Numerical Analysis of Asymmetric First Price Auctions with Reserve Prices [†]

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Preliminary: Comments Welcome

Abstract

We develop a powerful and user-friendly program for numerically solving first price auction problems where an arbitrary number of bidders draw independent valuations from heterogenous distributions and the auctioneer imposes a reserve price for the object. The heterogeneity in this model arises both from the specification of ex-ante heterogenous, non-uniform distributions of private values for bidders, as well as the possibility of subsets of these bidders colluding. The technique extends the work of Marshall, Meurer, Richard, and Stromquist (1994), where they applied backward recursive Taylor series expansion techniques to solve two-player asymmetric first price auctions under uniform distributions. The algorithm is also used to numerically investigate whether revenue equivalence between first price and second price auctions in symmetric models extend to the asymmetric case. In particular, we simulate the model under various environments and find evidence that under the assumption of first order stochastic dominance, the first price auction generates higher expected revenue to the seller, while the second price auction is more susceptible to collusive activities. However, when the assumption of first order stochastic dominance is relaxed, and the distributions of private values cross once, the evidence suggests that the second price auction may in some cases generate higher expected revenue to the seller.

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1 Introduction

In this paper, we develop a powerful and computationally tractable algorithm for numerically solving first price single object auction problems where bidders draw independent and private values from heterogenous distributions, groups of symmetric bidders collude, and the auctioneer imposes a reserve price for the object. The algorithm is then implemented in a simulation exercise to investigate revenue (non-) equivalence between the first and second price sealed bid auctions in this asymmetric environment. We also investigate the stability of collusive behavior among bidders. Thus, this study contributes to three broad but not unrelated branches in auction theory. We thus discuss the literature and open issues on these areas.

There is a large body of work in auction theory that assumes symmetry in the beliefs of bidders regarding the value of the object being sold. The typical environment studied is one in which buyers have common underlying preferences and draw their signals from a symmetric uniform probability distribution (see Maskin and Riley (1984), Mathews (1983), Milgrom and Weber (1982a), and Riley and Samuelson (1981) for key contributions in this line of research). The structure of the symmetric auction model is attractive to theorists because in most cases, bid functions and expected revenues can be obtained analytically. It is widely known however that the assumption of bidder symmetry is restrictive and in many cases unrealistic in most applications. In practice, there are usually compelling reasons to think that bidders are ex-ante asymmetric, in that their beliefs may be drawn from heterogenous distributions of private values.

Relaxing the assumption of bidder symmetry typically renders the computation of equilibrium bids analytically intractable. This unfortunate fact has inhibited revenue comparison since expected revenue calculations for the asymmetric case typically require knowledge of the bid functions. However, contributions of Lebrun (1991, 1996, 1999) and Maskin and Riley (2000a,b) relax the assumption of bidder symmetry and establish existence and uniqueness of the equilibrium at a first price auction when bidders draw independent and private valuations from heterogenous distributions. Capitalizing on this progress, Maskin and Riley (2000a) provide some comparison of the seller revenue and bidders bidding patterns under the assumption of bidder asymmetry. In particular, they assume that the distribution of private values for one bidder first order stochastically dominates the distribution of the other bidder. Under this assumption, they are able to show that the high bid auction dominates the open auction in terms of seller revenue, and that the stronger bidder (the bidder with the first order stochastically dominant distribution of valuations) shades his bid further below his valuation than the weak bidder. Maskin and Riley (2000a) also showed that under different characterizations on the distributions (where one of the bidder's distribution of private values has a probability mass at the lower end point), the revenue to the seller may be higher in open auctions, than in high bid auctions. An important contribution in this area was made by Marshall, Meurer, Richard, and Stromquist (1994) (MMRS hereafter) where they developed a numerical algorithm for computing the equilibrium bid functions for the asymmetric first price sealed bid auction.

A common feature of these models is that they all assume that heterogeneity in distributions are restricted to two types (a weak type and a strong type). Furthermore, specific calculations rely heavily on the assumption that individual valuations are drawn from uniform distributions. First order stochastic dominance is either an artifact of the distributions used in these models (MMRS) or it is explicitly assumed (Maskin and Riley (2000a)). The algorithm developed in this paper allows for richer form of heterogeneity than the two type case. Indeed, the algorithm allows for there to be as many types as there are bidders. Furthermore, the algorithm is not restricted to the assumption of type being drawn from uniform distributions. In fact, the algorithm can be implemented under the assumption of the investigator's preferred distribution function for types, as long as the function is (at least numerically) invertible. No specific ordering of the distributions of private values is necessary for implementation of the algorithm. The ability to relax the assumption of first order stochastic dominance with this algorithm turns out to be important as we find that results from revenue comparisons under the assumption of first order stochastic dominance may not necessary carry over to cases where first order stochastic dominance no longer hold.

The study of MMRS was also important in that it demonstrated the potential for numerical analysis to contribute significantly to the advancement of theoretical auction. Their key interest was to study the viability of bidding rings in asymmetric first and second price auctions. They found (numerical) evidence to support the conjecture that collusive agreements are easier to reach in second price auctions than in first price auctions. This result holds under the assumption that all bidders are endowed with the same uniform distribution for beliefs, and a subset of these bidders form a coalition and bid against the other symmetric individual bidders. The algorithm developed in this paper is flexible enough to study the case where a set of distributionally homogenous bidders form a coalition and bid against potentially heterogenous individual bidders. The restriction that the coalition can only be formed by distributionally homogenous bidders can be relaxed with some work. However, in reality, the evidence suggests that this is likely to be the case. For example, in antique auctions, experts are empirically more likely to collude for reasons of keeping information private.

The algorithm developed in this paper draws on the principle developed in MMRS. In particular, we compute approximate solutions to the system of ordinary differential equations (ODE) that characterize the first order conditions for the existence of an asymmetric Nash equilibrium by applying a recursive piecewise low order Taylor series expansions technique. Similar to MMRS, the solution belongs to a class of "two-point boundary value problems", but they suffer from problems caused by a singularity of the system at the origin. This singularity eliminates the option of forward extrapolation procedures as the solution will not satisfy endpoint boundary conditions. It also complicates backward shooting procedures because the recursion becomes unstable in a neighborhood of the origin.

The variety of distributions that the algorithm allows for introduces two key additional difficulties. The first is that strong curvature in the distribution of private values result in strong (amplified) curvature in the key auxiliary function used in the solution of the system. This strong curvature makes Taylor series approximations less precise, but the major difficulty is that the system may shoot too fast to the "x-axis", which generates a different form of singularity of the system. The problems could possibly be avoided by implementing smarter routines that adjust the step size in these regions. However, with the complexity of the solution method itself, we opt for simpler methods so that attention is focused on the key actual solution methodology. Also, it is very cheap to refine the grid on which the routine "steps", thus reducing the significance of these pathologies. Second, the algorithm requires that the density of distribution of private values be sufficiently bounded away from zero on its support. In fact, this problem did not appear in MMRS since the density of the uniform distribution is trivially bounded away from zero on its support. Despite these pathologies, in practice the algorithm turns out to be remarkably stable. Indeed, once the condition that the density be sufficiently bounded away from zero is respected by the user, our experience is that the algorithm always produces stable and precise bid functions, even with quite extreme curvature of the distribution of private values.

Marshall and Schulenberg (1998) extended the numerical algorithm of MMRS to accommodate reserve prices set by the auctioneer. Their numerical analysis provides evidence that once optimal reserve price is introduced, second price auctions dominate first price auctions in terms of expected revenue of the seller. Again, these algorithms are written for uniform private value distributions and can only analyze the case of two types of bidders, the coalition on one hand and the fringe of symmetric individual bidders on the other, or in the case of coalition versus coalition. In summary, to further enrich the scope of investigation of independent asymmetric first price auction framework, we propose a solution algorithm much in the spirit of MMRS. This algorithm allows for (1) ex-ante heterogeneity in the distribution of private values for bidders, (2) heterogeneity that results from a (homogenous) subset of bidders engaging in collusive activities, (3) non-uniform distribution of private values, (4) arbitrary number of types of bidders, and (5) reserve prices to be set by the auctioneer.

In Section 2 we present the model. The model is solved in section 3. Section 4 presents samples of the numerical results for the case where private values are of distributed exponential. Section 5 offers a brief discussion.

2 The Model

We describe the environment of concern in which a single object is sold in a first price auction. Specifically, bidders simultaneously submit sealed bids for a single object where the highest bidder wins and pays his bid price. The group of potential bidders comprises of n risk neutral individuals. Each of the n individuals belong to one of r types, where each type i $(i = 1, \cdots, r)$ draw their valuations independently from a distribution $F_i^\star(\cdot)$ on $[\underline{v}_i, \overline{v}_i]$.

Consider an arbitrary type i from the set of r types. This group of individuals is divided into k_i coalitions of size u_i each. Thus the number of type i individuals is given by $k_i u_i$ and the total number of participants is given by $n = \sum_{i=1}^r k_i u_i$. We assume that each coalition acts as one bidder who draws a valuation from the cumulative $F_i = [F_i^\star(\cdot)]^{u_i}$. Hence we have $K = \sum_{i=1}^r k_i$ possible bidders. Without loss of generality, we call each possible bidder a coalition. For example, consider the auction consisting of $n = 10$ individuals grouped in $r = 3$ types. There are 4 individuals of type 1, 3 of type 2 and 3 of type 3. The 4 individuals of type 1 decide to form 2 coalitions of size 2 each. Thus $k_1 = 2$ and $u_1 = 2$. All 3 individuals of type 2 collude, thus we have $k_2 = 1$ and $u_2 = 3$. Finally, the players of type 3 decide to play individually, thus making $k_3 = 3$ and $u_3 = 1$. This setup is sufficiently flexible to describe and study a very large variety of auction environments.

Bid functions are denoted by the Greek letter ϕ_i, $i = 1, \cdots, r$. Bidders are assumed to be risk neutral with utility from winning the auction with a bid b given a type v_i defined as $U_i(v_i -$

$b) = v_i - b$. Clearly, utility from winning the auction is increasing in the individual's signal. Under these assumptions, proposition 5 of Maskin and Riley (2000b) establishes the existence of a monotonic pure-strategy equilibrium in the standard first price auction. Indeed, Lebrun (1996) has shown that these bid functions are strictly monotone and increasing, therefore, invertible. Inverse bid functions are denoted by the Greek letter λ_i, $i = 1, \cdots, r$. Uniqueness of such equilibrium is well established in the case with two types (Lebrun (1996)). However in the general n player game, equilibrium may not be unique in that we may end up with "non-essential" equilibria (Briesmer et al. (1967)).

