An Economics-based Energy Account for Classical Mechanics*

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Abstract

Simple operations transform Hamilton's equations for particle motion in classical mechanics into energy units. Then one obtains a single equation in location, location-changes, momenta and momenta-changes with the interpretation: income from capital, in units of energy, balances with current investment expenditure on location changes and momenta changes, also in units of energy. For the special case of periodic motion, the inflow-useflow sub-accounts for distinct position variables and for distinct momenta variables balance over the period of motion.

1 Introduction

"Output" is particle motion for many problems in classical mechanics and this "output" can be expressed, using the Hamilton equations, as an energy value-sum of current position-change for the particle and current velocity-change for the particle. This energy expression appears as prices multiplied

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by quantities, where a price translates a quantity into units of energy. The quantities are time-changes in current co-ordinates of location for the particle and time-changes of current momenta[1] for the particle. "Output" is then expressable as an energy value of current simultaneous position-change and momentum-changes for the particle in motion. Balancing the current value of output is an energy value of current inputs (location co-ordinates "carrying" forces and momenta "driving" current particle motion). This energy value of current "input" is also easily derivable from the Hamilton equations, characterizing current motion of the particle and this energy value takes the form of rental-prices multiplied by quantities. A rental-price translates a unit of quantity into units of energy here. Quantities here are the inputs which operate as capital goods in the sense standard in economics.[2] Hence "small" manipulations of the Hamilton equations for a particle in motion in classical mechanics, changes which place the equations in units of energy, yield a quite standard instantaneous balance of input value with output or expenditure value. This we demonstrate below. The energy representations we make use of or the energy representations we derive from the Hamilton equations are roughly speaking forms from the Virial Theorem, a long-standing energy

[1] Momentum in classical mechanics is defined as a particle's mass, m multiplied by its current velocity, v. Velocity can be positive or negative. Momentum-change is then $m\frac{dv}{dt}$ or mass mutiplied by acceleration (velocity time-change). In classical mechanics, one takes mass as unchanging and the co-ordinate system as fixed, independent of current particle motion.

[2] Location co-ordinate capital has a flow of services defined by current, local "force" and this service flow results in or yields current investment or disinvestment in momentum, this latter a distinct capital good. Momentum-change is interpreted as current investment or disinvestment in momentum capital. Current momentum has a service flow which results in or yields current position-change (co-ordinate change) for the particle in motion. Co-ordinate change becomes investment or disinvestment in the co-ordinate capital goods. The new position (new co-ordinate capital levels) is usually then associated with a new level of force and in turn a new level of momentum change. This is brief overview of what is capital and what is investment and how they relate to particle motion. Our concern here is the energy accounting associated with this view of particle motion.

balance result for periodic motion. Hence one might refer the energy representations as Virial Theorem Energies (VTEs). If one is going to check the Virial Theorem or establish it for a special case, one is pretty well obliged to proceed via the Hamilton equations and in this proceeding one constructs and makes use of energy representations of forces, momenta, momenta-changes and position-changes. It is these Virial Theorem representations which we focus on here and from which we construct an economics-type energy account for classical mechanics. Also, the economics interpretation of the Virial Theorem has co-state variables operating as shadow prices of capital inputs.

Our value balance of inputs and outputs is a standard economics concept and stands in contrast the traditional energy value balance in classical mechanics, namely that in "the work energy theorem" where current output is the current "excess" in kinetic energy of the particle in motion over a base value and current input in the current "shortfall" in the potential energy, from a base value, of the particle. This notion of value balance is explicit first in Newton's *Principia* and is the one we are in a sense replacing in this analysis. Our complaint against the traditional concept of value balance is that it is not like anything in economics, whereas what we present is standard economics. Clearly in setting out our alternative energy account, we bring classical mechanics and economic dynamics under a single tent, perhaps for the first time. Whether our new energy account leads to new substantive physics is open, though we do list three items of apparently new physics at the end, albeit small items, it seems. The traditional value balance is a version of "conservation of energy" in classical mechanics and has been immensely productive as an organizing principle.

In Hartwick [2004], we developed this line of interpretation for classical mechanics for the special case of periodic or sustained motion. We restate that view here somewhat more compactly and inquire about particle motion of a non-periodic sort. We observe at each instant value balance between "capital incomes" and "expenditure on investment" but of a cross-subsidization sort. Some energy inflow from say location variables ends up currently funding some investment in momenta variables or *vice versa*. For periodic motion there was no such cross-subsidization, period by period. So we observe that general motion indeed exhibits the fundamental value balance of "income" with "expenditure" in units of energy at every instant but does not exhibit the income-expenditure INDEPENDENCE for input type that we observe per period for periodic motion. A straightforward intuition here is that over a very long interval, periodic motion is associated with distinct location and momenta variables being self-sustaining on average but non-periodic motion will exhibit some cross-subsidization of one input type by the other. It turns out that the general instantaneous balance reduces to the Hamilton equations quite directly and thus is capturing the Hamilton equations in disguise. It is the reverse that is of interest however - complicated equations characterizing equilibrium motion have a natural expression as a basic, value balance relation. Alternatively, the equations of equilibrium motion (Hamilton equations) can easily be written as balance relations between capital income and expenditure on capital investment, at each instant.

Periodic motion has the property of "input restoration" over each period. Location variables and momentum variables get returned to their initial values over each period. This seems obvious. Our observation here is that there is a straightforward income-expenditure balance relation over the period which sustains or drives this restoration process. The energy inflow (capital goods rental income) per period equals the energy use-flow (investment expenditure) per period; hence input restoration. Distinctive for periodic motion is OWN capital good funding per period. Location variable incomes balance with location variable expenditures per period (the Virial Theorem[3]) and momentum variable incomes balance with momentum variable expenditures per period (the complementary virial theorem (Hartwick [2004])). We can say that for sustained motion, input value from location variables precisely fund the expenditure value associated with restoration of the location variables, over the period. Similarly for momentum variables. There is energy inflow balanced with use-flow per period in aggregate as well as by "sector".

Our formal result is then: "small" manipulations of the Hamilton equations in classical mechanics transforms them into two families of energy balance relations and these relations combine to form a straight-forward, complete, instantaneous energy inflow-useflow account. This instantaneous energy account has the interpretation of income from inputs in units of energy balances with expenditure on input-changes, in units of energy.

We review.

(1) Particle motion in classical mechanics is "system output" and motion "reduces to" or is represented by current position-change for the particle and current velocity-change for the particle. Velocity-change equates with momentum-change in classical mechanics.

(2) There is a natural representation of current position-change for a particle in motion in classical mechanics in units of energy and there is a

[3] Goldstein, et. al. [2002; pp. p. 86]. Most physics textbooks do not mention the Virial Theorem but a few do. It is a result for periodic and quasi-periodic motion relating a measure of energy input from forces per period to twice kinetic energy per period. Inspection reveals that the inflow side is a sum over the period of input rental income, in the terminology of economics, and the "consumption" side of the account is the sum of expenditures on investment in the location variables, again in the terminology of economics. Of course the balance is in units of energy, not dollars.

natural representation of current velocity-change for the same particle in units of energy.

(3) For the special case of periodic motion of a particle (eg. the motion of the bob of an undamped simple pendulum), the balance of energy inflow, linked to particle position, with energy use-flow associated with position-change of the particle, per period is captured in the Virial Theorem.

(4) With regard to (3), there is a complementary virial theorem associated with periodic motion of a particle, dealing with energy balance and velocity-change of the particle. In this account, there is energy inflow linked to the current momentum levels of the particle in motion and energy use-flow linked to current momentum change of the particle in motion.

(5) Position variables operate as capital inputs in one energy account for periodic motion and momentum variables operate as capital inputs in the other energy account. Position-change variables operate as investment terms in one energy account and momentum-change variables operate as investment terms in the other energy account.[4] Energy balance per period has the interpretation of income from capital in units of energy equal to investment expenditures over the same interval. There are two generic accounts here, one for location variables as capital inputs and another for momentum variables as capital inputs. There is income-investment expenditure balance then for each type of input, per period. One might say that each input type sustains itself, period by period, with its own income stream (energy inflow stream).

[4] This curious role-reversal of variables in the two accounts can be attributed to the fact that particle momentum is so closely linked to particle position-change. Particle position-change seems to generate particle momentum. We will adopt the other view: particle momentum causes particle position-change. Nevertheless, the entwining of momentum and position-change for a particle results in an entwined pair of energy accounts. Roughly speaking, position-change and momentum-change for a particle in motion are joint products, linked in a subtle, yet elegant way – in a way unlike that for any standard economics problem.

There is no cross-subsidization over the period.

