b$.

In line with the motivation of the introduction, these three scenarios can be investigated in two directions. First, for any of the two benchmark steady states it is possible to derive three instrument-specific sets of feedback coefficients π_k and π_b which ensure locally stable dynamics. A simple geometric representation of these sets can be achieved if one recognizes that for each instrument the linearized local dynamics around any steady state are two-dimensional in k_t and b_t , giving rise to characteristic polynomials $p(\lambda)_i$, $i = g, \eta, \theta$, and that stability requires

$$p(1)|_i > 0, \quad p(-1)|_i > 0, \quad p(0)|_i < 1, \quad \text{with: } i = g, \eta, \theta.$$

As illustrated below, for each instrument these constraints (at equality) have at any steady state a linear representation in $\pi_b - \pi_k$ -space, giving rise to instrument-specific stability regions. Intuitively, it is clear that these three regions are not identical, since variations in g_t leave the intertemporal accumulation equation (15) unaffected, while variations in η_t and θ_{t+1} affect this equation through different margins, as to be inferred from (20) and (24).

Second, it can be investigated whether there exists a particular debt targeting rule, characterized by a particular pair of π_k and π_b , which can be implemented under all instruments. This amounts to check whether the regions of stabilizing feedback coefficients associated with the three instruments do overlap. Whenever this is the case such a debt targeting rule may be considered as the basis of a common fiscal

framework. Loosely speaking, under such a framework there are no tensions between stabilization issues and the unrestricted choice of fiscal instruments. As shown in the following two subsections, such a common framework may not always exist, depending on the characteristics of the particular steady state under consideration.

3.1 Underaccumulation steady states

This subsection shows that underaccumulation steady states, as derived in Section 2, have the particular feature that, without imposing further restrictions, all three instrument-specific sets of stabilizing feedback coefficients have a common intersection. This feature is directly linked to the absence of the interest rate effect on debt at this steady state, i.e. since $b = 0$ any deviation of the capital stock from its steady-state value does not have by itself destabilizing effects on government debt dynamics. In other words, the instability of debt dynamics comes entirely from the snowball effect which can be fully corrected by an appropriate choice of π_b . Accordingly, for all three instruments, if the debt targeting rule is characterized by $\pi_k = 0$ government debt dynamics are independent of the accumulation equation and for all three linearized dynamic systems the two eigenvalues are identically given by

$$\lambda_1 = A_2/A_1 \in (0, 1), \quad \lambda_2 = \frac{R(k) - \pi_b}{1 + n},$$

as one can directly infer from the three systems (18) and (19), (22) and (23), and (26) and (27), respectively. Evidently, if $\pi_k = 0$ and $\pi_b \in (R(k) - (1 + n), R(k) + 1 + n)$ dynamics will be locally stable under all three instruments. Moreover, if one considers the subset characterized by $\pi_k = 0$ and $\pi_b \in (R(k) - (1 + n), R(k))$ dynamics will not only be locally stable but also exhibit monotone adjustment for all three instruments. This reasoning leads to

Proposition 1 *Consider the three instrument-specific sets of feedback coefficients π_k and π_b which ensure local stability at the underaccumulation steady state under the debt targeting rule (14). These three sets have a joint intersection, i.e. there exist values for π_k and π_b such that the debt targeting rule can be implemented under all three instruments with a common set of feedback coefficients. Moreover, for some values in the joint intersection local adjustment dynamics are monotone under all three instruments.*

Proposition 1 is not confined, however, to the special assumption of $\pi_k = 0$. Generally speaking, whenever $\pi_k \neq 0$ this creates a policy-induced interdependence between capital stock dynamics and government debt dynamics. This interdependence differs between the three instruments. Yet, a debt targeting rule with $\pi_k \neq 0$

may nevertheless be implementable under all three instruments. To demonstrate this we turn now to a more detailed discussion of the instrument-specific stability regions and consider for illustration

Example 1: $F(K, L) = zK^\alpha L^{1-\alpha}$, $U(c, d) = \phi \ln c + \beta \ln d$
 $\delta = 1$, $z = 15$, $\alpha = 0.4$, $\phi = 1$, $\beta = 0.5$ (i.e. $s_w = \beta/(\phi + \beta) = 1/3$), $\eta = -\theta = 2.91$,
 $g = 4.1$, $1 + n = 2.43$. Assuming a period length of 30 years, this implies an annual population growth of 0.03. Assuming $b = 0$, one obtains $R = 6.29$ which corresponds to an annual real interest rate of 0.06, consistent with an underaccumulation steady state, i.e. $R(k) > 1 + n$. Moreover, $k = 0.92$, $y = 14.53$, where y denotes per capita output, leading to $g/y = 0.28$, $\eta/y = 0.2$, $\theta/y = -0.2$, i.e. agents have a similar steady-state tax burden in both periods, measured in terms of per capita output.

It is instructive to discuss first in some detail the stabilizing variations in government spending g_t because of their non-distortionary character. Illustrating Example 1, Figure 1a plots the stability region in $\pi_b - \pi_k$ -space for adjustments in g_t . Points inside the triangle in bold line are associated with two stable eigenvalues. The first dynamic benchmark of a permanently balanced primary budget discussed in the previous section has coordinates $\pi_b = \pi_k = 0$ and lies, by construction, outside the stability triangle. The triangle reflects that there are potentially two margins for stabilizing adjustments of the primary balance, π_b and π_k , which can be used in isolation or in combination. To further understand the shape of the triangle depicted in Figure 1a it is important to realize that there is one key difference between these two channels. Specifically, in the vicinity of any underaccumulation steady state with $k_0 \neq k$ and $b_0 \neq 0$, only debt imbalances, because of the snowball effect, destabilize on impact government debt dynamics, and consolidations according to π_b react immediately to this instability. By contrast, consolidations according to π_k respond with the delay of one period to the snowball effect and only to the extent that it leads to the crowding out of capital. Because of the different timing of the reactions under the two channels, stabilization can always be achieved if the primary surplus exclusively reacts to the debt imbalance, i.e. if $\pi_k = 0$ local stability is ensured if $\pi_b \in (R(k) - (1 + n), R(k) + 1 + n)$. By contrast, if the primary surplus exclusively reacts to the capital stock imbalance stabilization may not be possible, i.e. if $\pi_b = 0$ there may not exist a range for π_k such that local stability is ensured, as illustrated in Figure 1a.¹⁶ Hence, the effectiveness of the two fiscal feedback

¹⁶However, this finding depends on the strength of the snowball effect. To further illustrate this, consider example 1 and assume, everything else being equal, $\alpha = 1/3$. Because of the higher wage income share this induces, ceteris paribus, higher savings and at $b = 0$ a lower return factor $R = 4.35$, implying an annual real interest rate of 0.05, i.e. the snowball effect will be smaller than in example 1. Then, $R - A_1/A_2(1 + n) < 0$, and full stabilization can be achieved if $\pi_b = 0$ and if π_k takes on appropriate negative values. Apart from the leftward shift of the stability triangle, however, the features of Figure 1a remain qualitatively unchanged.

margins, when considered in isolation, is different, reflecting the general principle that, from a stabilization perspective, imbalances should be addressed directly at their source rather than indirectly and with some delay.

In any case, there exists a wide range of combinations of the two feedbacks which are consistent with locally stable dynamics. In particular, assume that $\pi_b < R(k) - (1 + n)$, i.e. there is no fully stabilizing direct reaction via π_b to the snowball effect. Then, if π_k is sufficiently negative (i.e. there is a sufficient reaction to the crowding out of capital) overall dynamics may nevertheless be stable. Conversely, assume that $\pi_b > R(k) + (1 + n)$, i.e. the direct reaction to debt imbalances overshoots and risks destabilizing fluctuations. Then, it may nevertheless be possible to have overall stable dynamics if π_k is sufficiently positive.

In general, stable pairs of feedback coefficients inside the stability triangle of Figure 1a are associated with a wide range of possible adjustment patterns. In particular, the area to the southwest of the hyperbola is associated with endogenously fluctuating adjustment dynamics because of complex eigenvalues. Note that only points within the small, shaded triangle ABD are associated with two stable and positive eigenvalues, ensuring monotone adjustment dynamics. Moreover, Figure 1a indicates that the combination of π_b and π_k associated with the highest speed of adjustment ($\lambda_1 = \lambda_2 = 0$), denoted by the point B , requires reactions along both margins. One easily verifies in (18) and (19) that the point B has coordinates $\pi_b = R(k)$ and $\pi_k = -A_2$. Intuitively, to fully offset any initial deviation from the steady-state values, the response of the primary balance should not only neutralize the debt imbalance ($\pi_b = R(k)$), but also fully correct for the disturbed savings behaviour of young agents ($\pi_k = -A_2$). More generally, the location of B indicates that for any given stabilizing direct reaction to debt imbalances, measured in terms of π_b , variations in π_k lead to different speeds of adjustments of the system. This is illustrated in Figure 1b which plots the impulse response of the system to an initial constellation $b_0 > 0$ and $k_0 < k$ for $\pi_b = R(k)$ and three distinct values of π_k : i) $\pi_k = -A_2$ (maximum speed of adjustment at point B , implying $\lambda_1 = \lambda_2 = 0$); ii) $\pi_k = 0$ (intermediate speed of adjustment at point C which, by construction, corresponds in terms of k_t to the Diamond model without debt dynamics and has $\lambda_1 = 0$ and $\lambda_2 = A_2/A_1 \in (0, 1)$); iii) $\pi_k = A_1 - A_2 - \varepsilon$ (slow speed of adjustment at a point close to D with $\lambda_1 = 0$ and $\lambda_2 = 1 - \varepsilon/A_1$, i.e. by choosing some small $\varepsilon > 0$ the second root can be made arbitrarily close to unity and Figure 1b uses for illustration $\varepsilon = 0.1$).¹⁷

Finally, Figure 1c completes the illustration of Example 1 and includes also the stability regions for the other two instruments which distort the accumulation equation. It is worth pointing out that for small values of π_b the area corresponding

¹⁷The initial conditions are such that k_0 is by 1% smaller than the steady state value of k . Since $b = 0$, b_0 is set somewhat arbitrarily at the level $b_0 = 0.05 > 0$.

to adjustments in η_t leaves less scope for compensating reactions of the primary balance to capital stock imbalances. Intuitively, this is the case since the disposable income of young agents depends negatively on first-period tax payments η_t . Hence, if the fiscal rule attempts to stabilize the unstable snowball effect, at least partly, via responses to capital stock imbalances this introduces not only a costly delay, but it also diminishes the disposable income of young agents and, hence, savings which are needed in the first place to support higher investments. Because of this effect, there is less scope to substitute delayed reactions via π_k for direct reactions via π_b than under non-distortionary variations of g_t .

By contrast, if the fiscal adjustment is instead achieved via reduced second-period transfers θ_{t+1} the same mechanism works in the opposite direction since this encourages savings. Consequently, for small values of π_b the stability region associated with variations in θ_{t+1} , is much wider than the one associated with both g_t and η_t . More specifically, as one infers from Figure 1c, the stability region associated with variations in θ_{t+1} is not bounded from below. This reflects that under this regime debt-stabilizing fiscal measures lead to higher savings, reinforcing thereby the overall stability of the system.

In sum, Figure 1c illustrates that the three instrument-specific stability regions, while having a common intersection, also have a clear idiosyncratic component. Specifically, acting as a counterpart to the existence of a common intersection, one can show that for each instrument there exists in general (i.e. beyond the particular functional forms used in the example) a stability region which does *not* lead to stability under the other two instruments.

Proposition 2 *For each of the three instruments there exist stabilizing feedback coefficients π_k and π_b which lie outside the stability regions of the other two instruments.*

Proof: see Appendix 1.

As the following subsection shows, at steady states with positive debt ($b > 0$) this idiosyncratic component may become the dominating force, precluding the existence of a common intersection of the three instrument-specific stability regions.

