Higher-Order "Perturbation" Solutions to Dynamic, Discrete-Time Rational Expectations Models

Methods and an Application to Optimal Monetary Policy

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Abstract

We present an algorithm and software routines for computing nth-order approximate solutions to dynamic, discrete-time rational expectations models around a nonstochastic steady state. We apply these routines to investigate the optimal monetary policy with commitment in an optimizing-agent model with nominal price rigidities, subject to a fiscal policy that is stochastic, suboptimal, and exogenous to the central bank.

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1. Introduction

An increasing number of authors have found the standard log-linearization procedure in macroeconomics insufficient for solution of interesting problems. Kim and Kim (2002) and Kollmann (2002a,b) note the importance of second-order approximations for measuring welfare gains from trade and international monetary policy coordination. Gaspar and Judd (1997), Judd (1999), and Woodford (2000) note the importance of second-order terms in the law of motion of the economy for calculating optimal policy in many situations that arise in practice.

In this paper, we present an algorithm and software that computes an nth order Taylor series approximation to the solution of a dynamic, discrete-time set of rational expectations equations around a nonstochastic steady state. Such approximate solutions are referred to as "perturbation" methods by Judd (1999). Our routines represent an improvement over other authors' work in this area in that we can approximate the true solution to arbitrarily high order.

We apply our routines to the calculation of optimal monetary policy for a practical problem of interest: the optimal monetary policy, and welfare gains from such a policy, in a closed economy with nominal price rigidities subject to a fiscal policy that is stochastic, suboptimal, and exogenous to the central bank. We are interested, in particular, in whether and to what extent the presence of suboptimal fiscal policy creates an incentive for monetary policy to be "asymmetric" and to stimulate the economy if the monopolistic and fiscal wedges in the model lead the economy to underproduce in steady state (relative to the Pareto optimum).

2. The Algorithm

We consider a system of dynamic, discrete-time rational expectations equations of the form:

$$E_t\, F(x_{t-\underline{\theta}}, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_{t+\overline{\theta}}\,;\, \varepsilon_t, \varepsilon_{t+1}, \ldots, \varepsilon_{t+\overline{\phi}}) = 0 \tag{1}$$

where F, x_t, and ε_t are vectors of dimensions n_F, n_x, and n_ε, respectively, E_t denotes the mathematical expectation conditional on all variables dated t and earlier, $\{\varepsilon_t\}$ is an exogenous stochastic process, and it is understood that the system of equations (1) is

satisfied at each time t, $t+1$, $t+2$, etc. The parameters $\underline{\theta}$, $\overline{\theta}$, and $\overline{\phi}$ denote the maximum number of lags and leads required to describe the equations in system (1).[1,2]

We will assume that the stochastic shocks ε_t are i.i.d. across time and that the components of ε_t (denoted ε_{it}, $i = 1, \ldots, n_\varepsilon$) are mutually independent as well. In other words, any cross-sectional or intertemporal correlation of the $\{\varepsilon_t\}$ process must be explicitly specified by the modeler as sums of individual components ε_{it}. We assume that $E[\varepsilon_{it}] = 0$, and we let $\text{Mom}_n(\varepsilon_{it})$ denote $E[\varepsilon_{it}^n]$, the nth moment of ε_{it}. When an nth moment is specified, we require that it exists. We require no other distributional assumptions regarding the ε_t, and denote the distribution function for ε_t by $\Psi(z)$.

We look for time-invariant, analytic, ergodic solutions to (1) of the form:[3]

$$x_t \;=\; b(x_{t-\underline{\theta}}, \ldots, x_{t-1}\,;\, \varepsilon_t) \tag{2}$$

In other words, variables dated $t-1$ and earlier are regarded as having been observed, and the vector x_t is to be solved as a function of these lagged values and the observed stochastic shock ε_t. In particular, we do not require the modeler to specify some components of x_t as being "state" or "predetermined" variables and the remaining components as being "co-state" or "jump" variables.[4] It should also be noted that the solution (2) depends as well on the coefficients and parameters of the model (1), and in particular on the moments of the stochastic disturbances ε_t.

Following Judd (1999), we let $\sigma \in [0, 1]$ denote an auxiliary, "scaling" parameter for the distribution of the stochastic shocks ε_t in (1). In particular, we consider a continuum of auxiliary models (1)$'$, parameterized by $\sigma \in [0, 1]$, each identical to (1) in every respect

[1] Lagged expectational terms can be incorporated into (1) by appropriate definition of auxiliary variables; e.g., by setting $b_t = E_t a_{t+1}$ and then considering b_{t-1}.

