A Continuous-Time Version of the Principal-Agent Problem.

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Abstract

This paper describes a new continuous-time principal-agent model, in which the output is a diffusion process with drift determined by the agent's unobserved effort. The risk-averse agent receives consumption continuously. An optimal contract, based on the agent's continuation value as a state variable, is computed by a new method using a differential equation. During employment the output path stochastically drives the agent's continuation value until it hits a low retirement point or a high retirement point. Unlike in related discrete-time models, one can use calculus to derive comparative statics and evaluate inefficiency. [1]

Keywords: Principal-agent model, hidden action, optimal contract, Brownian motion

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1 Introduction.

This paper explores a continuous-time analogue of the repeated principal-agent model. The continuous-time formulation leads to a new method for deriving an optimal contract, one that is simpler computationally than the widely used discrete-time method developed by Phelan and Townsend (1991). It also yields a clearer intuitive understanding of the optimal contract's nature. The optimal contract comes from a solution of a differential equation, which allows us to derive various comparative statics results.

We consider a dynamic setting, in which an agent controls the drift of an output process by his choice of effort. The principal does not observe the agent's effort, but sees the total output: a Brownian motion with a drift that depends on effort. The principal offers a contract to an agent at time 0 and commits to it. The contract consists of a flow of consumption at every moment of time contingent on the entire past output path. The agent is risk-averse and the principal is risk-neutral. The agent receives utility from consumption and disutility from effort. In our basic model, the agent demands an initial reservation utility from the entire contract in order to begin, and the principal will offer a contract only if he can derive positive profit from it. If the contract is offered and accepted, the agent cannot quit, and the principal cannot replace the agent. After we solve the basic model, we consider extensions where the agent can quit, and the principal can replace the agent at a cost. It is one of the advantages of the continuous-time formulation that such extensions can be analyzed with ease, using the same differential equation that applies to the basic model.

In our setting, the optimal contract can be described in terms of the agent's continuation value as a single state variable. At any moment of time, the agent's continuation value is the total utility, which the agent expects from the future if he acts optimally. In our continuous-time setting, the dynamic evolution of the agent's continuation value is naturally described by its drift and volatility. The volatility of the agent's continuation value has an analogue in discrete-time models, where the observed realization of output makes the agent's continuation value follow a random walk. In discrete time, it is the random walk's step size that determines the agent's incentive to work hard. That is, step size expresses the sensitivity of the agent's continuation value to realized output, and hence a large step size motivates the agent to work hard for high output. Volatility is the analog of step size in a continuous time model, and as one would expect, it is by manipulating the volatility of the agent's continuation value that the principal creates incentives in our setting.

The drift in the agent's continuation value can be understood as follows. From a contractual point of view, the agent's continuation value is the expected present value promised to the agent by the principal. Promises must be fulfilled. Given a particular promise, the principal can pay to the agent less now and escalate future promises, or pay more now and owe to the agent less in the future. The principal's choice of how to fulfill his promise induces a drift in the agent's continuation value. We find that it is optimal for the principal to pay to the agent at a rate, at which the marginal cost of delivering utility to the agent is a martingale whenever the agent receives positive consumption. We call this phenomenon *compensation smoothing.*

Here is the form of an optimal contract. Depending on the agent's participation constraint, the principal will choose a starting promised value W_0 to maximize his profit. The starting promised value can be equal to or greater than the agent's initial demand $\hat{W}$. As time goes on, the principal will recompute a new promised value W_t at every point of time using a stochastic differential equation driven by the output path. The agent remains employed as long as his continuation value stays in the interval $(0, W_{gp})$. When the agent's value reaches an endpoint of this interval, the agent is retired: he stops putting effort and receives his continuation value by consuming a constant stream of consumption payments from the principal. Before the agent is retired, the principal's payment to the agent at every point of time is uniquely determined by W_t. The larger W_t, the greater the payment. An interesting feature of the optimal contract is the existence of a probationary interval $[0, W^*]$ of continuation values, where $W^* < W_{gp}$. When the the agent's continuation value enters the probationary interval, the principal stops paying him, but keeps adjusting W_t based on the current path of output.

Why does the optimal contract exhibit these features? The low retirement point is necessary because the only way to deliver to the agent a value 0 is by giving him constant consumption 0, and allowing him to not work. The high retirement point exists due to the income effect: when the agent's consumption is high, it costs the principal too much to compensate him for positive effort. In the probationary interval, the principal is punishing the agent at his own cost, that is, the principal's profit is *increasing* in the agent's value in this interval. During probation, the least-cost way of punishing the agent is by paying him zero.[2]

Using the differential equation that characterizes the optimal contract, we can derive new comparative statics results about how the principal's profit and the optimal contract

[2] We assume that the utility from zero consumption is bounded and normalize it to zero.

depend on the parameters of the model. By far the most powerful effect on the principal's profit comes from the productivity of the agent. Increase in productivity improves the principal's profit directly through higher output and indirectly by making effort more detectable. We prove that increase in productivity improves the principal's profit even in the face of comparable adverse changes in both the cost of effort and the volatility of output.

The continuous-time formulation allows us to estimate how much efficiency is lost due to informational problems. In the limit as the discount rate goes to zero, the *total* loss of efficiency in terms of the principal's profit is $\frac{\delta\alpha^2\sigma^2}{2}$, where δ is the agent's coefficient of absolute risk aversion, α is the piece-rate (the fraction of the risk borne by the agent) required to induce proper incentives, and σ is the volatility of output. This result elaborates upon what is implied by the Folk Theorem: the "per period" inefficiency converges to 0 (proportionally to the discount rate r), but the total inefficiency accumulated over time converges to a strictly positive constant.

1.1 Related Literature.

Let us discuss how this paper is related to existing literature. The idea that the agent's promised value is a sufficient state variable to build optimal contracts is not new. Promised value has been used in many discrete-time models due to the theoretical developments of Abreu, Pearce and Stacchetti (1986 and 1990). Our paper will show that in continuous time, promised value sufficiently describes the past history to specify an optimal contract.

The research in our paper was inspired by the discrete-time model of Phelan and Townsend (1991). They develop a method of computing optimal long-term contracts in a discrete time setting. Their method relies on linear programming to solve the principal's problem in a given period based on a guess of a profit function for the continuation of the game. When this problem is solved for every continuation value, one obtains a new guess for a profit function. Multiple iterations lead to convergence to the true profit function. One can see a direct parallel between their discrete-time solutions and the continuous-time solutions in this paper. The main advantage of their approach is its applicability to a very wide range of settings, even those that require more than one state variable. Also, because of discretization of the set of feasible consumptions and continuation values, their method does not require any assumptions on the form of the agent's utility function. However, the method of Phelan and Townsend is much more computationally intensive than the method of solving differential equations suggested in this paper. Also, this paper contributes to our understanding of the theory of optimal contracts between a principal and an agent. A lot of features of an optimal contract are illuminated more clearly by the continuous-time solutions, e.g. drift and volatility of the promised value, existence of a probationary period and the two retirement points on the boundary. Finally, we are able to find analytical comparative statics.

Brownian motion has been first applied to the principal-agent problem in a paper by Holmstrom and Milgrom (1987). In their model, the agent also controls the drift of a diffusion process for total output. Unlike in our continuous-time model, in their model the agent receives only one payment in the end of a finite time interval, when the interaction with the principal ends. Their paper pursues two main results: that an optimal contract depends only on aggregate output and that the final payment to the agent is linear in aggregate output. Holmstrom and Milgrom show that these results hold given the following assumptions: the agent has an exponential utility of consumption, and the disutility of effort is computed by subtracting a fixed amount from the agent's consumption, independently of the total level of wealth. In particular, their form of utility function has no income effect: it takes the same monetary incentives to induce the agent's effort when his income is high or low.

In our model aggregation and linearity disappear on a global time scale. When we consider a wide range of utility functions, the principal needs the flexibility to adjust the agent's piece-rate depending on the agent's income level. In our model the agent's marginal utility and risk aversion are changing in a complex way with income level, and the optimal piece-rate reflects the incentives that the agent needs and the trade-off between incentives and insurance. However, both linearity and aggregation are present on a small time scale. In our model during a small interval of time Δt, the change in the agent's value is proportional to the change in total output ΔX. Speaking loosely, it does not matter by what sequence of wiggles this change ΔX has occurred; only the aggregate change determines the evolution of the agent's continuation value. The model of Holmstrom and Milgrom applies for the kinds of employment, where the agent's income level does not change significantly during the course of employment; therefore income effect does not matter. Our model applies to situations where the agent's income level changes during the course of a long term contract, e.g. when the agent expects to move up the employment ladder.[3]

[3]In our model, for discount rates very close to 0, the agent's income level and piece-rate do not change significantly for the effective duration of the contract. In this limiting case, the form of an optimal contract is approximately linear.

The asymptotic efficiency result, i.e. that the principal's profit converges to first best as the discount rate converges to 0, is not new. Among others, Fudenberg, Holmstrom and Milgrom (1990) derive an asymptotic efficiency result in a discrete-time setting. They argue that the reason for this result is that as the discount rate converges to 0, the agent effectively becomes less risk averse because the principal can smooth consumption for the agent. We show that even though the average "per-period profit" inefficiency converges to zero, the total inefficiency added up over time converges to a strictly positive constant. This result is new.

Recently, Williams (2003) has developed another new and very general principal-agent model in continuous time. The model is based on a stochastic state process X, whose evolution depends on the principal's choices, the agent's choices, X itself, and time t. The agent's choice is unobservable to the principal. The procedure to find an optimal contract involves solving a PDE, and forward an backward SDEs. The resulting contract can be written recursively using several state variables: time t, state X and the agent's value. When hidden states are allowed in the model, the contract involves an additional state variable. Williams considers a finite time horizon. Unlike Williams, we design a continuous-time model in infinite time horizon and analyze the specific issue of unobserved effort. In our setting, we are able to derive an optimal contract in terms of a single state variable, the agent's continuation value. The simplicity of the contract allows us to investigate its defining features, give intuitive meaning to the components of the contract, and derive comparative statics results.

The paper is organized as follows. Section 2 provides the setting and formulation of the principal's problem. Section 3 presents an optimal contract and discusses its interesting features: the drift and volatility of the agent's continuation value, retirement points, and the probationary interval. Section 4 provides rigorous mathematical derivation and justification for the form of an optimal contract. Section 5 presents some comparative statics results. Section 6 characterizes asymptotic contracts for interest rate r close to 0. Section 7 presents several alternative formulations of the model that can be solved by the same differential equation. Section 8 concludes the paper.