3.2 Golden rule steady states

Let us now assume that savings in this economy are sufficiently high such that the economy can settle down at a golden rule steady state with a lower interest rate (such that $R(k) = 1 + n$) and a positive debt level, satisfying

$$\frac{s(w - \eta + \frac{\theta}{1+n}, 1 + n) - \frac{\theta}{1+n}}{1 + n} - k > 0.$$

Compared to Section 3.1., higher savings may reflect structural reasons (like a higher propensity to save out of current wage income because of differences in preferences) or the response to different government policies (like lower transfer payments in the second period).

As discussed in Section 2 for the special case of a permanently balanced primary budget, at the golden rule steady state with $b_{gr} > 0$ the interest rate effect on debt is a key margin of instability. This subsection shows that it is precisely this margin which makes it difficult to find for such steady states a debt targeting rule that can be implemented under all three instruments. The weight of this margin rises in the level of debt, leading to increasingly distinct stabilization profiles under the three instruments. Hence, the implementability problem (which requires a common set of feedback coefficients) becomes increasingly severe as the level of debt rises.

Generally speaking, the presence of the interest rate effect on debt gives rise to two distinct features. First, it is impossible to address this instability directly by means of adjustments via π_k and, at the same time, to maintain a recursive dynamic structure which insulates the accumulation equation of the Diamond-model under all three instruments against the stabilization of debt dynamics. Second, to address this instability indirectly through adjustments via π_b comes with a delay and, depending on the strength of the interest rate effect on debt, this delay can be costly in terms of destabilizing dynamics. In combination, these two features give rise to the general result:

Proposition 3 *Consider the three instrument-specific sets of feedback coefficients π_k and π_b which ensure under the debt targeting rule (14) local stability at the golden rule steady state. These three sets do not necessarily have a joint intersection, i.e. it is possible that the debt targeting rule cannot be implemented under all three instruments with a common set of feedback coefficients.*

Before we further operationalize Proposition 3 by linking it explicitly to the level of steady state debt b_{gr} , we offer some intuition by discussing two examples. First, varying Example 1, we choose a parametrization which leads to a golden rule steady state with a ‘small’ debt ratio of 0.02.¹⁸ To this end, by lowering α , Example 2 chooses a slightly higher wage income share which raises effectively the propensity to save out of total income. This structural variation is sufficient to shift the economy to a golden rule steady state:¹⁹

Example 2: Consider Example 1, but let $\alpha = 0.2$, $\eta = -\theta = 3.16$, $g = 4.46$. Assuming $b = 0$, one obtains $R = 1.76 < 1 + n = 2.43$. At the golden rule steady

¹⁸To put this number into perspective it should be stressed that throughout the paper government debt is expressed as *net debt*.

¹⁹Moreover, to allow for comparability with Example 1, we adjust the levels of η , θ and g such that the corresponding ratios in terms of income are the same as in Example 1.

state, $R = 1 + n = 2.43$, yielding an annual real interest rate of 0.03, $b_{gr} = 0.36 > 0$, $k = 1.3$, $y = 15.8$ and a debt ratio of $b_{gr}/y = 0.02$. Moreover, $g/y = 0.28$, $\eta/y = 0.2$, $\theta/y = -0.2$, i.e. agents have a relative tax burden in both periods as in Example 1.

Figure 2 illustrates the stability regions associated with all three instruments for Example 2. By construction, the second benchmark discussed in Section 2 with coordinates $\pi_b = \pi_k = 0$ lies outside all three instrument-specific stability regions. Moreover, reflecting the low level of debt, Figure 2 shares with Figure 1c the feature that the three regions have a common intersection.²⁰

Alternatively, Example 3 varies Example 1 by allowing for a more substantial increase in savings through structural factors (by further lowering α and by raising the savings rate via a higher value of β) as well as policy-related factors (by considering a tax-transfer system which shifts the tax burden more strongly to second period income). In sum, this leads to a substantially higher steady-state debt ratio of 0.14.

Example 3: Consider Example 1, but let now $\alpha = 0.15$ and $\beta = 1$ (i.e. $s_w = \beta/(\phi + \beta) = 1/2$). Moreover, $\eta = 0.74$, $\theta = -6.63$, $g = 3.47$. Assuming $b = 0$, one obtains $R = 0.32 < 1 + n = 2.43$. At the golden rule steady state, $R = 1 + n = 2.43$, yielding $b_{gr} = 2.11 > 0$, $k = 0.89$, $y = 14.74$ and a debt ratio of $b_{gr}/y = 0.14$. Moreover, $g/y = 0.24$, $\eta/y = 0.05$, $\theta/y = -0.45$, i.e. the tax burden of agents in the second period is now substantially higher than in the first.

Figure 3 illustrates the stability regions associated with all three instruments for Example 3. The key result to be inferred from Figure 3 says that at sufficiently high debt level the three instrument-specific stability regions may no longer have a common intersection.²¹ More specifically, for the particular functional forms and baseline parameter values used in Example 3 the two stability regions associated with variations in g_t and η_t cease to have a common intersection as b_{gr} exceeds some threshold value.²² This finding reflects that the high interest rate effect on

²⁰However, as far as the region associated with variations in g_t is concerned, there is one important difference with Figure 1c. Since the interest rate effect on debt is now a key margin of instability, stabilization in Figure 2 can always be achieved if the primary surplus exclusively reacts to the capital stock imbalance, i.e. if $\pi_b = 0$ local stability is always ensured for some values $\pi_k < 0$. By contrast, if the primary surplus exclusively reacts to the debt imbalance (i.e. if $\pi_k = 0$) stabilization can only be achieved if debt is low (like in Example 2). This holds no longer true in a high debt regime (Example 3), indicating the implementability problems if debt is high.

²¹When interpreting the parameter values related to the effective savings rate in Examples 1 – 3 one should recall that the model set-up, counterfactually, does not allow for bequest motives. Hence, in every period all assets need to be refinanced out of the savings of young agents.

²²The fact that in Figure 3 one side of the the η -triangle (corresponding to $p(-1)|_\eta = 0$) falls exactly onto a demarcation line of the θ -type stability region (corresponding to $p(-1)|_\theta = 0$) is not generic but caused by the particular numerical assumption $\beta = 1$ (i.e. $s_w = 1$), as one can verify from the conditions stated in Appendix 1. If $\beta < 1$, $p(-1)|_\eta = 0$ slopes upward, while $p(-1)|_\theta = 0$

debt requires a stabilizing response via π_k which leads to quite different accumulation equations under the g_t -regime and the η_t -regime. For an intuitive explanation, it helps to realize that, compared with Figure 2, the stability triangle associated with the η_t -regime in Figure 3 has shifted to the southeast of the stability triangle associated with the g_t -regime. Under the g_t -regime points to the southeast of the g_t -triangle have one unstable eigenvalue. For the sake of the argument consider an initial constellation with $b_0 > b_{gr}$ and $k_0 < k$. Then, under the g_t -regime for feedback coefficients to the southeast of the g_t -triangle there is, for given savings, too much stabilization of debt dynamics, i.e. there is too little emission of new bonds b_{t+1} . This implies that the composition of next period's assets ($k_{t+1} + b_{t+1}$) becomes too productive, relative to the capacity of the economy to absorb investments in capital. However, points to the southeast of the g_t -triangle may nevertheless be consistent with fully stabilizing dynamics under the η_t -regime. The reason for this is that under the η_t -regime total savings will be lower because of the tax burden imposed on young agents. Because of this there is less scope that a strong reduction of b_{t+1} can trigger 'overinvestment' in physical capital k_{t+1} . This reasoning shows that the instrument-specific reactions to imbalances may not only be different, but also mutually exclusive if one wishes to maintain over time the knife-edge portfolio composition between government bonds and physical capital at the golden rule steady state.

If one attempts to make the role of the steady-state level of debt in Proposition 3 more precise one faces the challenge that b_{gr} , in general, is a function of both structural and policy parameters. In the simple example economy introduced above, the sets corresponding to these two types of parameters amount to $S = \{\alpha, \beta, \phi, n, z, \delta\}$ and $P = \{\eta, \theta, g\}$. From a policy perspective, it seems preferable to isolate implementability problems which can be cured by policy changes. To control for this aspect requires to keep the structural parameters fixed and to look only at those variations in b_{gr} which are policy-induced. For any set S consistent with the existence of a golden rule steady state it is possible to increase steady-state debt by shifting the tax burden more strongly to second period income. Yet, the maximum amount of debt that can be achieved with such a policy experiment depends itself on S and this debt level may not always be high enough to prevent the existence of a debt targeting rule which ensures implementability under all three instruments. This non-trivial interaction between structural and policy parameters is acknowledged in Proposition 4. A fairly tractable discussion of this interaction can be achieved by variations of the example economy, as shown in Appendix 1. Hence, to make the role of b_{gr} in Proposition 3 more operational, Proposition 4 draws directly on properties of the example economy.

slopes downward. Conversely, if $\beta > 1$, $p(-1)|_\eta = 0$ slopes downward and $p(-1)|_\theta = 0$ slopes upward.

Proposition 4 Consider the example economy discussed in examples 1 – 3, characterized by $F(K, L) = zK^\alpha L^{1-\alpha}$ and $U(c, d) = \phi \ln c + \beta \ln d$. Then, at any golden rule steady state the per capita debt level $b_{gr} > 0$ is a function of both structural parameters $S = \{\alpha, \beta, \phi, n, z, \delta\}$ and policy parameters $P = \{\eta, \theta, g\}$. Consider golden rule steady states which are characterized by the same set S , but different policy-induced debt levels b_{gr} because of differences in the set P . Then, many (although not all conceivable) sets of S have the property that the debt targeting rule cannot be implemented under all three instruments if b_{gr} exceeds some policy-induced threshold value $b_{gr}^* > 0$.

Proof: For a proof of Propositions 3 and 4, see Appendix 1.

4 Extensions

4.1 Alternative representations of the debt targeting rule

It is worth pointing out that there exist alternative representations of the debt targeting rule which lead to the same results summarized in Propositions 1 – 4. We consider two particularly intuitive alternatives. As a starting point, we repeat the flow budget constraint of the government

$$(1 + n) \cdot b_{t+1} = R(k_t) \cdot b_t - \pi_t,$$

and maintain the assumption that at the steady states under consideration the primary balance is zero, i.e. $\pi_t = 0$.

First, let us assume that the debt targeting rule is now expressed in terms of stabilizing reactions of the overall deficit Δ_t (i.e. inclusive interest payments), using

$$\begin{aligned} (1 + n) \cdot b_{t+1} &= b_t + \Delta_t, \\ \Delta_t &= \Delta(k_t, b_t) = (R(k_t) - 1) \cdot b_t - \pi_t \end{aligned}$$

Note that the deficit Δ_t , at any moment in time, consists of a predetermined component linked to interest payments on debt, and a policy component linked to the primary balance. Only the latter part can actively react to the two predetermined states of the economy, b_t and k_t . Accordingly, a debt targeting rule with stabilizing reactions of the deficit to the states of the economy, in linearized form, needs to be established from

$$(1 + n) \cdot db_{t+1} = \Delta_k \cdot dk_t + (1 + \Delta_b) \cdot db_t, \quad (28)$$

$$\text{with : } \quad \Delta_k = R'(k) \cdot b - \pi_k, \quad \text{and } \Delta_b = R(k) - 1 - \pi_b. \quad (29)$$

Consider the three linearized dynamic systems (18)-(19), (22)-(23), and (26)-(27) which were derived above for the three instruments. Using (28) as the second equation in these 3 systems and replacing π_k and π_b by the terms $R'(k) \cdot b - \Delta_k$ and $R(k) - 1 - \Delta_b$ in the first equation of the 3 systems, respectively, these systems can be transformed into three new systems, all exhibiting two-dimensional dynamics in k_t and b_t . However, since $R(k)$, $R'(k)$, and b are all evaluated at constant steady-state values, the switch from the representation in $\pi_b - \pi_k$ -space to a representation in $\Delta_b - \Delta_k$ -space amounts to an affine transformation, which leaves the results of Propositions 1 – 4 unaffected.