[2] In most macroeconomic models, $\overline{\phi}$ is either 0 or 1—in other words, no shocks dated later than $t+1$ are required to describe the system of equations (1). Stochastic shocks dated $t-1$ and earlier can be regarded as variables x—e.g., by setting $a_t = \varepsilon_t$, and considering a_{t-1}.

[3] The notation b generalizes Anderson and Moore (1985) and the AIM procedure, which produce a linear solution matrix B to a linear system of equations. By time-invariant, the solution function b is understood to yield $x_{t+k} \;=\; b(x_{t+k-\underline{\theta}}, \ldots, x_{t+k-1}\,;\, \varepsilon_{t+k})$ for all $k >= 0$. Because our solution algorithm relies on a local dth-order Taylor approximation, we require b to have a d derivatives; for the algorithm to converge to the true solution b as $d \to \infty$, we require b to be analytic. The ergodicity requirement rules out "bubbles" and solution functions b that are globally explosive.

[4] This is just as in the linear case: see Anderson and Moore (1985) and Sims (2000). Intuitively, the computer can figure out what linear combinations of variables are "predetermined" from the fact that variables dated $t-1$ or earlier are known. For example, a clear "predetermined" variable, such as $K_t = (1-\delta)K_{t-1} + I_{t-1}$, falls out of the solution algorithm into the form (2) trivially.

except that the distribution function for ε_t in these auxiliary models is given by $\Psi(z/\sigma)$ instead of by $\Psi(z)$. Thus, $\sigma = 1$ corresponds to the original model (1), which is to be solved, while small values of σ correspond to versions of the model with relatively little uncertainty. The case $\sigma = 0$ is taken to mean $\varepsilon_t = 0$ with probability 1—i.e., a deterministic version of the model.

Thus, we are considering a family of models of the form:

$$E_t\, F(x_{t-\underline{\theta}}, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_{t+\overline{\theta}}\,;\, \varepsilon_t, \varepsilon_{t+1}, \ldots, \varepsilon_{t+\overline{\phi}}\,;\, \sigma) = 0$$
$$\varepsilon_s \sim iid\ \Psi(z/\sigma),\ s > t$$

or, equivalently,

$$E_t\, F(x_{t-\underline{\theta}}, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_{t+\overline{\theta}}\,;\, \varepsilon_t, \sigma\varepsilon_{t+1}, \ldots, \sigma\varepsilon_{t+\overline{\phi}}\,;\, \sigma) = 0 \qquad (1)'$$
$$\varepsilon_s \sim iid\ \Psi(z),\ s > t$$

to which we are looking for a family of solutions indexed by σ:

$$x_t \;=\; b(x_{t-\underline{\theta}}, \ldots, x_{t-1}\,;\, \varepsilon_t\,;\, \sigma) \qquad (2)'$$

We have recycled the letters F and b here, but there is no risk of confusion as we will henceforth only refer to the generalized family of equations $(1)'$ and $(2)'$, and specify $\sigma = 1$ when we wish to refer to the original model (1) and solution (2).

Note in particular that we do *not* scale the value of the shock ε_t in $(1)'$ and $(2)'$ by σ, because ε_t is known at time t and because it is often the case in practice that the modeler wishes to shock a deterministic or "perfect foresight" model—i.e., a model for which $\sigma = 0$. Specifications $(1)'$ and $(2)'$ are the proper parameterizations that allow the researcher to perform this kind of "counterfactual" experiment in which agents are completely surprised by a shock that they did not think could occur.[5]

2.1 Approximate Solutions to the Model

Finding the nonlinear solution function b is difficult in general. As is standard practice,

[5] There is an earlier literature on "perfect foresight" solutions of nonlinear rational expectations models (e.g., Anderson (1993), Fuhrer and Madigan (1997), and the Troll software package), which solve the model to numerical precision imposing the constraint that $\sigma = 0$. In these algorithms, the modeler can still see how the perfect foresight solution reacts to a shock to the system, graph impulse responses to a shock, and so on. Our perturbation approach nests this older literature very naturally.