2 The Setting.

Consider the following dynamic principal-agent model in continuous time. A standard Brownian motion $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F}, \mathcal{Q})$ drives the output process. The total output produced up to time t, denoted by X_t, evolves according to

$$dX_t = A_t\, dt + \sigma dZ_t,$$

where A_t is the agent's choice of effort level and σ is a constant. The agent's effort level process $A = \{A_t \in \mathcal{A}, 0 \leq t < \infty\}$ is adapted to $\mathcal{F}_t$, where the set of feasible effort levels $\mathcal{A}$ is compact with the smallest element 0. We will pay particular attention to the binary action case $\mathcal{A} = \{0, a\}$. The agent experiences cost of effort $c(A_t)$, measured in the same units as the utility of consumption, where $c : \mathcal{A} \to \Re$ is continuous, increasing and convex. Assume that $c(0) = 0$ and there is $\epsilon > 0$ such that $c(a) \geq \epsilon a$ for all $a \in \mathcal{A}$.

For convenience of notation we denote by $\mathcal{Q}^A$ the probability measure over paths of output X induced by effort choice A. The expectation under $\mathcal{Q}^A$ will be denoted by E^A.

The output process X is publicly observable by both the principal and the agent. The principal does not observe the agent's effort A, and uses observations of X to give the agent incentives to make costly effort. Before the agent starts working for the principal, the principal offers him a contract that specifies a flow of consumption utility $U_t(X_s; 0 \leq s \leq t) \in [0, U_H]$ based on the principal's observation of X_t. The principal can commit to any such contract. It costs the principal $g(u)$ to deliver to the agent consumption utility u, where $g : [0, U_H] \to [0, C_H]$ is the inverse of the agent's utility of consumption function. Assume that g is increasing, convex and C^2. Normalize $g(0) = 0$ and assume $g'(0) = 0$, i.e. that the agent's marginal utility of consumption is infinite at zero consumption. Section 7 will present an extension in which $g'(0) > 0$.

For simplicity, assume that both the principal and the agent discount the flow of profit and utility at a common rate r. If the agent chooses effort level $A_t, 0 \leq t < \infty$, his total expected utility is given by

$$E\left[\int_0^\infty e^{-rt}(U_t - c(A_t))\, dt\right],$$

and the principal gets profit

$$E\left[\int_0^\infty e^{-rt}\, dX_t - \int_0^\infty e^{-rt} g(U_t)\, dt\right].$$

We say that an effort level process $A_t, 0 \leq t < \infty$ is *incentive compatible* with respect to U_t if it maximizes the agent's total value given U_t.

2.1 Formulation of The Principal's Problem.

The problem of the principal is to offer a contract for the agent: a stream of consumption utility $U_t(X_s, 0 \leq s \leq t)$ contingent on realized output up to time t for all t, and an incentive-compatible advice of effort level $A_t, 0 \leq t < \infty$ that maximizes the principal's profit

$$E\left[\int_0^\infty e^{-rt}\, dX_t - \int_0^\infty e^{-rt} g(U_t)\, dt\right]$$

subject to delivering to the agent a required initial value of at least $\hat{W}$:

$$E\left[\int_0^\infty e^{-rt}(U_t - c(A_t))dt\right] \geq \hat{W}. \tag{1}$$

We assume that the principal can choose not to employ the agent, so we are only interested in contracts that generate nonnegative profit for the principal.

3 Optimal Contract.

In this section, we will heuristically derive an optimal contract. In section 4 we will formally show that an optimal contract takes the form presented in this section. Only for this section, assume that an optimal contract can be written in terms of the agent's promised value W_t as a single state variable. The promised value W_t is the total utility that the principal expects the agent to derive from the future after a given moment of time t. Promised value will play the role of the unique state descriptor that determines how much the agent gets paid, what effort level he is supposed to choose, and how the promised value changes due to the realization of output. The principal must design a contract that specifies functions $u(W)$, the flow of consumption utility to the agent, $a(W)$, the recommended effort level, and the law of motion of W_t driven by the output path X_t. Three objectives must be met. First, the agent must have sufficient incentives to choose the recommended effort levels. Second, payments, recommended effort and the law of motion must be consistent, so that the state descriptor W_t represents the agent's *true* continuation value. Lastly, the contract must maximize the principal's profit.

Before we describe the dynamic nature of the contract, note that the principal has the option to retire the agent with any value $W \in [0, W_H] = [0, U_H/r]$. To retire the agent with value W, the principal offers him constant consumption utility rW and allows him to choose zero effort. Denote the principal's profit from retiring the agent by

$$F_0(W) = -\frac{g(rW)}{r}.$$

Note that the principal cannot deliver any value less than 0, because the agent can guarantee himself nonnegative utility by always taking effort 0. In fact, the only way to deliver value 0 is through retirement. Similarly, retirement is the only way to deliver value W_H, which is the highest value that the principal can deliver to the agent. We will call F_0 the principal's *retirement profit.*

The optimal contract will consist of an interval $(0, W_{gp})$ of continuation values,[4] where the agent is actively employed and W_t evolves as a diffusion process driven by X_t. Mechanically, the principal will adjust the agent's promised value according to equation

$$dW_t = (rW_t - u(W_t) + c(a(W_t)))\ dt + \frac{\gamma(a(W_t))}{\sigma}\ \underbrace{(dX_t - a(W_t)dt)}_{\sigma dZ_t} \tag{2}$$

until the retirement time when W_t hits 0 or W_{gp}. Typically, W_{gp} is less than W_H.

We want to point out two facts about expression (2). First, the drift $rW_t - u(W_t) + c(a(W_t))$ of the agent's promised value accounts for promise keeping. In order for W_t to correctly describe the principal's debt to the agent, it should grow at an interest rate r and fall due to the flow of repayments $u(W_t) - c(a(W_t))$. Note that when the agent follows effort recommendation, term $dX_t - a(W_t)\,dt$ is driftless. Second, function $\gamma(a)$, which gives the minimum volatility of the promised value required to induce effort level a, is defined by

$$\gamma(a) \equiv \min\{y \in [0, \infty) : \frac{ya}{\sigma} - c(a) \geq \frac{ya'}{\sigma} - c(a') \text{ for all } a' \in \mathcal{A}\}.$$

What is the intuition behind this definition? Speaking loosely, the agent will choose effort level maximize the expected change of his future promised value due to effort minus the cost of effort. In equation (2), only the drift of X is affected by the agent's effort. Therefore, the agent will choose effort level a' to maximize

$$\frac{\gamma}{\sigma}a' - c(a').$$

From concavity of $c(a)$, function $\gamma(a)$ is increasing in a. For the binary action case with

[4]Subscript gp stands for "golden parachute."

$\mathcal{A} = \{0, a\}$, $\gamma(a) = c(a)\sigma/a$.

We come to the crucial part where the principal has to compute the main features of an optimal contract: payments $u(W)$ and recommendations of effort level $a(W)$. Also, the principal has to find the best retirement point W_{gp}.

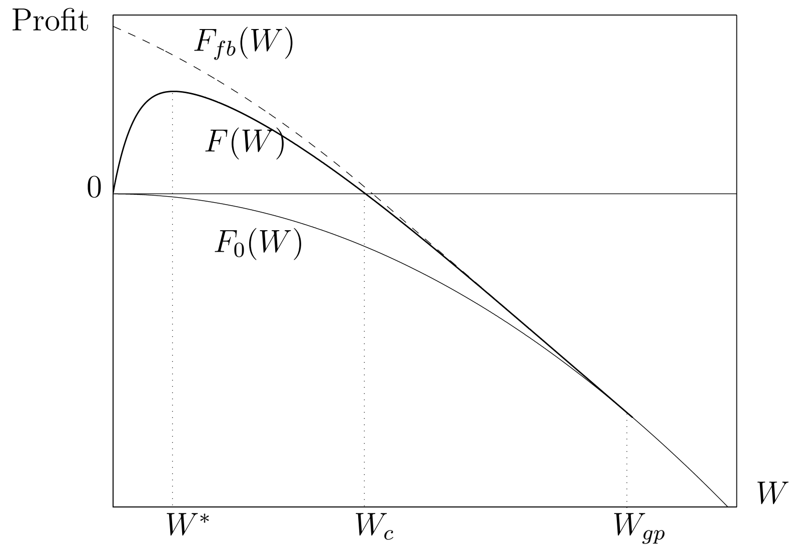

Figure 1: Typical form of F.

Denote by $F(W)$ the highest profit that the principal can derive when he delivers to the agent value W. To maximize profit, the principal must optimally choose $u(W)$ and $a(W)$ for each value $W \in [0, W_{gp}]$. Function F should satisfy equation

$$rF(W) = \max_{a>0,u} a - g(u) + F'(W)(rW - u + c(a)) + F''(W)\gamma(a)^2/2 \tag{3}$$

The principal is maximizing the expected current flow of profit $a - g(u)$ plus the expected change of future profit due to the drift and volatility of the agent's promised value.

Equation (3) can be rewritten in the following form suitable for computation:

$$F''(W) = \min_{a>0,u} \frac{rF(W) - a + g(u) - F'(W)(rW - u + c(a))}{\gamma(a)^2/2} \tag{4}$$

To compute the optimal contract, the principal must solve this differential equation by setting $F(0) = 0$ and choosing the largest value of $F'(0) > 0$ such that solution F reaches F_0 at some point $W_{gp} > 0$, as shown in a computed example on Figure 1. Denote by $u(W)$ and $a(W)$ the minimizing values of u and a in (4) for $W \in (0, W_{gp})$. Function $F(W)$, which is concave, gives the optimal profit that the principal can earn while delivering to the agent value $W \in [0, W_{gp}]$. Functions $u(W)$ and $a(W)$ give the optimal consumption utility and effort recommendation, which the agent receives when his continuation value is W.