Second, the debt targeting rule can be reinterpreted as a rule which expresses the issuance of new per capita debt directly in terms of stabilizing reactions to the states of the economy, according to

$$\begin{aligned} (1+n) \cdot b_{t+1} &= h_t, \\ h_t &= h(k_t, b_t) = R(k_t) \cdot b_t - \pi_t. \end{aligned} \tag{30}$$

After linearizing (30) and substituting out the relevant terms in the three systems, it is clear that a switch from a representation in $\pi_b - \pi_k$ -space to a representation in $h_b - h_k$ -space amounts to another affine transformation, leaving, again, the results of Propositions 1 – 4 unaffected.

4.2 Endogenous labour supply and distortionary taxes

The purpose of this subsection is to show that all the central findings of Section 3, as summarized by Propositions 1 – 4, prevail qualitatively in a richer setting which is characterized by an endogenous labour supply and distortionary taxes. Yet, there is an interesting twist to the results of this richer setting which is worth pointing out. To this end, we assume now that preferences are described by the more general expression

$$U(c_t - \varphi(l_t), d_{t+1}),$$

where l_t denotes the variable labour supply of the representative young agent and the function $\varphi(l_t)$ captures the disutility of work, with $\varphi(0) > 0$, and $\varphi'(l_t) > 0$, $\varphi''(l_t) > 0$ for all $l_t > 0$. As discussed in Greenwood et al. (1988), this labour supply specification has the convenient feature that it can be solved independently from the intertemporal consumption and savings decisions, allowing for easy comparability with the analysis of the previous section. Moreover, also for simple comparability, it is assumed that total tax revenues have a lump-sum and a distortionary component

$$\eta_t = \eta + \tau_t w_t l_t, \tag{31}$$

where τ_t denotes the wage income tax rate. We maintain the assumption that for the steady-state tax-transfer system, characterized by $g \equiv \eta - \frac{\theta}{1+n}$, only lump-sum taxes

matter. In contrast to the previous analysis, however, the entire out-of-steady-state adjustment burden falls on variations of the distortionary component τ_t , whenever tax adjustments are the preferred instrument to stabilize the debt level around the steady-state target. Accordingly, the objective can be replaced by

$$U(w_t l_t - \varphi(l_t) - \eta_t - s_t, R_{t+1} s_t + \theta_{t+1}),$$

giving rise to the pair of first order conditions

$$\varphi'(l_t) = (1 - \tau_t) w_t \quad (32)$$

$$U_1 = R_{t+1} U_2. \quad (33)$$

With an endogenous labour supply, the labour market equilibrium condition becomes $L_t = l_t N_t$ and the first-order conditions from the profit maximization of firms are given by

$$R_t = 1 - \delta + F_K(k_t, l_t) \quad (34)$$

$$w_t = F_L(k_t, l_t). \quad (35)$$

Combining (31), (32), and (35) yields

$$l_t \varphi'(l_t) = l_t F_L(k_t, l_t) - (\eta_t - \eta),$$

which implicitly defines the equilibrium labour supply

$$l_t = l(k_t, \eta_t)$$

in the vicinity of some steady state $l = l(k, \eta)$, with partial derivatives $l_k(k, \eta) = \frac{F_{LK}(k, l)}{\varphi''(l) - F_{LL}(k, l)} > 0$ and $l_\eta(k, \eta) = \frac{-1}{l \cdot [\varphi''(l) - F_{LL}(k, l)]} < 0$.²³ To keep the structure of the analysis as similar as possible to Section 2, we define the adjusted gross wage income net of the disutility term $\varphi(l_t)$ as

$$\tilde{w}_t \equiv w_t l_t - \varphi(l_t) = F_L(k_t, l_t) \cdot l_t - \varphi(l_t) = \tilde{w}(k_t, \eta_t). \quad (36)$$

Using (36), the savings function reduces to

$$s_t = s(\tilde{w}_t - \eta_t + \frac{\theta_{t+1}}{R_{t+1}}, R_{t+1}) - \frac{\theta_{t+1}}{R_{t+1}}.$$

²³For clarification, note that partial derivatives use the notation $l_k(k, \eta) = \frac{\delta l_t}{\delta k_t} \Big|_{k_t=k}$ and $l_\eta(k, \eta) = \frac{\delta l_t}{\delta \eta_t} \Big|_{\eta_t=\eta}$, i.e. in the latter case the derivative is taken with respect to η_t and then evaluated at the steady state characterized by $\eta_t = \eta$.

Key features of the factor price frontier remain qualitatively unchanged compared with Section 2 if one expresses the factor prices in terms of the ‘adjusted return rates’

$$\begin{aligned} R_t &= 1 - \delta + F_K(k_t, l_t) = R(k_t, \eta_t) \\ \tilde{w}_t &= \tilde{w}(k_t, \eta_t). \end{aligned}$$

In particular, R_t falls in k_t , and \tilde{w}_t rises in k_t , since²⁴

$$R_k = F_{KK} + F_{KL} \cdot l_k = F_{KK} \cdot \frac{\varphi''}{\varphi'' - F_{LL}} < 0 \quad (37)$$

$$\tilde{w}_k = l \cdot (F_{LK} + F_{LL} \cdot l_k) = l \cdot F_{LK} \cdot \frac{\varphi''}{\varphi'' - F_{LL}} > 0. \quad (38)$$

Similarly, one obtains

$$R_\eta = F_{KL} \cdot l_\eta < 0 \quad (39)$$

$$\tilde{w}_\eta = l \cdot F_{LL} \cdot l_\eta > 0, \quad (40)$$

i.e. the adjusted return rates move upon a change in the distortionary labour tax in different directions, reflecting that the labour supply itself falls in η .²⁵ Then, the set of intertemporal equilibrium conditions can be summarized as

$$(1+n)(k_{t+1} + b_{t+1}) = s(\tilde{w}(k_t, \eta_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})}, R(k_{t+1}, \eta_{t+1})) - \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})} \quad (41)$$

$$(1+n)b_{t+1} = R(k_t, \eta_t)b_t - \pi_t \quad (42)$$

$$\pi_t = \eta_t - \frac{\theta_t}{1+n} - g_t \quad (43)$$

The system (41)-(43) is structurally similar to the system (4)-(6) discussed in the previous section. Specifically, let

$$\begin{aligned} A_1^* &= 1 + n - R_k \cdot [s_R + (1 - s_w) \frac{\theta}{R(k)^2}] \\ A_2^* &= s_w \tilde{w}_k > 0 \end{aligned}$$

²⁴To establish (37) we exploit $F_{KK} \cdot F_{LL} = (F_{KL})^2$ which follows from the linear homogeneity assumption made in (A 4).

²⁵Note that $R_\eta < 0$ implies that the gross wage rate $w = F_L(k, l)$ rises in η_t , i.e. $w_\eta = F_{LL} \cdot l_\eta > 0$. To see why the adjusted wage term \tilde{w} satisfies $\tilde{w}_\eta > 0$ note first that the gross wage bill (wl) falls in η_t , despite the rise in w , if the wage elasticity of employment is larger than one. However, the disutility of labour decreases because of the reduced labour supply, and the latter effect always dominates, ensuring $\tilde{w}_\eta > 0$. Finally, it is worth pointing out that the net wage rate $(1 - \tau_t)w = w - (\eta_t - \eta)/l$, evaluated at the steady state, falls in η_t , since $w_\eta - 1/l < 0$ if $-F_{LL}/[\varphi''(l) - F_{LL}] < 1$, and the latter inequality must always be satisfied.

and assume $A_1^* > A_2^*$. Then, in the light of (37) and (38), dynamics remain qualitatively unchanged for variations in g_t and θ_{t+1} , since η_t will be held constant in these two scenarios. If, however, adjustments in the primary balance are achieved via distortionary wage income taxes, as embodied in η_t , the dynamic system behaves qualitatively differently from Section 3 because of the additional partial effects resulting from (39) and (40). The key difference is that variations in the wage income tax affect the labour supply and, hence, the (pre-tax) factor return rates. As indicated by (39) and (40), the reduced labour supply decreases the equilibrium interest factor R_t and increases both the wage rate w_t and the adjusted wage term \tilde{w}_t . This feature implies that the distortionary wage income tax acts in terms of factor prices like a built-in-stabilizer which moderates the destabilizing interest rate effect on debt. In other words, whenever wage income taxes are changed to address unstable debt dynamics this has the convenient implication that the interest rate effect on debt will be endogenously dampened through the mechanics of the factor-price frontier, assuming competitive factor markets and an elastic labour supply. As we show in the final two subsections, this feature does not affect the assessment of underaccumulation steady states, but it somewhat moderates the assessment of golden rule steady states.

To conclude this subsection, it is worth pointing out that this analysis equivalently (and probably more naturally) could have been carried out in terms of state-contingent variations of the direct tax instrument τ_t , using

$$\tau_t = \tau_k(k_t - k) + \tau_b(b_t - b),$$

with associated values $\pi_k = \tau_k w l$ and $\pi_b = \tau_b w l$ in the linearized budget constraint, evaluated at the steady-state value $\tau = 0$. The instead considered variations in tax revenues η_t take implicitly the reaction of the labour supply to changes in τ_t into account, allowing for easy comparability with the set-up introduced in Section 3. In any case, with τ_t and η_t evaluated at their respective steady-state values, one can show that the linearized dynamic systems are identical for the two approaches. Hence, none of the results depends on this notational choice.

4.2.1 Underaccumulation steady state

At any underaccumulation steady state the interest rate effect on debt is zero and the moderation of factor prices under the η_t -regime is therefore, qualitatively, without consequence for the structure of the dynamic system. Recall from Section 3.1. that

$$\eta_t = \tilde{\eta}(k_t, b_t) = \eta + \pi_k(k_t - k) + \pi_b(b_t - b),$$

with $\tilde{\eta}_k = \pi_k$ and $\tilde{\eta}_b = \pi_b$. Appendix 2 summarizes for all three instruments key features of the linearized dynamic equations. Because of the structural similarities

between the systems (41)-(43) and (4)-(6) the main result of this section, however, can be entirely inferred from the linearized version of (42)

$$(1+n) \cdot db_{t+1} = [(R_k + R_\eta \pi_k)b - \pi_k] \cdot dk_t + [R(k) - \pi_b + R_\eta \pi_b b] \cdot db_t, \quad (44)$$

using $\tilde{\eta}_k = \pi_k$ and $\tilde{\eta}_b = \pi_b$. Since $b = 0$, (44) turns under the particular assumption of $\pi_k = 0$ into the one-dimensional dynamic equation in b_t and b_{t+1}

$$db_{t+1} = \frac{R(k) - \pi_b}{1+n} \cdot db_t.$$

Hence, for all three instruments g_t , η_t , and θ_{t+1} the two eigenvalues of the respective dynamic systems are identically given by

$$\lambda_1 = A_2^*/A_1^* \in (0, 1), \quad \lambda_2 = \frac{R(k) - \pi_b}{1+n}.$$

if one assumes $\pi_k = 0$. Because of this feature, the classification of dynamic equilibria under the three instruments is as in Section 3.1 and Propositions 1 and 2 remain unaffected.