In this paper, we assume that the distributions of private values have common support, that is $\underline{v}_i = \underline{v}$ and $\overline{v_i} = \overline{v}$ for all i. We also assume that F_i is twice continuously differentiable, with first derivative (the corresponding pdf) bounded away from zero on $[\underline{v}, \overline{v}]$. Under these assumptions, Lebrun (1999) proves in the general n bidder case that the equilibrium is unique, and that the inverse bid functions have a common support $[\underline{v}, t_\star]$, where $t_\star$ is the bid associated with the valuation $\overline{v}$, and $\underline{v}$ is the reserve price set by the auctioneer. We show in this paper that this equilibrium is amenable to numerical analysis, and presents itself as a natural extension to the methods proposed in MMRS. As such the (numerical) determination of $t_\star$ is a critical component of the problem to be solved.

3 Model Solution

3.1 The Differential Equations

Let $t = \phi_i(v)$ denote the Nash equilibrium bid submitted by coalition i when its highest valuation is v. Hence t is given by

$$t = Arg \max_{u \in (\underline{v}, \overline{v})} (v - u) F_i^{-1}(\lambda_i(u)) \prod_{j=1}^{K} [F_j(\lambda_j(u))]^{k_j}. \tag{3.1}$$

The first-order condition generates the following differential equation:

$$\begin{aligned} \prod_{j=1}^{K} [F_j(\lambda_j(t))]^{k_j} &= (\lambda_i(t) - t) \Big\{ \sum_{j=1}^{K} \big[k_j f_j(\lambda_j(t)) \lambda_j'(t) \prod_{s=1, s\neq j}^{K} F_s(\lambda_s(t)) \big] \\ &- f_i(\lambda_i(t)) \lambda_i'(t) \prod_{s=1, s\neq i}^{K} F_s(\lambda_s(t)) \Big\}, \quad i = 1, \cdots, K, \end{aligned} \tag{3.2}$$

where $f_i(\cdot)$ is the density function corresponding to $F_i(\cdot)$. Simplifying gives:

$$1 = (\lambda_i(t) - t) \left\{ \sum_{j=1}^{K} k_j f_j(\lambda_j(t)) \lambda_j'(t) F_j^{-1}(\lambda_j(t)) - f_i(\lambda_i(t)) \lambda_i'(t) F_i^{-1}(\lambda_i(t)) \right\} \tag{3.3}$$

The initial conditions are

$$\lambda_i(\underline{v}) = \underline{v}, \qquad i = 1, \cdots, K, \tag{3.4}$$

and the terminal conditions require the existence of a number $t_\star \in [\underline{v}, \overline{v}]$ such that

$$\lambda_i(t_\star) = \overline{v}, \qquad i = 1, \cdots, K. \tag{3.5}$$

3.2 Numerical Solution

Define $l_i(t) = F_i(\lambda_i(t))$. Differentiating gives $l_i'(t) = f_i(\lambda_i(t))\lambda_i'(t)$. Then equation (3.3) can be written as:

$$1 = [F_i^{-1}(l_i(t)) - t][\sum_{j=1}^{K} k_j \frac{l_j'(t)}{l_j(t)} - \frac{l_i'(t)}{l_i(t)}], \qquad i = 1, \cdots, K. \tag{3.6}$$

This transformation of the system of equations reduces the dimension of the problem, and makes it far more tractable than otherwise. The initial and terminal conditions for these new functions are:

$$l_i(\underline{v}) = 0, \qquad i = 1, \cdots, K, \tag{3.7}$$

$$l_i(\overline{v}) = 1, \qquad i = 1, \cdots, K. \tag{3.8}$$

Let l_t^0 denote the right derivative of l_i at $\underline{v}$,

$$l_i^0 = \lim_{t \to \underline{v}} l_i'(t). \tag{3.9}$$

It is straightforward to show that:

$$l_i^0 = f_i(\underline{v}) \frac{\sum_{j=1}^{K} k_j}{\sum_{j=1}^{K} k_j - 1}. \tag{3.10}$$

Successive derivations reveal that all higher order derivatives of l_i are 0 at $\underline{v}$. It follows that any attempt to evaluate numerically a forward solution of the first order differential equations (3.6) produces a linear solution. This problem is a manifestation of that found by MMRS. We thus follow their recommendation and solve the system of equations (3.6) backward starting from an assumed terminal point using the initial condition (3.4) as an indicator of whether or not we have used the correct value of $t_\star$.

The solution technique proceeds by approximating the l's by piecewise (low-order) polynomial expansions. For notational convenience, we switch to matrix representation of the first order differential equations (3.3). The matrix representation also convenient for implementation in FORTRAN 90. Define the following vectors:

$$\begin{aligned} l(t) &= \begin{bmatrix} l_1(t) \\ l_2(t) \\ \vdots \\ l_r(t) \end{bmatrix} = \textstyle\sum_{j=0}^{\infty} a_j (t - t_o)^j, \\ l'(t)/l(t) &= \begin{bmatrix} l_1'(t)/l_1(t) \\ l_2'(t)/l_2(t) \\ \vdots \\ l_r'(t)/l_r(t) \end{bmatrix} = \textstyle\sum_{j=0}^{\infty} \alpha_j (t - t_o)^j, \end{aligned} \tag{3.11}$$

where

$$a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{rj} \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{rj} \end{bmatrix}. \tag{3.12}$$

Given that $l'(t) = \sum_{j=0}^{\infty}(j+1)a_{j+1}(t-t_o)^j$, we can represent the equality $l'(t) = (l'(t)/l(t)) \times l(t)$ in terms of its power series expansion:

$$\begin{aligned} \sum_{j=0}^{\infty}(j+1)a_{j+1}(t-t_o)^j &= \left[\sum_{j=0}^{\infty}\alpha_j(t-t_o)^j\right]\left[\sum_{j=0}^{\infty}a_j(t-t_o)^j\right] \\ &= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}a_{is}\alpha_{i,j-s}\right)(t-t_o)^j. \end{aligned} \tag{3.13}$$

We can define a recursively as

$$a_{i,J+1} = \frac{\sum_{j=0}^{J} a_{ij}\alpha_{i,J-j}}{J+1}, \qquad i = 1,\cdots,r \tag{3.14}$$

There are two key recursive formulas that constitute our numerical solution. Equation (3.14) is the first. To derive the second, define

$$F_i^{-1}(l_i(t)) - t = \sum_{j=0}^{\infty} p_{ij}(t-t_o)^j, \qquad i = 1,\cdots,r \tag{3.15}$$

To evaluate (3.15), we implement an efficient recursive chain rule for such Taylor series expansions developed in appendix C of MMRS. Specifically, the routine takes as inputs the Taylor series approximation of $l_i(t)$ and (user supplied) Taylor series expansion of $F_i^{-1}(x)$ and returns as output the approximation of $F_i^{-1}(l_i(t))$. One significant advantage of this routine is that for complicated functions, one can break the function in to smaller, more manageable components that are easier to approximate by Taylor series expansion, and then use the routine to compose these different parts and recover a Taylor series approximation of the function.

Clearly, the algorithm requires that the inverse function $F_i^{-1}(x)$ is defined and well behaved. If there exists a point (v) on the support of the distribution function F_i such that $f_i(v) = 0$, then the second term in the Taylor series expansion of $F_i^{-1}(F_i(v))$ will be undefined. Thus the algorithm requires that $f_i(v) > 0$ for all v in the interval $[\underline{v},\ \overline{v}]$.

The system of first order differential equations (3.6) can be written as

$$\begin{aligned}
1 &= \left[\sum_{j=0}^{\infty} p_{ij}(t-t_o)\right]\left[\sum_{s=1}^{r} k_s\left(\sum_{j=0}^{\infty}\alpha_{sj}(t-t_o)^j\right)-\sum_{j=0}^{\infty}\alpha_{ij}(t-t_o)^j\right] \qquad (3.16)\\
&= k_1\sum_{j=0}^{\infty}\left(p_{i0}\alpha_{1j}+\sum_{s=1}^{j}p_{is}\alpha_{1,j-s}\right)(t-t_o)^j+\cdots\\
&+ (k_i-1)\sum_{j=0}^{\infty}\left(p_{i0}\alpha_{ij}+\sum_{s=1}^{j}p_{is}\alpha_{i,j-s}\right)(t-t_o)^j+\cdots\\
&+ k_r\sum_{j=0}^{\infty}\left(p_{i0}\alpha_{rj}+\sum_{s=1}^{j}p_{is}\alpha_{r,j-s}\right)(t-t_o)^j, \qquad i=1,\cdots,r
\end{aligned}$$

This system of equations can be used to recursively evaluate the values of α. To see this, note that that by evaluating (3.16) implies the following recursion in matrix form:

$$\begin{bmatrix} (k_1-1)p_{10} & k_2p_{10} & \cdots & k_rp_{10} \\ k_1p_{20} & (k_2-1)p_{20} & \cdots & k_rp_{20} \\ \vdots & & \ddots & \\ k_1p_{r0} & k_2p_{r0} & \cdots & (k_r-1)p_{r0} \end{bmatrix} \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \vdots \\ \alpha_{r0} \end{bmatrix} = 1, \qquad (3.17)$$

for $j=0$ and for $j=J\geq 1$,,

$$\begin{bmatrix} (k_1-1)p_{10} & k_2p_{10} & \cdots & k_rp_{10} \\ k_1p_{20} & (k_2-1)p_{20} & \cdots & k_rp_{20} \\ \vdots & & \ddots & \\ k_1p_{r0} & k_2p_{r0} & \cdots & (k_r-1)p_{r0} \end{bmatrix} \begin{bmatrix} \alpha_{1J} \\ \alpha_{2J} \\ \vdots \\ \alpha_{rJ} \end{bmatrix} = \qquad (3.18)$$

$$- \begin{bmatrix} (k_1-1)\sum_{j=1}^{J}p_{1j}\alpha_{1,n-j} & + & k_2\sum_{j=1}^{J}p_{1j}\alpha_{2,n-j} & + & \cdots & + & k_r\sum_{j=1}^{J}p_{1j}\alpha_{r,n-j} \\ k_1\sum_{j=1}^{J}p_{2j}\alpha_{1,n-j} & + & (k_2-1)\sum_{j=1}^{J}p_{2j}\alpha_{2,n-j} & + & \cdots & + & k_r\sum_{j=1}^{J}p_{2j}\alpha_{r,n-j} \\ \vdots & & & & & & \\ k_1\sum_{j=1}^{J}p_{rj}\alpha_{1,n-j} & + & k_2\sum_{j=1}^{J}p_{rj}\alpha_{2,n-j} & + & \cdots & + & (k_r-1)\sum_{j=1}^{J}p_{rj}\alpha_{r,n-j} \end{bmatrix}.$$