Our central result here for the case of general conservative particle-motion in classical mechanics is: (a) capital income is balanced with investment expenditure at each instant, in the sum over different capital inputs, but (b) cross-subsidization of one use-flow by an income flow from a different type of capital input at each instant is the rule. If all this sounds reasonable to an economist, a physics outsider, there are some subtleties. The complete instantaneous energy account is the sum of two sub-accounts and in one of these sub-accounts co-state variables operate as capital goods shadow prices and state variables as capital inputs to current "production". This is what an economist would anticipate since classical mechanics problems can be posed as dynamic optimization problems (problems in "action"-sum minimization over an interval). However the other half of the energy account has the same co-state variables now operating as capital inputs (momenta) and the same state variables now operating as capital goods shadow prices. The same variables appear in each energy sub-account, but with essentially dual roles. A capital good in one sub-account is operating as a capital good price in the other sub-account and *vice versa*. This sort of duality has no counterpart in economic dynamics.[5] Roughly speaking the root of this duality lies in the fact that current output or particle-motion in classical mechanics comprises simultaneous position or co-ordinate change and velocity or momenta-changes and momenta are represented by co-state variables in the Hamilton equations. Physicists, I believe, see current kinetic energy or twice

[5] In the Appendix we set out a problem from textbook capital theory which has shadow prices as capital goods prices and the location and momentum variables as capital goods. In other words the seemingly strange situation of co-state variables operating both as shadow prices and as capital goods in fact has a standard expression in a multi-good capital theory model. What is strange in a problem of dynamic optimization is in fact not strange in a different formulation of classical mechanics.

current kinetic energy as the measure of current "output" in a problem of particle-motion in classical mechanics. In our view twice current kinetic energy is measuring only current position-change of the particle in motion and we of course exposit this view below. A complete account must also deal with current velocity or momenta-changes as well. Hence our accounting system below.

More guidance to economists. We view periodic motion as essentially capital goods restoration over the period of motion. Sustained motion involves restoring inputs to their initial productive state over each period. Phelps [1961] model of golden rule investment reduces to this same scenario when labor force growth is removed. At each instant decayed capital must be replaced in order for consumption to be sustained at its golden rule value. Alternatively, the golden rule level of capital must be restored, period by period by replacement investment.[6] A period in this Phelps, continuous-time world is an instant, rather than a discrete interval. Nevertheless, we argue that Phelps model with no labor force growth provides an economics template for interpreting periodic motion in classical mechanics. There is of course no consumption sink for part of current product in classical mechanics. In classical mechanics all output over the period is soaked up as replacement investment whereas in Phelps's model some current output "evaporates" as current consumption by people. The rest of output goes to replacement investment. Detail on this is provided below.

Though the two energy accounts in question are very similar to look at, momentum and location variables are rather different. Energy is needed to increase the velocity of a particle (increase its momentum) as well as to de-

[6]We present a multi-sector extension of this argument in the Appendix. In the Appendix we start with a capital theory model and "specialize" it to approximate the behavior of particle motion in classical mechanics.

crease the velocity (decrease its momentum). With regard to location, energy is needed simply to change the position of the particle, whether "forward" or "backwards". And with regard to levels, location variables "yield" energy because they capture the relative intensity of forces at various locations whereas momentum variables "yield" energy because momentum itself is intrinsically "motion preserving" (Newton's first law). We deal with an energy representation of momentum in our energy account (in the complementary virial theorem). In general then, local particle motion involves velocity-change occuring simultaneously with position-change and velocity-changes are also energy using. Velocity-change "consumes" or uses up energy. One must think then of two simultaneous energy accounts associated with particle motion in classical mechanics, one associated with position-change (here one has the Virial Theorem for the case of periodic motion) and another with velocity-change. Velocity-change in classical mechanics takes the more formal representation as momentum-change and for the special case of periodic motion, momentum varibles must get returned to their initial values over the period. Hence there is a sequence of energy usings or investments associated with momentum changes over the period when each momentum variable gets returned to its intial value over the period of motion.

2 Energy Accounting for Particle Motion in Classical Mechanics

In brief, our argument is the following. Particle motion satisfies the Hamilton equations

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\text{and } \frac{\partial H}{\partial p_i} = \dot{q}_i, \; i = 1, ..., 3$$

for H the Hamiltonian, q_i a component of a vector of location (state) variables, and p_i a component of a vector of momentum (co-state) variables. We are comfortable with reading the Hamilton equations as: first, net force is causing current momentum-change and secondly, velocity is causing current position-change for the particle. Lemons [2002; p. 53] for example speaks of particle motion characterized by "force" causing "change of velocity", a particular case of the first equation, and "velocity" causing "change of position", a particular version of the second equation.[7] We multiply the first by q_i and the second by p_i to get equations in units of energy[8].

$$q_i \frac{\partial H}{\partial q_i} = -q_i \dot{p}_i$$
$$\text{and } p_i \frac{\partial H}{\partial p_i} = p_i \dot{q}_i.$$

$q_i \frac{\partial H}{\partial q_i}$ is "generalized net work" in units of energy.[9] It is a measure of the net effect of force on the particle in its current location. (We illustrate with examples below.) We interpret this "generalized work" as a measure or current energy inflow, attributable to location, on the particle in motion. $p_i \frac{\partial H}{\partial p_i}$ is an energy measure of the effect of the particle's momentum on itself. We interpret this as a measure in units of energy of the particle's current

[7] "Newton's second law identifies the net force F(t) per unit particle mass, M with the rate at which the particle changes its velocity V(t). This velocity, in turn, describes the rate at which the particle changes its position."

[8] These energy representations are motivated by a derivation of the Virial Theorem. Hence a label, "Virial Theorem Energies" or VTEs.

[9] Work is force multiplied by distance and is in units of energy. Work usually refers to energy involved in the horizontal movement of a mass. By generalized work, we mean "force multiplied by distance" in an abstract setting.

momentum on its motion.[10] Again we see this as an energy measure of current energy inflow to current particle motion. The right hand sides are respective measures of energy useflow associated respectively with momentum-change for the particle and position-change for the particle. A verbal rendering of these equations is: first, net force, in units of energy is causing current momentum-change, in units of energy, and secondly, momentum, in units of energy is causing current particle position-change, measured in units of energy.

Our aggregate capital income-expenditure balance, per instant, energy equation is then

$$\sum_{i=1}^{3}\left\{\left\{p_i\dot{q}_i - q_i\frac{\partial H}{\partial q_i}\right\} + \left\{-q_i\dot{p}_i - p_i\frac{\partial H}{\partial p_i}\right\}\right\} = 0, \tag{1}$$

where q_i is a capital good in the first bracket and p_i is a capital good in the second bracket. $\dot{q}_i$ and $\dot{p}_i$ operate as respective investment terms. $\frac{\partial H}{\partial q_i}$ and $\frac{\partial H}{\partial p_i}$ are operating as respective rental prices, which translate "quantities" into units of energy. p_i in the first bracket is a capital goods price which translates investment, $\dot{q}_i$ into units of energy and q_i in the second bracket is a capital goods price which does the analogous thing for investment, $\dot{p}_i$. $\frac{\partial H}{\partial q_i}$ and $\frac{\partial H}{\partial p_i}$ operate as rental prices for capital goods. The physics interpretation here is that instantaneous energy inflow from forces and momenta balance with current energy "demands" made by current position-change of the particle in motion AND current momenta-changes. What we should infer from this equation is that "equilibrium motion" has a representation as an instantaneous value inflow-useflow balance at each instant. Clearly one could not

[10] We like the interpretation of generalized work causing momentum change, in units of energy, and momentum, in units of energy, causing particle position-change. Our approach may part with standard textbook classical mechanics most when we define and make use on an energy representation of current momentum. But we also depart from conventional energy accounting by substituting generalized work for potential energy, roughly speaking.

look at this equation and infer the appropriate Hamilton equations. Hence the equation is capturing or representing the full Hamilton equations weakly, at best. Being aware that this equation was equal to zero at every point in time is weakly informative about the equations of motion for a particle. Buried in this equation are a number of "zero relations", including the Euler-Lagrange equations, which constitute the equations of motion for the particle. We do not contend that this inflow-useflow balance relations "captures" the key Euler-Lagrange equations.[11] The interesting aspect is "the opposite" so to speak. The key equations of motion have an inflow-useflow value balance representation which translates very directly into economics. Periodic motion captures a special case of this equation. We report on this below.