4.2.2 Golden rule steady state

At any golden rule steady state, Proposition 3 of Section 3.2 remains valid, but the stabilizing reactions of factor prices under the η_t -regime make it in a certain sense less likely that the instrument-specific sets of stable feedback coefficients π_k and π_b have no joint intersection. To operationalize this insight, it is convenient to reconsider the example economy used so far with a more general preference structure. Specifically, let

$$U(c_t - \varphi(l_t), d_{t+1}) = \phi \ln \left[c_t - \frac{\xi}{1+\chi} l_t^{1+\chi} \right] + \beta \ln d_{t+1}, \quad (45)$$

where $\chi > 0$ denotes the inverse of the constant elasticity of the labour supply. As shown in Appendix 2, when combined with a Cobb-Douglas production function, (45) implies for the two crucial partial effects on return rates (39) and (40):

$$R_\eta = F_{KL} \cdot l_\eta = -\frac{\alpha}{\alpha + \chi} \cdot \frac{1}{k} < 0 \quad (46)$$

$$\tilde{w}_\eta = l \cdot F_{LL} \cdot l_\eta = \frac{\alpha}{\alpha + \chi} \in (0, 1), \quad (47)$$

where $\alpha \in (0, 1)$ denotes the Cobb-Douglas share of capital. Evidently, the parameter χ is of key importance for the reactions of the factor prices to changes in η .

Specifically, as χ becomes large the labour supply becomes inelastic and the economy behaves qualitatively like the benchmark scenario discussed in Section 3, since $R_\eta \rightarrow 0$ and $\tilde{w}_\eta \rightarrow 0$. For illustration, Example 4 sets $\chi = 1000$ and uses numerical values for the other parameters which reproduce Example 3, i.e. the g_t -regime and the η_t -regime have no common intersection in terms of stabilizing feedback coefficients. By contrast, Example 5 drops the assumption of an inelastic labour supply and uses instead a much lower value of $\chi = 2$.

Example 4: $F(K, L) = zK^\alpha L^{1-\alpha}$, $U(c, l, d,) = \phi \ln[c - \frac{\xi}{1+\chi} l^{1+\chi}] + \beta \ln d$. $\chi = 1000$, $\delta = 1$, $z = 15$, $\alpha = 0.15$, $\phi = 1$, $\beta = 1$, $R = 1 + n = 2.43$. Moreover, $\eta = 0.74$, $\theta = -6.63$, $g = 3.46$, implying $b_{gr} = 2.11 > 0$, $k = 0.88$, $y = 14.72$, $b_{gr}/y = 0.14$, $g/y = 0.24$, $\eta/y = 0.05$, $\theta/y = -0.45$. The ‘free’ parameter ξ is set at $\xi = 30$, normalizing the labour supply to $l = 1$.

Example 5: Consider Example 4, but let $\chi = 2$. We maintain $g/y = 0.24$, $\eta/y = 0.05$, $\theta/y = -0.45$. Everything else being equal, this implies: $R = 1 + n = 2.43$, $b_{gr} = 0.81 > 0$, $k = 0.57$, $y = 9.54$, $b_{gr}/y = 0.08$, $l = 0.6$.

Figures 4 and 5 show for the two example economies the stability regions associated with all three instruments. The key finding is that under the elastic labour supply of Example 5 the stability triangle of the η_t -regime is no longer strictly to the southeast of the g_t -triangle, but allows instead for a common intersection. In other words, Figures 4 and 5 illustrate that under distortionary wage income taxes there is scope for stabilizing reactions of factor prices which moderate the strong results of Propositions 3 and 4.

5 Conclusion

This paper studies the stabilization of government debt dynamics under a number of different fiscal instruments from a comparative perspective. Specifically, the paper addresses the question of whether a state contingent debt targeting rule which links the stabilization of long-run debt to the underlying state of the economy can be implemented under all available fiscal instruments with a common set of feedback coefficients. Using a fully tractable overlapping generations framework, the main analytical result of the paper says that the answer to this question cannot be given without reference to the level of long-run debt around which the economy is stabilized. Intuitively, this finding reflects that different fiscal instruments (like government spending, public transfers, and the menu of available taxes) typically affect the economy through instrument-specific margins which are associated with rather different distortions (related, for example, to the labour-leisure decision, investment decisions, or consumption decisions). The steady-state level of debt to be stabilized

determines the weight of these margins within the set of intertemporal conditions. As the level of debt rises the importance of these distinct margins increases, implying that for any particular debt targeting rule the instrument-specific adjustment paths become increasingly diverse. Exploiting this feature, the paper shows that there can easily exist a threshold value of long-run debt beyond which the instrument-specific adjustment paths become so diverse that there exists no longer a debt targeting rule which can be implemented under all instruments.

As the paper stands, these results are derived in a deliberately small and fully tractable model of a closed economy. Yet, the policy implications can probably best be seen in the context of a monetary union with decentralized fiscal policies, subject to certain provisions of a common fiscal framework. The analysis of this paper does not add any new arguments why such a framework is necessary. Instead, it indicates that within any such framework high levels of average debt are likely to create tensions between the necessary provisions of a common framework which tracks deficit developments and the unrestricted choice of fiscal instruments at the national level. This paper implicitly assumes that the latter feature is by itself of considerable value. Therefore, if one wishes to preserve this value under the conditions of a monetary union the results of this paper indicate that the union's fiscal framework should be organized around a sufficiently ambitious target level of debt. We leave it for future work to further explore this mechanism, also with a focus on quantitative issues, within a modelling framework that explicitly allows for features which are characteristic of a set-up with multiple countries.

References

- [1] Annicchiarico, B. and Giammarioli, N., Fiscal Rules and Sustainability of Public Finances in an Endogenous Growth Model, *European Central Bank Working Paper Series*, No. 381, 2004.
- [2] Aiyagari, R., Marcet, A., Sargent, T. and Seppälä, J., Optimal taxation without state contingent debt, *Journal of Political Economy*, 110, 6, 1220-54, 2002.
- [3] Aiyagari R. and McGrattan, E., The optimum quantity of debt, *Journal of Monetary Economics*, 447-69, 42, 1998.
- [4] Azariadis, C., *Intertemporal Macroeconomics*. Cambridge, MA and Oxford: Blackwell, 1993.
- [5] Benigno, P. and Woodford, M., Optimal monetary and fiscal policy, in : M. Gertler and K. Rogoff (eds.), *NBER Macroeconomics Annual 2003*, Cambridge/USA, 2003.

- [6] Bryant, R. and Zhang, L., Intertemporal fiscal policy in macroeconomic models: introduction and major alternatives, *Brookings Discussion Papers in International Economics*, 123, 1996.
- [7] Chalk, N., The sustainability of bond-financed deficits, An overlapping generations approach, *Journal of Monetary Economics*, 45, 293-328, 2000.
- [8] Chari, V. and Kehoe, P., On the needs for fiscal constraints in a Monetary Union, Working Paper, 1998.
- [9] Chari, P. and Kehoe, P., Optimal fiscal and monetary policy, in: Taylor, J. and Woodford, M. (eds.), *Handbook of Macroeconomics*, Vol. 1c, Amsterdam, 1999.
- [10] De la Croix, D. and Michel, P., *A Theory of Economic Growth, Dynamics and Policy in Overlapping Generations*, Cambridge University Press, 2002.
- [11] Diamond, P., National debt in a neoclassical growth model, *American Economic Review*, 55/5, 1126-50, 1965.
- [12] Díaz-Giménez, J., Giovanetti, G., Marimon, R., and Teles, P., Nominal debt as a burden on monetary policy, Working paper, 2004.
- [13] Farmer, R., Deficits and cycles, *Journal of Economic Theory*, Volume 40, Issue 1, 77-88, 1986.
- [14] Fatás, A., von Hagen, J., Hughes Hallett, A., Strauch, R., and Sibert, A., *Stability and growth in Europe: towards a better pact*, CEPR, 2003.
- [15] Fernandez-Huertas Moraga, J. and Vidal, J-P., Fiscal Sustainability and Public Debt in an Endogenous Growth Model, *European Central Bank Working Paper Series*, No. 395, 2004.
- [16] Galor, O. and Ryder, H., Existence, Uniqueness and Stability of Equilibrium in Overlapping Generations Model with Productive Capital, *Journal of Economic Theory*, 49, 2, 374-87, 1989.
- [17] Giannitsarou, C., Balanced budget rules and aggregate instability: the role of consumption taxes, Working paper, 2004.
- [18] Greenwood, J., Hercowitz, Z., and Huffman, G., Investment, capacity utilization and the real business cycle, *American Economic Review*, 78, 3, 402-17, 1988.

- [19] Guo, J. and Harrison, S., Balanced-budget rules and macroeconomic (in)stability, *Journal of Economic Theory*, 119, 357-363, 2004.
- [20] Judd, K., A dynamic theory of factor taxation, *American Economic Review*, 77, 2, 42-48, 1987.
- [21] Kaas, L. and von Thadden, L., Budgetary policy and unemployment dynamics in an OLG model with collective bargaining, *Economic Journal*, 114, 727-49, 2004.
- [22] Kollmann, R., Welfare maximizing operational monetary and fiscal policy rules, Working paper, 2004.
- [23] Lambertini, L., Fiscal rules in a monetary union, Working paper, 2004.
- [24] Lucas, R. and Stokey, N., Optimal fiscal and monetary policy in an economy without capital, *Journal of Monetary Economics*, 12, 55-93, 1983.
- [25] Mankiw, G. and Weinzierl, M., Dynamic scoring: a back-of-the-envelope guide, *NBER Working paper*, 11000, 2004.
- [26] Marin, J., Sustainability of Public Finances and Automatic Stabilisation under a Rule of Budgetary Discipline, *European Central Bank Working Paper Series*, No. 193, 2002.
- [27] Martin, F., A positive theory of government debt, Working paper, 2004.
- [28] Mitchell, P., Sault, J. and Wallis, K., Fiscal policy rules in macroeconomic models: principles and practice, *Economic Modelling*, 17, 171-193, 2000.
- [29] Pérez, J. and Hiebert, P., Identifying endogenous fiscal policy rules for macroeconomic models, *European Central Bank Working Paper Series*, No. 156, 2002.
- [30] Schmitt-Grohé, S. and Uribe, M., Balanced-budget rules, distortionary taxes and aggregate instability, *Journal of Political Economy*, 976-1000, 105, 1997.
- [31] Schmitt-Grohé, S. and Uribe, M. Optimal simple and implementable monetary and fiscal rules, Working paper, 2004.
- [32] Tirole, J., Asset Bubbles and Overlapping Generations, *Econometrica*, 53, 6, 1499-1528, 1985.
- [33] Turnovsky, S., The effects of taxes and dividend policy on capital accumulation and macroeconomic behaviour, *Journal of Economic Dynamics and Control*, 14, 491-521, 1990.

- [34] Uhlig, H., One money, but many fiscal policies in Europe: what are the consequences?, *CEPR Discussion paper*, No. 3296, 2002.
- [35] Woodford, M., Fiscal requirements for price stability, *Journal of Money, Credit, and Banking*, 33, 3 669-728, 2001.

Appendix 1: Fixed labour supply and lump sum taxes

Preliminaries to the proofs of Propositions 2 – 4 :

For further reference, we derive for the three instruments the characteristic polynomials $p(\lambda)_i$ and the critical stability conditions $p(1)|_i = 0$, $p(-1)|_i = 0$, $p(0)|_i = 1$, for: $i = g, \eta, \theta$.