Inspection of the first term on the right hand side of equations (3.17) and (3.18) reveal that partition inverse techniques in matrix algebra can be applied to get a closed form inverse of this matrix. Thus this equation can be efficiently solved for recursively for α_j, $j\geq 1$. For ease of notation, write equations (3.17) and (3.18) as $A\alpha_{J+1}=B_J$, where B_J is 1 for $J=0$, and a function of $(\alpha_0,\cdots,\alpha_J)$ for $J\geq 1$. Then α_{J+1} can be calculated as

$$\alpha_{J+1}=A^{-1}B_J. \qquad (3.19)$$

Equation (3.19) is the second recursive formula that makes up our numerical solution to the system of differential equations defined in equation (3.6).

3.3 The Algorithm

A single run of computation requires initializing certain parameters, evaluating the corresponding numerical solution, and then deciding upon whether or not another run is needed.

1. *Initialization*. The parameters to be initialized are below.

   (i) $t_\star$: Unlike in MMRS, finding refined interval in which $t_\star$ belongs is not possible in our model. It is clear however, that $t_\star \in (\underline{v}, \overline{v})$

   (ii) N: The number of equal length intervals of $(\underline{v}, t_\star)$ to be considered.

   (iii) J: The order of the Taylor series expansions.

   (iv) ε: A small positive number to be used in evaluation of our convergence criterion.

2. *Numerical Evaluation*. Approximate values of $l_i(t_{j-1})$ are computed recursively by means of Taylor series expansions or order J around t_s, for s running backward from $s = N+1$ to $s = 1$. At any step s, the process of the Taylor series expansion can be described as two steps

   (i) Set $a_0 = l_{t_s}$. Calculate the Taylor series expansion of $F^{-1}(l(t_s))$ and compute p_0. With these quantities in hand, the matrix A in equation (3.17) can be computed. From these initializations, we can calculate α_0 from equation (3.19). Given α_0 and a_0, a_1 is calculated from equation (3.14).

   (ii) At step $j \geq 1$ of the Taylor series expansion, use the expansion of $F^{-1}(l(t_s))$ as well as a_j to compute p_j. Then use $(p_0, \cdots, p_j)$ and $(\alpha_0, \cdots, \alpha_j)$ to compute B_j in equation (3.18). With these, α_{j+1} can be computed from equation (3.19), and then a_{j+1} can be computed from equation (3.14). We then use our vector of coefficients $(a_0, \cdots, a_J)$ to compute $l(t_{s-1})$, and repeat step (i) for $s = s-1$.

3. *Convergence Criteria*. MMRS presents in their Appendix B a robust stopping criteria for their algorithm. However due to the high level on nonlinearity in our model, these stopping criteria prove to be unsatisfactory. The main reason for this is that for some distributions, l_i^0 will be highly nonlinear, and may shoot rapidly to zero. We thus adopt the stopping criteria that is implied by the initial conditions (3.7). This criteria provides the following objective function:

   $$\min_t S(t) = \sum_{i=1}^{K} [\hat{l}_i(\underline{v};t) - \underline{v}]^2 \tag{3.20}$$

   By construction, this objective function has a unique minimum at $\hat{l}_i(\underline{v};t_\star) = \underline{v}$. We employ the simplex search algorithm AMOEBA to find this minimum. The stopping rule is to stop if the improvement in the objective function is less that ε. If (3.20) is minimized, then the corresponding sequence of $\hat{l}_i(t_o)$'s constitute our (approximate) numerical solution.

4 Some Examples

The solution methodology and algorithm developed in 2 allow for the analysis of a large variety of auction environments. As an illustration, we select an arbitrary auction environment and study the corresponding bid functions that result from our computations. The bid functions are computed for an auction with $n = 5$ initial participants grouped into three types ($r = 3$). The distribution of private values belong to the standard two parameter Weibull family:

$$\tilde{F}_i(v) = 1 - \exp\left\{-\left(\frac{v}{\lambda_i}\right)^{\gamma_i}\right\}, \quad i = 1, \cdots, r. \tag{4.1}$$

The Weibull distribution is an appealing choice for private values in this example because of the flexibility of the distribution with respect to its parameters. Specifically, if the shape parameter is less than or equal to 1, the shape of the corresponding pdf is everywhere decreasing. However, for the values of the shape parameter greater than 1, the shape of the corresponding pdf becomes unimodal. The reservation price is chosen to be $\underline{v} = 1.0$ and the upper truncation point of the distribution is chosen to be $\bar{v} = 4.0$. The actual distribution used in the computation is therefore given by:

$$F_i(v) = \left[\frac{\tilde{F}_i(v) - \tilde{F}_i(\underline{v})}{\tilde{F}_i(\bar{v}) - \tilde{F}_i(\underline{v})}\right]^{u_i} \quad i = 1, \cdots, r, \tag{4.2}$$

where, u_i is the number of individuals in coalition i. Note that individual bidders automatically have a coalitions parameter of $u_i = 1$. The truncation here also serves the purpose of ensuring that the resulting density is bounded away from zero, with is necessary for the algorithm to work. The first six figures involve the situation presented below:

Characteristic		Type 1	Type 2	Type 3
Shape parameter	(γ)	0.5	1.0	2.5
Mean	(μ)	2.5	2.5	2.5
k		1	1	1
u:				
Figure 1		3	1	1
Figure 2		1	3	1
Figure 3		1	1	3
Figure 4		2	2	1
Figure 5		2	1	2
Figure 6		1	2	2

Because of the instability of the algorithm at the origin and the possible severe curvature of the function $l_i(t)$, numerical accuracy is essential. Therefore, key steps are taken to ensure high numerical precision. In the solution, N, the number of grid points is chosen to be 10000. Since the support of beliefs is the interval $[1,\ 4]$, this means that the step size is 0.0003. All the real variables needed to calculate $l_t(t)$ are declared in double precision, and all Taylor series expansions are done up to the fifth order. Finally, the convergence criterion is take to be $\varepsilon = 10E - 12$. Experimentation with the algorithm clearly indicates that these criteria can be significantly relaxed at a low cost of accuracy.

The algorithm converges relatively fast. For a three type auction environment, the algorithm converges in 2 minutes and 5 seconds on a 3GHz Pentium 4 laptop computer. The maximum of the objective function in any of the following computations in this and the rest of the paper is $4E - 4$. Despite the pathologies discussed in section 2, in practice the algorithm turns out to be remarkably stable. Indeed, once the condition that the density be bounded away from zero is respected by the user, our experience is that the algorithm always produces stable and precise bid functions, even with quite extreme curvature of the distribution of private values.

The first three figures plot the equilibrium bid functions that result from environments where one coalition of three individuals competes against two individuals of different types. Inspection of the bid functions indicate that the coalition bids least aggressively. This is consistent with the findings of Maskin and Riley (2000a) concerning the inverse relationship between the strength of the participant and aggression of its bid. However, their results were derived in the case of two types. The computed bid functions here strongly suggest that this result extends to cases where there are more than two types. Figures 4 to 6 explore the situation where there are two coalitions of two individuals each, bidding against a single individual. The gap between the bid functions narrow relative to the first three figures. This narrowing is due to the fact that in these cases the bidders are more equally matched in terms of their optimism, thus leading to more competition among them.

Figures 7 to 9 represent a different auction environment, given in the following table

Characteristic		Type 1	Type 2	Type 3
Shape parameter	(γ)	0.5	1.0	2.5
Mean	(μ)	2.5	2.5	2.5
u		1	1	1
k:				
Figure 7		3	1	1
Figure 8		1	3	1
Figure 9		1	1	3

In these environments there are no collusive agreements ($u_1 = u_2 = u_3 = 1$). However, unlike the models of MMRS and Marshall and Schulenberg (1998), there is still asymmetry among the bidders, due to the different values of the parameters indexing their distributions of private values. The bid functions are close to each other in these cases because of the fact that the parameters were chosen close to each other. These three graphs show the power of the algorithm, in that we can now numerically study asymmetric auctions without assuming that asymmetry comes from a subset of symmetric bidders deciding to collude.

Our final example is one of particular interest. Welfare analysis of auction models such as what is done in Maskin and Riley (2000a) assumes that the strong buyer's valuation first order stochastically dominates that of the weak buyer. This example indicates what happens when this condition is violated in a specific sense. In this example, there are 12 initial bidders, 6 of type 1 , 4 of type 2, and 2 of type 3. The shape parameters of the private value distributions for each type are: $\gamma_1 = 2.0$, $\gamma_2 = 1.5$, and $\gamma_3 = 1.2$. The corresponding means are given by: $\mu_1 = 1.063$, $\mu_2 = 1.174$, $\mu_3 = 1.317$. We also have that bidders of the same type collude. Inspection of figure (6) shows that the cdf of type 1 coalition starts out below those of coalitions 2 and 3. But then it increases and eventually rises above those of its competitors until the point $v = 3$ where they are all equal to 1. Interestingly, the bid functions exhibit very similar pattern. Maskin and Riley (2000a) show in the case of two bidders that the stronger bidder shades his bids more than the weaker bidder. The stronger bidder is defined as the bidder whose valuation first order stochastically dominates the other bidder. Figure (6) shows that in comparing two bidders, if the fist bidder's valuation is first stochastically dominated by that of the second bidder, and then eventually dominates the valuation of the second bidder, then the fist bidder bids more aggressively in an interval at the lower end of the support, and bids less aggressively in an interval at the upper end of the support. In short, the graph in (6) indicates that when the distribution of private values cross once in the interior of their common support, the corresponding equilibrium bid functions also cross (at most) once.