3 Momenta as Capital Inputs

Our treatment of momentum variables as capital goods might strike an economist as odd. One can obtain Hamilton's equations from a dynamic optimization problem as "action sum" minimization over a finite interval.[12] In such a problem, the location variables are natural state variables and the momentum variables are natural co-state variables. Hence an economist might see it as natural to interpret location variables as capital goods and co-state variables as capital goods shadow prices. This goes through well in the in-

[11] Early on we jumped to the conclusion that since we were dealing with an equation equal to zero, that there must be a conserved quantity in a time-integral. Clearly this is not the case. An equation equal to zero may be an Euler-Lagrange equation and in this case the conserved quantity is not of the usual "integral" sort, as with conservation of energy. I benefited from conversations with Alexei Cheviakov on this matter.

[12] Hamilton's principle is that our "aciton sum" is extremized by a path of equilibrium motion of a particle. We label action, current potential energy minus current kinetic energy. This appears not standard. More standard is referring to our "action sum" as "action".

terpretation of the Virial Theorem as a statement of energy inflow balanced with energy useflow, per period. But in physics, the co-state variables capture much more than being prices or "translators" of quantities into units of energy. For example, when a simple pendulum swings through its low point, there is no external force acting to move the bob. All force is acting to create tension on the rod holding the bob and yet the bob (the particle in motion) is moving most rapidly and is exhibiting most kinetic energy. The answer to this paradox is of course that it is the momentum of the bob that is causing the "large" motion at this low point. This momentum "driving" the bob has an energy representation analogous to force having an energy representation. Energy inflow from current momentum is captured above in the term $p_i \frac{\partial H}{\partial p_i}$. This energy inflow is the analogue of net generalized work, namely $q_i \frac{\partial H}{\partial q_i}$. One might refer to the latter as the energy inflow associated with local net force. We see a large symmetry here with momentum having an energy inflow representation and location, standing in for local force, having an energy inflow repesentation. This may be our central departure from textbook physics, namely the symmetric treatment of location variables and momentum variables as capital goods or inputs, in units of energy.[13] This view stems from our distinction between velocities and position-changes[14] since velocity becomes a rental price associated with a momentum input. We will comment on signs of inputs, outputs, prices and rentals below.

Newton's First Law can be paraphrased as: "a particle's current momentum, in the absence of an "outside" force, is responsible for its current "linear" position-change". And his Second Law can be paraphrased as: "cur-

[13]We still retain the interpretation of co-state variables as shadow prices of location co-ordinates in one energy account but these same co-state variables operate as momenta (capital inputs) in the complementary energy account.

[14]This may be our second large departure from textbook physics, our "separating out" velocity from position-change for the particle.

rent force on a particle in motion is responsible for its current change in momentum". One can argue that Newton's mathematical theory of forces and particle motion was built up from this second law while the first law remained in the background, unformalized. Our contention is that Hamilton's formalization of classical mechanics incorporates Newton's first law in the equation: "velocity causes current position-change of the particle" and mass multiplied by velocity is momentum. Our preferred approach to this Hamilton equation is to multiply both sides by current momentum of the particle to formalize, in units of energy, the idea that current momentum causes current position-change of the particle, with position-change as $p\dot{q}$, which ends up aggregated to twice current kinetic energy of the particle. Twice current kinetic energy, central to Newton's analysis, is the energy measure of current position change in the particle. And there remains another energy account dealing with force and current momentum-change.

4 Three Illustrations from Classical Mechanics

(i) Particle Free-fall.

This classic Galilean problem in a locally-unchanging gravitational field has potential energy, mgz for potential energy normalized to be zero when z is zero. m is particle mass, g is "gravitational acceleration", and z is a "vertical" co-ordinate. The particle falls from some initial positive value of z toward z equal to zero. The Galilean (constant acceleration) "work-energy theorem" can be illustrated with an apple of mass m (in kilograms) dropped from an outstretched hand at height z_0 (in meters).[15] Potential energy is mgz_0 for

[15] A feather and cannon-ball descended in free fall at the same speed or accelerated at the same rate. For free fall, we have $d = \frac{1}{2}gt^2$, where g is the constant acceleration

g the gravitational acceleration (9.8 meters per second squared).[16] Kinetic energy is zero, $\frac{1}{2}m\dot{z}_0^2$ for $\dot{z}_0 = 0$. After a lapse of a small interval, the particle is at z_1, with potential energy mgz_1 and possesses kinetic energy, $\frac{1}{2}m\dot{z}_1^2$ with velocity, $\dot{z}_1$ (a negative number). The work-energy theorem indicates that the change in potential energy, $mgz_0 - mgz_1$ equals the change in kinetic energy,

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factor, t is time elapsed, and d is distance. Here the initial velocity is zero. This formula formalizes Galileo's famous Leaning Tower of Pisa experiments in free fall. In fact, it is doubtful that Galileo learned much from such experiments or even performed them at the Tower. His notes reveal that he tested uniform acceleration with balls rolling down inclined planes because he could time a slower descending body more accurately. In the Two New Sciences, we read of his experimentation with uniform acceleration:

"A piece of wooden moulding or scantling, about 12 cubits long, half a cubit wide, and three finger-breadths thick, was taken; on its edge was cut a channel a little more than one finger in breadth; having made this groove very straight, smooth and polished, and having lined it with parchment, also as smooth and polished as possible, we rolled along it a hard, smooth round bronze ball. Having placed this board in a sloping position, by lifting one end some one or two cubits above the other, we rolled the ball, as I was just saying, along the channel, noting, in a manner presently to be described, the time required to make the descent. We repeated this experiment more than once in order to measure the time with an accuracy such that the deviation between two observations never exceeded one-tenth of a pulse beat. Having performed the operation and having assured ourselves of its reliability, we now rolled the ball only one-quarter of the length of the channel; and having measured the time of its descent, we found it precisely one-half of the former. Next we tried other distances, comparing the time for the whole length with that for the half, or with that for two-thirds, or three-fourths, or indeed for any fraction; in such experiments, repeated a full hundred times, we always found that the spaces traversed were to each other as the squares of the times, and this was true for all inclinations of the... channel along which we rolled the ball..."

[16]Energy in every day life is power-flow as in watts multiplied by time. The units of measurement are watt-hours or kilowatt-hours or horsepower-hours. One also hears of so many BTU's of energy being used (BTU standing for British Thermal Unit). A BTU is 252 calories and a calorie is 4.184 joules. (The calories in food are actually kilo calories.) A joule is the energy associated with 1 watt of power acting for one second. One kilowatt hour is 3.6 million Joules. Joule (1818-89) is the English scientist who established the heat equivalent of mechanical energy. He very precisely measured the rise in the temperature of water in a beaker when a particular amount of agitation was directed to the water, agitation being associated with mechanical energy. The modern notion of energy dates from Joule and other scientists who were concerned about the physics of steam engines. A steam engine turns a heat flow into a mechanical power flow or heat energy into work flow.

$\frac{1}{2}m\dot{z}_1^2 - \frac{1}{2}m\dot{z}_0^2$.[17] Observe that if we differentiate both sides with respect to time, at z_1 we get $-g = \frac{d\dot{z}_1}{dt}$, which says that local acceleration downward of the apple is g, the constant gravitational acceleration in the neighborhood of the surface of the earth. $\frac{d\dot{z}_1}{dt}$ is negative.

Consider some illustative examples.[18] How far does a body, near the surface of the earth, fall in 3 seconds in free fall? The formula is $z = v_0 t + \frac{1}{2}gt^2$ or $0 + \frac{1}{2} \times 9.8\frac{m}{s^2}t^2 = 9.8 \times 3^2 = 44$ meters. This is Galileo's formula. The mass of the body has no effect on distance travelled in free-fall. What speed does the body achieve after 70 meters? Here we use the work energy formula: $v_t^2 = v_0^2 + 2gz$ or $v_t^2 = 0 + 2 \times 9.8\frac{m}{s^2} \times 44m$ yielding $v_t = 37\frac{m}{s}$. How long does it take for the body to reach a speed of $25\frac{m}{s}$? We use the formula, $v_t = v_0 + at = 0 + 9.8\frac{m}{s^2}t = 25\frac{m}{s}$ which implies $t = 7.8$ seconds. How much time elapses by the time the particle has fallen 300 meters? We use the formula $z = v_0 t + \frac{1}{2}gt^2$ or $300m = 0 + \frac{1}{2} \times 9.8\frac{m}{s^2} \times t^2$ to get $t = 7.8$ seconds. This last formula was known to medieval thinkers as Merton's Theorem in the form $\frac{z}{t} = v_0 + [t/2]a$ for $\frac{z}{t}$ average speed over the interval in question. ■

Here change in kinetic energy is work done.[19] This simple link between work and change in potential energy is true only because potential energy is linear in the location variable. In general one cannot equate change in potential energy to work so directly. Central to our analysis below is "work" and the energy inflow that it corresponds with. Generalized work takes the

[17] This is the basic physics idea that current output value should balance with current input value. This is found in Newton's Principia. Here output is current change in kinetic energy and input is current change in potential energy. Our view is that economics "offers" a different notion of current input value balanced with current output value, one where twice current kinetic energy is a principal component of current output value and linearized potential energy is a principal component of current input value. In addition output in our view comprises an energy representation of current momentum-change for the particle in motion and input comprises an energy representation of current momentum.

