Price and Wealth Dynamics in a Speculative Market with an Arbitrary Number of Generic Technical Traders

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February 20, 2005

Abstract

We consider a simple pure exchange economy with two assets, one riskless, yielding a constant return, and one risky, paying a stochastic dividend, and we assume trading to take place in discrete time inside an endogenous price formation setting. Traders demand for the risky asset is expressed as a fraction of their individual wealth and is based on future prices forecast obtained on the basis of past market history.

We describe the evolution of price and wealth distribution in the general case where any number of heterogeneous traders is allowed to operate in the market and any smooth function which maps the infinite information set to the present investment choice is allowed as agent's trading strategy.

We give a complete characterization of equilibria and derive stability conditions analyzing a dynamical system of arbitrary large dimension. We show that this system can only possess isolated generic equilibria where a single agent dominates the market and continuous manifolds of non-generic equilibria where many agents hold finite wealth shares. Irrespectively of agents number and of their behavior, we show that all possible equilibria returns belong to a one dimensional "Equilibria Market Line".

JEL codes: G12, D83.

Keywords: Asset pricing, Price and wealth dynamics, Optimal selection principle.

Contents

1 Introduction

This work analyzes a simple asset pricing model where an arbitrary number of heterogeneous traders participate in a speculative activity. We consider a simple, pure exchange, two-asset economy. The first asset is a riskless security, yielding a constant return on investment. This security is chosen as the numéraire of the economy. The second asset is a risky equity, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the aggregate demand function. Agents participation to the market is described in terms of their individual demand for the risky asset.

We impose two restrictions on the way in which the individual demand of traders is formed. First, we assume that the amount of risky security demanded by a trader is proportional to his wealth. This assumption is consistent with, but not limited to, the maximization of an expected utility function with constant relative risk aversion (CRRA). Second, since the present work is mainly concerned with the effect of speculative behaviors and not of asymmetric evaluation and/or knowledge of the underlying fundamental of the economy, we assume that all traders, when making their investment choices, possess the same public information set. This set is naturally defined to contain the past price returns and the complete characterization of the dividend stochastic process.

Concerning the first assumption, the CRRA-type traders behavior seems to better fit empirical and experimental evidence than the CARA (constant absolute risk averse) behavior[1]. Nonetheless, the literature, typically referred to as "agent based", devoted to the analysis of market models with heterogeneous traders in an endogenous price formation setting (see LeBaron (2000) for a review) seems to have preferred the CARA to CRRA framework. The reason essentially rests, we believe, in the necessity, introduced by the choice of the latter, of accounting, along the evolution of the economy, for the dynamics of each individual portfolio. This necessity can lead, when many different traders operate in the market, to an enormous increase in the dimension of the system and, consequently, to a seeming increase in complexity.

By contrast a series of recent papers by Chiarella and He (Chiarella and He, 2001, 2002) started to analytically explore the dynamics and asymptotic properties of CRRA agent based models while another group of contributions (Levy et al., 1994, 2000; Zschischang and Lux, 2001) performed a numerical investigation of these properties.

These contributions share a common approach: they study the aggregate dynamics resulting from the interaction of a small group of particular trading strategies. The individual demand of traders is usually obtained through the maximization of a CRRA expected utility[2] using different estimators to forecast the future price movements in order to reflect different stylized speculative behaviors, like "fundamental", "trend chaser" or "contrarian" attitude. In these contributions, typically only one or two of these strategies are considered present at the same time in the market. Moreover, the requirement of keeping the resulting dynamical system to a relatively low degree forces the authors to consider only a small class of forecasting

[1]See, for instance the discussions of this issue in Levy et al. (2000), Campbell and Viceira (2002) and references therein.

[2] They frequently use logarithmic utility function, but the generalization of their analyses and conclusions to the more generic power utility is, often, trivial.

functions, based on a small subset of the whole information set available to agents.

In the present paper, we extend these early investigations and analyze the aggregate dynamics and asymptotic behavior of the market when an arbitrary number of different traders, each with his own trading strategy, participate to the trading activity. Furthermore, we do not restrict in any way the procedure used by agents in order to build their forecast about future prices starting from their information set. Even if considering generic agent behaviors leads us to study dynamical systems of an arbitrary large dimension, we are able to extend to this general case the main theoretical results presented by Chiarella and He. We provide a complete characterization of market equilibria and a description of the stability conditions based exclusively on few parameters characterizing the traders investment strategies. We show that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only two types of equilibria are possible: generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy and non generic equilibria, associated with continuous manifolds of fixed points, where many agents possess a finite share of the total wealth. We are also able to show, in total generality, that a simple function, the "Equilibria Market Line", can be used to obtain both a geometric characterization of the location of all possible equilibria and the condition of their stability.