5 Second Price versus First Price Auctions

In this section, we employ the proposed algorithm to provide an insight into the revenue equivalence between first price and second price auctions when bidders are ex-ante asymmetric, and when asymmetric bidders collude. Since the Dutch descending price auction is strategically equivalent to the first price auction, and under the assumption of private values, the English ascending price auction and the second price auction has the same optimal strategies, the results of this analysis are also extended to comparison of Dutch and English auctions. First we study the seller revenue and bidder surplus in a case where symmetric bidders collude. Then we perform the analysis for the case where asymmetric bidders all compete. Finally, we perform the same analysis for the case where symmetric subsets of the bidders collude. In the cases of bidder collusion, the auction environment is characterized as follows. The membership of the coalition is determined ex ante. The size of each coalition, as well as the types of each coalition member is common knowledge within the coalition and to all other bidders. If the coalition wins the item, the coalition member with the highest valuation is awarded the item. Coalition members with values that are not the highest within the coalition do not submit a bid at the main auction. Also, there are no side payments within the coalition. MMRS discusses the effects of relaxing the assumption that valuations are known within the coalition. Their conclusion is that the first price coalition calculations are as favorable as possible from the point of view of the coalitions. Also they argue that the coalition is not disadvantaged at the first price auctions by not allowing for side payments among coalition members. Their arguments and conclusions carry over in this study.

5.1 The Simulation Technique

Now that we have a method of computing equilibrium bid functions for general asymmetric first price auctions models, we can simulate expected seller revenue and bidder surplus for comparison across first and second price auctions. The technique is essentially the following. First, private values are drawn randomly from the distributions for each coalition. Then the corresponding equilibrium bids are computed for each coalition. With these in hand, the winning bid is recorded as well as the identity of the winner along with its surplus and the revenue to the auctioneer. This process is repeated *NSIM* times. The resulting average revenue to the seller is the estimated expected revenue. Also, the average surplus of each coalition divided by the number of individuals in the coalition is the estimated expected surplus to each bidder. There are two important issues that have to be addressed in this Monte Carlo study.

In reality, what the algorithm generates is the equilibrium bid for a grid of private values. In the simulation, the random draw from the distribution of private values may not lie on the pre-specified grid. Therefore, interpolation techniques have to be implemented in order to attain approximate values of the equilibrium bid at points off the grid. For this we fit the data of bids to a high order orthogonal polynomial regression on private values. The order of the polynomial used in the reported results is 15. This ensures very high numerical accuracy. In fact, the square-root mean squared deviation of the predicted bid from the actual bids is typically indistinguishable. This high level of accuracy is due the smoothness of the bid functions.

In the simulation of expected revenue to the seller and expected profits to the bidders, the variance of these random variables are of the magnitude of the random variables themselves. Thus the raw simulation means and standard errors are of little use since the resulting difference in the means are rarely statistically different from zero for reasonable simulation numbers (up to 2 million). Increasing the simulation number solves this problem, but the resulting computation time makes this solution unappealing. In this paper, our variance reduction strategy is to extensively employ the regression approach to linear control variates. For completeness, a brief description of this technique presented.[1] Consider the problem of estimating the expectation of a random variable $\alpha = EY$ by Monte Carlo simulation, where Y is a random variable that is independently and identically distributed (i.i.d.). The natural point estimate of α is the average $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$, computed from n draws from the distribution of Y. Suppose the investigator has available a random vector $\mathbf{C} \in \Re^d$ with known mean μ_c that is correlated with Y. The method consists of using $\bar{\mathbf{C}}_i - \mu_c$ to control for $\bar{Y}_n$ via the linear transformation:

$$\bar{Y}_n(\lambda) = \bar{Y}_n - \lambda' \left(\bar{\mathbf{C}}_i - \mu_c\right), \tag{5.1}$$

where λ is a $d \times 1$ vector chosen to minimize $\mathrm{Var}(\bar{Y}_n(\lambda))$. Clearly the λ that achieves this minimum is the coefficient $(\hat{\lambda})$ in the linear regression of $\{Y_i\}_{i=1}^{n}$ on $\{\mathbf{C}_i'\}_{i=1}^{n}$ and a constant. The resulting reduction in the variance is given by the ratio:

$$\frac{Var(\bar{Y}_n(\hat{\lambda}))}{Var(\bar{Y}_n)} = 1 - R_{YC}^2, \tag{5.2}$$

where R_{YC}^2 is the usual coefficient of determination from the regression of Y on $\mathbf{C}$.

In the present paper, corresponding draws from the private value distribution for each type $F_j^{u_j}$ $j = 1, \cdots, r$, as well as draws from the private value distribution when all n individuals are of type j, F_j^n, $j = 1, \cdots, r$, provide an 'efficient' set of control variates. This requires that we have analytic or numeric means of these distributions. However, the mean of a two parameter truncated Weibull distribution is easily computed numerically with very high precision. In particular, the computation of the means require a one dimensional integration, which we perform using Gauss-Hermite quadrature with 40 abscissas.

[1] For a concise discussion of the various control variates techniques, see Szechtman (2003).

In computing expected revenue in the asymmetric first and second price auctions, the reduction in variance from the control variates technique is striking. In all results presented, we achieve reductions in variance from as low as 62% to as high as 91%. The reduction in variance achieved in the calculation of expected bidder surplus is more modest, but still quite significant. We achieve reduction in variance from as low as 19% to as high as 69%. The variance reduction technique here, along with clever choice of control variates results in all the revenue comparisons and all the bidder surplus comparisons that follow be highly statistically significant.

5.2 Symmetric Bidders Colluding

Consider the auction environment where there are $N = 5$ potential bidders, each drawing valuations from the same truncated two parameter Weibull distribution. The shape parameter is given by $\gamma = 1.0$ (thus an exponential distribution), and the mean of the distribution is 2.0. The distribution is truncated between 0.5 (the reserve price) and 3.0.

TABLE 3
Auctioneer's Expected Revenue and Bidders' Expected Surplus (per capita) at a First and a Second Price Auction (N=5, $\gamma = 1$, $\mu = 2$, Lower end point=0.5, Upper end point=3.0

					First Price			Second Price	
k_1	k_2	u_1	u_2	Auct.	k_1	k_2	Auct.	k_1	k_2
1	1	4	1	1.4758	0.1572	0.2205	1.3941	0.2161	0.1022
				(0.0001)	(0.0004)	(0.0012)	(0.0002)	(0.0003)	(0.0013)
1	2	3	1	1.7078	0.1204	0.1394	1.6878	0.1561	0.1023
				(0.0001)	(0.0008)	(0.0010)	(0.0002)	(0.0007)	(0.0011)
1	3	2	1	1.8099	0.1064	0.1119	1.8069	0.1234	0.1023
				(0.0002)	(0.0011)	(0.0009)	(0.0002)	(0.0011)	(0.0009)
1	1	3	2	1.6545	0.1353	0.1483	1.6460	0.1560	0.1231
				(0.0001)	(0.0007)	(0.0010)	(0.0002)	(0.0007)	(0.0010)
2	1	2	1	1.7668	0.1171	0.1236	1.7654	0.1232	0.1024
				(0.0002)	(0.0007)	(0.0016)	(0.0002)	(0.0007)	(0.0017)
5	0	1	0	1.8498	0.1022	0.1022	1.8496	0.1022	0.1022
				(0.0002)	(0.0006)	(0.0006)	(0.0002)	(0.0006)	(0.0006)

Note. Computed by Monte Carlo using 1,000,000 drawings.

Table 3 reports the expected revenues and surpluses for first and second price auctions. Note that the bid functions in this analysis satisfy the (weak) first order stochastic dominance condition assumed by Maskin and Riley (2000a). Conditional on the size of the coalition, the auctioneer's expected revenue at the first price auction is always greater than or equal to the expected revenue at the second price auction. Similar results are found in MMRS where their analysis is based on private values being drawn from the standard uniform distribution. MMRS argue that this is evidence that first the price auction is less susceptible to collusion than the second price auction.

Maskin and Riley (2000a) also finds that the high bid (or first price) auction generate higher expected revenue than the open (or second price) auction under shifts and stretches the private value distribution. These transformations of the distributions result in a violation of the assumption made in our analysis that all private value distributions have common support. The evidence here suggests that the results found in MMRS and Maskin and Riley (2000a) are not artifacts of the assumption that beliefs are drawn from uniform distribution. However, Maskin and Riley (2000a) find that the open auction is superior to the high bid auction when probability mass is shifted to the lower end point for one type. This transformation of the private value distribution violates the assumption in this paper that the distributions are twice-continuously differentiable everywhere on the interior of their common support. It also violates their assumption that distributions are ordered in the first order stochastic dominance sense. We thus take this as evidence supporting the conjecture that the first price auction is superior to the second price auction whenever the distributions of private values satisfy the first order stochastic dominance property and are twice-continuously differentiable on the interior of their supports. Table 3 also suggests that strong bidders prefer the second price (open) auction and weak bidders prefer the first price (high bid) auction. This result is also consistent with Maskin and Riley (2000a).

We now compare the expected buyer surplus to gain some insight into the relative profitability and susceptibility of collusion in first and second price auctions. The simulation results in Table 3 seem to support the conjecture that coalitions are more profitable in second price auctions that they are in first price auctions. To see this we compare the first three rows of Table 3 where we find that conditional on the size of the coalition, the expected surplus to the coalition k_1 is always greater in the second price auction that it is in the first price auction. Again, a similar conclusion is draw in MMRS.

The simulation results also shed light on another interesting related question. In the case where all the bidders form separate coalitions, are the individuals in the larger coalition better off than those in smaller coalitions? The results in Table 3 suggest that this is indeed the case in first price auctions, but not so for second price auctions. Looking at column 4 of Table 3, we see that the per-capita surplus in the coalition of 3 bidders is higher than the per-capita surplus in the coalition of 2 bidders in the second price auction, but lower in the first price auction.