we assume that the modeler can solve $(1)'$ for a nonstochastic steady state $\overline{x}$:

$$F(\overline{x},\ldots,\overline{x},\ldots,\overline{x}\,;\,0,\ldots,0\,;\,0)=0 \tag{3}$$

so that we have:

$$\overline{x}\;=\;b(\overline{x},\ldots,\overline{x}\,;\,0\,;\,0) \tag{4}$$

We then assume that there exists a neighborhood of this steady state for which the solution function b exists and is unique. Following the method described by Judd (1999), we can proceed to calculate the second- and higher-order derivatives of the unknown function b at the known point $(\overline{x},\ldots,\overline{x}\,;\,0\,;\,0)$ by repeatedly applying the implicit function theorem to the known system $(1)'$ at the known point $(\overline{x},\ldots,\overline{x},\ldots,\overline{x}\,;\,0,\ldots,0\,;\,0)$. In particular, we substitute the relationships:

$$x_t\;=\;b(x_{t-\underline{\theta}},\ldots,x_{t-1}\,;\,\varepsilon_t\,;\,\sigma) \tag{2}'$$

$$x_{t+1}\;=\;b\big(x_{t-\underline{\theta}+1},\ldots,x_{t-1},b(x_{t-\underline{\theta}},\ldots,x_{t-1};\varepsilon_t;\sigma)\,;\,\sigma\varepsilon_{t+1}\,;\,\sigma\big) \tag{2}''$$

etc.

into $(1)'$ and use the implicit function theorem to compute the derivatives of b with respect to $x_{t-\underline{\theta}},\ldots,x_{t-1}$, ε_t, and σ at the point $(\overline{x},\ldots,\overline{x}\,;\,0\,;\,0)$.

In this sense, our solution to $(1)'$ is only valid when the lagged values $x_{t-\underline{\theta}},\ldots,x_{t-1}$ lie within a sufficiently small neighborhood of $\overline{x}$, and ε_t and σ lie within a sufficiently small neighborhood of 0. We also require F and b to be sufficiently smooth within these neighborhoods.

It is important to emphasize, however, that so long as the solution function b is analytic, then the algorithm above is guaranteed to converge to b everywhere within the domain of convergence (the multidimensional analog of the radius of convergence) of the Taylor series expansion of b around the point $(\overline{x},\ldots,\overline{x}\,;\,0\,;\,0)$.[6] This domain is potentially very large—in some models it can even be all of $\mathbb{R}^{\underline{\theta}n_x+n_\varepsilon+1}$. Thus, there is a very rigorous sense in which our algorithm is *globally*—and not just locally—valid. In the limit, as we let the order of approximation d tend to infinity, we arrive at the true nonlinear solution function b over the entire domain of convergence.

[6] Moreover, this convergence is uniform on compact subsets, no matter how large the subsets. This is a standard result in complex analysis—see, e.g., Ahlfors (1979).

In contrast to other researchers (e.g., Schmitt-Grohe and Uribe (2001), Kim et al. (2002)), our algorithm and computer code allow for approximation of the function b to arbitrary order d, as opposed to simply the second order. We are indebted to Judd (1999) for making the essential point that, once the first derivatives of b have been calculated, one needs only to solve a straightforward, recursive linear problem to calculate the derivatives of b to each successive order. We use AIM, a generalization of the Blanchard-Kahn (1980) algorithm developed by Anderson and Moore (1985), to arrive at the first derivatives of b. This requires that the solution function b be first-order stable local to the nonstochastic steady state, a standard assumption in the literature.[7]

2.2 Computer Implementation of the Algorithm

We implemented the solution procedure described above in Mathematica. While second-order approximations can be done fairly easily in any computer language, generalization to higher orders is dramatically simpler in Mathematica, and the large library of built-in routines for symbolic differentiation and solving linear systems of equations in Mathematica has the following advantages:

A. Allows for much simpler and intuitive code (less than 200 lines), making the algorithm easy to understand and more likely to be free of bugs.

B. Eliminates the need for user-written differentiation or linear solution routines, which are likely to be more amateur than built-in Mathematica routines and hence more likely to suffer from bugs and numerical inaccuracies.

C. Allows for easy symbolic differentiation, improving numerical accuracy, particularly at higher orders.

D. Allows the algorithm to proceed in exacty the same way and produce exactly the same output as if the user were performing it by hand, making the algorithm and results simpler and more intuitive.