[18] From Halpern [1988].

[19] See for example Moore [1983; p. 16].

form of a force multiplied by a distance. Clearly change in kinetic energy (work) is an outcome or "product" of an "action". The cause here involves some energy inflow. Our focus is on (a) the change in position of a particle and (b) on its change of velocity at an instant and on the energy representations of these outcomes and the energy representation of the causes of these outcomes.

The Hamiltonian formalism can be taken up here. The Hamiltonian function has current potential energy on the particle minus current kinetic energy of the particle plus pv, for p the co-state variable (here momentum), and $\dot{z} = v$, an equation of motion. $\dot{z}$ is negative. That is, $H(t) = mgz - \frac{1}{2}mv^2 + pv$. There are rules, in the form of Hamilton equations, for operating on the Hamiltonian. The Hamilton equations are

$$\begin{aligned} -\frac{\partial H}{\partial z} &= \dot{p} \text{ or } -mg = \dot{p} \\ \frac{\partial H}{\partial p} &= \dot{z} \text{ or } v = \dot{z} \\ \text{and } \frac{\partial H}{\partial v} &= 0 \text{ or } p = mv \text{ and } \dot{p} = m\dot{v}. \end{aligned}$$

One sees directly that the central equation is $\frac{d\dot{z}}{dt} = -g$, as we observed above. We read the first equation as: force is causing current momentum-change for the particle in motion. The second equation we read as: current velocity is causing position change for the particle. One can find this interpretation of particle motion in textbooks but it is not common. Since gravitational force is the only "outside agent" here, one thinks of force causing position-change or "force is moving the particle". In fact in addition to the operation of force is the operation of momentum. Recall that Newton's first law says that momentum can keep a moving particle in motion indefinitely in a straight line, provided no force is intervening. We go so far as to suggest a missing law in Newton: the first law should have two components - that momentum causes a moving particle to possess velocity of a linear sort, and velocity in turn causes local linear position-change in the moving particle, in the absence of an intervening force.

Our last Hamilton equation is so to speak definitional. It gives us the definition of momentum, p. It derives from the fact that at each instant the Hamiltonian is extremized by the control variable, here v. Implicit for the moment is the fact that our Hamiltonian set up is part of a control theory problem in dynamic optimization (action sum minimization over an interval of time). Physics textbooks tend to start with a Lagrangian and derive the definition of p, as above, from this function and then substitute to express the Hamiltonian as a function of location and co-state (momentum) variables alone, i.e. as $H = mgz + \frac{p^2}{2m}$. The Hamilton equations are then worked using this reduced-form H. More on this below. We proceed somewhat differently. We have been inspired by the use of control theory in dynamic optimization in economics. We consider the control theory approach to the Hamiltonian as "structural" or less "reduced-form" than is standard in physics textbooks. In particular we find it valuable to maintain a distinction between a velocity variable, v and a position-change variable, $\dot{z}$. Physicists do not maintain this distinction. They proceed as if these two variables were capturing the same physical phenomenon. We maintain the notion throughout that velocity is causing position-change in the particle in motion.[20] We go so far as to suggest that Newton over-looked the law: velocity causes position-change. Newton's first law can be formalized as momentum causes "linear, particle

[20]For example, Lemons [2002; p. 53] comments: "Newton's second law identifies the net force F(t) per unit particle mass, M with the rate at which the particle changes its velocity V(t). This velocity, in turn, describes the rate at which the particle changes its position."

motion" in the absence of force. We suggest refining this to momentum causes velocity AND velocity causes "linear" position-change in the absence of force. Force on the other hand is causing velocity-change or momentum-change. In classical mechanics velocity-change and momentum-change can be treated almost synonymously.

From the Hamilton equations we have energy balance relations[21]

$$\begin{aligned} -z\frac{\partial H}{\partial z} &= z\dot{p} \text{ or } -mgz = \dot{p}z \\ p\frac{\partial H}{\partial p} &= p\dot{z} \text{ or } pv = p\dot{z} \end{aligned}$$

Net investment here is

$$\{p\dot{z} - zmg\} - \{z\dot{p} + pv\}$$

which is zero at each instant of motion. The energy associated with current changes in location and momentum (investment expenditure, in units of energy), namely $p\dot{z} - z\dot{p}$,associated with z and p, namely $zmg + pv$, is supplied by capital rentals.[22] The first pair of terms in brackets will be negative (and "large") when the particle is initally dropped while the second pair of terms will be positive (and "large") when the particle is initially dropped from z_0. Near the ground the signs will be reversed since force and momentum-change will approach zero near the ground and kinetic energy and energy inflow from momentum will be "large" and positive. There will be a cross-over point $\widehat{z}$ when each pair of terms in each of the brackets is zero. And

[21] We change the Hamilton equations into equations in units of energy by multiplying the first by z and the second by p. We proceed this way because these "transformations" lead to the Virial Theorem for periodic motion. More on the Virial Theorem below.

[22] In the Appendix we start with a text book capital theory model with two capital goods and, with it, approximate the solution to the problem of particle free-fall.

the sum $\int_{t_0}^{T}\{p\dot{z}-zmg\}\,dt-\int_{t_0}^{T}\{z\dot{p}+pv\}\,dt$ equals zero. What distinguishes PERIODIC motion here is that each integral equals zero on average over the LONG RUN.[23] More on this below.

(ii) The simple undamped pendulum.

In this case of periodic motion, the is a single location (state) variable, θ, the pendulum rod angle with respect to the vertical "origin". p is the single co-state variable. Potential energy is mgh for $h=[1-\cos\theta]$.[24] This normalizes the potential energy function to have value zero when the particle or bob swings through its low point. Kinetic energy is $\frac{1}{2}mgv^2$ for $v=l\dot{\theta}$, l being the length of the pendulum rod holding the bob. The Hamiltonian is $H=mg[1-\cos\theta]-\frac{1}{2}mgv^2+pv$. p get defined here by $\frac{\partial H}{\partial v}=0$ or $p=mv$. The Hamilton equations are

$$\frac{\partial H}{\partial\theta}=-\dot{p}\text{ and }\frac{\partial H}{\partial p}=\dot{\theta}$$

and in units of energy are[25]

$$\theta\frac{\partial H}{\partial\theta}=-\theta\dot{p}\text{ and }p\frac{\partial H}{\partial p}=p\dot{\theta}.$$

At an instant, net investment is

$$\left\{p\dot{\theta}-\theta\frac{\partial H}{\partial\theta}\right\}+\left\{-\theta\dot{p}-p\frac{\partial H}{\partial p}\right\}$$

[23] Physicists also take up quasi-periodic cases in which the length of the period varies but again our separate integrals would equal zero in the long run.

[24] Simple triganometry yields this definition of h, the vertical height of the bob above the baseline, this latter being the horizontal defined by the low-point in the swing of the bob.

[25] Generalized work here is $\theta\partial H/\partial\theta$. Our analysis turns on the symmetric object, also in units of energy, namely $p\partial H/\partial p$. This latter is the energy representation of current momentum. A label might be the work representation of current momentum. There is of course work exerted and a task defined by work expended. Physics seems to choose the last view; i.e. work as change in kinetic energy. We are thinking of work as input and energy used up in "production" activity as use-flow.

which is zero. Net investment expenditure on current position-change, namely $\left\{p\dot{\theta} - \theta\frac{\partial H}{\partial \theta}\right\}$, plus net investment expenditure on current momentum-change, namely $\left\{-\theta\dot{p} - p\frac{\partial H}{\partial p}\right\}$, is zero.

The Virial Theorem (Goldstein et. al. [2002; p. 86]) for this problem[26] is,

$$\int_0^B \left\{p\dot{\theta} - \theta\frac{\partial H}{\partial \theta}\right\} dt = 0$$

for B the period of motion.[27] One also has

$$\int_0^B \left\{\theta\dot{p} - p\frac{\partial H}{\partial p}\right\} dt = 0$$

which we refer to as the complementary virial theorem, a zero energy relationship, per period, for the momentum variable. Each equation has the interpretation of investment expenditure per period, balancing with own capital rentals per period. The sum over the period of energy inflow and useflow is POSITIVE in each sub-account above. This is an accounting of particular energy inflows sustaining distinctive OWN motion over the period: position change and momentum change. This was the central message of our earlier paper. We should emphasize that this "rentals supporting own investment" should not be interpreted as causal. It is more like an accounting "co-incidence" but is highly relevant for energy accounting balance or consistency over the very long run, for periodic motion.