We now turn to the question of whether coalitions are sustainable in first and second price auctions. Our strategy is to check if it is individually rational for an existing coalition to accept an outsider, and if it is individually rational for the outsider to join the coalition. Comparing rows one to three of Table 3 (in reverse order) we see that in both the first and second price auctions, increasing the number of participants in the coalition increases the per-capita expected surplus. For example, moving from row 3 to 2, we see that in the first price auction, increasing the number of participants from 2 to 3 increases the per-capita expected surplus from 0.1064 to 0.1204. Thus it seems individually rational for existing coalitions to accept an outsider. But is it the case that individual bidders will want to join the coalition?

Comparing the first 3 rows of Table 3 from bottom up, we see that for the first price auction, the per-capita expected surplus is always higher outside the coalition than inside it. Furthermore, as the size of the coalition increases, the remaining individual bidders do better in terms of expected surplus, and the difference grows as the size of the coalition grows. However, the individual bidders outside the coalition in the second price auction do just as well regardless of the size of the coalition, and always do worse than individuals inside the coalition. Thus the evidence suggests that as the size of the coalition grows, the remaining individual bidders are more likely to join the coalition. This suggests then that the second price auction is susceptible to collusion and the first price auction is not. Similar results are found in MMRS.

5.3 Asymmetric Bidders Colluding

The conclusion drawn above that the first price auction is superior to the second price auction whenever the distributions of private values satisfy the first order stochastic dominance property and are twice-continuously differentiable on the interior of their supports leads us to our next issue. Is the first price auction still superior to the second price auction when we relax the assumption of first order stochastic dominance? To shed some light on this question, we simulate an asymmetric auction environment where there are $N = 5$ potential bidders partitioned into 2 types characterized by different distributions of private values. The first three bidders draw valuations from a truncated two parameter Weibull distribution with shape parameter given by $\gamma = 1.5$, and the mean of the distribution is 1.0, where the truncation is between 0.5 and 3.0. The last two bidders draw valuations from the a truncated two parameter Weibull distribution with shape parameter is given by $\gamma = 1.5$, and the mean of the distribution is 3.0, again where the truncation is between 0.5 and 3.0.

TABLE 4

Auctioneer's Expected Revenue and Bidders' Expected Surplus (per capita) at a First and a Second Price Auction ($N = 5$, $\gamma = 1.5$, $\mu_1 = 1.0$, $\mu_2 = 3.0$, Lower end point=0.5, Upper end point=3.0)

					First Price			Second Price	
k_1	k_2	u_1	u_2	Auct.	k_1	k_2	Auct.	k_1	k_2
3	2	1	1	1.7627	0.0449	0.1994	1.7552	0.0389	0.2140
				(0.0002)	(0.0009)	(0.0010)	(0.0002)	(0.0010)	(0.0010)
1	2	3	1	1.7219	0.0481	0.2165	1.7241	0.0491	0.2145
				(0.0002)	(0.0009)	(0.0009)	(0.0002)	(0.0009)	(0.0010)
1,1	2	2,1	1	1.7510	0.0458,0.0464	0.2044	1.7462	0.0433,0.0389	0.2144
				(0.0002)	(0.0012),(0.0017)	(0.0010)	(0.0002)	(0.0012),(0.0017)	(0.0010)
3	1	1	2	1.6393	0.0614	0.2231	1.5869	0.0388	0.2983
				(0.0002)	(0.0008)	(0.0008)	(0.0041)	(0.0004)	(0.0007)

Note. Computed by Monte Carlo using 1,000,000 drawings.

Row 2 of Table 4 sheds some light on this question. The bid functions in this environment cross one at around 1.5 (Figure 9). What is remarkable is that this is the only setup in which the expected revenue to the auctioneer is higher in the second price auction. The difference is also statistically significant. This result does suggest that in the case where first order stochastic dominance no longer holds, the first price auction may no longer be superior to the second price auction. In particular, in the case where the CDF's cross once, leading to the resulting bid functions crossing once, it may be the case that the second price (open) auction is superior to the first price (high bid) auction. This result is close to the result in proposition 4.5 of Maskin and Riley (2000a) where the weak bidder is characterized by some of the mass of the distribution being shifted to the lower end point. As smoothed version of their setup would lead to the two cdf's crossing once. Thus, according to our results, the bid functions cross once. These results then suggest that whenever the bid functions cross once, the second price auction may in some cases generate higher revenue to the auctioneer than the first price auction.

It is important to note that we are not claiming that if the bid functions cross then the second price auction always generates higher expected revenue than the first price auction. The point of this exercise is to show that dominance of the first price auction may simply be an artifact of the assumption of first order stochastic dominance. To illustrate the point, we present a case where the bid functions cross, but the first price auction is still superior the second price auction. In this environment, there are 4 bidders, 3 of type 1 and 1 of type 2. Type one bidders draw private values from the truncated two parameter Weibull distribution with mean 1.0 and shape parameter 2.0. Type two bidders draw private values from the truncated two parameter Weibull distribution with mean 3.0 and shape parameter 1.5. The distributions are truncated between 0.5 and 3.0. The resulting equilibrium bid functions along with their corresponding private value distributions are presented in Figure 12. The key distinctions between the function in Figure 11 and 12 are that the bid functions in Figure 12 cross further to the left of the support and they deviate further apart. The simulated expected revenues in this environment are 1.2999 for the first price auction and 1.2984 for the second price auction. Again these differences are statistically significant. Thus this is an example where even though the bid functions cross, the first price auction is still superior to the second price auction in terms of expected seller revenue.

6 Conclusion

We propose an algorithm for numerically solving first price auction problems where bidders draw independent valuations from heterogenous distributions and the auctioneer imposes a reserve price for the object. The heterogeneity in this model arises both from the specification of ex-ante heterogenous, non-uniform distributions of private values for bidders, as well as the possibility of subsets of these bidders colluding. We simulate the model under various environments. The simulation results suggest that stronger bidders shade their bids more in the asymmetric first price auctions. The results also indicate that collusive activities are more profitable and sustainable in asymmetric second price auctions. We find evidence that under the assumption of first order stochastic dominance, the first price auction generates higher expected revenue to the seller. However, when the assumption of first order stochastic dominance is relaxed, and the distributions of private values cross once, the evidence suggests that the second price auction may in some cases generate higher expected revenue to the seller.

Possible extensions of the algorithm include, allowing for bidders of different types to collude, extending the algorithm to heterogenous affiliated distributions, and allowing for risk aversion in bidder preferences. The latter extension, though theoretically important will essentially require an additional Taylor series expansion, to evaluate the chosen utility function. The other extensions present a more formidable challenge of efficiently computing order statistics from complex, convoluted joint distribution functions.

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Figure 1: $k_1 = k_2 = k_3 = 1$, $u_1 = 3$, $u_2 = 1$, $u_3 = 1$