1) Regarding g_t , the characteristic equation associated with (18) and (19) satisfies

$$\begin{vmatrix} A_2 - A_1\lambda & -(1+n)\lambda \\ R'(k)b - \pi_k & R(k) - \pi_b - (1+n)\lambda \end{vmatrix} = 0, \text{ i.e.}$$

$$p(\lambda)|_g = \lambda^2 - \left[\frac{A_2}{A_1} + \frac{R(k) - \pi_b}{1+n} - \frac{R'(k)b - \pi_k}{A_1} \right] \cdot \lambda + \frac{A_2}{A_1(1+n)}(R(k) - \pi_b)$$

$$p(0)|_g = \frac{A_2}{A_1} \cdot \frac{R(k)}{1+n} - \frac{A_2}{A_1(1+n)}\pi_b$$

$$p(1)|_g = \left(1 - \frac{A_2}{A_1}\right)\left(1 - \frac{R(k) - \pi_b}{1+n}\right) + \frac{R'(k)b - \pi_k}{A_1}$$

$$p(-1)|_g = \left(1 + \frac{A_2}{A_1}\right)\left(1 + \frac{R(k) - \pi_b}{1+n}\right) - \frac{R'(k)b - \pi_k}{A_1}$$

2) Regarding η_t , the characteristic equation associated with (22) and (23) satisfies

$$\begin{vmatrix} A_2 - s_w\pi_k - A_1\lambda & -s_w\pi_b - (1+n)\lambda \\ R'(k)b - \pi_k & R(k) - \pi_b - (1+n)\lambda \end{vmatrix} = 0, \text{ i.e.}$$

$$p(\lambda)|_\eta = p(\lambda)|_g + s_w \frac{\pi_k}{A_1} \cdot \lambda + s_w \frac{R'(k)b\pi_b - R(k)\pi_k}{A_1(1+n)} \quad (48)$$

$$p(0)|_\eta = p(0)|_g + s_w \cdot \frac{R'(k)b\pi_b - R(k)\pi_k}{A_1(1+n)}$$

$$p(1)|_\eta = p(1)|_g + s_w \cdot \frac{R'(k)b\pi_b + [(1+n) - R(k)]\pi_k}{A_1(1+n)}$$

$$p(-1)|_\eta = p(-1)|_g + s_w \cdot \frac{R'(k)b\pi_b - [(1+n) + R(k)]\pi_k}{A_1(1+n)}$$

3) Regarding θ_{t+1} , the characteristic equation associated with (26) and (27) satisfies

$$\left| \begin{array}{cc} A_2 - [A_1 - (1 - s_w) \frac{(1+n)\pi_k}{R}] \cdot \lambda & -[1 + n - (1 - s_w) \frac{(1+n)\pi_b}{R}] \cdot \lambda \\ R'(k)b - \pi_k & R(k) - \pi_b - (1 + n)\lambda \end{array} \right| = 0$$

Let $\tilde{A}_1 = A_1 - (1 - s_w) \frac{1+n}{R(k)} \pi_k$. Note that $\tilde{A}_1 = A_1$ if $\pi_k = 0$. Then:

$$\begin{aligned} p(\lambda)|_\theta &= \lambda^2 - \theta_1 \lambda + \theta_2, \text{ with:} \\ \theta_1 &= \frac{A_2}{\tilde{A}_1} + \frac{R(k) - \pi_b}{1 + n} - \frac{R'(k)b - \pi_k}{\tilde{A}_1} + \frac{(1 - s_w) \frac{1+n}{R(k)} \pi_b (R'(k)b - \pi_k)}{\tilde{A}_1 (1 + n)} \\ \theta_2 &= \frac{A_2}{\tilde{A}_1 (1 + n)} (R(k) - \pi_b), \\ p(0)|_\theta &= \frac{A_2}{\tilde{A}_1} \frac{R(k)}{1 + n} - \frac{A_2}{\tilde{A}_1 (1 + n)} \pi_b \\ p(1)|_\theta &= (1 - \frac{A_2}{\tilde{A}_1}) (1 - \frac{R(k) - \pi_b}{1 + n}) + \frac{R'(k)b - \pi_k}{\tilde{A}_1} - (1 - s_w) \frac{\pi_b}{\tilde{A}_1 R(k)} (R'(k)b - \pi_k) \\ p(-1)|_\theta &= (1 + \frac{A_2}{\tilde{A}_1}) (1 + \frac{R(k) - \pi_b}{1 + n}) - \frac{R'(k)b - \pi_k}{\tilde{A}_1} + (1 - s_w) \frac{\pi_b}{\tilde{A}_1 R(k)} (R'(k)b - \pi_k) \end{aligned}$$

Proof of proposition 2:

Consider Figure 1c. Then, independent of the particular functional forms underlying Example 1, the stability constraints at the underaccumulation steady state with $R(k) > 1 + n$ and $b = 0$ satisfy:

i) If $\pi_k = 0$, then $p(0)|_g = p(0)|_\eta = p(0)|_\theta = 1$ jointly intersect at $\pi_{b,0} = R(k) - (1 + n)A_1/A_2$. If $\pi_k = 0$, then $p(1)|_g = p(1)|_\eta = p(1)|_\theta = 0$ jointly intersect at $\pi_{b,1} = R(k) - (1 + n) > \pi_{b,0}$. If $\pi_k = 0$, then $p(-1)|_g = p(-1)|_\eta = p(-1)|_\theta = 0$ jointly intersect at $\pi_{b,-1} = R(k) + (1 + n) > \pi_{b,1}$.

ii) $p(0)|_g = 1$ is vertical in $\pi_b - \pi_k$ -space. Moreover, $p(0)|_\eta = 1$ slopes downward and $p(0)|_\theta = 1$ slopes upward, since

$$\begin{aligned} p(0)|_\eta = 1 &\Leftrightarrow \pi_k = A_2 R(k) - A_1 (1 + n) - \frac{A_2}{s_w R(k)} \pi_b \\ p(0)|_\theta = 1 &\Leftrightarrow \pi_k = \frac{R(k)}{(1 - s_w)(1 + n)} (A_1 - A_2 \frac{R(k)}{1 + n}) + \frac{A_2}{(1 - s_w)(1 + n)} \frac{R(k)}{1 + n} \pi_b \end{aligned}$$

iii) $p(1)|_g = 0, p(1)|_\eta = 0, p(1)|_\theta = 0$ all slope upward in $\pi_b - \pi_k$ -space, with

$0 < \text{slope}|_{p(1)|_\eta=0} < \text{slope}|_{p(1)|_g=0} < \text{slope}|_{p(1)|_\theta=0}$, since

$$\begin{aligned} p(1)|_g = 0 &\Leftrightarrow \pi_k = (A_1 - A_2)\left(1 - \frac{R(k)}{1+n}\right) + \frac{A_1 - A_2}{1+n}\pi_b \\ p(1)|_\eta = 0 &\Leftrightarrow \pi_k = \frac{(A_1 - A_2)(1+n - R(k))}{(1-s_w)(1+n) + s_w R(k)} + \frac{A_1 - A_2}{(1-s_w)(1+n) + s_w R(k)}\pi_b \\ p(1)|_\theta = 0 &\Leftrightarrow \pi_k = \frac{(A_1 - A_2)R(k)\left(1 - \frac{R(k)}{1+n}\right)}{(1-s_w)(1+n) + s_w R(k)} + \frac{A_1 - A_2}{1+n} \frac{R(k)}{(1-s_w)(1+n) + s_w R(k)}\pi_b \end{aligned}$$

Then, combining i) with the slope conditions established in ii) and iii) implies that for each of the three instruments there exist stabilizing feedback coefficients π_k and π_b which lie outside the stability regions of the other two instruments. \square

Proof of propositions 3 and 4:

The proof considers, for simplicity, only the g -regime and the η -regime and shows for a particularly tractable example that the intersection of the two stability regions becomes empty if the golden rule level of steady-state debt exceeds some policy-induced threshold value $b_{gr}^* > 0$. Specifically, a constellation is derived where

$$\text{i) } p(0)|_\eta < 1, \text{ ii) } p(1)|_\eta > 0, \text{ and iii) } p(-1)|_g > 0 \quad (49)$$

are not jointly satisfied. For easy reference, consider Figure 6 which plots the three conditions at equality. Let $F(K, L) = zK^\alpha L^{1-\alpha}$ and, assuming $\phi = 0$, let $U(c, d) = \ln d$, implying that all disposable wage income is saved, i.e. $s_w = 1$. Hence, steady-state savings conveniently satisfy

$$s = s(w - \eta + \frac{\theta}{R}, R) - \frac{\theta}{R} = w - \eta = (1 - \alpha) \cdot zk^\alpha - \eta.$$

Let $\delta = 1$, $z > 0$, $n > 0$, $\alpha \in (0, 1)$ and assume $g > 0$, with g in steady state being calibrated such that $g = g_s \cdot zk^\alpha$. Similarly, let η being calibrated such that $\eta = \eta_s \cdot zk^\alpha$. Assume, without any further restriction, $g_s \in (0, 1)$. Then, feasibility implies $\eta_s \in (0, 1 - \alpha)$, and θ can be recursively obtained from $\theta = (1 + n)(\eta - g)$.²⁶ In line with the main text, let $S = \{\alpha, \beta, \phi, n, z, \delta\}$ and $P = \{\eta, \theta, g\}$. We derive subsequently three critical conditions purely in terms of the variables $\alpha \in S$ and $\eta \in P$.²⁷ The three conditions, when jointly satisfied, are sufficient to ensure that

²⁶If $g_s \in (0, \alpha)$ the lower bound of η_s can be made negative (and maximum debt higher), contingent on g_s . Then, both η and g would enter the critical conditions derived below, without affecting, however, the logic of the proof.

²⁷At the expense of considerably more tedious algebra it is straightforward to extend the proof to the more general utility function $U(c, d) = \phi \ln c + \beta \ln d$, with $\phi > 0$ and $s_w = \beta / (\phi + \beta) \in (0, 1)$, thereby enlarging the number of critical variables in the set S . In any case, assuming $\phi = 0$ the following analysis indicates that (A 1) and (A 2) in Section 2 act only as widely used *sufficient* conditions that need not to be satisfied for the existence of a golden rule steady state.

there exists a golden rule steady state which has the property that the intersection of the two stability regions becomes empty if the level of steady-state debt exceeds a policy-induced threshold value $b_{gr}^* > 0$.

Step 1) Consider a steady state with $b = 0$ and $k > 0$, satisfying

$$k = \frac{(1 - \alpha)zk^\alpha - \eta}{1 + n} = \frac{(1 - \alpha - \eta_s)zk^\alpha}{1 + n} \Leftrightarrow k = \left[\frac{(1 - \alpha - \eta_s)z}{1 + n} \right]^{\frac{1}{1-\alpha}}.$$

A golden rule steady state satisfies

$$1 + n = R(k_{gr}) = \alpha z k_{gr}^{\alpha-1} \Leftrightarrow k_{gr} = \left(\frac{\alpha z}{1 + n} \right)^{\frac{1}{1-\alpha}}.$$

At any such steady state, the output level is independent of the set P . Hence, variations in η_s lead one-to-one to variations in η . Moreover, $b_{gr} > 0$ at the golden rule steady state if $k > k_{gr} \Leftrightarrow \eta_s < 1 - 2\alpha$. Note that $b_{gr} + k_{gr} = \frac{1}{1+n}(1 - \alpha - \eta_s)zk_{gr}^\alpha$, implying

$$b_{gr} = \left(\frac{1 - 2\alpha - \eta_s}{\alpha} \right) \cdot k_{gr} = \left(\frac{1 - 2\alpha - \eta_s}{\alpha} \right) \cdot \left(\frac{\alpha z}{1 + n} \right)^{\frac{1}{1-\alpha}}.$$

Evidently, for a given set S the level of b_{gr} declines in η_s , i.e. a more front-loaded funding of g via higher η_s reduces savings and thereby b_{gr} . For further reference, $R'(k_{gr}) \cdot b_{gr} = (1+n) \frac{(\alpha-1)}{\alpha} (1-2\alpha-\eta_s)$, $A_1 = 1+n$, and $A_2 = w'(k_{gr}) = (1+n)(1-\alpha)$. To sum up, combined with the feasibility condition, a golden rule steady state with $b_{gr} > 0$ exists if

$$\eta_s \in (0, 1 - 2\alpha), \quad (50)$$

and the upper bound of b_{gr} can be calculated as $\bar{b}_{gr} = \left(\frac{1-2\alpha}{\alpha} \right) \cdot k_{gr}$.