Given the form of F, what initial value will the principal offer to the agent given that the agent requires value at least $\hat{W}$? Denote by W^* the maximum of F, and by W_c the critical value, where $F(W_c) = 0$, as shown in Figure 1. Then the starting value that maximizes the principal's profit subject to the agent's participation constraint (1) is

$$W_0 = \begin{cases} W^* & \text{if } \hat{\mathrm{W}} < \mathrm{W}^* \\ \hat{W} & \text{if } \hat{\mathrm{W}} \in [\mathrm{W}^*, \mathrm{W_c}] \\ \text{no contract} & \text{if } \hat{\mathrm{W}} > \mathrm{W_c} \end{cases}$$

Let us summarize the optimal contract. The principal will give the agent a starting promised value W_0 and keep adjusting it according to equation (2) until a stopping time τ when W_t hits a retirement point 0 or W_{gp}. Until time τ, the principal will provide the agent with consumption utility $u(W_s)$ and advise him to take action $a(W_s)$, which are found by solving equation (4). If W_t hits 0, the principal will retire the agent by giving him constant consumption utility of 0 after that forever. If W_t hits W_{gp} the principal will also retire the agent by giving him rW_{gp}. When the agent gets retirement, he is allowed to take action 0 forever.

One interesting feature of the optimal contract is the existence of retirement points. The principal must retire the agent when W hits 0 because the only way to deliver to the agent value 0 is to pay him 0 forever. Why is it optimal for the principal to retire agent at some point W_{gp}, which is typically less than W_H? This happens because when the flow of payments to the agent is large enough, it costs the principal too much to compensate the agent for his effort. This is true even if the agent's effort is perfectly observed. Figure 2 shows the principal's first-best profit F_{fb}, which is the upper envelope of unconstrained profit functions $F_a(W) = F_0(W + c(a)/r) + a/r$ for each individual effort level. Note the interval $[W^*_{gp}, W_H]$, on which $F_0(W)$ is first-best profit. On this interval, the expected flow of output a from any positive effort level is smaller than the cost $c(a)g'(rW)$ of compensating the agent for that effort level. Therefore, it is optimal to retire the agent for $W \in [W^*_{gp}, W_H]$ even if effort is observable. In summary, the main reason to have a high retirement point is that the agent's marginal utility decays to 0 when his consumption increases, and the marginal disutility of effort remains bounded above 0. With unobservable effort, $W_{gp} < W^*_{gp}$ because the principal must compensate the agent not only for the effort, but also for the

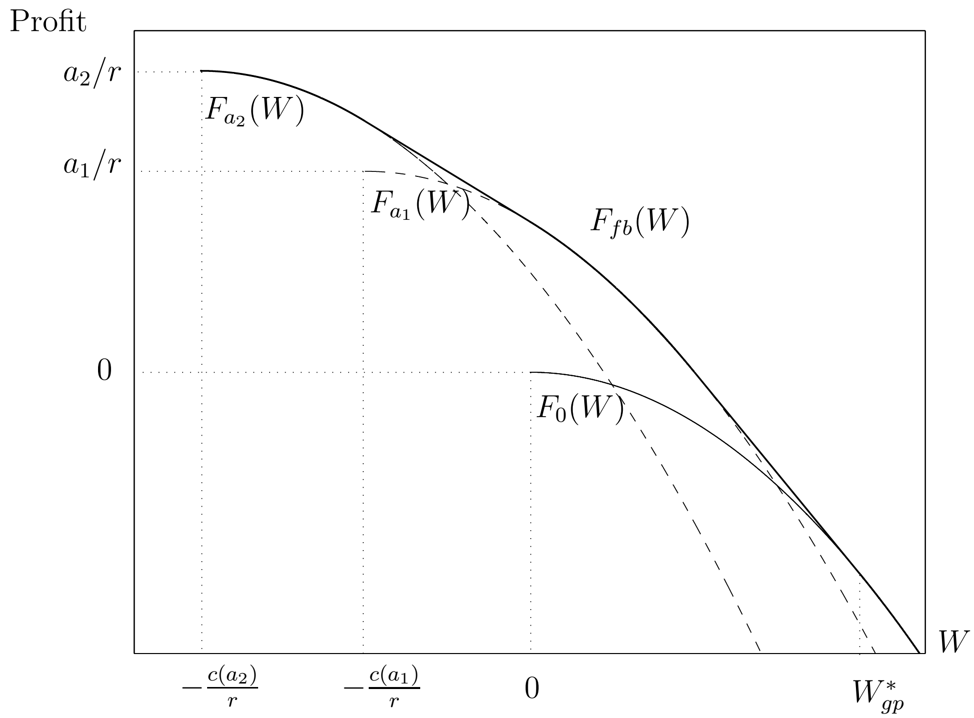

Figure 2: First best profit.

risk caused by incentives to induce any positive effort level.

Another interesting feature of the optimal contract is a probationary interval $[0, W^*]$, on which $F(W)$ is increasing. From equation (4), the agent receives consumption 0 when his value is in this interval. Intuitively, the principal punishes the agent at his own cost when the agent is on probation. Providing the agent with positive consumption would not be the least-costly way to punish the agent.

Let us discuss more intuition behind the principal's choice of the recommended effort level and the choice of payments to the agent. We can see from (3) that the principal will choose effort level $a(W)$ to maximize

$$a + c(a)F'(W) + \frac{\gamma(a)^2}{2}F''(W), \tag{5}$$

where a is the expected flow of output, $-c(a)F'(W)$ is the cost of compensating the agent for his effort, and $-\frac{\gamma(a)^2}{2}F''(W)$ is the cost of compensating the agent for the income uncertainty caused by incentives.

To get intuition behind the choice of u, let us interpret the agent's value as the principal's debt to the agent. The principal will choose the cheapest way to repay his debt to the agent. Theorem 1 assumes that the contract derived in this section is optimal, and shows that under this contract $g'(u(W_t))$ is a martingale whenever the agent is not on probation. We can call this phenomenon *compensation smoothing.*

Theorem 1. *If $W_t > W^*$, then $g'(u(W_s))$ is a martingale for $t \leq s \leq \tau_{W^*}$ under the optimal contract, where W^* is the maximum of F.*

Proof. Certainly this is true when $W_t = W_{gp}$, because after that $u(W_s)$ is constant. When $W_t \in (W^*, W_{gp})$, F must satisfy (4) at W_t; the first-order condition of minimization with respect to u implies that $g'(u) = -F'(W)$. Let us show that $F'(W)$ has drift 0.

From (4) and the Envelope Theorem, we have

$$F'''(W) = \frac{-F''(W)(rW - u + c(a))}{\gamma(a)^2/2},$$

By Ito's Lemma and (2), the drift of $F'(W)$ is

$$F''(W)(rW - u + c(a)) + F'''(W)\gamma(a)^2/2,$$

which by the previous expression is zero. QED.

The next section justifies why the optimal solution really takes this form.

Remark 1. We assume that the principal will refuse to offer a contract to the agent, unless the principal gets nonnegative profit from some contract. If there is no value $F'(0) > 0$ such that the corresponding solution to (4) reaches F_0 at some point $W_{gp} > 0$, then every contract with positive value to the agent gives the principal negative profit. In this case, the principal will refuse to offer a contract.

Remark 2. We assume in this model that the agent cannot save or borrow, and is restricted to consume what the principal pays him at every moment of time. What would happen if the agent actually *could* save and borrow at rate r, but the principal did not know it? Which would he do? By Theorem 1, $g'(u(W_s))$ is a martingale, so the agent's marginal utility of consumption $1/g'(u(W_s))$ must be a submartingale. Since the agent's marginal utility increases in expectation, he is tempted to save for the future.

Remark 3. We can call $-F'(W)\gamma(a)/\sigma$ the agent's piecerate, i.e. the risk borne by the agent measured in the units of the principal's profit rather than the agent's value. This expression is meaningful if $F'(W) < 0$, i.e. when the agent is not on probation and his flow consumption is positive. Since the principal maximizes (5), it can be shown that the piecerate is always less than 1 when the agent is not on probation.

4 Justification.

In this section we do not assume that an optimal contract can be written in terms of promised value as a single state variable, but derive this property of an optimal contract. Consider an arbitrary contract $(U, A) = \{U_t, A_t; 0 \leq t < \infty\}$. Define the the agent's continuation value at time t by

$$W_t(U, A) = E^A \left[\int_t^\infty e^{-r(s-t)}(U_s - c(A_s))\, ds | \mathcal{F}_t \right], \tag{6}$$

Notice that process $W(U, A)$ can be associated with any contract. Define the value that the agent expects from an entire strategy A given the information at time t by

$$V_t(U, A) = E^A \left[\int_0^\infty e^{-rs}(U_s - c(A_s))\, ds | \mathcal{F}_t \right] = \int_0^t e^{-rs}(U_s - c(A_s))\, ds + e^{-rt} W_t(U, A) \tag{7}$$

Note that, since both U_s and A_s are bounded, V_t is a bounded martingale under $\mathcal{Q}^A$ with last element $V_\infty(A, U) = \int_0^\infty e^{-rs}(U_s - c(A_s))\, ds$.

Lemma 1. *Assume that filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Then process $V_t(U, A)$ has a RCLL modification.*

Proof. Since V is a martingale, the function $t \to EV_t$ is constant, thus right-continuous. Then by Theorem 1.3.13 of Karatzas and Shreve (from now on K-S), a RCLL modification exists. QED

Proposition 1. Representation of the agent's value as a diffusion process. *There exists a progressively measurable process $Y = \{Y_t, \mathcal{F}_t; 0 \leq t < \infty\}$ such that*

$$E^A \left[\int_0^t (e^{-rs} Y_s)^2\, ds \right] < \infty$$

for every $0 \leq t < \infty$ and

$$V_t(U, A) = \int_0^t e^{-rs} Y_s\, dZ_s^A + V_0(U, A); \quad 0 \leq t < \infty \tag{8}$$

Proof. This result follows immediately from the Theorem on Representations of Brownian, Square-Integrable Martingales as Stochastic Integrals (K-S, p.182). The factor e^{-rt} in front of Y_t is just a convenient rescaling. QED

From (7) and (8) it follows that

$$dV_t(U,A) = e^{-rt} Y_t \, dZ_t^A = e^{-rt}(U_t - c(A_t))\, dt - re^{-rt} W_t(U,A)\, dt + e^{-rt}\, dW_t(U,A) \quad \Rightarrow$$

$$dW_t(U,A) = (rW_t(U,A) - U_t + c(A_t))\, dt + Y_t \, dZ_t^A, \tag{9}$$

This is a useful equation for the continuation value process W.