$\frac{\partial H}{\partial \theta}$ is $\frac{\partial U}{\partial \theta}$ or force here and $\frac{\partial H}{\partial p}$ is a measure of velocity of the particle here. Hence energy inflow reduces to $\theta\frac{\partial U}{\partial \theta}$ in the first equation. $\theta\frac{\partial U}{\partial \theta}$ is "generalized work" as distinct from $U(\theta)$, potential energy. Hence "generalized work" be-

[26] The Virial Theorem is usually expressed as a time-average or in energy per instant rather than energy per period.

[27] We suggest that the natural normalization of the potential energy function for this problem yields $\int_0^B U(\theta)dt = \int_0^B \theta U_\theta(\theta)dt$.

comes a central concept in our formulation of energy accounting for classical mechanics.[28]

In addition to satisfying these "sustained motion" equations immediately above, one has capital income-expenditure balance per instant, in the sense of equation (1). Particle motion is generally then: capital income sustaining current expenditure on capital investment and disinvestment. The general case requires a summing over different types of capital. For the special case of periodic motion, each type of capital stands on its own bottom, so to speak; there is no cross-subsidization of investment in either location or momentum.

(iii) The Closed Orbit Kepler Problem

The Hamilton equations specialize to

$$\begin{aligned} \frac{dU}{dr} - \frac{dT}{dr} &= -\dot{p}_r \\ 0 &= \dot{p}_\theta \\ v_r &= \dot{r} \\ v_\theta &= \dot{\theta} \end{aligned}$$

for $H = U(r) - T + p_r v_r + p_\theta v_\theta$, $U(r) = -\chi/r$,[29] $T = \frac{1}{2}mv_r^2 + \frac{1}{2}mr^2v_\theta^2$, $v_r = \dot{r}$, and $v_\theta = \dot{\theta}$. $\frac{dT}{dr}$ is the centripetal force "holding" the particle in orbit. The periodicity property here is $\int_0^B -r\dot{p}_r dt = \int_0^B p_r \dot{r} dt$. The Virial Theorem can be expressed as $\int_0^B r\left\{\frac{dU}{dr} - \frac{dT}{dr}\right\} dt = \int_0^B \dot{r}p_r dt$ for B the period of orbital motion.[30] This can be re-written as $\int_0^B r\left\{\frac{dU}{dr}\right\} dt = \int_0^B \left\{\dot{r}p_r + \dot{\theta}p_\theta\right\} dt$ which has the interpretation of location capital rentals, $r\left\{\frac{dU}{dr}\right\}$ funding investments

[28] In our earlier paper we emphasized that one could always make $\int_0^B \theta \frac{\partial U}{\partial \theta} dt$ equal to $\int_0^B U(\theta) dt$ by an appropriate normalization of $U(\theta)$.

[29] Potential energy is negative for this problem. Convention has the potential energy function normalized so that potential energy tends to zero, far from the central force.

[30] Here generalized work is $r\left\{\frac{dU}{dr}\right\}$, distance multiplied by force. The Virial Theorem is usually presented in this context as a time average, i.e. as $\int_0^B r\left\{\frac{dU}{dr}\right\} dt/B =$

in location variables.[31] The complementary virial theorem is $\int_0^B p_r v_r dt = \int_0^B -r\dot{p}_r dt$, which as the interpretation of rentals associated with capital good p_r funding investment in p_r, over the period of motion.

Capital good p_θ supplies energy in amount $p_\theta v_\theta$. This energy can be equated to the flow required to hold p_θ unchanging, i.e. to the energy representation of centripetal force, namely $r^2 v_\theta^2$. Hence p_θ does indeed have an energy balance with its need for restoration over the period, but in this case it is "restoratioń" at each instant of time.

The full energy account is (a) position variable r supplies energy to restore both position variables to their initial values over the period (the Virial Theorem), (b) momentum variable p_r supplies energy to restore p_r to its initial value over the period, and (c) momentum variable p_θ supplies energy to hold p_θ unchanging at each instant or variable p_θ supplies energy corresponding to the centripetal force, at each instant.

In the special case of CIRCULAR motion around a central force, all of the central force is centripetal. And this centripetal force, rT_r is an energy useflow with corresponding energy inflow, $p_\theta \frac{\partial H}{\partial p_\theta}$ just as we observe for the elliptical orbit here. This is very striking. For both the closed Kepler orbit and the circular orbit cases, the energy representation of centripetal force equals the energy representation of "rotation" activity, $\dot{\theta} p_\theta$. This seemed striking for the case of the circular orbit alone, but the same property shows up for the case of the closed Kepler orbit. The Hamilton equations for the

$\int_0^B \left\{ \dot{r}p_r + \dot{\theta}p_\theta \right\} dt/B$. Also $\left\{ \dot{r}p_r + \dot{\theta}p_\theta \right\}$ is twice kinetic energy and is thus usually written as $2T$.

[31] $r\left\{\frac{dU}{dr}\right\} = -U(r)$ under the traditional normalization of the potential function. We identify $\int_0^B -U(r)dt$, with total energy associated with the action of central force on the particle in motion, per period.

case of a circular orbit are then

$$\begin{aligned} rU_r - rT_r &= 0 \\ \text{and } p_\theta v_r &= p_\theta \dot{\theta}. \end{aligned}$$

$U(r) = -\chi/r$ is potential energy, for χ a positive constant. The so-called Huygens-Newton result is $rU_r = p_\theta\dot{\theta}$. And the complementary energy account is $p_\theta v_r = rT_r$, that is the energy corresponding to centripetal force is supplied by momentum, p_θ in $p_\theta v_r$.

Some exercises.[32]

(a) Constancy of Angular Momentum in the Kepler Problem: An Implication

For the Kepler problem, p_θ unchanging is constancy of angular momentum and p_θ and the ends of the orbit (perihelion and aphelion (long end)) have $\dot{r} = 0$ or $p_\theta = mv_a r_a = mv_p r_p$. Hence motion at the ends satisfies

$$\frac{v_p}{v_a} = \frac{r_a}{r_p}$$

or relative speed on the orbit is faster at the perihelion in accord with $v_p = [\frac{r_a}{r_p}]v_a$. Since $v = r\dot{\theta}$, we also infer that $\dot{\theta}_p = [\frac{r_a^2}{r_p^2}]\dot{\theta}_a$.

(b) Mass of the Earth Using the Huygens-Newton Force Formula for Circular Orbits

Centripetal force, according to the Huygens-Newton formula ($\dot{\theta}p_\theta = 2T$), is $\frac{mr^2\dot{\theta}^2}{r} = \frac{mv^2}{r}$. And the gravitational force between two masses is $\frac{GMm}{r^2}$. The moon has an approximately circular orbit with radius $3.8\times10^8 m$. G=$6.7\times10^{-11}\frac{Nm^2}{kg^2}$. in SI units. The period of the moon's rotation is 27 days or $\dot{\theta} = 2.7\times10^{-6} rad/s$. Hence equating forces yields the earth's mass in

$$M = \frac{v^2 r}{G} = \frac{\dot{\theta}^2 r^3}{G} = 6.0 \times 10^{24} kg.$$

[32] From Halpern [1988].

Note that the mass of the moon washes out of the calculation.

(c) Mass of the Earth Using the Weight of an Object at the surface

Radius of the earth is 6370km and an object at the surface weighs mg or $m \times 9.8m/s^2$. The gravitational force between the earth and this test mass is $\frac{GMm}{r^2}$. Equating weight of the object and force of gravity involves

$$\begin{aligned} 9.8m/s^2 &= \frac{M \times 6.67 \times 10^{-11} Nm^2/kg^2}{[6.37 \times 10^6 m]^2} \\ &= 6.0 \times 10^{24} kg. \end{aligned}$$

(d) Weight of an Object on Mars

From the above problem we have

$$w = \frac{GmM}{r^2}, \text{ for } w = mg.$$

Hence $\frac{w_e}{w_m} = \frac{M_e}{M_m} = \frac{r_e^2}{r_m^2}$. For an object that weighs $200N$ on earth, we have

$$w_m = 0.11 \times [\frac{6370}{3440}]^2 \times 200N = 75N$$

for radius of earth at 6370km and the radius of mars at 3440km and mars having 0.11 the mass of the earth. Since

$$\frac{w_e}{w_m} = \frac{g_e}{g_m}$$

we have $g_m = 9.8N \times [\frac{75}{200}] = 3.7m/s^2$.