Step 2) Consider Figure 6. We derive a condition which ensures that both $p(0)|_\eta = 1$ and $p(1)|_\eta = 0$ slope downward in $\pi_b - \pi_k$ -space and have an intersection with coordinates $\pi_b^{**} > 0$ and $\pi_k^{**} < 0$. From the preliminaries to the proofs, using $s_w = 1$, one obtains

$$\begin{aligned} p(1)|_\eta > 0 &\Leftrightarrow \pi_k < R'(k_{gr}) \cdot b_{gr} + \frac{R'(k_{gr})b_{gr} + A_1 - A_2}{1 + n} \pi_b \\ \text{with } \pi_k^- &= R'(k_{gr}) \cdot b_{gr} < 0 \\ p(0)|_\eta < 1 &\Leftrightarrow \pi_k > A_2 - A_1 + \frac{R'(k_{gr})b_{gr} - A_2}{1 + n} \pi_b \\ \text{with } \pi_k^+ &= A_2 - A_1 < 0, \end{aligned}$$

where π_k^- and π_k^+ denote the intercepts if $\pi_b = 0$, respectively. Evidently, $p(0)|_\eta = 1$ slopes always downward in $\pi_b - \pi_k$ -space, while $p(1)|_\eta = 0$ slopes downward if

$\pi_k^+ > \pi_k^-$. Moreover, upon substituting out, the intersection of the two conditions at equality has coordinates

$$\begin{aligned}\pi_b^{**} &= A_2 - A_1 - R'(k_{gr}) \cdot b_{gr} = \pi_k^+ - \pi_k^- \\ \pi_k^{**} &= R'(k_{gr}) \cdot b_{gr} - \frac{(\pi_k^+ - \pi_k^-)^2}{1+n} < 0\end{aligned}$$

Hence, $\pi_b^{**} > 0 \Leftrightarrow \pi_k^+ > \pi_k^-$, and inserting the expressions established above gives

$$\begin{aligned}\pi_k^+ &> \pi_k^- \Leftrightarrow -\alpha(1+n) > (1+n)\frac{(\alpha-1)}{\alpha}(1-2\alpha-\eta_s) \\ &\Leftrightarrow \alpha^2 - 3\alpha + 1 - \eta_s(1-\alpha) > 0 \\ &\Leftrightarrow \eta_s < 1 - 2\alpha - \frac{\alpha^2}{1-\alpha}.\end{aligned}\tag{51}$$

Step 3) Finally, building on step 2, we derive a condition which ensures that parts i) and ii) of (49) and part iii) cannot be jointly satisfied. Using the notation of Figure 6, if $\pi_k^*(\pi_b^{**}) > \pi_k^{**}(\pi_b^{**})$ the intersection of $p(0)|_\eta = 1$ and $p(1)|_\eta = 0$ is to the southeast of $p(-1)|_g = 0$, since the latter equation slopes upward in $\pi_b - \pi_k$ -space. To see this, note that

$$p(-1)|_g > 0 \Leftrightarrow \pi_k > R'(k_{gr}) \cdot b_{gr} - 2(A_1 + A_2) + \frac{A_1 + A_2}{1+n}\pi_b.$$

Moreover, combining the expressions as inequalities implies that parts i) and ii) of (49) and part iii) cannot be jointly satisfied if $\pi_k^*(\pi_b^{**}) > \pi_k^{**}(\pi_b^{**})$. To establish this condition, evaluating $p(-1)|_g = 0$ at π_b^{**} yields

$$\pi_k^* = R'(k_{gr}) \cdot b_{gr} - 2(A_1 + A_2) + \frac{A_1 + A_2}{1+n}(A_2 - A_1 - R'(k_{gr})b_{gr}),$$

and by comparing the two terms one obtains $\pi_k^*(\pi_b^{**}) > \pi_k^{**}(\pi_b^{**}) \Leftrightarrow$

$$\left(\frac{A_2 - A_1 - R'(k_{gr})b_{gr}}{1+n}\right)^2 + \frac{A_1 + A_2}{1+n} \cdot \frac{A_2 - A_1 - R'(k_{gr})b_{gr}}{1+n} - \frac{2(A_1 + A_2)}{1+n} > 0.$$

Inserting the steady-state relations derived above, one obtains $\pi_k^*(\pi_b^{**}) > \pi_k^{**}(\pi_b^{**}) \Leftrightarrow$

$$\left(\frac{\alpha^2 - 3\alpha + 1 - \eta_s(1-\alpha)}{\alpha}\right)^2 + (2-\alpha) \cdot \frac{\alpha^2 - 3\alpha + 1 - \eta_s(1-\alpha)}{\alpha} > 2 \cdot (2-\alpha).\tag{52}$$

Using $\eta = \eta_s \cdot y_{gr}$, all three critical conditions (50), (51), (52) are solely expressed in terms of the variables $\alpha \in S$ and $\eta \in P$. By combining them appropriately it is possible to prove Propositions 3 and 4. To this end, we show that for a broad range

of plausible values of α all three conditions (50), (51), and (52) can be satisfied upon appropriate variations in η . Specifically, consider $0 < \alpha < \bar{\alpha} < 1/2$, with the critical bound $\bar{\alpha}$ being derived below. Then, there exists a unique $\eta_s^*(\alpha) \in (0, 1-2\alpha)$ which is associated with a unique $b_{gr}^* \in (0, \bar{b}_{gr})$ such that the golden rule steady state cannot be implemented under all three instruments if $\eta_s \in (0, \eta_s^*(\alpha)) \Leftrightarrow b_{gr} \in (b_{gr}^*, \bar{b}_{gr})$. To derive $\bar{\alpha}$ we address (51) and (52) in turn.

First, consider condition (51). Assume $\eta_s = 0$. Then, considering $\alpha \in (0, 1/2)$, condition (51) as a strict equality is satisfied by a unique $\bar{\alpha}_1 = 0.382$ and (51) holds for any $\alpha \in (0, \bar{\alpha}_1)$. The LHS of (51) falls in η_s and condition (51) defines for any $\alpha \in (0, \bar{\alpha}_1)$ a unique upper bound $\eta_s(\alpha) \in (0, 1-2\alpha)$.

Second, consider condition (52). Assume $\eta_s = 0$. Then, considering $\alpha \in (0, \bar{\alpha}_1)$, condition (52) as a strict equality is satisfied by a unique $\bar{\alpha} = 0.254$ and (52) holds for any $\alpha \in (0, \bar{\alpha})$. The LHS of (52) falls in η_s and condition (52) defines for any $\alpha \in (0, \bar{\alpha})$ a unique upper bound $\eta_s^*(\alpha) \in (0, \eta_s(\alpha))$, implying $\eta_s^*(\alpha) \in (0, 1-2\alpha)$.

Finally, from this reasoning it is clear that the implementability of the debt targeting rule under all three instruments even under high policy-induced debt levels may not be a problem for *all* sets S . If $\alpha \in (\bar{\alpha}, 1/2)$ then the upper bound of η_s defined in (51) and (52) can never be larger than the lower bound of zero defined in (50). \square

Appendix 2: Endogenous labour supply and distortionary taxes

Key features of the linearized dynamics under all three instruments:

As derived in the main text, consider the equations (41)-(43)

$$\begin{aligned} (1+n)(k_{t+1} + b_{t+1}) &= s(\tilde{w}(k_t, \eta_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})}, R(k_{t+1}, \eta_{t+1})) - \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})} \\ (1+n)b_{t+1} &= R(k_t, \eta_t)b_t - \pi_t \\ \pi_t &= \eta_t - \frac{\theta_t}{1+n} - g_t \end{aligned}$$

and let $A_1^* = 1+n - R_k \cdot [s_R + (1-s_w)\frac{\theta}{R(k)^2}]$ and $A_2^* = s_w \tilde{w}_k$. For variations in g_t and θ_{t+1} the taxation term η_t will be held constant. Hence, because of the structural similarity to the analysis given in Appendix 1, it is clear that the characteristic polynomials in these two cases are given by

$$p(\lambda)|_g = \lambda^2 - \left[\frac{A_2^*}{A_1^*} + \frac{R(k) - \pi_b}{1+n} - \frac{R'(k)b - \pi_k}{A_1^*} \right] \cdot \lambda + \frac{A_2^*}{A_1^*(1+n)} (R(k) - \pi_b)$$

and

$$\begin{aligned}
p(\lambda)|_{\theta} &= \lambda^2 - \vartheta_1^* \lambda + \vartheta_2^*, \text{ with:} \\
\vartheta_1^* &= \frac{A_2^*}{\tilde{A}_1^*} + \frac{R(k) - \pi_b}{1+n} - \frac{R'(k)b - \pi_k}{\tilde{A}_1^*} + \frac{(1-s_w)\frac{1+n}{R(k)}\pi_b(R'(k)b - \pi_k)}{\tilde{A}_1^*(1+n)} \\
\vartheta_2^* &= \frac{A_2^*}{\tilde{A}_1^*(1+n)}(R(k) - \pi_b), \\
\text{with: } \tilde{A}_1^* &= A_1^* - (1-s_w)\frac{1+n}{R(k)}\pi_k
\end{aligned}$$

Regarding η_t , the linearized dynamics, similar to (18) and (19) and by using $\eta_k = \pi_k$ and $\eta_b = \pi_b$, can be summarized as

$$\begin{aligned}
(A_1^* - \psi_1 \pi_k) \cdot dk_{t+1} + (1+n - \psi_1 \pi_b) \cdot db_{t+1} &= (A_2^* + \psi_2 \pi_k) \cdot dk_t + \psi_2 \pi_b \cdot db_t \\
\psi_1 &= R_\eta \left[s_R + (1-s_w) \frac{\theta}{R(k)^2} \right] \\
\psi_2 &= s_w (\tilde{w}_\eta - 1)
\end{aligned}$$

$$(1+n) \cdot db_{t+1} = [(R_k + R_\eta \pi_k)b - \pi_k] \cdot dk_t + [R(k) - \pi_b + R_\eta \pi_b b] \cdot db_t,$$

giving rise to the characteristic polynomial

$$\begin{aligned}
p(\lambda)|_{\eta} &= \lambda^2 - \zeta_1^* \lambda + \zeta_2^*, \text{ with:} \tag{53} \\
\zeta_1^* &= \frac{A_2^* + \psi_2 \pi_k}{A_1^* - \psi_1 \pi_k} + \frac{R(k) - \pi_b + R_\eta \pi_b b}{1+n} - \frac{[(R_k + R_\eta \pi_k)b - \pi_k](1+n - \psi_1 \pi_b)}{(A_1^* - \psi_1 \pi_k)(1+n)} \\
\zeta_2^* &= \frac{(A_2^* + \psi_2 \pi_k)[R(k) - \pi_b + R_\eta \pi_b b] - [(R_k + R_\eta \pi_k)b - \pi_k]\psi_2 \pi_b}{(A_1^* - \psi_1 \pi_k)(1+n)}.
\end{aligned}$$

Note that (53) reduces to (48) if $R_\eta = 0$ (and, hence, $\psi_1 = 0$) and $\tilde{w}_\eta = 0$ and if one replaces A_1^* and A_2^* by A_1 and A_2 .