4.1 A Condition for the Optimality of the Agent's Effort.

An agent's strategy A is optimal with respect to U if it maximizes his total expected utility $V_0(U,A)$. To identify a condition for the optimality of agent's effort, consider two alternative strategies A and A^*. Strategies A and A^* induce probability measures over output paths denoted by $\mathcal{Q}^A$ and $\mathcal{Q}^{A^*}$ respectively. We denote expectations under these measures by E^A and E^{A^*}. The standard Brownian motions under measures $\mathcal{Q}^A$ and $\mathcal{Q}^{A^*}$ are given by

$$Z_t^A = \frac{1}{\sigma}\left(X_t - \int_0^t A_s \, ds\right) \quad \text{and} \quad Z_t^{A^*} = \frac{1}{\sigma}\left(X_t - \int_0^t A_s^* \, ds\right) \tag{10}$$

Lemma 2. *Define $\hat{V}_t$ to be the entire value that the agent expects to obtain if he followed strategy A until time t, and plans to continue by following strategy A^**

$$\hat{V}_t = \int_0^t e^{-rs}(U_s - c(A_s))\, ds + E^{A^*}\left[\int_t^\infty e^{-rs}(U_s - c(A_s^*))\, ds | \mathcal{F}_t\right] \tag{11}$$

If $\hat{V}_t$ is a supermartingale under measure $\mathcal{Q}^A$, then strategy A^* is at least as good for the agent as A. If $\hat{V}_t$ is a submartingale, but not a martingale, then A^* is strictly worse than A.

Proof. Note that $\lim_{t\to\infty} \hat{V}_t = \int_0^\infty e^{-rs}(U_s - c(A_s))\, ds = V_\infty(U,A)$. Because $\hat{V}_t$ are uniformly bounded, by Dominated Convergence Theorem

$$\lim_{t\to\infty} E^A[\hat{V}_t] = E^A[V_\infty(U,A)] = V_0(U,A).$$

If $\hat{V}_t$ is a supermartingale, then

$$V_0(U,A) = \lim_{t\to\infty} E^A[\hat{V}_t] \le \hat{V}_0 = V_0(U,A^*),$$

so strategy A^* is at least as good as A.

If $\hat{V}_t$ is a submartingale, but not a martingale, then

$$V_0(U,A) = \lim_{t\to\infty} E^A[\hat{V}_t] > \hat{V}_0 = V_0(U,A^*),$$

so strategy A^* is worse than A. QED

Proposition 2. Necessary and sufficient condition for agent's optimality. *For a given strategy A^*, suppose that Y_t^* is the volatility of $W_t(U,A^*)$ given by Proposition 1. A^* is optimal if and only if the following condition holds for all alternative strategies A almost surely:*

$$Y_t^* A_t^* - \sigma c(A_t^*) \ge Y_t^* A_t - \sigma c(A_t), \quad 0 \le t < \infty \tag{12}$$

PROOF. Recall that $\hat{V}_t$, defined by (11), is the value from following strategy A until time t and then switching to A^*. Let us identify the drift of $\hat{V}_t$.

$$\hat{V}_t = \int_0^t e^{-rs}(c(A_s^*)-c(A_s))ds + V_t(U,A^*) = \int_0^t e^{-rs}(c(A_s^*)-c(A_s))ds + \int_0^t e^{-rs} Y_s^* dZ_s^{A^*} + V_0(U,A^*)$$

$$= \int_0^t e^{-rs}\left(c(A_s^*) - c(A_s) + Y_s^* \frac{A_s - A_s^*}{\sigma}\right) ds + \int_0^t e^{-rs} Y_s^* \, dZ_s^A + V_0(U,A^*),$$

where, by (10), the Brownian motions under $\mathcal{Q}^A$ and $\mathcal{Q}^{A^*}$ are related by $Z_t^{A^*} = Z_t^A + \int_0^t \frac{A_s - A_s^*}{\sigma}\, ds$.

Thus $\hat{V}_t$ is a diffusion process under $\mathcal{Q}^A$ with drift

$$e^{-rt}\frac{(Y_t^* A_t - \sigma c(A_t)) - (Y_t^* A_t^* - \sigma c(A_t^*))}{\sigma}.$$

Suppose (12) holds for strategy A^*. Then for any alternative strategy A the drift of $\hat{V}_t$ under $\mathcal{Q}^A$ is never positive, so $\hat{V}_t$ is a supermartingale. From Lemma 2 we conclude that strategy A^* is at least as good as any alternative strategy A.

If (12) does not hold on a set of positive measure, choose A_t that maximizes $Y_t^* A_t - c(A_t)$

for all $t \geq 0$. Then the drift of $\hat{V}_t$ under $\mathcal{Q}^A$ is everywhere nonnegative and positive on a set of positive measure. Therefore, by Lemma 2, strategy A is strictly better than A^*. We conclude that a candidate strategy A^* of the agent is optimal if and only if (12) holds almost surely. QED

From Proposition 2 it follows that the minimal volatility of the agent's continuation value required to induce him to take action $a > 0$ is

$$\gamma(a) = \min\{y \in [0, \infty) : ya - \sigma c(a) \geq ya' - \sigma c(a') \text{ for all } a' \in \mathcal{A}\}.$$

4.2 The Principal's Problem.

Consider the problem of maximizing the principal's profit subject to delivering to the agent a specific value $W_0 \in [0, W_H]$. The optimality differential equation, which is reproduced below, is the key tool for finding an optimal contract.

$$F''(W) = \min_{a>0,u} \frac{rF(W) - a + g(u) - F'(W)(rW - u + c(a))}{\gamma(a)^2/2} \tag{13}$$

We first explore solutions to equation (13). Then, based on an appropriate solution to this equation, in Proposition 3 we identify a sufficient condition, under which a contract is optimal. Based on this condition, we construct an optimal contract in Proposition 4 using a strong solution of equation (2).

Lemma 3. *Given initial conditions $F(0) = 0$ and $F'(0) = F_p$, a solution to equation (13) exists and is unique. Such solutions are concave and continuous in F_p.*

Proof. See Appendix.

Among solutions to (13) with $F_p > 0$ that reach F_0 at some $W_{gp} > 0$, consider the solution with the largest F_p, call it the *highest feasible solution* and denote it by $F : [0, W_H] \to \Re$. Figure 3 shows several typical solutions of equation (13) marked by A, B and C. On the figure, solution B is the highest feasible solution. For now, assume that such a solution exists. Then $F(W) \geq F_0(W)$ for all $W \in [0, W_H]$. Otherwise, if we increase F_p slightly, the corresponding solution would still reach F_0 at some positive value of W, which means that we did not choose F_p correctly. Since F is concave and $F(W) \geq F_0(W)$ for all W, any line tangent to F never goes below F_0, i.e.

$$\forall\, W, W', \quad F'(W)(W' - W) + F(W) \geq F_0(W') \tag{14}$$

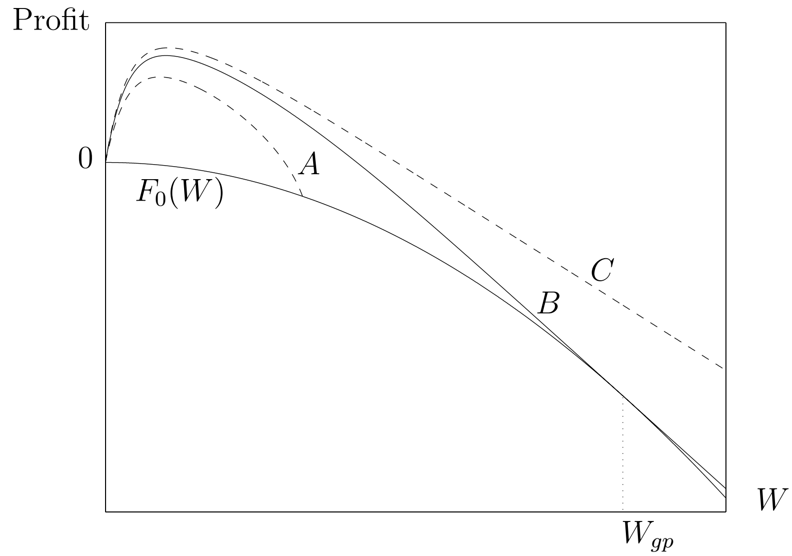

Figure 3: Typical solutions of the optimality equation.

Define $u(W)$ and $a(W)$ to be the minimizing values associated with the highest feasible solution F of equation (13).

Proposition 3. Sufficient condition for optimality of a contract. *Consider the highest feasible solution F of the optimality equation, as defined above. Consider a contract (U, A) that induces continuation values $W_t = W_t(U, A)$ with $W_0 \in [0, W_{gp}]$. Denote the volatility of W_t by Y_t, and define $\tau = \inf\{t : W_t = 0 \text{ or } W_{gp}\}$. This contract is optimal in delivering value W_0 if*

$$\begin{array}{ll} U_t = u(W_t),\;\; A_t = a(W_t),\;\; \text{and}\;\; Y_t = \gamma(A_t) & \text{for } t \in [0, \tau) \\ U_t = rW_\tau,\;\; A_t = 0,\;\; \text{and}\;\; Y_t = 0 & \text{for } t \geq \tau \end{array} \tag{15}$$

To prove Proposition 3, we will need the following lemma:

Lemma 4. *Suppose (14) holds for a function $F : [0, W_H] \to \Re$ that solves (13). For an arbitrary incentive compatible contract (U, A) define*

$$G_t(U, A) = \int_0^t e^{-rs}\, dX_s - \int_0^t e^{-rs} g(U_s)\, ds + e^{-rt} F(W_t(U, A)), \quad 0 \leq t < \infty.$$

Then $G_t(U,A)$ is a $\mathcal{Q}^A$-supermartingale with a last element

$$G_\infty(U,A) = \int_0^\infty e^{-rs}\, dX_s - \int_0^\infty e^{-rs} g(U_s)\, ds \tag{16}$$

Moreover, if a contract satisfies (15), G_t is a martingale.