(e) Kepler's Third Law

The Huygens-Newton formula for centripetal force for a circular orbit ($\frac{mv^2}{r}$) can be written as $\frac{m4\pi^2 r}{T^2}$. Equating this to $\frac{GMm}{r^2}$ yields period, $T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$. Hence for M the mass of the sun, we have the orbits of planets satisfying T^2 proportional to r^3, which is known as Kepler's Third Law.

Since the mass of a spherical object is $\rho 4\pi r^3/3$ for ρ the density, we can substitute for M above to obtain

$$T = \sqrt{\frac{3\pi}{\rho G}}.$$

Hence for a low altitude orbit (one near the surface of the spherical object), we have the period independent of the radius of the object. The period reflects only the density, ρ. ■

Clearly with periodic motion, one can look at a very long interval and see average energy inflow from location varibles being just enough for "restoring" location variables "on average" and similarly for momentum variables. (This is a kind of time homogeneity.) This provides a particular definition of periodic motion, a particular form of variable restoration per period. It happens to be an economics type of definition – a balance between inflow and useflow in units of energy, per period, per capital good type in the long run. What about non-periodic motion? Clearly this can be viewed as non-restoration of each and every variable at regular intervals in the long run. Of interest is that per instant we observe value balance between energy inflows and useflows (income from capital and expenditure on investment in capital), even though over the long run we are not seeing this value balance BY INPUT TYPE. Roughly speaking the key equations of equilibrium motion (slightly amended Hamilton equations) assume a basic zero value-balance form at each instant. This suggests to us the strong economics quality of classical mechanics. Classical mechanics can be expressed in different types of capital income-expenditure balance relations. We first noted this for periodic motioin "via" the Virial Theorem and we now observe the balance more generally.

5 Phelps's Golden Rule and Sustained Motion

A template for understanding periodic motion in classical mechanics is Phelps [1961] model without labor growth. The current physical output is Q equal to $F(K)$. At each moment output gets divided between replacement investment δK and consumption, C. There is no growth in this model, i.e. $\dot{K} = 0$. There is no discounting of the utility of consumption. Maximum C at each moment (in each period) is satisfied by steady state, K^*, $F_K(K^*) = \delta$ or the value of capital rentals, $K^* F_K(K^*)$ equal to expenditure on investment in replacement capital, δK^*. Note that capital is restored per "period" here by own rental income from K^*.[33][34] Hence this model is analogous to those exhibiting periodic motion in classical mechanics.

The Phelps model provides a template for interpreting what is going on in periodic models in classical mechanics. Attention gets directed to investment for capital restoration being funded by own rentals. Periodic motion also has capital income equal to investment expenditure, per period. One can see this above for the pendulum and Kepler problems and we discussed it in more detail in our earlier article. Periodic motion has location variable rentals funding location variable restoration or replacement investment and momentum variable rentals funding momentum variable restoration or replacement investment per period. Own rental income is funding own investment. There is no "cross-subsidization". Above we observed that in general, at each in-

[33] We have left open the question of how the economy moves to the Phelps level of K, given some initial stock, K_0. One might start with an optimal savings formulation with a concave utility function for consumption and a positive utility discount rate, ρ, and then let ρ tend to zero.

[34] In the Appendix we present a multi-sector version (two capital goods, one for location and one for momentum) of this golden rule result.

stant of time, particle motion does exhibit "cross subsidization", though total investment expenditure equals total rental income from capital. Hence, periodic motion is a special case of capital income-expenditure balance, with no cross-subsidization per period. In contrast, our free-fall problem exhibited no cross-subsidization at point $\widehat{z}$ but this was in a sense an anomaly. Over the period, the distinct inputs, the location co-ordinate and momentum were not independently self-sustaining in an energy sense.

6 Problems with the Economics Interpretation of Classical Mechanics

We have put forth the view that classical mechanics has a natural economics interpretation. We see input flows balanced in value terms with output flows. We see capital inputs with capital goods prices and rentals and we see a production function (potential energy funtion) that works like a production function in economics. We also see co-state variables operating as capital goods prices. What is wrong then with the economics interpretation of classical mechanics? Prices, rentals, and inputs can be negative and this seems to violate economics sense. We do not dispute this "problem". However we see periodic motion as essentially economic because energy flows PER PERIOD are positive. Hence inflow and useflows or input value and product value per period are of the correct sign from the standpoint of economics. The negative sign "problems" suggest to us that classical mechanics is a somewhat more general type of economic system, given its attendant sign "problems". Some might argue that economics is the general case since it makes such tight demands on the signs of prices and goods. We see no problem with this view.

Secondly, classical mechanics departs from economics with its special duality. Particle motion involves the simultaneous production of position-change and momenta-changes (velocity changes) and these joint products are convolved in a special way. Co-state variables work as capital goods prices in one energy account and the same variables work as capital inputs in another energy account. This may not be too surprising because momentum is mysteriously bound up with particle motion. A particle takes on momentum because it is in motion but a careful analysis suggests that it is this same momentum which is "causing" current position-change in the particle. Hence we should not be surprised that the production of both position-change and velocity-change should be convolved in the energy accounts somehow. The way they are convolved is certainly elegant and in its own way quite reasonable. The two energy accounts for position variables and momenta variables are distinct and inter-connected. One would be hard-pressed to find or create such a complicated production system in economics. If someone had come up with such a production system, they might well have worked up to Hamilton equations as the basic valuation accounts. Hamilton, it seems, did not come up with his formalization of classical mechanics by thinking about conventional problems in classical mechanics. He first developed his Hamiltonian formalism in his analysis of optical phenomena and then realized that what he had developed could be used to describe standard phenomena in classical mechanics.

Economic dynamics is to many, dynamics with an explicit discount rate. In many ways this is post Cass-Koopmans economics. It was quite respectable to think about economic dynamics before Cass-Koopmans without taking discounting into account. The Phelps model above is of course a leading example of economic dynamics without discounting. Hence our appeal to it. We do not deny that economic dynamics with discounting is very interesting but our analysis here suggests that as economists, we should keep our perspective broad and not gloss over action-sum minimization in classical mechanics because there is no discounting involved. Hence our view that one can distil a fairly clear economics story in classical mechanics, even though there is no discount rate in classical mechanics (classical dynamics).

7 Is Classical Mechanics Intuitive?

Generalized work on the input side has an energy value the same as "product value" at each instant. The notion of "generalized work" must be expanded to include energy inflows from current momentum values as well as current location values, where these latter capture the values of local forces. "Product value" is the energy representation of current particle position-change and momenta-changes. A system with friction, a non-conservative system, will not possess this value balance relation since some energy input will not show up in current position-change and momenta-changes. Some energy inflow will be lost to heat or the manifestation of the friction.

Our energy account and view of what is going on with particle-motion in classical mechanics is very different from the standard textbook view in physics. The standard view sees the current diminution in potential energy as "input" and the current increase in the particle's kinetic energy as "output". This is "the work-energy theorem" and was set out by Newton in Proposition 39, Book 1 of his *Principia.* He was not focusing on energy relations there but did produce "the work-energy theorem" in his analysis of particle free-fall in a general force field. Though "the work-energy theorem" and the related "conservation of energy"[35] has served physics extremely well since Newton, it is not a conceptual scheme that connects well to ways of conceptualizing in say economics. Once one asks about the nature of current "product" in classical mechanics and current "input", one is drawn back to the drawing board to, we suggest here, an account such as we have developed. It may not be an account that yields new physics but it allows for a different conceptual framework for what is going on in classical mechanics.

The great and iconclastic Feynman suggested somewhat indirectly that there is a need for a re-conceptualization of what the equations of classical mechanics are saying. Consider this passage from Feynman's well-known lectures. This passage follows his own explanation or reporting of particle motion on a closed Kepler orbit.

"What is gravity? ...All we have done is to describe **how** the earth moves around the sun, but we have not said **what makes it go**. Newton made no hypotheses about this; he was satisfied to find what it did without getting into the machinery of it. **No one has since given any machinery**. It is characteristic of physical laws that they have this abstract character. The law of conservation of energy is a theorem concerning quantities that have to be calculated and added together, with no mention of the machinery, and likewise the great laws of mechanics are quantitative mathematical laws for which no machinery is available. Why can we use mathematics to describe nature without a mechanism behind it? No one knows." (Feynman,

[35] One way to view this is to write down the Hamiltonian for a problem in conservative motion, as in $H(t) = U(q(t)) - \frac{1}{2}v(t)^2 + p(t)v(t)$, and then write down the derivative, $\dot{H}(t)$ and observe that the Hamilton equations of equilibrium motion imply $\dot{H}(t) = 0$. This is a proof of: equilibrium motion implies conservation of energy, where $H(t)$ is the so-called current total energy (the sum of current kinetic and potential energies) for the problem under consideration.