Derivation of (46) and (47):

Consider $\varphi(l) = \frac{\xi}{1+\chi} l^{1+\chi}$, with $\varphi'(l) = \xi l^\chi$. At the steady state, $w = \varphi'(l)$, since $\tau_t = 0$. Hence, $\tilde{w} = l\varphi'(l) - \varphi(l) = \xi \frac{\chi}{1+\chi} l^{1+\chi}$. Using $l_\eta(k, \eta) = \frac{-1}{l \cdot [\varphi''(l) - F_{LL}(k, l)]} < 0$, (39) and (40) turn into:

$$\begin{aligned}
R_\eta &= F_{KL} \cdot l_\eta = \frac{-F_{KL}(k, l)}{l \cdot [\varphi''(l) - F_{LL}(k, l)]} < 0 \\
\tilde{w}_\eta &= l \cdot F_{LL} \cdot l_\eta = \frac{-F_{LL}(k, l)}{\varphi''(l) - F_{LL}(k, l)} > 0.
\end{aligned}$$

Consider $F(K, L) = zK^\alpha L^{1-\alpha}$, with $F_L(k, l) = z(1-\alpha)k^\alpha l^{-\alpha}$. Using the steady-state relationship $\varphi'(l) = w = F_L(k, l)$ one obtains upon differentiation, $R_\eta = -\frac{\alpha}{\alpha+\chi} \cdot \frac{1}{k}$ and $\tilde{w}_\eta = \frac{\alpha}{\alpha+\chi} \in (0, 1)$. \square

Figure 1a: Underaccumulation steady state

Stability triangle associated with g_t -regime (example 1)

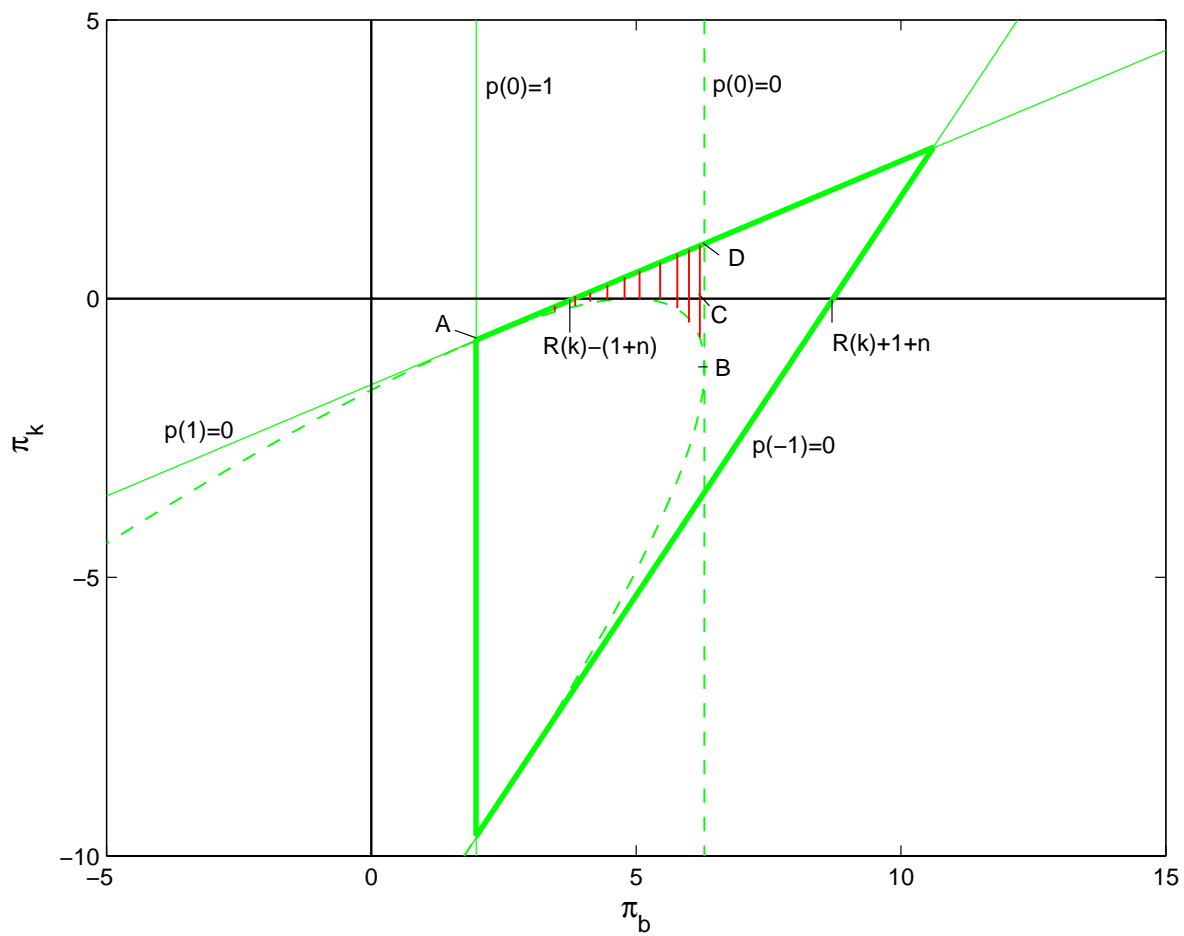


Figure 1b: Underaccumulation steady state

Adjustment paths under g_t -regime for different feedback coefficients (example 1)

- B:* dashed (blue) line
- C:* solid (black) line
- D:* dashed-dotted (red) line

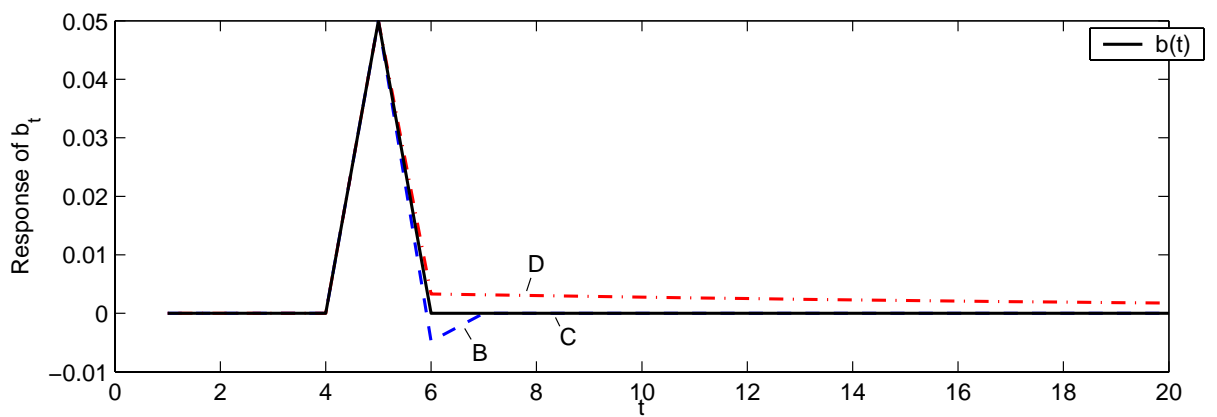
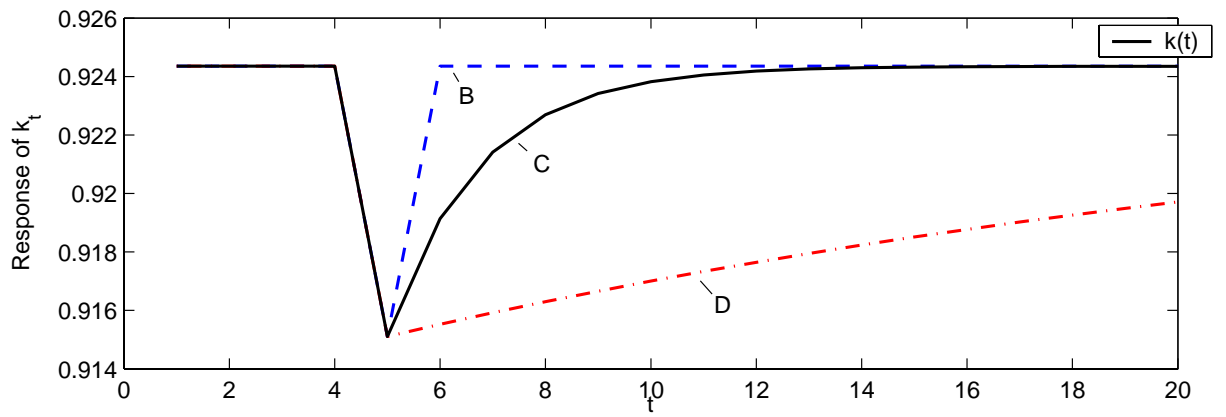


Figure 1c: Underaccumulation steady state

Stability regions for all three instruments (example 1)

Shaded area: common intersection

g_t -regime: solid (green) line

η_t -regime: dashed (blue) line

θ_{t+1} -regime: dashed-dotted (red) line

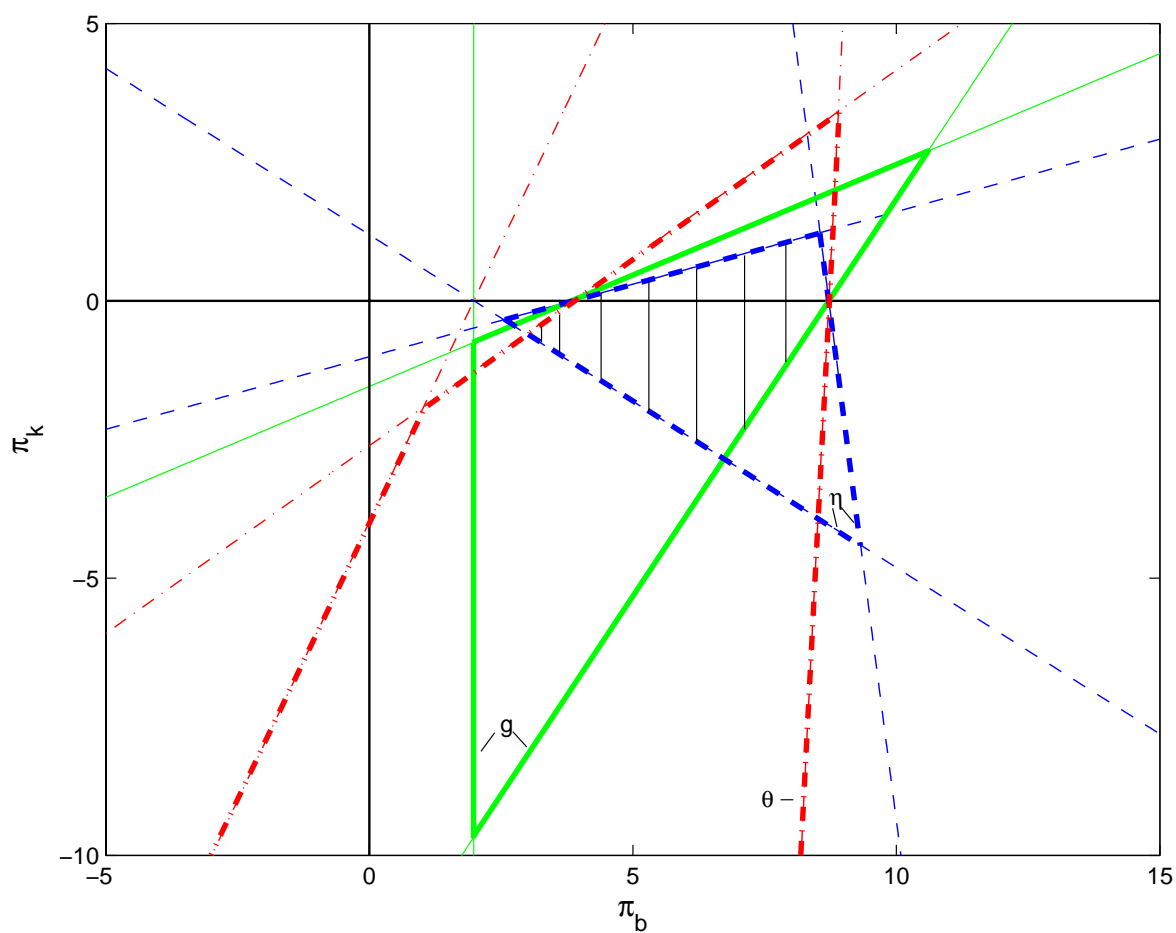


Figure 3: Golden rule steady state

Stability regions for all three instruments (example 3)

g_t -regime: solid (green) line
 η_t -regime: dashed (blue) line
 θ_{t+1} -regime: dashed-dotted (red) line