Proof. By exploring the drift of G, let us show that G is a supermartingale for an arbitrary contract, and a martingale for a contract that satisfies (15) for $t \in [0, \infty)$. For an arbitrary contract (U, A), according to equation (9) $W(U, A)$ satisfies

$$dW_t = (rW_t - U_t + c(A_t))\, dt + Y_t\, dZ_t^A.$$

By Ito's formula

$$dF(W_t) = F'(W_t)\, dW_t + F''(W_t)\frac{Y_t^2}{2}\, dt = \left(F'(W_t)(rW_t - U_t + c(A_t)) + F''(W_t)\frac{Y_t^2}{2}\right) dt + F'(W_t)Y_t\, dZ_t^A.$$

Thus,

$$\frac{dG_t(U,A)}{e^{-rt}} = dX_t - g(U_t)\, dt - rF(W_t)\, dt + dF(W_t) =$$

$$\left(A_t - g(U_t) - rF(W_t) + F'(W_t)(rW_t - U_t + c(A_t)) + F''(W_t)\frac{Y_t^2}{2}\right) dt + (\sigma + F'(W_t)Y_t) dZ_t^A.$$

From Proposition 2, we know that since the contract (U, A) is incentive compatible, $Y_t \geq \gamma(A_t)$ for all t. If $A_t > 0$, then using (13) and $F'' \leq 0$ we obtain

$$A_t - g(U_t) - rF(W_t) + F'(W_t)(rW_t - U_t + c(A_t)) + F''(W_t)\frac{Y_t^2}{2} \leq$$

$$a(W_t) - g(u(W_t)) - rF(W_t) + F'(W_t)(rW_t - u(W_t) + c(a(W_t))) + F''(W_t)\frac{\gamma(a(W_t))}{2} = 0.$$

Equality is reached when the contract satisfies (15).

If $A_t = 0$, then using (14) and $F'' \leq 0$ we obtain

$$A_t - g(U_t) - rF(W_t) + F'(W_t)(rW_t - U_t + c(A_t)) + F''(W_t)\frac{Y_t^2}{2} \leq$$

$$rF_0(U_t/r) - rF(W_t) + F'(W_t)(rW_t - U_t) \leq 0.$$

Equality holds when $Y_t = 0$, $rW_t = U_t$ and $F(W_t) = F_0(W_t)$.

We conclude that the drift of G is negative or zero for an arbitrary incentive compatible contract, and zero for a contract that satisfies (15).

Using (K-S, 1.3.19), to show that G has a last element G_∞ defined by (16), we need to check that $G_t, 0 \le t < \infty$ are uniformly integrable. Terms $e^{-rt}F(W)$ and $\int_0^t e^{-rs} g(U_s)\, ds$ are uniformly integrable since they are uniformly bounded. Term $\int_0^t e^{-rs}\, dX_s$ has a uniformly bounded second moment, therefore it is also uniformly integrable. Since $e^{-rt}F(W)$ converges to 0, G_t converges to G_∞ pointwise and in L^1.

This concludes the proof that $G_t, 0 \le t \le \infty$ is a supermartingale for an arbitrary incentive-compatible contract (U, A) and a martingale for a contract that satisfies (15). QED

Proof of Proposition 3. Consider a candidate contract (U^*, A^*) with volatility of continuation values Y^* that satisfies (15), and an arbitrary alternative incentive-compatible contract (U, A). Let us show that (U^*, A^*) gives the principal profit greater or equal than (U, A). Using Lemma 4

$$E^{A^*}[G_\infty(U^*, A^*)] = G_0(U^*, A^*) = F(W_0) = G_0(U, A) \ge E^A[G_\infty(U, A)].$$

QED

What if for all $F'(0) = F_p > 0$, the corresponding solution to (13) stays above F_0 for all $W > 0$? Then there is no contract that gives the principal nonnegative profit for any $W_0 > 0$. Indeed, consider F that solves (13) with initial conditions $F(0) = 0$ and $F'(0) = 0$. Since F is concave, $F(W) < 0$ for all $W > 0$. From the continuity of solutions in initial conditions, $F(W) \ge F_0(W)$ for all W. Therefore, (14) holds and Lemma 4 applies for an arbitrary incentive-compatible contract (U, A). Therefore, the principal's profit from contract (U, A) is

$$E^A[G_\infty(U, A)] \le G_0(U, A) = F(W_0) < 0.$$

Proposition 4. Existence of an optimal contract. *If $a(W)$ has bounded variation, then equation*

$$dW_t = \left(rW_t - u(W_t) + c(a(W_t)) - \frac{a(W_t)\gamma(a(W_t))}{\sigma} \right) dt + \frac{\gamma(a(W_t))}{\sigma}\, dX_t \qquad (17)$$

with an initial condition $W_0 \in [0, W_{gp}]$ has a unique strong solution until the time $\tau =$

$\inf\{t : W_t = 0 \,\text{or}\, W_{gp}\}$. *Then the the contract* (U, A) *defined by*

$$
\begin{array}{ll}
U_t = u(W_t), \;\text{ and }\; A_t = a(W_t), & \text{for } t \in [0, \tau) \\
U_t = rW_\tau, \;\text{ and }\; A_t = 0, & \text{for } t \geq \tau
\end{array}
\tag{18}
$$

is optimal in delivering to the agent value W_0.

Whenever we talk about a solution W_t until a stopping time τ, assume for convenience that W_t stays constant from τ onwards. The following lemma, which follows from Theorem 4 of Veretennikov (1979), gives sufficient conditions for existence and uniqueness of a solution to a one-dimensional SDE.

Lemma 5. *Let* $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *be a one-dimensional Brownian motion. If* b *and* y *are Borel and bounded,* y *is bounded away from zero and has bounded variation, then equation*

$$dW_t = b(W_t)\, dt + y(W_t)\, dX_t$$

has a unique strong solution.

Lemma 6. *If* $a(W)$ *has bounded variation, a strong solution to (17) until the stopping time* τ *exists and is unique.*

Proof. Define $b(W) = rW - u(W) + c(a(W))$ and $y(W) = \gamma(a(W))$. Clearly, both b and y are bounded for $W \in [0, W_{gp}]$. Since $a(W)$ has bounded variation, $y(W)$ also has bounded variation, and since $\gamma(a)$ is bounded above 0, $y(W)$ is also bounded above 0. It follows from Lemma 5 that equation (17), which can be written as

$$dW_t = b(W_t)\, dt + \frac{y(W_t)}{\sigma}\, dX_t,$$

has a unique strong solution until the stopping time τ. QED

Lemma 7. *Let* W *be the unique strong solution to (17). Define* (U, A) *by (18). Then* W_t *indeed represents agent's continuation value associated with the contract* (U, A)*, i.e.* $W_t = W_t(U, A)$ *a.s.*

Proof. Consider

$$V_t = e^{-rt} W_t + \int_0^t e^{-rs}(U_s - c(A_s))\, ds,$$

and let us show that $V_t = V_t(U, A)$. Note that V satisfies

$$dV_t = e^{-rt}\left(dW_t - (rW_t - U_t + c(A_t))\,dt\right) = e^{-rt}\left(\frac{\gamma(A_t)}{\sigma}\,dX_t - \frac{A_t\gamma(A_t)}{\sigma}\,dt\right) = e^{-rt}\gamma(A_t)dZ_t^A,$$

so V is a martingale under $\mathcal{Q}_A$. Also, note that V is uniformly bounded, and $\lim_{t\to\infty} V_t = V_\infty(U,A)$. Therefore, using the Dominated Convergence Theorem,

$$V_t = \lim_{s\to\infty} E[V_s|\mathcal{F}_t] = E[V_\infty(U,A)|\mathcal{F}_t] = V_t(U,A).$$

for $0 \le t \le \infty$. Then,

$$e^{-rt}W_t = V_t - \int_0^t e^{-rs}(U_s - c(A_s))\,ds = V_t(U,A) - \int_0^t e^{-rs}(U_s - c(A_s))\,ds = e^{-rt}W_t(U,A),$$

so $W_t = W_t(U,A)$ almost surely. QED

Proof of Proposition 4. A strong solution W to (17) exists by Lemma 6. Define a contract (U,A) by (18). By Lemma 7, W_t is the true continuation value. Therefore, (U,A) satisfies (15), so by Proposition 3 it is the optimal way to deliver value W_0. QED

Propositions 1 through 4 give us the form of an optimal contract to deliver a particular value W_0 to the agent, and verify that such a contract is indeed optimal. To devise an optimal contract that satisfies agent's participation constraint (1), the principal must choose an initial promised value $W_0 \in [\hat{W}, W_H]$ that maximizes his profit $F(W_0)$. Since F is a concave function with maximum at some point W^*, we have

$$W_0 = \begin{cases} W^* & \text{if } \hat{\mathrm{W}} < \mathrm{W}^* \\ \hat{W} & \text{if } \hat{\mathrm{W}} \in [\mathrm{W}^*, \mathrm{W_c}] \\ \text{no contract} & \text{if } \hat{\mathrm{W}} > \mathrm{W_c} \end{cases}$$

5 Comparative Statics.

For the rest of the paper, let us consider the binary-action case with $\mathcal{A} = \{0, a\}$. By Proposition 2, the volatility of continuation values required to induce effort a is $\gamma(a) = c(a)\sigma/a$. The following change of variables is useful for comparative statics analysis: Denote average value by $\omega = rW$, average profit $f(\omega) = rF(W)$ and average retirement profit $f_0(\omega) = rF_0(W) = -g(\omega)$. Let $\omega_{gp} = rW_{gp}$ and $\omega^*_{gp} = rW^*_{gp}$. Then equation (13) can be conveniently rewritten as

$$f''(\omega) = \min_u \frac{f(\omega) + f'(\omega)(u-\omega) - f_0(u) - a - f'(\omega)c(a)}{r(c(a)\sigma/a)^2/2}. \tag{19}$$

We are interested in the effect of the following parameters on the principal's profit and the optimal contract: effort level a, which reflects productivity, cost of effort $c(a)$, discount rate r and the volatility of output σ, which reflects the size of informational problem.

The first result is that changes in volatility of output are equivalent to changes in the discount rate in terms of their effect on the principal's profit and the optimal contract. The intuition behind this result is that when discounting is slower, the principal has longer time to observe output and detect the agent's effort, which is equivalent to having lower volatility of output for faster discounting rates.

Result 1. *Changing σ^2 by a factor of α has the same effect on the principal's profit as changing r by a factor of α.*

PROOF. Equation (19) is the same whether we change σ^2 or r by a factor of α. Therefore, both of these changes have the same effect on the principal's profit. QED

The following lemma is the key engine in proving a lot of other comparative statics results:

Lemma 8. *Let $h(f, f', \omega)$ be the right hand side of (19) for parameters $(a, c(a), \sigma, r)$, and $\tilde{h}(f, f', \omega)$ be the right hand side of (19) for parameters $(\tilde{a}, \tilde{c}(\tilde{a}), \tilde{\sigma}, \tilde{r})$. Let $f : [0, U_H] \to \Re$ be the principal's average profit for parameters $(a, c(a), \sigma, r)$. If*

$$\tilde{h}(f(\omega), f'(\omega), \omega) < h(f(\omega), f'(\omega), \omega)$$

for all ω, then the principal's profit for the second set of parameters is at least as great as for the first set of parameters for all ω.