Leighton, and Sands [1963, p. 7-9], their emphasis)

Feynman is actually making a number of different observations here but he is explicit in asserting that conservation of energy is not intuitive. He also seems to be saying that the detailed equations of motion do not pay the effort at intuiting. Our view is that the reason the equations are counter-intuitive is that force and momentum are generally scrambled together in the equations. In a sense the Hamilton formalism allows for the separation of the roles of location and momentum in particle motion in classical mechanics. This is of course not Feynman's explicit observation here. He is not rejecting the view that conservation of energy provides a valuable benchmark for organizing thinking about problems in physics but the law itself possesses no compelling intuition, directly. Indirectly however, conservation of energy is linked to the "time symmetry" or "time autonomousness" of the laws of physics.

A further critique of physics textbooks. The Virial Theorem for periodic motion has twice the sum of kinetic energy over the period as the implicit measure of "ouput". And it has the sum of "generalized work" over the period as the implicit measure of "input". This to us is fine but we would like to see the labels energy inflow over the period and energy "consumption" or use-flow over the period attached to the two sides of the energy equation. What is missing is what we call "the complementary virial theorem" dealing with energy inflow linked to momentum variables and energy use-flows linked to momentum changes over the period. The energy representation of a current momentum, $p_i \frac{\partial H}{\partial q_i}$, requires a label, a counterpart to the generalized work associated with force. And the energy representation of current momentum-change, namely $-\dot{p}_i q_i$, also requires a label. Kinetic energy is the term used for energy associated with current particle position-change. We are not asking for new physics, here. We are asking for new labels of terms in a new energy account, an account built around the idea of current "product" factoring back into current "input".

We recognize that one can read key equations backwards. For example when one sees kinetic energy on the right hand side in the Hamilton equations, one has a left hand side with "energy from momentum". Hence the energy used up in position-change for the particle is not independent of the energy representation of momentum. The energy of position change seems convolved with that of momentum "inflow". We ask for a conceptual separation here, namely the view that particle position-change is energy using AND particle velocity change is independently energy using. This does not seem to be a standard intuitive take on what is going on in classical mechanics in large part we believe because problems end up being solved in terms of location variables. This makes a "location variable view" of what is going on the "view" that dominates textbook expositions. This is easy to understand but we suggest that this fails to communicate the fact that particle motion is a production system with simultaneous position-change and momenta-changes as current outputs.

Newton created modern classical mechanics from Galileo's treatment of local, particle free-fall and Kepler's three laws. In his own words, Newton created a mathematical theory of force, both local and celestial. Newton very explicitly asked that his theory generate observed trajectories of particle motion. He said (we paraphrase) on many occasions in the *Principia*: "Proposition i involves solving for the position of a particle and the interval of its motion, relative to its initial position." He then solved for equations generating trajectories and time lapses of motion. Never did he say "Proposition i involves solving for the trajectory and the corresponding path of velocity-changes of a particle and its interval of motion, relative to its initial position and velocity". Such a formulation would have emphasized our point about particle-motion being a production system, with current "product" being defined by current particle position-change and velocity-change. Newton was not wrong. He seems to have been intellectually cautious. He was able to analyze motion to his satisfaction with his mathematical theory of force but it seems that he was aware that force was a fairly controversial entity.[36] He chose not to digress into a larger theory of motion involving energy accounting or "input" and "output" detail. There is a sense in which the *Principia* contains the bare-bones theory of particle motion and that suited Newton's purposes. Significant "extensions" were made by Euler, Lagrange and Hamilton. Hamilton's view of particle motion, implicit in "the Hamilton equations" is, in our view, a significant refinement of Newton's view since it treats particle motion as "equally weighted", simultaneous position-change and velocity-change. Newton focused his analysis of particle motion on position-change and the interval of motion. But Hamilton's view has not displaced Newton's view in collge-level textbooks. In Hartwick [2004] we presented an energy accounting based on Hamilton's view for periodic motion and here we re-present that accounting, with extension to general particle motion, not simply periodic motion. We did not start out to construct an energy account for particle motion based on the Hamilton equations but our inquiry about current "product" and current "input" in classical mechanics led us to develop such an account.

[36]In a recent short essay, Frank Wilczek [2004] takes issue with the wide use of the concept of force. To him "force" is not a fundamental concept in modern physics and as such is somewhat vague. "While force itself does not appear in the foundational equations of modern physics, energy and momentum certainly do, and force is very closely related to them: Roughly speaking, it's the space derivative of the former and the time derivative of the latter (and $F = ma$ just states the consistency of those definitions.)" p. 12.

8 Concluding Remarks

We have been motivated to investigate the nature of motion in classical mechanics from the standpoint of economic accounting. One has a system with a product, namely particle motion, and one is motivated to factor back output into inputs. And one has the large question of current value balance: the value of current output being reducible to the value of current inputs. We have succeeded in a re-conceptualization of what is going on in particle motion in classical mechanics. One can tell the story with physics concepts alone as current inflow of energy from a location variable "going" to support a particular momentum-change, but the appeal to economics thinking and terminology makes our re-conceptualization very natural. There is income (in units of energy) going to investment expenditure and there is input or capital good restoration over a period. Distinct from this economics-based re-conceptualization, we have presented bits of new physics, it seems. First we have suggested a natural normalization for the potential function for periodic problems. Of interest here is the seeming, somewhat accidental normalization for the Kepler problem is the one we would endorse from the perspective of energy accounting. Secondly, we have dwelt on a basic complementary virial theorem. This seems like a very valuable "result" but is by no means based on an excursion into deep technical realms of physics or mathematics. It is easy to see, once it is pointed out. And thirdly, we have observed a striking energy account associated with centripetal force in the Kepler problem. The energy representation of centripetal force has the same value as the energy representation of current angular momentum. Perhaps some more technical new physics will emerge from our conceptualization in the future.

(1) The starting point of our analysis is the view that at each instant of

time, particle motion in classical mechanics is a system with a "product" and "inputs". We have observed that this intuition can be made concrete, particularly by drawing upon economics ideas. The "output" associated with particle motion is in general a simultaneous bit of position-change for the particle and momenta-changes for the particle. In units of energy these outputs can be "factored back" into values of inputs, at a moment in time. Twice current kinetic energy is the energy measure of current particle position change, and $-\dot{p}_i q_i$ is the energy value of current change in momentum i.

(2) The Virial Theorem in classical mechanics for periodic and quasi-periodic motion provides a concrete case of "input value" (rental income from capital inputs) in units of energy balanced with "product value" (investment expenditure) in units of energy per period. However the Virial Theorem deals only with the energy balance of position variables.

(3) There is a complementary virial theorem for periodic motion dealing with energy inflow per period associated with momentum variables balanced with energy use-flows associated with change-in-momentum values. This balance relation holds for momentum variables in a precisely analogous way as does the Virial Theorem for location variables.

(4) For non-periodic motion, we observe at each instant of time a balance of the energy value of "product" with the energy value of "input", with "product" comprising the SUM of the energy values of current position-change and momenta-changes for a particle in motion. There is in general cross-subsidization of energy from one type of input to the other over an arbitrary, long interval. Periodic motion, in contrast, exhibits no cross-subsidization, on average, over an arbitrary, long interval. Periodic motion corresponds with an economics steady state and resembles economics in the sense that

each "product" factors back in value terms to the value of "own inputs". Non-periodic motion possesses the striking property of current input value equalling current product value, but in aggregate over the "products" involved, not "product by product" as with periodic motion, dealt with on a period basis. The balance of instantaneous input value with product value for motion in general in classical mechanics suggests a deep constant returns to scale property in classical mechanics, akin to that for so-called balanced growth in economic dynamics. Again, we emphasize that it is input, "product" value balance in aggregate rather than on a "product" by "product" basis.

(5) Position-change and momenta changes in their energy representations have the interpretation of current investment expenditure in economics. The energy inflow associated with force or with the position of the particle has the interpretation of capital good incomes or rental flows. And the analogous energy inflow associated with momenta variables has the interpretation of capital good incomes or rental flows. The main instantaneous energy account in classical mechanics reads: the sum of capital input rentals at an instant balances with investment expenditures on capital, all in units of energy.

(6) Periodic motion is motion involving the restoration of inputs, location and momenta variables in "physical" units, over the period and there are simultaneous inflow-useflow value relations, in units of energy, sustaining the restoration process, over each period. Restoration of inputs can be referred to as capital input maintenance, over the period.