E. Allows the option of computing all coefficients to arbitrarily high precision (although working with arbitrary-precision rather than machine-precision numbers has a significant cost in terms of computation time).

[7] The AIM algorithm treats unit roots as stable, so unit roots in the first-order approximation satisfy this stability requirement. Anderson (2001) shows that AIM is significantly faster and more numerically robust than the numerous other alternatives available. An additional advantage of AIM is that it does not require the modeler to designate some variables as being "predetermined" variables and the rest as being "jump" variables, as discussed previously. Software implementations of AIM in a variety of languages can be obtained from the Federal Reserve Board's public web site.

The specific implementation of the algorithm proceeds as follows:

1. Find the nonstochastic steady state solution $(\overline{x}, \ldots, \overline{x}, \ldots, \overline{x}\,;\, 0, \ldots, 0\,;\, 0)$ to $(1)'$.

2. Compute the first-order solution: Differentiate the function F with respect to each of its arguments and evaluate them at steady state. Write out the first-order Taylor series approximation to F as a linear model in its arguments and find the solution to this model using AIM. This yields the first derivatives of b with respect to each of its arguments.

3. Define unknown solution function b^{x_i} for each element x_i of x, $i = 1, \ldots, n_x$, with arguments $(x_{t-\underline{\theta}}, \ldots, x_{t-1}\,;\, \varepsilon_t\,;\, \sigma)$. Let b denote $(b^{x_1}, \ldots, b^{x_{n_x}})'$.

4. Substitute out for x_{t+k}, $k > 0$, in $(1)'$ using the relationships $(2)'$, $(2)''$, etc. For notational simplicity, denote the result by:

$$E_t\,(F \circ b) = 0 \tag{5}$$

Note that $F \circ b$ has arguments $(x_{t-\underline{\theta}}, \ldots, x_{t-1}\,;\, \varepsilon_t, \ldots, \varepsilon_{t+\max(\overline{\theta},\overline{\phi})}\,;\, \sigma)$.

Now, for each order $d > 1$, calculate the dth derivatives of b as follows:

5. Calculate all the dth order derivatives of $F \circ b$ with respect to its arguments and save the result for differentiation to higher orders. The derivatives of F are known (and are symbolic), the derivatives of b up through degree $d-1$ are unknown in general (and are symbolic) but are known at the steady state (and are numerical). The dth order derivatives of b are unknown (and are symbolic).

6. Evaluate the first derivatives of $F \circ b$ at steady state $(\overline{x}, \ldots, \overline{x}; 0, \ldots, 0; 0)$. This yields numerical values for the (known) derivatives of F, and write out the result as a first-order Taylor approximation to $F \circ b$ with undetermined coefficients.

7. Take the expectation implied by (5) for this approximation, which converts terms involving ε_{t+k}, $k > 0$ into the (known) corresponding moments. Solve for the undetermined dth order derivatives of b by setting each coefficient of the Taylor series equal to zero (i.e., the method of undetermined coefficients) and solving the resulting linear system of equations.

Repeat steps 5–7 for each successive order d, as desired.

The Mathematica code implementing this algorithm is quite succinct (less than 200 lines) and can be downloaded from http://www.mindspring.com/∼eswanson, or is available from the authors upon request. All that is required of the modeler is to specify a "model file" with the equations of F written out in standard Mathematica notation, using time

indices $t-1$, t, $t+1$, etc. to convey to the software the correct intertemporal relationship of the variables in the model. Simple examples of model files are downloadable along with the algorithm.

2.3 Example: Stochastic Growth Model

A simple model that illustrates the timing conventions and convenience of the algorithm is the standard stochastic growth model:

$$Y_t = A_t K_t^{\alpha} \tag{6}$$

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t \tag{7}$$

$$K_t = (1-\delta)K_{t-1} + I_{t-1} \tag{8}$$

$$Y_t = C_t + I_t \tag{9}$$

$$C_t^{-\gamma} = \beta E_t\left[(1+r_{t+1})\, C_{t+1}^{-\gamma}\right] \tag{10}$$

$$r_t = \alpha A_t K_t^{\alpha-1} - \delta \tag{11}$$

$$\text{Welf}_t = \frac{C_t^{1-\gamma}}{(1-\gamma)} + \beta E_t \text{Welf}_{t+1} \tag{12}$$

We desire a solution to (6)–(12) for A_t, C_t, I_t, K_t, r_t, Y_t, and Welf_t as functions of variables dated $t-1$, ε_t, and σ. (In fact, the only time $t-1$ variables in the solution will be A_{t-1}, I_{t-1}, and K_{t-1}, as the algorithm recognizes that these are sufficient, along with ε_t and the auxiliary parameter σ, to completely characterize the inherited economic state.)