PROOF. See Appendix.

Result 2. *The principal's profit is increasing in a, decreasing in $c(a)$, decreasing in σ, and decreasing in r.*

PROOF. It is easy to check that increasing a, decreasing $c(a)$, decreasing σ, or decreasing r decreases the right hand side of (19). Therefore, using Lemma 8, we conclude that the principal's profit must increase. QED

Note that increase in a increases the principal's profit it two ways: by increasing productivity and by making it easier for the principal to detect effort. Also, increase in $c(a)$ has two consequences: it decreases the flow of utility to the agent and makes it more difficult for the principal to give the agent incentives. The decrease in the flow of utility to the agent has a positive effect on the principal's profit when he is punishing the agent with consumption 0, but the increased difficulty to give the agent incentives always overweighs this positive effect. Thus, when $c(a)$ increases, the principal's profit decreases overall.

The following result is quite nontrivial. It shows that increase in productivity is worthwhile even if it causes an equivalent increase in the cost of effort, and even if it exacerbates informational problems.

Result 3. *If we increase a, $c(a)$ and σ^2 by a factor of $\alpha > 1$, then the principal's profit increases.*

Proof. Let $h(f, f', \omega)$ be the right hand side of (19) for parameters $(a, c(a), \sigma, r)$, and $\tilde{h}(f, f', \omega)$ be the right hand side of (19) for parameters $(\alpha a, \alpha c(a), \sqrt{\alpha}\sigma, r)$. Denote by $f : [0, \omega_{gp}] \to \Re$ the principal's average profit for the first set of parameters. For all $\omega \in [0, \omega_{gp})$,

$$f'(\omega) > f'(\omega_{gp}) = f_0'(\omega_{gp}) \geq f_0'(\omega_{gp}^*) = -a/c(a) \quad \Rightarrow \quad a + f'(\omega)c(a) > 0.$$

Consider an arbitrary $\omega \in [0, \omega_{gp})$. From (14), $b = \min_u f(\omega) + f'(\omega)(u - \omega) - f_0(u) \geq 0$. Therefore, we have

$$h(f(\omega), f'(\omega), \omega) = \frac{b - (a + f'(\omega)c(a))}{r\gamma(a)^2/2} > \frac{b - \alpha(a + f'(\omega)c(a))}{\alpha r\gamma(a)^2/2} = \tilde{h}(f(\omega), f'(\omega), \omega).$$

Therefore, by Lemma 8, the principal's profit must be higher for parameters $(\alpha a, \alpha c(a), \sqrt{\alpha}\sigma, r)$ than for parameters $(a, c(a), \sigma, r)$. QED

6 Asymptotic Contracts as $r \to 0$.

It is best to formulate the theorem about optimal contracts in the limit as $r \to 0$ in terms of averages. Let f_{fb} be the principal's first best average profit, the upper envelope of $f_0(\omega)$ and $f_0(\omega + c(a)) + a$. We know from earlier analysis that $\omega_{gp} \leq \omega_{gp}^*$, where $f_0'(\omega_{gp}^*) = -a/c(a)$. Note that $u(\omega/r)$ is the flow of consumption utility that the agent receives when his average promised value is ω.

Theorem 2. *As $r \to 0$, $\omega_{gp} \to \omega^*_{gp}$ and the principal's average profit f converges to first best pointwise on $(0, \omega^*_{gp})$. The agent's flow of consumption utility under the optimal contract $u(\omega/r)$ converges to $\omega + c(a)$ pointwise on $(0, \omega^*_{gp} - c(a))$ and to ω^*_{gp} pointwise on $[\omega^*_{gp} - c(a), \omega^*_{gp})$. For $\omega \in (0, \omega^*_{gp} - c(a))$, $f(\omega) = f_{fb}(\omega) + f''_{fb}(\omega) r\gamma(a)^2/2 + o(r)$.*

Proof. See Appendix.

According to Theorem 3, the principal's average profit converges to first best, which is consistent with the Folk Theorem. The total loss of efficiency accumulated over time is approximately a constant, which equals

$$-\frac{f''_{fb}(\omega)\gamma(a)^2}{2} = \frac{g''(u)c(a)^2\sigma^2}{2a^2} = \frac{\delta\alpha^2\sigma^2}{2},$$

where $\delta = g''(u)/g'(u)^2$ is the agent's coefficient of absolute risk aversion and $\alpha = c(a)g'(u)/a$ is the piece-rate.

Let us discuss the optimal contract. The contract features several prominent intervals of the agent's values. On the main interval $(0, \omega^*_{gp} - c(a))$, which we can call *regular employment*, the agent receives compensation for the cost of his effort and the annuity value of W_t in the form of consumption utility. Therefore, on this interval the agent's value is asymptotically driftless.

There is a small *probationary interval* of values $[0, \epsilon)$, where the agent receives consumption 0. In the probationary interval the agent's value has a strong upward drift. The principal compensates the agent for his effort exclusively with continuation values. This is optimal for the principal, because his profit actually increases in the agent's value. The low retirement point is extremely costly for the principal. A simple calculation shows that even if the agent chooses no effort in the probationary interval, his value would still have a slight upward drift.

There is also a special *pre-retirement interval* $(\omega^*_{gp} - c(a), \omega^*_{gp})$ near the high retirement point. In that interval the agent receives approximately constant consumption utility, which is equal to his "golden parachute" consumption utility. The agent's value has an *upward* drift towards the high retirement point. Unlike near the low retirement point, the principal is not trying to prevent high retirement. The principal chooses retirement with a golden parachute, whereas he is forced to retire the agent if the low retirement point is reached. Throughout the pre-retirement interval, the agent is supposed to put effort. If he chose not to, a simple calculation shows that his value would drift down towards the regular employment interval. The agent is compensated for his effort partly with consumption, and partly with the increasing chance of a golden parachute.

7 Extensions.

The basic model presented above has great flexibility, which allows us to include new features with ease, according to the situation that we are trying to illustrate. Here are three possible extensions. First, for legal reasons, it may be impossible to implement contracts that bind the agent to the principal forever. What if the agent can walk away? Second, we assumed that when the principal retires the agent, he shuts down the factory and does not make any production profit or loss anymore. What if, upon retiring an agent, the principal could hire a new one? Third, we assumed that $g'(0) = 0$, i.e. the marginal utility of consumption is infinite at consumption 0. How does the optimal contract change if we allow for bounded marginal utility ($g'(0) > 0$)?

7.1 What if the Agent Can Quit?

Suppose that the agent can quit working for the principal at any time, and replace his continuation value from contract with the principal by an alternative outside value $\tilde{W} \leq \hat{W}$. The alternative value $\tilde{W}$ can be interpreted as value from a new employment minus the search cost. What is the optimal contract in this situation?

Let us precisely describe the principal's problem. The principal has to specify a stream of consumption utility U_t, $0 \leq t \leq \tau$, an incentive compatible advice of effort level A_t, $0 \leq t \leq \tau$, and a stopping time τ, at which the agent is allowed to take alternative employment. The principal's objective is to maximize his profit

$$E\left[\int_0^\tau e^{-rt}\, dX_t - \int_0^\tau e^{-rt} g(U_t)\, dt\right].$$

subject to

$$E\left[\int_0^\tau e^{-rt}(U_t - c(A_t))dt + e^{-r\tau}\tilde{W}\right] \geq \hat{W}$$

and for all $t < \tau$

$$E\left[\int_t^\tau e^{-r(s-t)}(U_s - c(A_s))ds + e^{-r(\tau-t)}\tilde{W}\right] \geq \tilde{W}.$$

By analogy with the basic model, it can be shown that the optimal contract in this setting has the following form: The principal has to solve the familiar optimality equation

$$F''(W) = \min_{a>0,u} \frac{rF(W) - a + g(u) - F'(W)(rW - u + c(a))}{\gamma(a)^2/2} \tag{20}$$

by setting $F(\tilde{W}) = 0$ and choosing the highest positive value for $F'(\tilde{W})$ such that the resulting solution will touch F_0 at some point $\tilde{W}_{gp} > \hat{W}$. If there is no such value for $F'(\tilde{W})$, then no contract can give positive profit to the principal. Denote by $u : [\tilde{W}, \tilde{W}_{gp}] \to [0, U_H]$ the consumption utility and by $a : [\tilde{W}, \tilde{W}_{gp}] \to \mathcal{A}$, the effort, which solve the minimization problem in (20).

The principal will compute the agent's continuation value according to equation (2), starting with a value W_0 which maximizes F on the set $[\hat{W}, \tilde{W}_{gp}]$, provided that $F(W_0) \geq 0$. If $F(W_0) < 0$, then there is no contract that gives the principal positive profit, so he will refuse to offer employment to the agent. If $F(W_0) \geq 0$, then the principal will offer to the agent a flow of utility $u(W_t)$ and suggest action $a(W_t)$ until the time when W_t hits $\tilde{W}$ or $\tilde{W}_{gp}$. If W_t hits $\tilde{W}$, payments stop and the agent quits. If W_t hits $\tilde{W}_{gp}$, the agent stops working and receives a lifetime flow of utility of $r\tilde{W}_{gp}$.

In this setting, we can show that the principal's profit and $\tilde{W}_{gp}$ are both decreasing in $\tilde{W}$ when there is any contract at all that gives the principal positive profit. Intuitively, the larger $\tilde{W}$, the less ability the principal has to punish the agent, the smaller his profit will be. Why does the agent get high retirement earlier when $\tilde{W}$ is larger? When W is large, the size of profit that the principal can make in case W drifts down contributes to the decision whether to retire the agent or not. When $\tilde{W}$ is larger, the principal's profit is smaller, so he has less incentive to keep the agent employed, and $\tilde{W}_{gp}$ must fall.

A typical contract is shown in Figure 4 for an interesting example when $\tilde{W} = \hat{W}$. In this case, if there is a contract that gives the principal positive profit, then he will always start off the agent with an initial promised value W_0 strictly greater than $\tilde{W}$. In this case, W_0 is the point at which F achieves a maximum.