Appendix: From Capital Theory to Classical Mechanics

GENERAL (NON-PERIODIC) MOTION

Here is a basic Dorfman, Samuelson, Solow [1957 p. 290 and 294] capital theory program:

max $K_1[C_1 + \Delta S_1] + K_2[C_2 + \Delta S_2]$

subject to:

$[C_1 + \Delta S_1] - (1 - a_{11})X_1 + a_{12}X_2 \leqq 0$

$[C_2 + \Delta S_2] + a_{21}X_1 - (1 - a_{22})X_2 \leqq 0$

$b_{11}X_1 + b_{12}X_2 \leqq S_1$

$b_{12}X_1 + b_{22}X_2 \leqq S_2$

with variables non-negative. C_i is a consumption flow, K_i is a capital goods price, S_i is a quantity of capital, ΔS_i is an increment in capital good i (an investment), and the $a's$ and $b's$ are technical coefficients, capturing given production or transformation relationships. Here the services to two capital goods are "creating" flows of consumption goods and investment goods with a fixed technology of "transformation". The $X_i's$ are activity levels.

Their dual program is

min $r_1S_1 + r_2S_2$

subject to:

$p_1 \geqq K_1$

$p_2 \geqq K_2$

$-p_1(1 - a_{11}) + p_2a_{21} + b_{11}r_1 + b_{12}r_2 \geqq 0$

$p_1a_{12} - p_2(1 - a_{22}) + b_{21}r_1 + b_{22}r_2 \geqq 0$

and non-negativity constraints. Here r_i is a capital goods rental price and p_i is a capital goods "stock" price. We can compress this problem by assuming that $p_1 \geqq K_1$ and $p_2 \geqq K_2$ solve with equalities. The compressed formulation becomes

max $K_1[C_1 + \Delta S_1] + K_2[C_2 + \Delta S_2]$

subject to

$B_{11}[C_1 + \Delta S_1] + B_{12}[C_2 + \Delta S_2] \leqq S_1$

$B_{12}[C_1 + \Delta S_1] + B_{22}[C_2 + \Delta S_2] \leqq S_2$

and non-negativity constraints. The dual problem compresses to

min $r_1 S_1 + r_2 S_2$

subject to

$B_{11} r_1 + B_{21} r_2 \geqq K_1$

$B_{12} r_1 + B_{22} r_2 \geqq K_2$

and non-negativity constraints.

To specialize this system to an approximation[37] of a conservative classical mechanics problem, one (1) requires capital good S_i to be the input for the production of ΔS_j and (2) eliminates any consumption (sets $C_1 = C_2 = 0$ a priori). The first restriction might be referred to as production reciprocity or the reciprocal production property of classical mechanics. That is, our system becomes

max $K_1[\Delta q] + K_2[\Delta p]$

subject to:

$B_{21}[\Delta q] \leqq p$

$B_{12}[\Delta p] \leqq q$

with variables non-negative.

The dual program is

min $r_1 q + r_2 p$

[37] The approximation we have in mind is that we have an essentially linear link between a capital input and its product whereas in classical mechanics this link is in general not linear. But our linear representation of classical mechanics illustrates key issues very well.

subject to:

$r_2 B_{21} \geqq K_1$

$r_1 B_{12} \geqq K_2$

and non-negativity constraints. The capital inputs have been relabelled q for S_1 and p for stock S_2.

RESULT: This specialized capital theory program, solved with equalities, has the property that

$$\begin{aligned} r_1 q &= K_2[\Delta p] \\ r_2 p &= K_1[\Delta q]. \end{aligned}$$

The demonstration involves solving for $r_1 q$ in the second system and for $K_2[\Delta p]$ in the first system and observed that they are the same. These are two equations of "capital rentals equal to reciprocal investment expenditure".

The other specialization of our capital theory problem that classical mechanics asks for is the substitution for capital goods prices K_1 and K_2 with values p and q respectively. Yes, as we emphasized in the text, capital goods have levels p and q and capital goods prices have levels q and p. This is the striking specialization which classical mechanics "asks" of our general capital theory problem. This might be referred to as the dual roles of co-state and state variables.[38] Hence we have capital rentals equal to investment expenditures in

$$\begin{aligned} r_1 q &= q[\Delta p] \\ r_2 p &= p[\Delta q]. \end{aligned}$$

[38] In economics we think of prices K_1 and K_2 as deriving from preferences of agents or a planner, preferences for a preferred current bundle of product. Classical mechanics endogenizes these preferences in the above curious and profound fashion. Preferences "evolve" in a special classical mechanics fashion.

These two equations are our analogues to the Hamilton equations, here in units of energy as they are often expessed in the text above. Particle motion induces systematic changes in both capital stocks and capital goods prices period to period.

Finally, classical mechanics does not require variables to be non-negative. Given these departures from the general model set out above first, we argue that the resulting program is a good approximation to say the problem of particle free-fall in classical mechanics. In summary, particle motion in classical mechanics is a specially structured problem in capital theory, with location and co-state variables operating as capital goods. And the same co-state variables and location co-ordinates also are operating as capital goods prices in the sense above.

Let us solve for our analogue to particle free fall. We revert to our earlier notation. We can treat the above model as solved with equalities and get $\Delta S_1 = S_2/B_{21}$ and $\Delta S_2 = S_1/B_{12}$. We depart from the non-negativity constraint and set out $\Delta S_1 = S_2/[-B_{21}]$. Then we can take S_2 as momentum and S_1 as location (vertical co-ordinate for the free fall problem). We take S_1 equal to 1000 meters initially and S_2 equal to an initial momentum of 0.001. If we "run" our model we will have S_1 declining in increasing increments at a constant rate over each period and S_2 increasing over each period, also at a constant rate. Though these behaviors are not quite those of classical mechanics, they are reasonable approximations. They emerge from a canonical, simple, purely economics model with two capital inputs.

PERIODIC MOTION

The special case of periodic motion involves capital good restoration at the "end" of each period. Hence periodic motion can be viewed as a problem in stationary capital stocks. We follow Dorfman, Samuelson and Solow again (p. 234). The specialization for periodic motion is set out with $\Delta q = \delta q$ and $\Delta p = \delta p$ for δ the exogenously-given, constant decay rate. That is, current product (investments) must exactly replace decayed capital and q and p levels must remain unchanging. This is the essence of periodic motion. The decay of capital corresponds to energy used up by motion of the particle over the period.

That is, our system becomes

max $p[\Delta q] + q[\Delta p]$

subject to:

$B_{21}[\Delta q] \leqq p$

$B_{12}[\Delta p] \leqq q$

with variables non-negative.

The dual program is

min $r_1 q + r_2 p$

subject to:

$r_2 B_{21} \geqq p$

$r_1 B_{12} \geqq q$

and non-negativity constraints.

The "maintain capital intact" problem here has $\Delta q = \delta q$ and $\Delta p = \delta q$ at each period and capital stocks, q and p unchanging. Dorfman, Samuelson and Solow argue that with a smooth technology, a particular consumption vector (Δq and Δp) corresponds to a "right" technology and an efficient vector of capital inputs.[39] The "right" technology will be that for which the own rate

[39]This section of Dorfman, Samuelson and Solow compels one to infer that they presented the golden rule, attributed to Phelps, well before Phelps, unless one believes that one must be working in a population growth (at a constant rate) context in order to obtain the correct golden rule investment condition. We see population growth as a somewhat

of interest for each sector is the same. This is a golden rule argument. We pursued this above. If the technology is such that own interest rates are equal in: $r_1/p = r_2/q$, then the technology satisfies the condition

$$\frac{K_1}{K_2} = \left(\frac{B_{21}}{B_{12}}\right)^{1/2}.$$

for K_1 and K_2 standing in for capital goods prices, p and q. And if each capital good is shrinking from use at the same rate δ over each period, then replacement capital, Δq and Δp are δq and δp respectively. In such a case we observe that the condition $\Delta q/q = \Delta p/p$ implies that

$$\frac{S_1}{S_2} = \left(\frac{B_{12}}{B_{21}}\right)^{1/2}.$$

for S_1 and S_2 standing in for capital stocks, q and p. Since $\frac{K_1}{K_2} = \frac{p}{q} = \frac{S_2}{S_1}$ in classical mechanics, we have the result that: Proportional investment replacement across sectors implies equal own interest rates and vice versa. In addition, such restrictions on the technology yield the condition that own rentals are funding own current investment or $K_1 \Delta S_1 = r_1 S_1$ and $K_2 \Delta S_2 = r_2 S_2$. This is a multi-good extension of our version of the golden rule of investment (Phelps with no labor force growth) in the text above. This is the condition we observe for periodic motion in classical mechanics, namely own funding of own investment over the period in question. Hence our simple capital theory model is picking up (replicating?) key conditions we noted for the classical mechanics "model".

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