Equations (6)–(12) can be entered into the model file essentially exactly as they are written above (although the explicit appearance of E_t should be dropped, because the algorithm automatically takes the time-t expectation of each equation in step 7). In particular, the researcher does not need to classify some variables as "predetermined" and others as being "jump" variables—the computer does this automatically—the researcher does not need to substitute out identities such as (9), and the researcher is free to tack on auxiliary equations such as (12), which have no effect on the solution to the other variables of the system, but nonetheless produce a result that is of interest. Moreover, the researcher can specify that some variables should be transformed into logarithms so as to log-linearize (and log-higher-order) rather than linearize the variables in question.

The explicit second-order solution to this model is provided in Appendix A.

3. Application: Optimal Monetary Policy in an Optimizing-Agent Model with Nominal Price Rigidity

We apply the above algorithm to the problem of optimal monetary policy. Before calculating the optimal policy, we must first specify a framework in which the monetary authority operates. We take as our baseline an optimizing-agent, Calvo-style sticky price framework along the lines of Yun (1996), taken from Erceg, Henderson, and Levin (2000) without the sticky wage assumption. Generalization to the case of nominal wage rigidity is straightforward but adds several additional equations and first-order conditions, complicating the presentation.

Readers are referred to Erceg, Henderson, and Levin (2000) for details of the model and derivations of the pricing equations; we provide only the essential elements here. The utility function of the representative agent is given by:

$$\text{Welf}_t \;=\; E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{C_s^{1-\varphi}}{1-\varphi} \;-\; \frac{\chi_0 L_s^{1+\chi}}{1+\chi} \;+\; \frac{\mu_0 (1 - M_s/P_s)^{1-\mu}}{1-\mu} \right]$$

where C_s denotes consumption, L_s labor, M_s nominal money balances, and P_s the price level. Like EHL, we regard the parameter μ_0 as being arbitrarily small, so that we can neglect it in calculations of utility, while still allowing the monetary authority to control nominal interest rates by setting the money supply.

There is a continuum of producers on the unit interval with identical, constant-returns-to-scale production functions, who set prices and produce to meet demand as in Calvo (1983). If a firm is not allowed to reset its price, its price is indexed at gross rate Φ (which could be 1, reducing to the standard Calvo model without indexation). Consumers have demand that is CES across the continuum of varieties, and P_t is the corresponding CES price index. We abstract away from capital accumulation and depreciation, hence consumption C_t equals output Y_t.

The equations of the model are as follows:

Pricing:

$$z_{n,t} \;=\; Y_t^{1-\varphi} \text{mcr}_t \;+\; \xi\beta\Phi^{-(1+\theta)/\theta} E_t \pi_{t+1}^{(1+\theta)/\theta} z_{n,t+1} \tag{13}$$

$$z_{d,t} \;=\; (1-\tau_t) Y_t (Y_t - G_t)^{-\varphi} \;+\; \xi\beta\Phi^{-1/\theta} E_t \pi_{t+1}^{1/\theta}\, z_{d,t+1} \tag{14}$$

$$p_{0t} \;=\; (1+\theta) z_{n,t} / z_{d,t} \tag{15}$$

$$\pi_t^{-1/\theta} = (1-\xi)(p_{0t}\pi_t)^{-1/\theta} + \xi\Phi^{-1/\theta} \tag{16}$$

$$\text{Disp}_t = (1-\xi)p_{0t}^{-(1+\theta)/\theta} + \xi(\pi_t/\Phi)^{(1+\theta)/\theta}\,\text{Disp}_{t-1} \tag{17}$$

Production:

$$Y_t = \text{Disp}_t^{-1}\bar{K}^{\alpha}L_t^{1-\alpha} \tag{18}$$

$$\text{mcr}_t = \frac{Y_t^{\varphi}\chi_0 L_t^{\chi}}{(1-\alpha)\,\bar{K}^{\alpha}L_t^{-\alpha}} \tag{19}$$