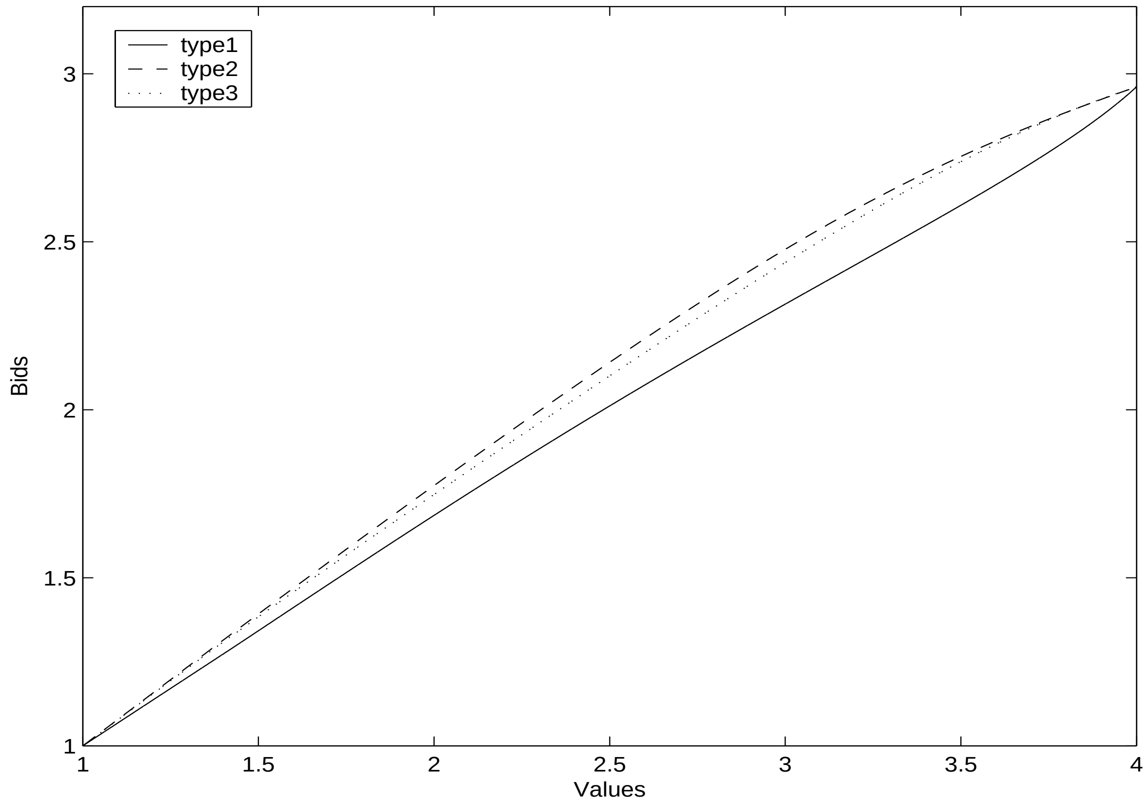

Figure 2: $k_1 = k_2 = k_3 = 1$, $u_1 = 1$, $u_2 = 3$, $u_3 = 1$

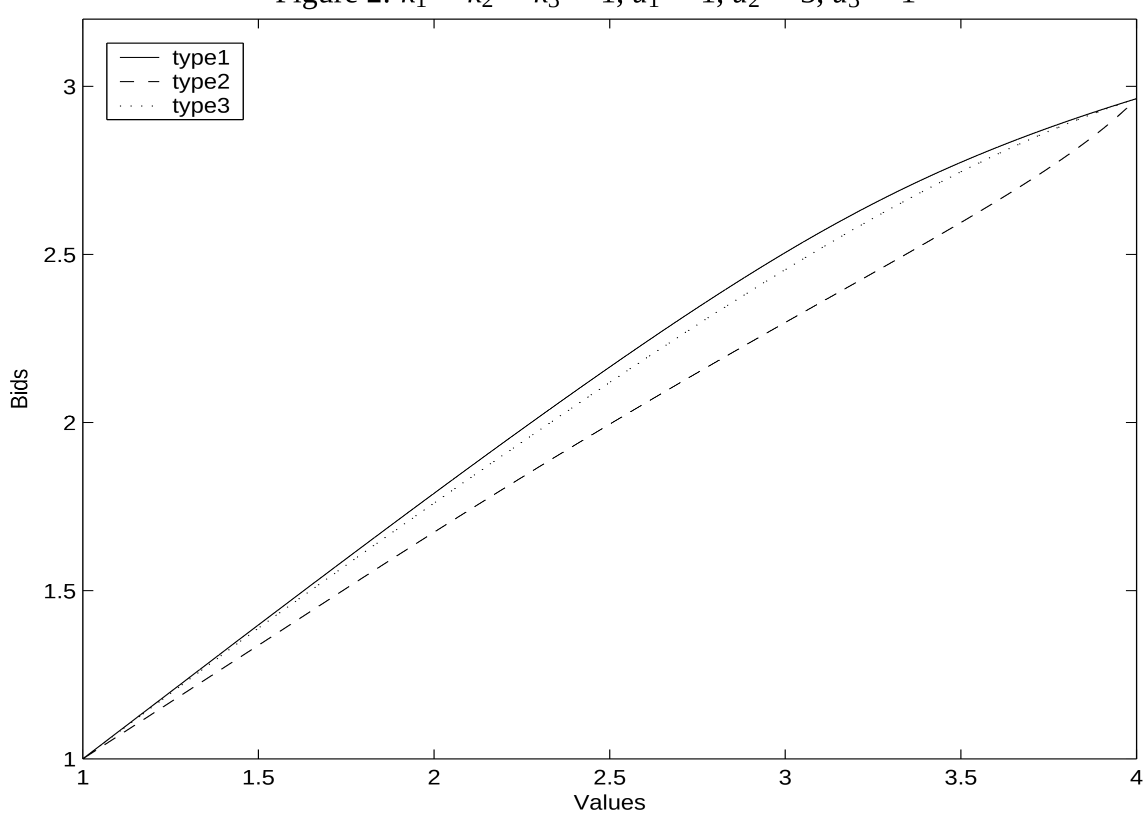

Figure 3: $k_1 = k_2 = k_3 = 1$, $u_1 = 1$, $u_2 = 1$, $u_3 = 3$

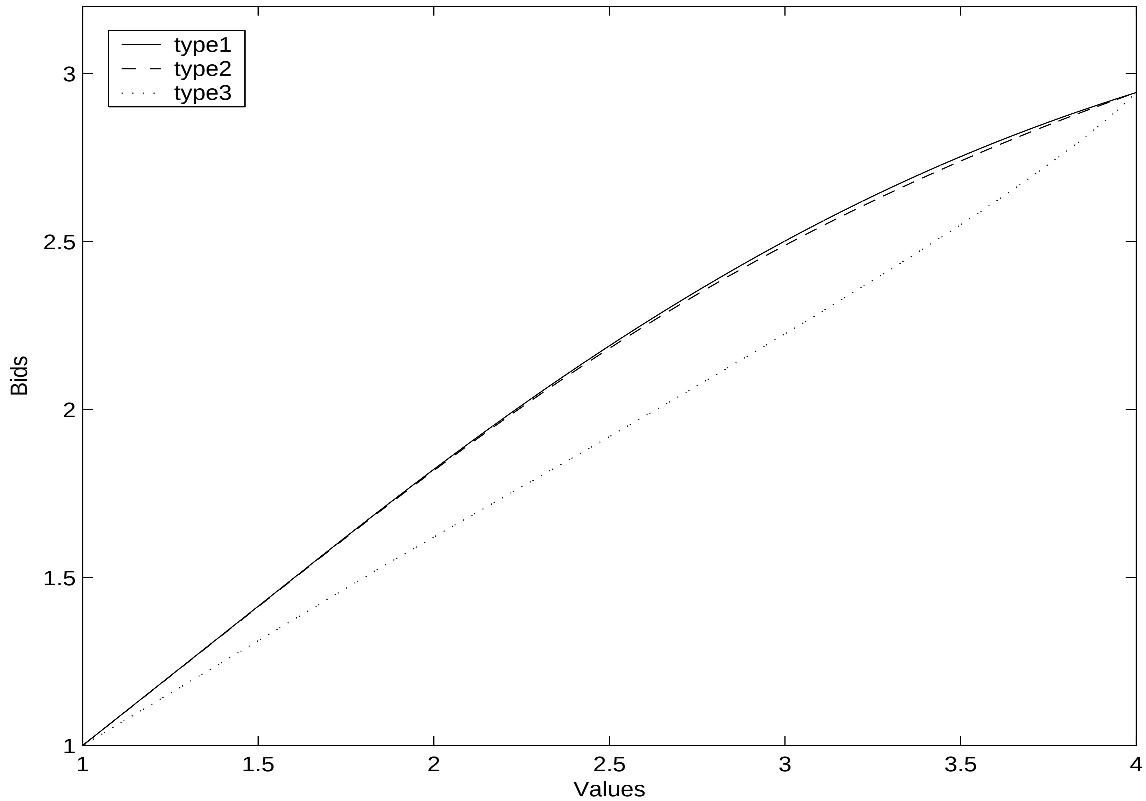

Figure 4: $k_1 = k_2 = k_3 = 1$, $u_1 = 2$, $u_2 = 2$, $u_3 = 1$

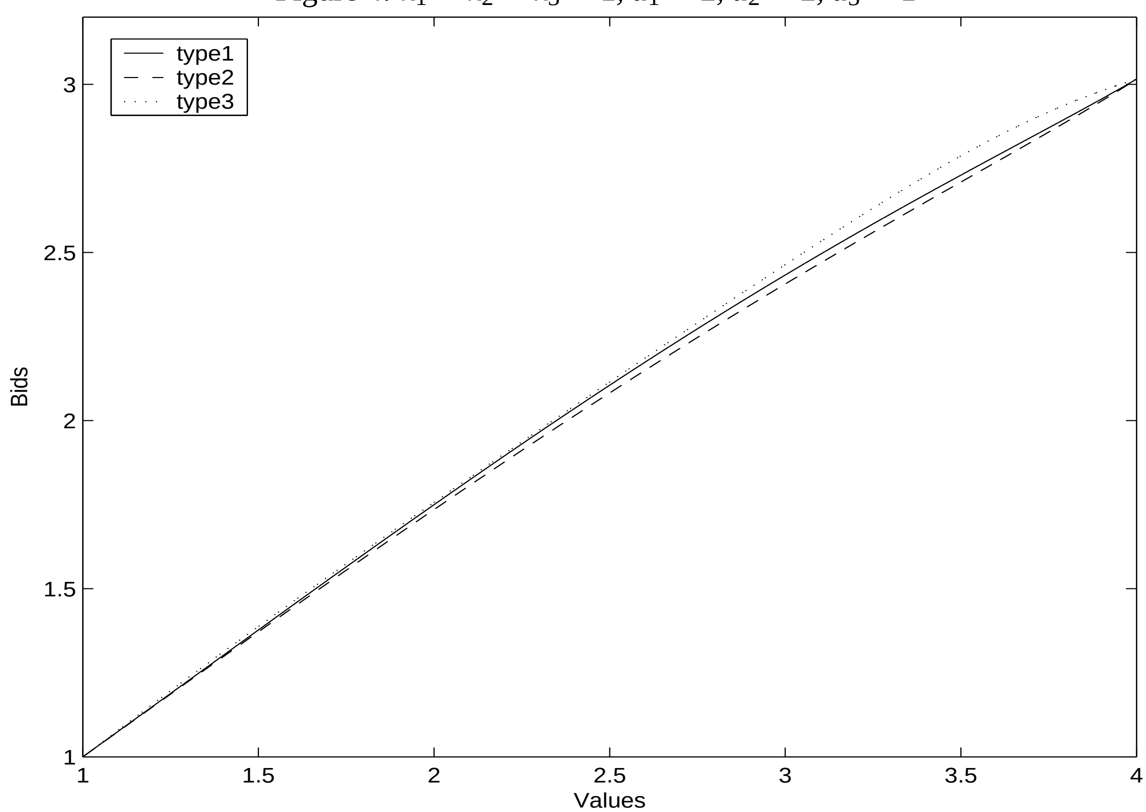