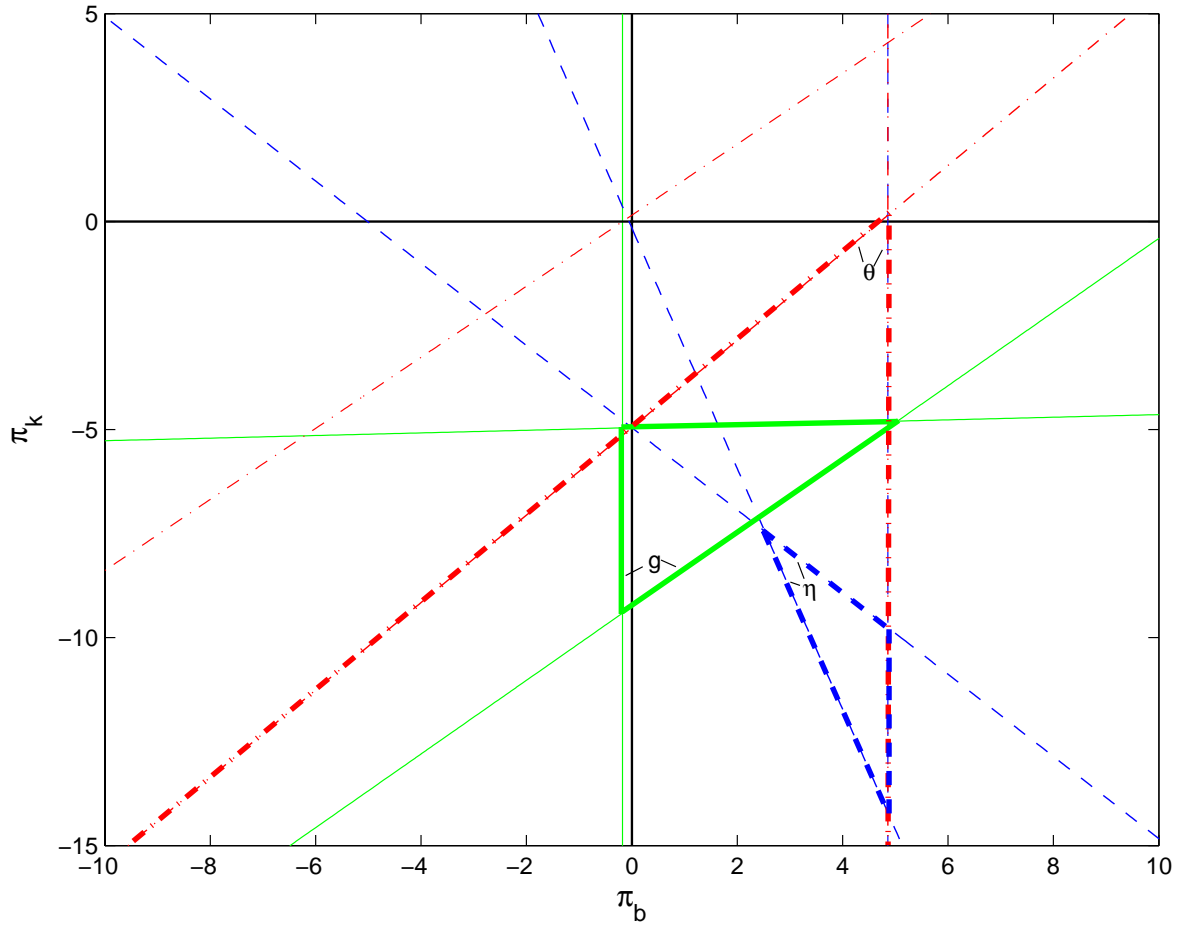


Figure 4: Golden rule steady state under distortionary taxation

Stability regions for all three instruments (example 4)

g_t -regime: solid (green) line
 η_t -regime: dashed (blue) line
 θ_{t+1} -regime: dashed-dotted (red) line

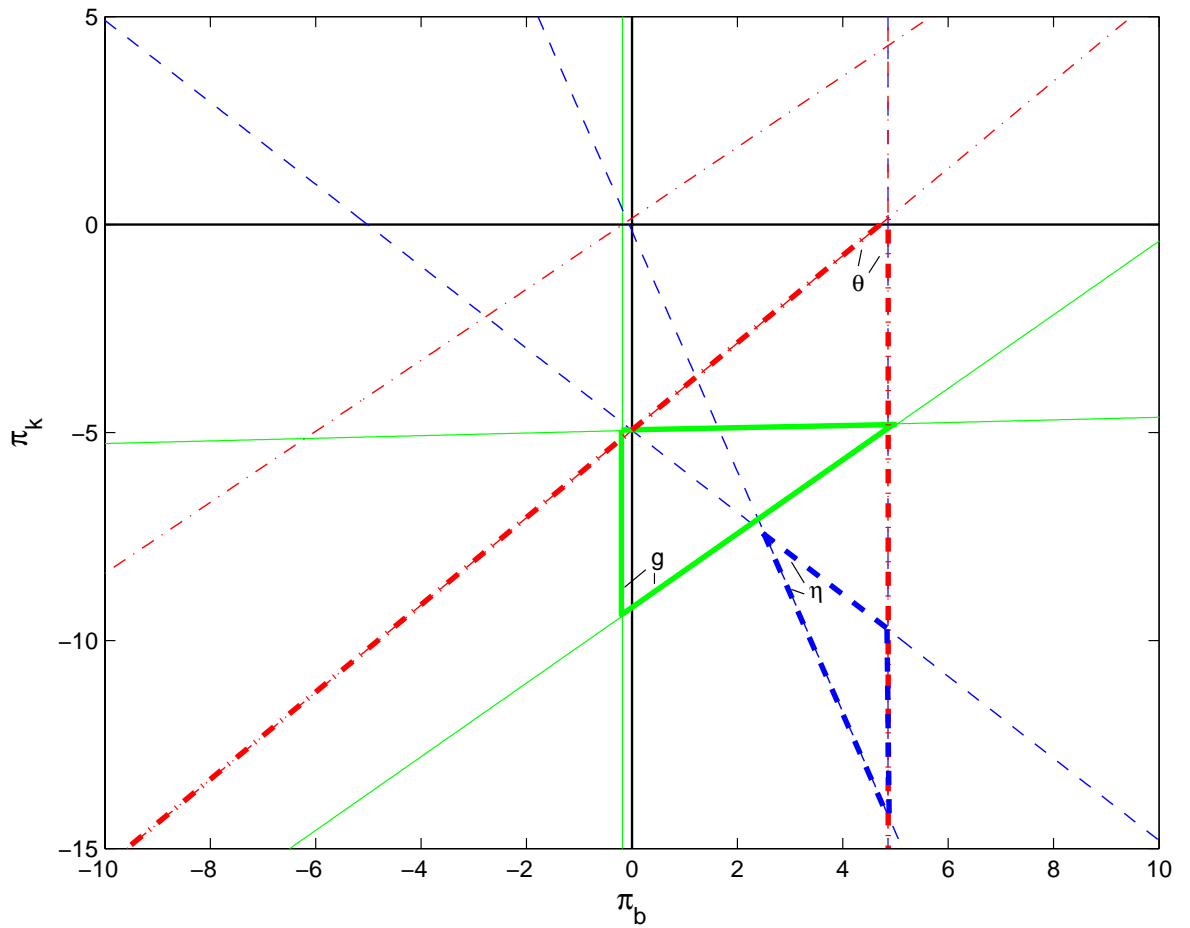


Figure 5: Golden rule steady state under distortionary taxation

Stability regions for all three instruments (example 5)

Shaded area: common intersection

g_t -regime: solid (green) line
 η_t -regime: dashed (blue) line
 θ_{t+1} -regime: dashed-dotted (red) line

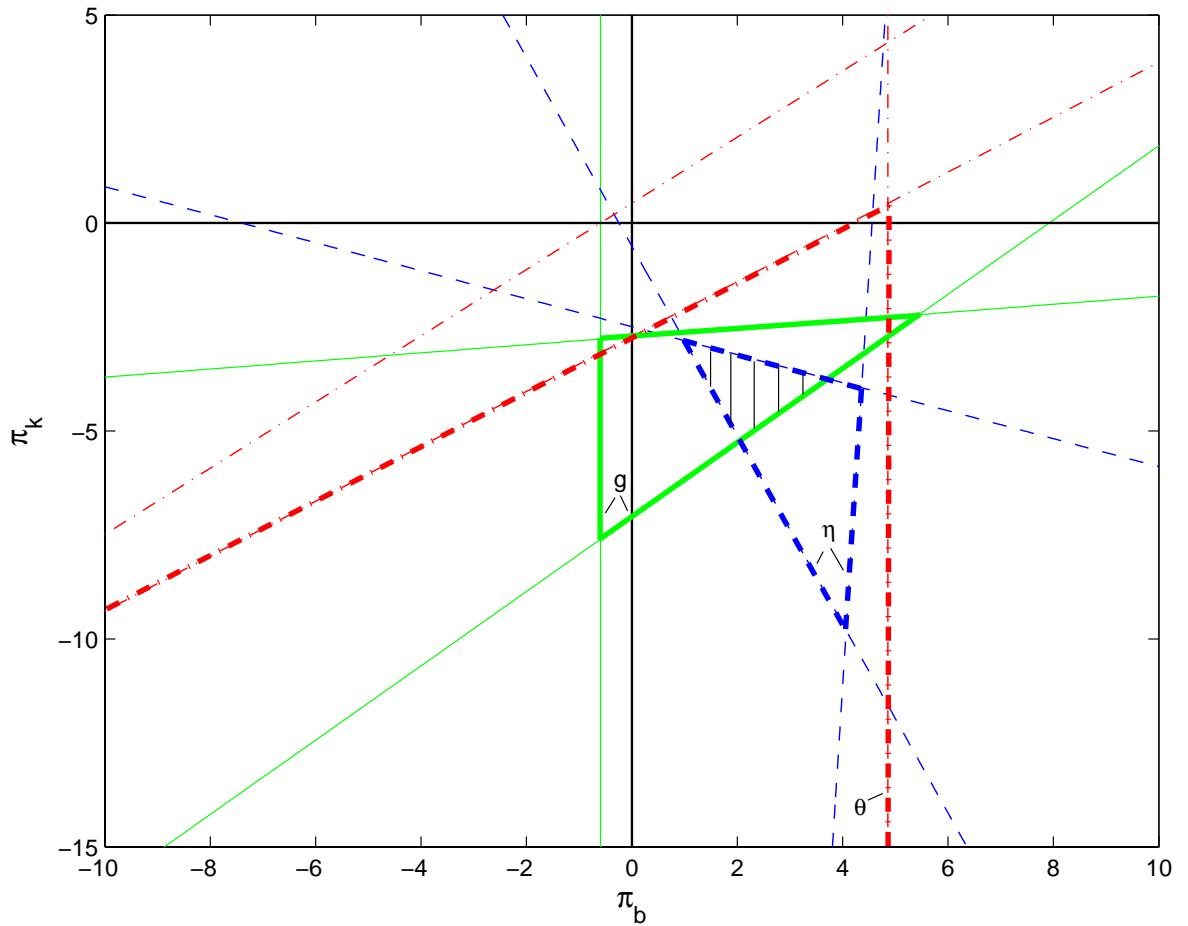


Figure 6: Golden rule steady state

Empty intersection of g_t -triangle and η_t -triangle (Proof of Proposition 3)

g_t -regime: solid (green) line
 η_t -regime: dashed (blue) line

Shaded areas represent necessary conditions for stability under the two regimes