Government:

$$\log(G_t/\bar{G}) = \rho_g \log(G_{t-1}/\bar{G}) + \varepsilon_t^G \tag{20}$$

$$\tau_t - \bar{\tau} = \rho_\tau(\tau_{t-1} - \bar{\tau}) + \varepsilon_t^\tau \tag{21}$$

Euler Equation & Welfare:

$$(Y_t - G_t)^{-\varphi} = \beta\,\text{Int}_t\,E_t(1/\pi_{t+1})\,(Y_{t+1} - G_{t+1})^{-\varphi} \tag{22}$$

$$\text{Welf}_t = \frac{(Y_t - G_t)^{1-\varphi}}{1-\varphi} - \chi_0\frac{L_t^{1+\chi}}{1+\chi} + \beta\,E_t\text{Welf}_{t+1} \tag{23}$$

where $\bar{K}$ the given level of capital, Y_t output, Int_t the gross nominal interest rate, π_t the gross inflation rate, mcr_t the marginal cost of a firm (which is the same across firms), p_{0t} the relative price set by firms who are allowed to set prices in period t, and z_{1t}, z_{2t}, and Disp_t are auxiliary variables that allow us to express infinite sums from the agents' first-order conditions in a recursive form. Equation (23) similarly expresses agents' welfare in a recursive form. The parameter ξ is one minus the probability of receiving the Calvo signal to change price, $1+\theta$ is the CES aggregator parameter (hence θ is the steady-state markup of price over marginal cost), and τ_t is the tax on output (or subsidy if negative), which is funded by a lump-sum tax (or rebate) on agents.

There are two sources of uncertainty in the model: government spending shocks and tax shocks, denoted by the ε_t^G and ε_t^τ. The presence of fiscal policy shocks introduces a tradeoff for the monetary authority between stabilizing prices and stabilizing output: On one hand, the economy is most efficient when all producers are charging the same price, so the central bank would like to focus policy on achieving this goal; on the other hand, fluctuations in government spending and the output tax (or subsidy) create temporary fluctuations in output and employment that the central bank would like to offset. The monetary authority cannot in general achieve both goals.

3.1 Optimal Monetary Policy

The optimal monetary policy with commitment chooses Int_t (and a state-contingent plan for Int_s, $s > t$) so as to maximize the representative agent's welfare subject to constraints (13)–(22).[8]

Writing out the monetary policymakers' Lagrangean and differentiating (Mathematica can do this step automatically, increasing convenience and reducing human error) yields first-order conditions:

$$\lambda_t^{z_n} = \lambda_{t-1}^{z_n}\xi\Phi^{-(1+\theta)/\theta}\pi_t^{(1+\theta)/\theta} + \lambda_t^p(1+\theta)/z_{dt} \tag{24}$$

$$\lambda_t^{z_d} = \lambda_{t-1}^{z_d}\xi\Phi^{-1/\theta}\pi_t^{1/\theta} - \lambda_t^p(1+\theta)z_{nt}/z_{dt}^2 \tag{25}$$

$$\lambda_t^p = \lambda_t^\pi(1-\xi)(-1/\theta)(p_{0t}\pi_t)^{-(1+\theta)/\theta}\pi_t + \lambda_t^{\text{Disp}}(1-\xi)\big(-(1+\theta)/\theta\big)p_{0t}^{-(1+2\theta)/\theta} \tag{26}$$

$$\lambda_t^{\text{Disp}} = \lambda_{t+1}^{\text{Disp}}\beta\xi(\pi_{t+1}/\Phi)^{(1+\theta)/\theta} + \lambda_t^{z_n}(-1/\text{Disp}_t^2)\chi_0 L_t^{1+\chi}/(1-\alpha) + \lambda_t^Y\big(-1/\text{Disp}_t^2\big)\bar{K}^\alpha L_t^{1-\alpha} \tag{27}$$

$$0 = \lambda_t^{\text{Euler}}\beta\,(1/\pi_{t+1})\,(Y_{t+1}-G_{t+1})^{-\varphi} \tag{28}$$