Figure 5: $k_1 = k_2 = k_3 = 1$, $u_1 = 2$, $u_2 = 1$, $u_3 = 2$

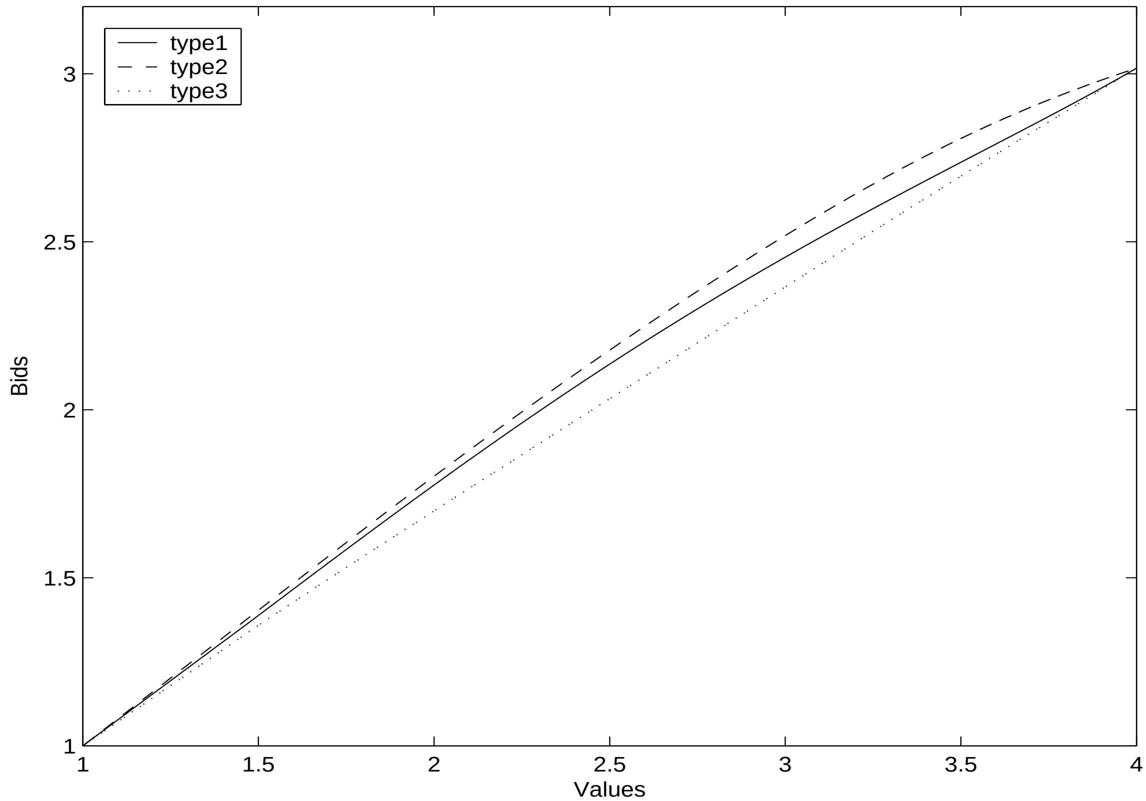

Figure 6: $k_1 = k_2 = k_3 = 1$, $u_1 = 1$, $u_2 = 2$, $u_3 = 2$

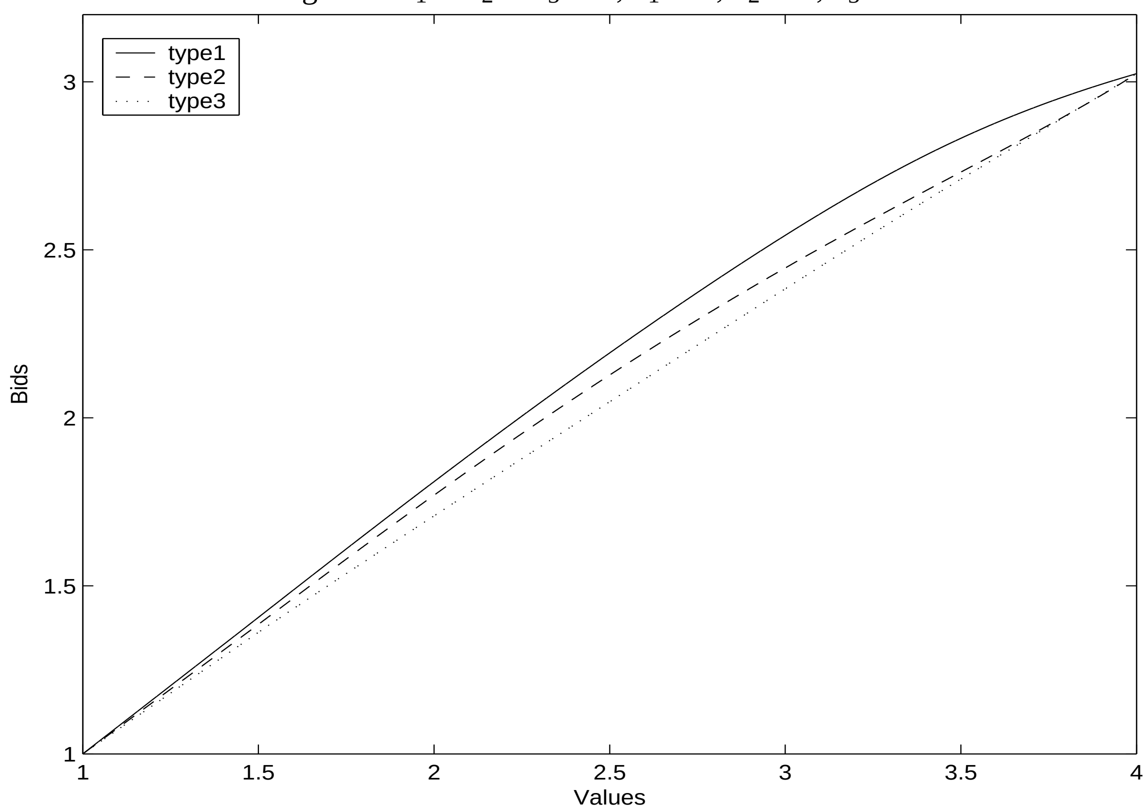

Figure 7: $u_1 = u_2 = u_3 = 1$, $k_1 = 3$, $k_2 = 1$, $k_3 = 1$

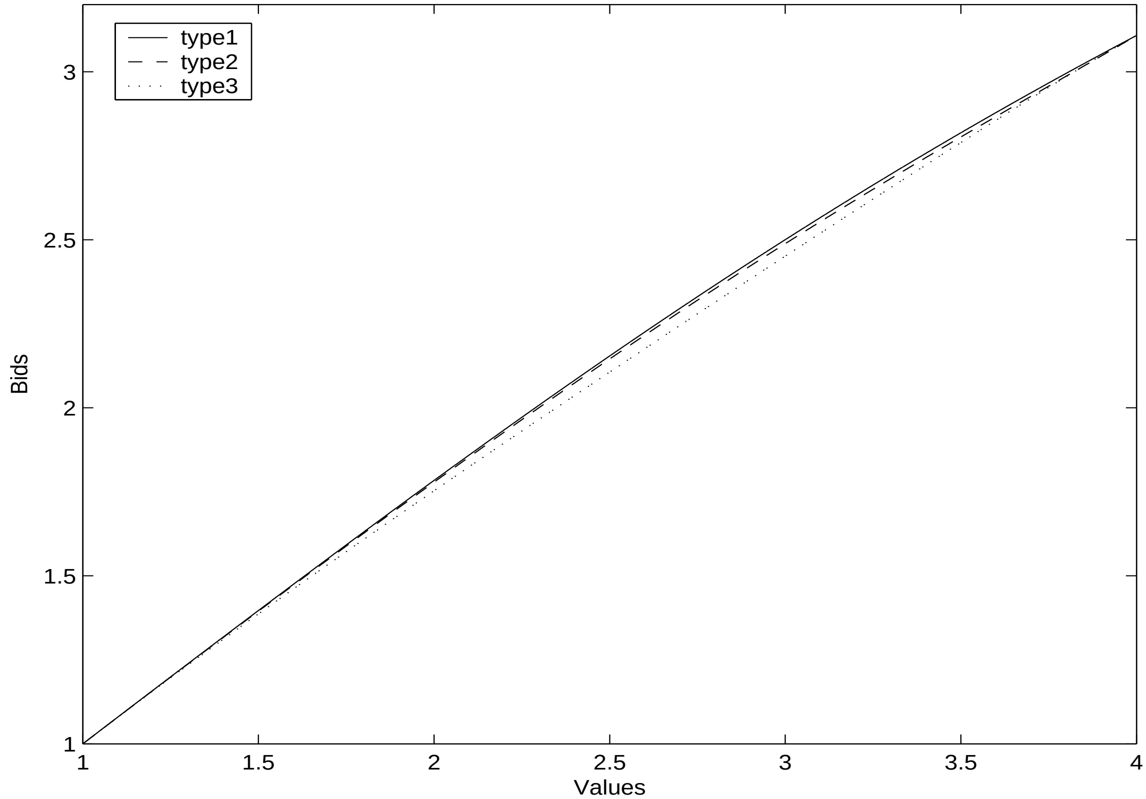

Figure 8: $u_1 = u_2 = u_3 = 1$, $k_1 = 1$, $k_2 = 3$, $k_3 = 1$