Remark. We have implicitly assumed that payments to the agent have to stop if the agent quits and that the agent will not seek re-employment if the high retirement point is reached. Both of these are natural assumptions. Indeed, if the low retirement point is reached, the agent is in the probationary region and receives payments of 0. The principal is keeping the agent's value down at his own cost. Payments after the agent quits would improve the agent's value, which is opposite to what the principal is trying to achieve.

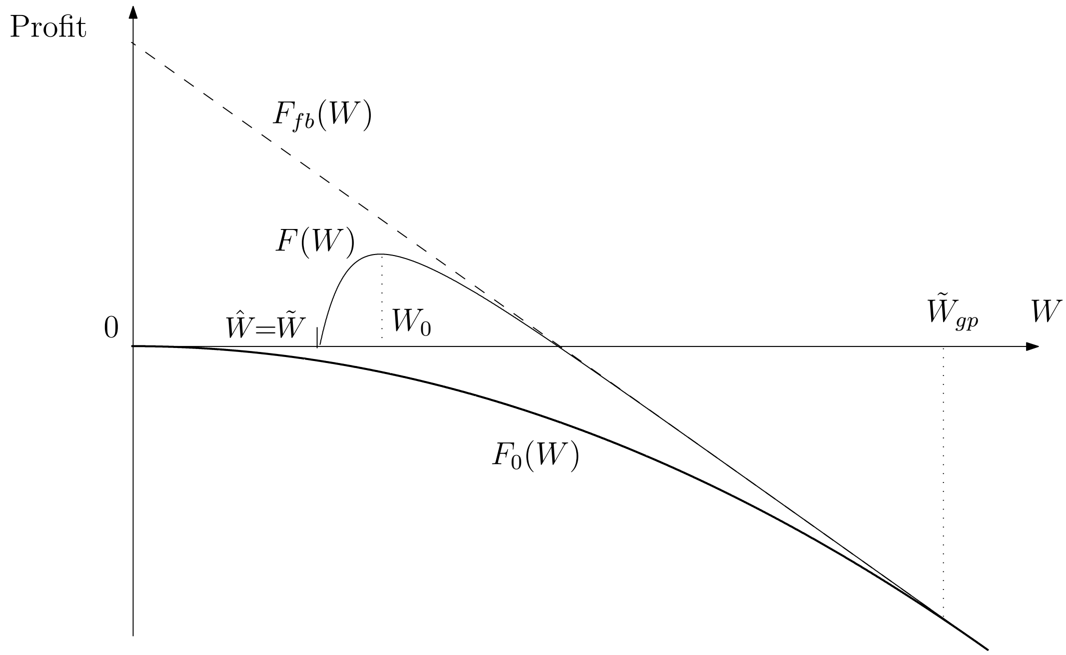

Figure 4: The principal's profit with an alternative participation constraint.

Therefore, it is optimal not to compensate the agent if he chooses to quit. On the other hand, suppose that all employment available to the agent is of the same nature. If it is efficient for one principal to retire the agent with a golden parachute, then the agent is so well off that nobody can efficiently make him work, so nobody will offer him employment.[5]

7.2 What if the Principal Can Replace the Agent?

Suppose instead that the agent cannot freely quit, but the principal can let him go at any moment of time and hire a new agent. Assume that all potential agents have the same reservation value $\hat{W}$ to start employment, and the principal faces a search cost C. A contract specifies when employment ends, and payments to the agent both before and after the termination of employment. After an agent stops working for the principal, he stops putting effort, but continues to consume payments from the principal, and does not seek reemployment. What is an optimal contract in this situation?

To compute an optimal contract, let us take a guess $D = F(W_0) - C$ about additional

[5] It is possible to imagine an alternative setting where, once the agent is retired with a golden parachute, he can accept employment of a different nature, which was not possible earlier. This would be a very interesting alternative extension.

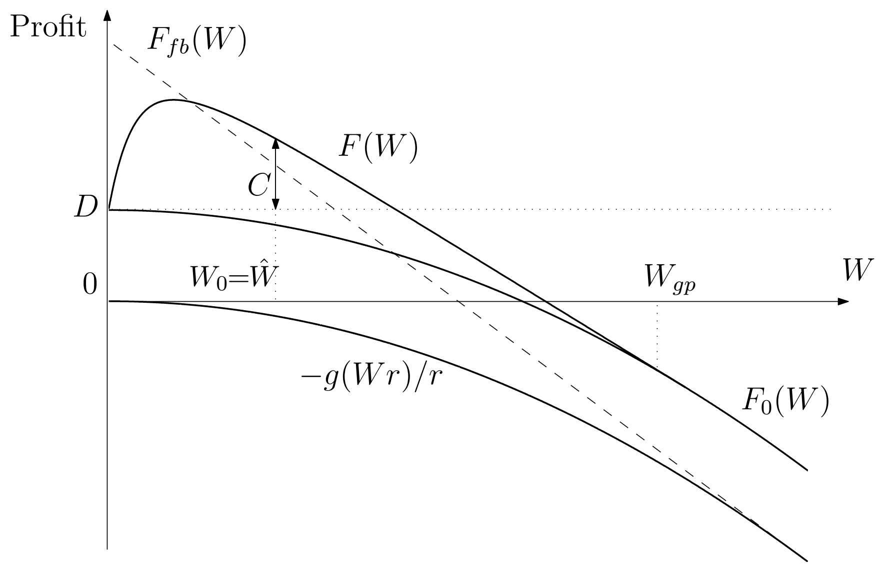

Figure 5: The principal's profit when the agent can be replaced at cost C.

profit that the principal can realize when he fires an agent and replaces him with another. Then the new retirement profit function is $F_0(W) = -g(Wr)/r + D$. To find the optimal contract, the principal must solve the familiar equation (20) with initial conditions $F(0) = F_0(0) = D$ and maximal $F'(0) > 0$ such that the resulting solution will reach F_0 at some point $W_{gp} > 0$. Choose W_0 to be the point that maximizes the resulting function F on the interval $[\hat{W}, W_{gp}]$, where $\hat{W}$ is the agent's reservation value. Our guess of D is correct if $F(W_0) = D + C$. If it happens that $F(W_0) < D + C$, then our guess of D is too large, and if $F(W_0) > D + C$, then our guess of D is too small. Once we find $F(W)$, the optimal contract will give the agent starting value W_0 (in Figure 5, $W_0 = \hat{W}$) and take the usual form. Note that due to the principal's ability to replace an agent, his profit may be greater than the first best profit with just one agent.

In this setting, we find that W_{gp} is increasing in C. This is a natural conclusion, since when it is less costly to replace an agent with a new one, the principal will retire the old agent sooner.

7.3 What if $g'(0) > 0$?

We assumed so far that $g'(0) = 0$, which means that the agent's marginal utility at consumption 0 is infinity. If the agent's marginal utility at consumption 0 is finite, then the

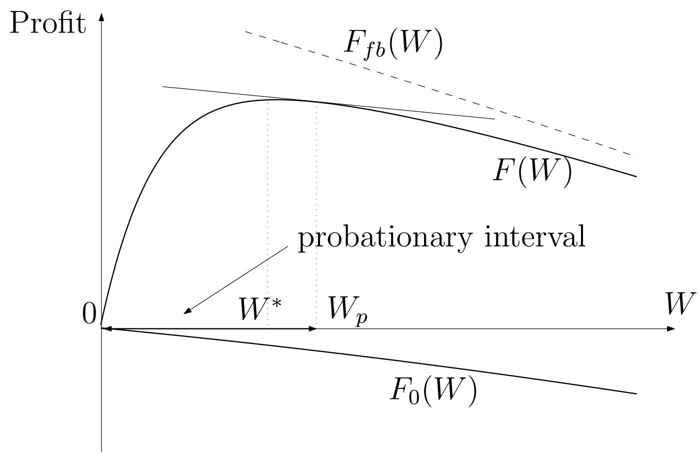

Figure 6: When $g'(0) > 0$, the probationary interval is greater than $[0, W^*]$.

optimal contract can be found by the same procedure as before. In this altered setting, point W^*, at which function F is maximized, will lie strictly inside the probationary interval $[0, W_p]$. Point W_p is defined by $F'(W_p) = F_0'(0) < 0$.[6] Therefore, if the agent's reservation value is $\hat{W} \leq W^*$, then the agent's starting value will be $W_0 = W^*$ and the agent will start strictly inside the probationary interval. A typical example is shown in Figure 6.

8 Conclusion.

This paper develops a new flexible method of analyzing long-term interaction between a principal and an agent. Continuous-time modeling allows us to better explore informational problems when the agent's effort is unobserved. Contracts in continuous-time can be characterized by the volatility and drift of the agent's promised value. The volatility of the promised value summarizes the agent's incentives. Higher volatility induces a higher effort level. The principal will choose a volatility that induces the optimal effort level, maximizing output minus the cost of effort and the cost of providing incentives. At the same time, the drift of the agent's continuation value depends on how the principal chooses to pay up his promises over time. The principal will choose a payment scheme under which the marginal cost of delivering utility to the agent is a martingale. We call this effect *compensation smoothing.* An optimal contract features an employment interval

[6] In the optimality equation, $u = 0$ will be chosen when $F'(W) \geq -g'(0) = F_0'(0)$. Otherwise, the minimizing choice of u is positive and satisfies the first-order condition $F'(W) = -g'(u)$.

with two retirement endpoints. Inside the employment interval the agent's continuation value follows a diffusion process with drift and volatility determined by the considerations above. Once the agent's value hits a retirement point, it is most efficient for the principal to compensate the agent with constant consumption and allow him to choose an effort level of zero forever.

The reason for existence of a low retirement point is that the principal cannot deliver to the agent any value below zero. The agent can always guarantee himself value zero by putting no effort. In other words, zero is the agent's minmax payoff. The existence of a high retirement point is more surprising. In many situations, which are not covered by our model, the agent gets promoted after good performance, because the agent's performance reflects his skill level. In our model, we show that even if all agents are of the same skill level, the optimal contract features a high retirement point due exclusively to the income effect: the fact that when the agent becomes wealthy, it costs the principal too much to compensate him for his effort. Under the optimal contract, once a retirement point is reached, the agent remains there forever.