$$\begin{aligned}-(\lambda_t^\pi/\theta)\pi_t^{-(1+\theta)/\theta} = {}& \lambda_t^\pi(1-\xi)(-1/\theta)\pi_t^{-(1+\theta)/\theta}p_{0t}^{-1/\theta} + \lambda_{t-1}^{\text{Euler}}(-1/\pi_t^2)\,\text{Int}_{t-1}(Y_t-G_t)^{-\varphi}\\ &+ \lambda_{t-1}^{z_n}\big((1+\theta)/\theta\big)\,\pi_t^{1/\theta}\xi\Phi^{-(1+\theta)/\theta}z_{nt} + \lambda_{t-1}^{z_d}(1/\theta)\pi_t^{(1-\theta)/\theta}\xi\Phi^{-1/\theta}z_{dt}\\ &+ \lambda_t^{\text{Disp}}\big((1+\theta)/\theta\big)\big)\pi_t^{1/\theta}\xi\Phi^{-(1+\theta)/\theta}\text{Disp}_{t-1}\end{aligned} \tag{29}$$

$$0 = -\chi_0 L_t^\chi + \lambda_t^Y(1-\alpha)L_t^{-\alpha}\text{Disp}_t^{-1}\bar{K}^\alpha + \lambda_t^{z_n}\frac{\chi_0(1+\chi)}{(1-\alpha)\text{Disp}_t}L_t^\chi \tag{30}$$

$$\begin{aligned}\lambda_t^Y = {}& (Y_t-G_t)^{-\varphi} + \lambda_t^{z_d}(1-\tau_t)\big((Y_t-G_t)^{-\varphi} - \varphi Y_t(Y_t-G_t)^{-\varphi-1}\big)\\ &+ \lambda_t^{\text{Euler}}\varphi\,(Y_t-G_t)^{-\varphi-1} - \lambda_{t-1}^{\text{Euler}}\varphi\,(Y_t-G_t)^{-\varphi-1}(\text{Int}_{t-1}/\pi_t)\end{aligned} \tag{31}$$

which can be simplified further. The optimal monetary policy under commitment is simply the solution to equations (13)–(31), setting the initial values of the lagged Lagrange multipliers equal to zero. Applying the algorithm described above to this system of equations yields the optimal monetary policy to arbitrarily high order.

[8] Equations (20) and (21) can either be treated as exogenous by the policymaker, or can be treated as constraints and G_t and τ_t throughout the system treated as choice variables. The first-order conditions below do the former since the resulting system of equations is a bit smaller. We have also substituted out for marginal cost before computing the first-order conditions, also to reduce the size of the system.

4. Results

5. Conclusions

Appendix A: Solution to Stochastic Growth Model

The second-order solution to equations (6)–(12) in the text is provided below. "Inv" is used instead of "I" to avoid confusion in Mathematica with the imaginary number i. The variables A_t, C_t, and Y_t are transformed to (natural) logarithms before the approximation is conducted. Parameter values are taken to be: $\alpha = 0.3$, $\beta = 0.99$, $\gamma = 2$, $\delta = 0.1$, and $\rho = 0.8$. Note that variables on the right-hand side denote deviations from steady state (and the logged variables A, C, and Y are log-deviations from steady state). Variables on the left-hand side are in levels (log-levels for A, C, and Y). The term $\text{mom}(\varepsilon, n)$ refers to the (known) nth moment of ε. The format of the output below is exactly as it comes out of Mathematica (using the "TeXForm" output option), with the exception of line breaks, which have been added.