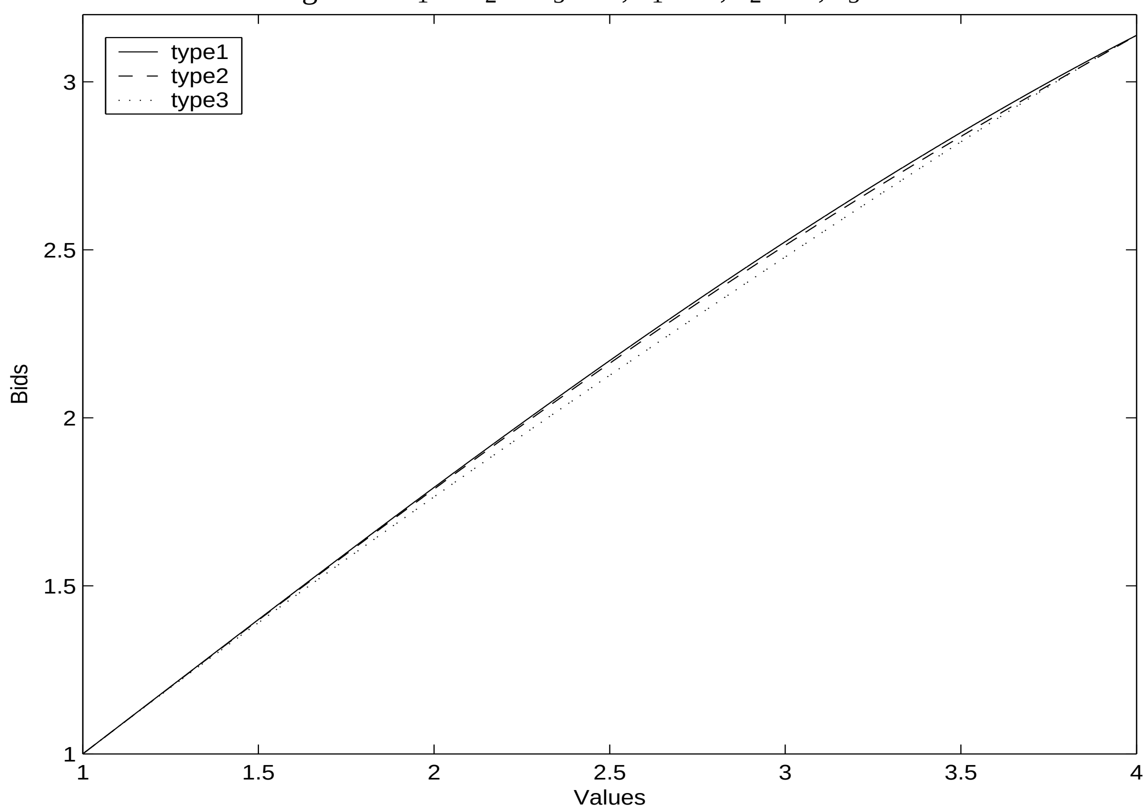

Figure 9: $u_1 = u_2 = u_3 = 1$, $k_1 = 1$, $k_2 = 1$, $k_3 = 3$

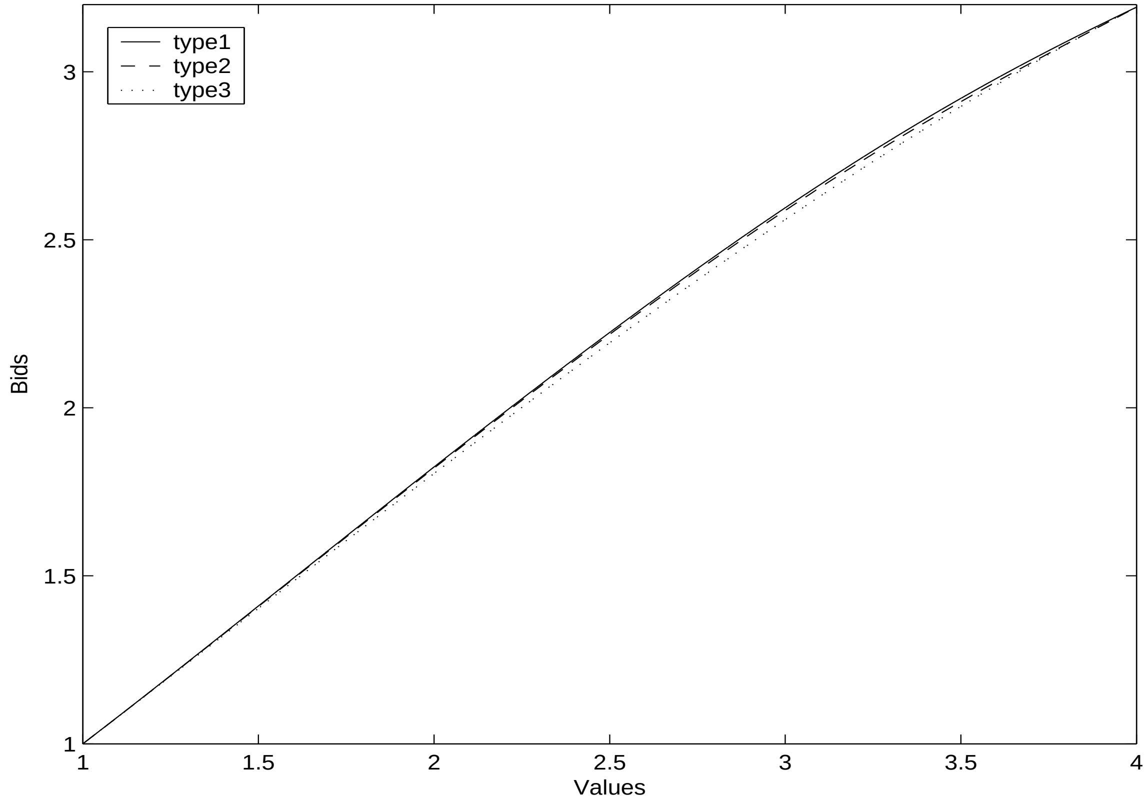

Figure 10: $k_1 = k_2 = k_3 = 1$, $u_1 = 6$, $u_2 = 4$, $u_3 = 2$

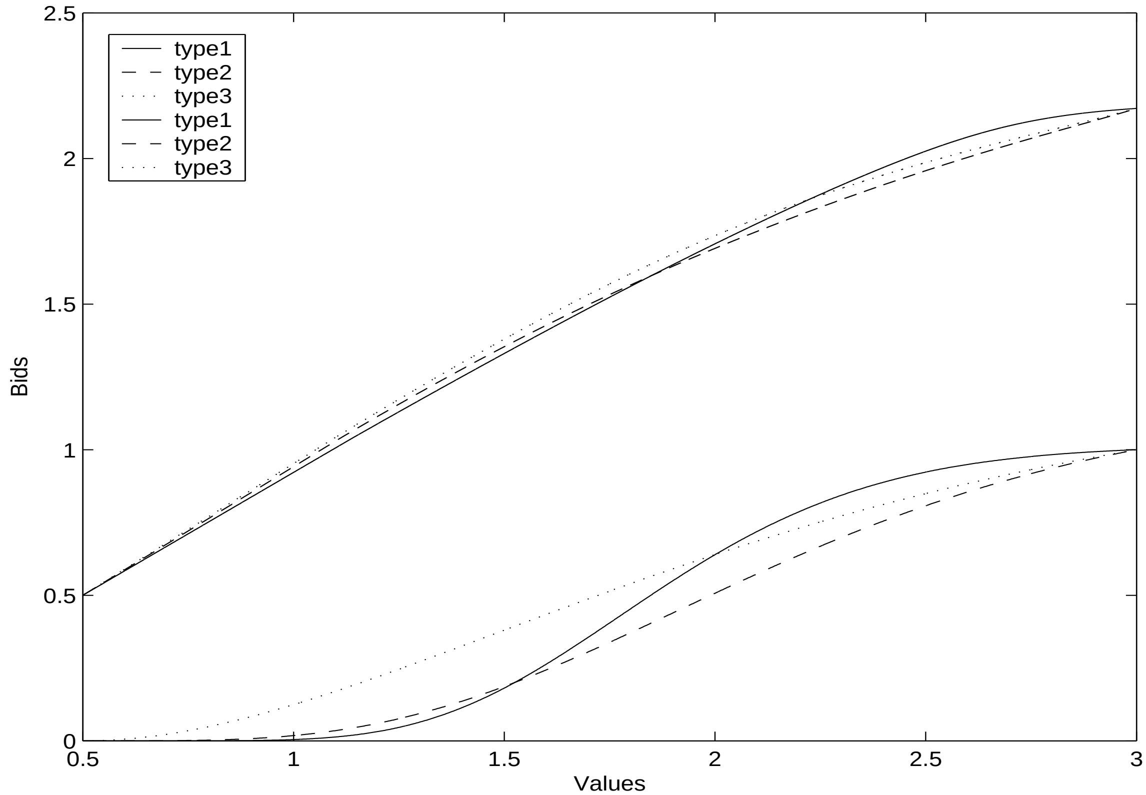

Figure 11: $u_1 = 3, u_2 = 1, k_1 = 1, k_2 = 2$

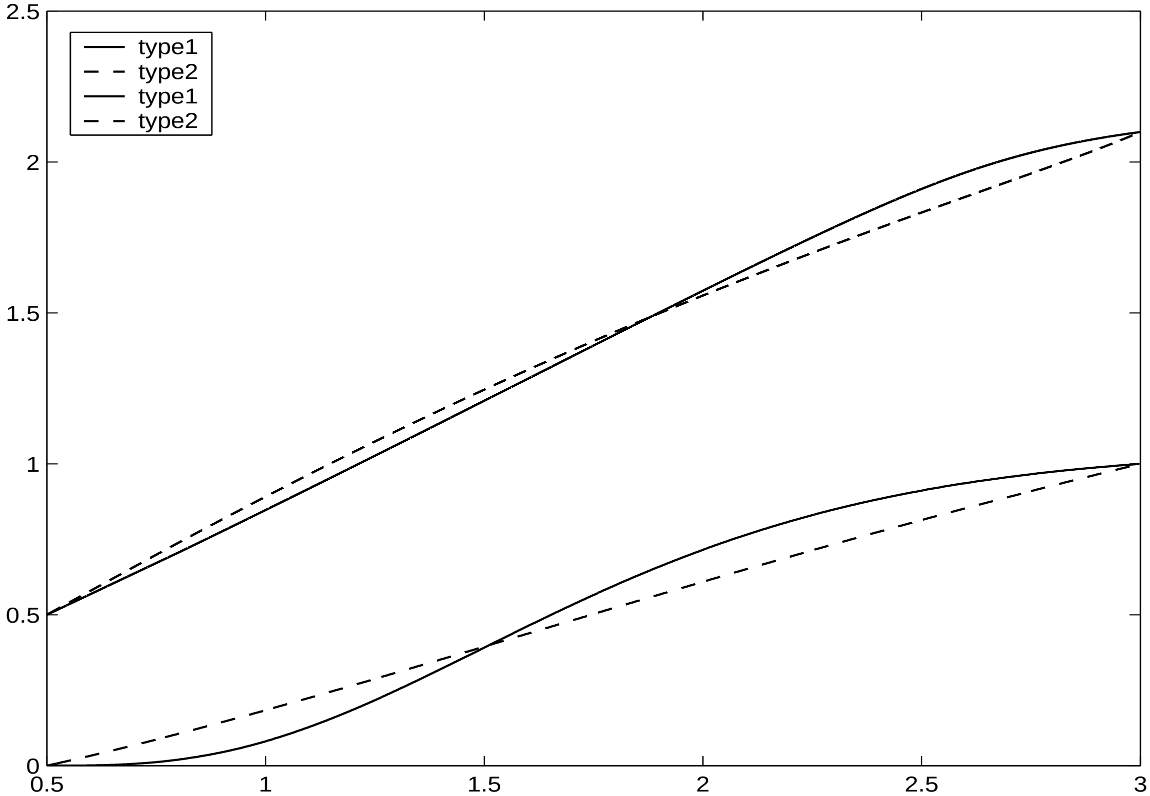

Figure 12: $k_1 = k_2 = 1, u_1 = 3, u_2 = 1$