Several open questions for further research come to mind from the basic continuous-time model presented in this paper. First, what do optimal contracts look like when the agent can save and borrow behind the principal's back?[7] This would be a much more realistic situation for many applications, but the answer to this problem remains extremely difficult. The contract proposed in this paper would be vulnerable to many deviations in the setting where the agent could save and borrow. The agent would save his income to insure himself against future manipulations by the principal. One way to approach this problem is to add a restriction on the contract that the payments to the agent must induce a martingale marginal utility of consumption. Then the agent would not be able to improve his welfare by deviating only with his effort, or only with his savings. This idea is investigated in a first-order approach taken up in discrete time in Werning (2002). His approach is very useful in providing an upper bound on the principal's profit, since the first-order conditions are necessary for incentive compatibility. However, does the first-order approach guarantee full incentive compatibility of contracts? First-order conditions probably do imply full incentive compatibility when the agent's cost of effort is sufficiently convex, but the necessary conditions on the cost of effort are hard to identify. When the agent's choice of effort is binary, the first order approach fails as the following verbal argument demonstrates:

[7] If the savings were observable and contractible, then the principal would be able to achieve the same profit as if the agent could not save or borrow.

under any scheme proposed by the first order approach, the agent's marginal utility of consumption is a martingale. When the agent is supposed to put positive effort, the first order condition implies that he is indifferent between positive effort and effort zero, given that he does not alter his consumption pattern. The agent's deviation to effort zero would modify the underlying probability measure, so that with the original consumption pattern his marginal utility of consumption will be a submartingale. Therefore, by saving appropriately the agent can strictly improve his utility. We conclude that under an optimal contract subject to just first order incentive compatibility conditions, the agent always has a profitable deviation, which involves choosing effort zero and increased savings. Kocherlakota (2003) shows that the first order approach is invalid when the agent's cost of effort is linear in the unemployment insurance problem, and develops a number of new elegant ideas to solve the problem. There is hope that the answer to the question of hidden savings can be found by applying the work of Williams (2003), who considers the possibility of a hidden state variable.

It would be interesting to apply the basic model developed in this paper to describe a market with multiple agents and principals. Due to the flexibility of a continuous-time model, this direction is promising. The optimality equation allows us to use calculus to determine how contracts depend on the parameters of the production technology. Also, we found that the same equation describes an optimal contract under alternative boundary conditions for various values of the agent's quitting value $\tilde{W}$ and the cost of replacement of an agent C. One can imagine a dynamic market with random entry and exit of principals and agents. For an added twist, one can imagine also that principals have different production technologies and agents have different observable skill levels. What happens in a dynamic equilibrium? How are values of $\tilde{W}$ and C determined endogenously, and how do they vary over time and across population?

9 Appendix.

Proof of Lemma 3. First, we need to verify linear growth and Lipschitz conditions. This will imply existence, uniqueness and continuity of solutions in initial conditions. Define

$$H(\Phi, \Phi', W) = \min_{u,a} \frac{r\Phi - a + g(u) - \Phi'(rW - u + c(a))}{\gamma(a)^2/2}.$$

We are exploring equation

$$F''(W) = H(F(W), F'(W), W). \tag{21}$$

There are constants K and L such that

$$\forall\, u, a, W \qquad \frac{r}{\gamma(a)^2/2} \leq K \quad \text{and} \quad \frac{|rW - u + c(a)|}{\gamma(a)^2/2} \leq L.$$

Therefore, the linear growth conditions hold. To verify a Lipschitz condition, consider a pair of points (Φ, Φ') and (Ψ, Ψ') in the phase space of F and F'. Assume without loss of generality that $H(\Psi, \Psi', W) \geq H(\Phi, \Phi', W)$, and that (u, a) are the minimizers of $H(\Phi, \Phi', W)$. Then

$$H(\Psi, \Psi', W) \leq \frac{r\Psi - a + g(u) - \Psi'(rW - u + c(a))}{\gamma(a)^2/2} \quad \text{and} \quad H(\Phi, \Phi', W) = \frac{r\Phi - a + g(u) - \Phi'(rW - u + c(a))}{\gamma(a)^2/2}$$

$$\Rightarrow \quad |H(\Psi, \Psi', W) - H(\Phi, \Phi', W)| = H(\Psi, \Psi', W) - H(\Phi, \Phi', W) \leq |\Psi - \Phi| K + |\Psi' - \Phi'| L.$$

Next, let us show that a solution that starts from initial conditions $F(0) = 0$ and $F'(0) \geq 0$ is concave. Let us show that if ever $H(\Phi, \Phi', W) = 0$ on the path of a solution, then then the corresponding solution must be a straight line $F(W') = \Phi + (W' - W)\Phi'$. We need to verify that $H(\Phi + (W' - W)\Phi', \Phi', W') = 0$ for all W :

$$H(\Phi + (W' - W)\Phi', \Phi', W') = \min_{u,a} \frac{r\Phi + (W' - W)\Phi' - a + g(u) - \Phi'(rW' - u + c(a))}{\gamma(a)^2/2} =$$

$$\min_{u,a} \frac{r\Phi - a + g(u) - \Phi'(rW - u + c(a))}{\gamma(a)^2/2} = H(\Phi, \Phi', W) = 0.$$

Because $H(0, F'(0), 0) < 0$, the solution that starts from initial conditions $F(0) = 0$ and $F'(0) \geq 0$ is not a straight line. Therefore, $H(F(W), F'(W), W)$ never reaches 0 on the path of F, and so $H(F(W), F'(W), W)$ must remain negative. This completes the proof that solutions to (13) with initial conditions $F(0) = 0$ and $F'(0) \geq 0$ are concave functions. QED

Proof of Lemma 8. Consider solutions to

$$\tilde{f}'' = \tilde{h}(\tilde{f}, \tilde{f}', \omega) \tag{22}$$

with initial condition $\tilde{f}(0) = 0$. Since $\tilde{h}(0, f'(0), 0) < h(0, f'(0), 0)$, the solution to (22) with $\tilde{f}'(0) = f'(0) + \epsilon$ just hit f at some point ω_ϵ for all small $\epsilon > 0$. As we increase ϵ, from the continuity of solutions in initial conditions, ω_ϵ increases until it reaches ω_{gp} where f hits f_0, or $\tilde{f}$ becomes tangent to f at ω_ϵ. The latter is impossible, however, since this would imply that

$$f'' \leq \tilde{f}'' \quad \Rightarrow \quad h(f(\omega_\epsilon), f'(\omega_\epsilon), \omega_\epsilon) \leq \tilde{h}(f(\omega_\epsilon), f'(\omega_\epsilon), \omega_\epsilon).$$

We conclude that $\tilde{\omega}_{gp} \geq \omega_{gp}$, and the profit curve $\tilde{f}$ for the second set of parameters stays above f on $[0, \omega_{gp}]$. QED

Proof of Theorem 2. We will go loosely through the argument behind the proof, to spare the reader of long precise calculations. Note that when the principal chooses to deliver to the agent consumption utility u, he gets a flow of profit $a - g(u) = f_{fb}(u - c(a))$.

First, to find a lower bound on the principal's profit, consider a scheme under which the principal induces the agent to work for $\omega \in (0, \omega^*_{gp} - c(a))$, and retires him when ω hits 0 or $\omega^*_{gp} - c(a)$. Suppose when the agent's average value is ω, the principal gives him consumption utility $u = \omega + c(a)$. Then the principal's profit under the optimal scheme is certainly greater or equal to the principal's profit under this scheme. Under this scheme the agent's average continuation value evolves according to

$$d\omega_t = r\gamma(a) dZ_t.$$

Denote by τ the retirement time. For all $\omega_0 \in (0, \omega^*_{gp} - c(a))$, $E[e^{-r\tau}]$ converges to 0 exponentially fast, so the contribution to profit from the agent's retirement becomes negligible. Ignoring retirement, we can evaluate the principal's average profit from this scheme as

$$rE\left[\int_0^\infty e^{-rt} f_{fb}(\omega)\, dt\right] = r\int_0^\infty e^{-rt} f_{fb}(\omega_0) + \frac{\mathrm{Var}[\omega_t]}{2} f''_{fb}(\omega_0)\, dt =$$
$$= r\int_0^\infty e^{-rt} f_{fb}(\omega_0) + \frac{r^2\gamma(a)^2 t}{2} f''_{fb}(\omega_0)\, dt = f_{fb}(\omega_0) + \frac{r\gamma(a)^2}{2} f''_{fb}(\omega_0) + o(r^2).$$

This gives us a lower bound on the principal's profit

$$f(\omega) \geq f_{fb}(\omega) + \frac{r\gamma(a)^2}{2} f''_{fb}(\omega).$$

Any line tangent to f lies above f, and f has to be above its lower bound for all $\omega \in (0, \omega^*_{gp} - c(a))$. Therefore, $f(\omega) + f'(\omega)(v - \omega)$ has to lie above

$$f_{fb}(v) + \frac{r\gamma(a)^2}{2} f''_{fb}(v) = f_0(v + c(a)) + a + \frac{r\gamma(a)^2}{2} f''_{fb}(v).$$

For $v = u - c(a)$ we obtain

$$f''(\omega) = \frac{f(\omega) + f'(\omega)(u - c(a) - \omega) - f_0(u) - a}{r\gamma(a)^2/2} \geq f''_{fb}(v).$$

The last inequality is strict when the corresponding line tangent f lies strictly above the lower bound. But $f''(\omega) < f''_{fb}(v)$ is impossible since then the corresponding solution f to (19) would have to end up above f_{fb}. We conclude that all lines tangent to f must be also tangent to the lower bound and, since f is strictly concave, this is only possible when f is approximately at the lower bound. Thus we have

$$f(\omega) \sim f_{fb}(\omega) + \frac{r\gamma(a)^2}{2} f''_{fb}(\omega).$$

From the optimality equation (19), we must have $f'(\omega) = f'_{fb}(u(\omega/r) - c(a))$, so $u(\omega/r) \sim \omega + c(a)$ for $\omega \in (0, \omega^*_{gp} - c(a))$. Since $u(\omega/r)$ is increasing in ω,

$$u(\omega_{gp}/r) = \omega_{gp} \geq u\left(\frac{\omega^*_{gp} - c(a)}{r}\right) \sim \omega^*_{gp},$$

so we conclude that $\omega_{gp} \to \omega^*_{gp}$ and $u(\omega/r) \to \omega^*_{gp}$ for $\omega \in (\omega^*_{gp} - c(a), \omega^*_{gp})$.

Finally, since $f(\omega) \to f_{fb}(\omega)$ on $(0, \omega^*_{gp} - c(a))$ and f is concave, the principal's profit must converge to first best on $(\omega^*_{gp} - c(a), \omega^*_{gp})$ as well. QED

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