$$
\begin{aligned}
A_t &= 0.8\, A_{t-1} + 1.\, \varepsilon_t, \\
C_t &= 0.111483 + \\
&\quad 0.281837\, A_{t-1} + 0.0276019\, {A_{t-1}}^2 + \\
&\quad 0.0906962\, \text{Inv}_{t-1} - 0.0227844\, A_{t-1}\, \text{Inv}_{t-1} - 0.00992342\, {\text{Inv}_{t-1}}^2 + \\
&\quad 0.341768\, K_{t-1} - 0.0858578\, A_{t-1}\, K_{t-1} - 0.0747883\, \text{Inv}_{t-1}\, K_{t-1} + 0.0299726\, {K_{t-1}}^2 - \\
&\quad 0.625874\, \sigma^2\, \text{mom}(\varepsilon, 2) + \\
&\quad 0.352296\, \varepsilon_t + 0.0690047\, A_{t-1}\, \varepsilon_t - 0.0284805\, \text{Inv}_{t-1}\, \varepsilon_t - 0.107322\, K_{t-1}\, \varepsilon_t + 0.0431279\, {\varepsilon_t}^2, \\
\text{Inv}_t &= 0.418697 + \\
&\quad 0.914231\, A_{t-1} + 0.416465\, {A_{t-1}}^2 + \\
&\quad 0.00870854\, \text{Inv}_{t-1} + 0.0849762\, A_{t-1}\, \text{Inv}_{t-1} - 0.00270785\, {\text{Inv}_{t-1}}^2 + \\
&\quad 0.0328161\, K_{t-1} + 0.320213\, A_{t-1}\, K_{t-1} - 0.0204078\, \text{Inv}_{t-1}\, K_{t-1} - 0.022043\, {K_{t-1}}^2 + \\
&\quad 0.699687\, \sigma^2\, \text{mom}(\varepsilon, 2) + \\
&\quad 1.14279\, \varepsilon_t + 1.04116\, A_{t-1}\, \varepsilon_t + 0.10622\, \text{Inv}_{t-1}\, \varepsilon_t + 0.400267\, K_{t-1}\, \varepsilon_t + 0.650727\, {\varepsilon_t}^2, \\
K_t &= 1.43198 + \\
&\quad 0.238836\, \text{Inv}_{t-1} - 0.0285214\, {\text{Inv}_{t-1}}^2 + \\
&\quad 0.9\, K_{t-1} - 0.214953\, \text{Inv}_{t-1}\, K_{t-1} + 0.045\, {K_{t-1}}^2, \\
r_t &= 0.010101 + \\
&\quad 0.0880808\, A_{t-1} + 0.0352323\, {A_{t-1}}^2 - \\
&\quad 0.0184073\, \text{Inv}_{t-1} - 0.0147258\, A_{t-1}\, \text{Inv}_{t-1} + 0.00373687\, {\text{Inv}_{t-1}}^2 - \\
&\quad 0.0693636\, K_{t-1} - 0.0554909\, A_{t-1}\, K_{t-1} + 0.0281631\, \text{Inv}_{t-1}\, K_{t-1} + 0.0183814\, {K_{t-1}}^2 + \\
&\quad 0.110101\, \varepsilon_t + 0.0880808\, A_{t-1}\, \varepsilon_t - 0.0184073\, \text{Inv}_{t-1}\, \varepsilon_t - 0.0693636\, K_{t-1}\, \varepsilon_t + 0.0550505\, {\varepsilon_t}^2, \\
\text{Welf}_t &= -89.4506 + \\
&\quad 4.72893\, A_{t-1} - 0.0632947\, {A_{t-1}}^2 + \\
&\quad 0.808223\, \text{Inv}_{t-1} - 0.385097\, A_{t-1}\, \text{Inv}_{t-1} - 0.080667\, {\text{Inv}_{t-1}}^2 + \\
&\quad 3.04561\, K_{t-1} - 1.45115\, A_{t-1}\, K_{t-1} - 0.607951\, \text{Inv}_{t-1}\, K_{t-1} + 0.377341\, {K_{t-1}}^2 - \\
&\quad 9.79089\, \sigma^2\, \text{mom}(\varepsilon, 2) + \\
&\quad 5.91117\, \varepsilon_t - 0.158237\, A_{t-1}\, \varepsilon_t - 0.481371\, \text{Inv}_{t-1}\, \varepsilon_t - 1.81394\, K_{t-1}\, \varepsilon_t - 0.0988979\, {\varepsilon_t}^2, \\
Y_t &= 1.53663 + \\
&\quad 1.22931\, A_{t-1} + 0.491722\, {A_{t-1}}^2 + \\
&\quad 0.110101\, \text{Inv}_{t-1} + 0.0880808\, A_{t-1}\, \text{Inv}_{t-1} - 0.00920364\, {\text{Inv}_{t-1}}^2 + \\
&\quad 0.414891\, K_{t-1} + 0.331913\, A_{t-1}\, K_{t-1} - 0.0693636\, \text{Inv}_{t-1}\, K_{t-1} + 0.0767548\, {K_{t-1}}^2 + \\
&\quad 1.53663\, \varepsilon_t + 1.22931\, A_{t-1}\, \varepsilon_t + 0.110101\, \text{Inv}_{t-1}\, \varepsilon_t + 0.414891\, K_{t-1}\, \varepsilon_t + 0.768316\, {\varepsilon_t}^2
\end{aligned}
$$

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