Our general results extend previous contributions and allow a better understanding of the principles governing the asymptotic market dynamics. In particular, we are able to discuss the validity and limits of the "quasi-optimal selection principle" formulated in Chiarella and He (2001) for linear demand functions, when more general traders behaviors are taken in to consideration. At the same time, the possible existence of multiple, isolated, locally stable equilibria and the ensuing local nature of traders relative performances can be interpreted as an "impossibility theorem" for the construction of a dominance order relation inside the space of trading strategies.

The present paper is organized as follows. In Section 2 we describe our simple pure-exchange economy, writing explicitly the traders inter-temporal budget constraint and laying down the equations (and constraints) governing the dynamics of the market. In Section 3 we present the simple case in which a single trader operates in the market. The Equilibria Market Line is derived, and its use is shortly discussed. The general case in which an arbitrarily large number of traders participates the trading activity is analyzed in Section 4. Our conclusions, and the directions our work will plausibly take in the future are briefly mentioned in Section 5.

2 Definition of the Model

Consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities take place in discrete time. The economy is composed by a risk-less asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the end of each period t. Let the risk-less asset be the numéraire of the economy, so that its price is fixed to 1. The price P_t of the risky asset is determined at each period, on the basis of its aggregate demand, through market-clearing condition. The resulting intertemporal budget constraint is derived below and the main hypotheses, on the nature of the investment choices and of the fundamental process, are discussed. These hypotheses will allow us to derive the explicit dynamical system governing the evolution of the economy.

2.1 Intertemporal Budget Constraint

Let $W_{t,n}$ be the wealth of trader n at time t and let $x_{t,n}$ stands for the fraction of this wealth invested into the risky asset. After the trading session at time $t-1$, agent n possesses $x_{t-1,n}\, W_{t-1,n}/P_{t-1}$ shares of risky asset and $(1-x_{t-1,n})\, W_{t-1,n}$ shares of risk-less security. At this moment he receives the payment of risk-less interest r_f on the wealth invested in the latter and dividends payment D_{t-1} per each risky asset. Therefore, at time t the wealth of agent n, for any notional price P, reads

$$W_{t,n}(P) = (1-x_{t-1,n})\, W_{t-1,n}\,(1+r_f) + \frac{x_{t-1,n}\, W_{t-1,n}}{P_{t-1}}\,(P + D_{t-1}) \tag{2.1}$$

and his individual demand for the risky asset becomes $x_{t,n}\, W_{t,n}(P)/P$. The actual price of the risky asset at time t is fixed at the level for which aggregate demand is equal to aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, the price P_t is defined as the solution of the equation

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n}(P_t) = P_t \quad . \tag{2.2}$$

At the end of period t the dividend D_t and the risk-free interest r_f are paid and the trading session at time $t+1$ can start.

The dynamics defined by (2.1) and (2.2) describes an exogenously growing economy due to the continuous injections of new riskless assets, whose price remains, under the assumption of totally elastic supply, unchanged. It is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce rescaled variables $w_{t,n} = W_{t,n}/(1+r_f)^t$, $p_t = P_t/(1+r_f)^t$ and $e_t = D_t/(P_t\,(1+r_f))$. Rewriting the market dynamics defined by (2.2) and (2.1) using these variables one obtains

$$\begin{cases} p_t = \displaystyle\sum_{n=1}^{N} x_{t,n}\, w_{t,n} \\ w_{t,n} = w_{t-1,n} + w_{t-1,n}\, x_{t-1,n}\left(\dfrac{p_t}{p_{t-1}} - 1 + e_{t-1}\right) \quad \forall n \in \{1,\dots,N\} \quad . \end{cases} \tag{2.3}$$

These equations give the evolution of state variables $w_{t,n}$ and p_t over time, provided that stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified. Such dynamics implies a simultaneous determination of the equilibrium price p_t and of the agents' wealths $w_{t,n}$. Due to this simultaneity, the $N+1$ equations in (2.3) define the state of the system at time t only implicitly. Indeed, the N variables $w_{t,n}$ defined in the second equation appear on the right-hand side of the first, and, at the same time, the variable p_t defined in the first equation appears in the right-hand side of the second. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

2.2 Dynamical System for Wealth and Return

The transformation of the implicit dynamics of (2.3) into an explicit one is not generally possible. In this subsection we formulate and briefly discuss the conditions that agents investment shares $x_{t,n}$ should meet, in order for such a transformation to be possible.

First, let $\mathbb{I}_{t-1} = \{P_{t-1}, P_{t-2}, \dots, D_{t-1}, D_{t-2}, \dots\}$ stands for the information set available to traders before the trading round at time t starts. In the present dynamical setting it is natural to make the following

Assumption 1. For each agent n there exists some deterministic function f_n which maps the present information set into its investment share:

$$x_{t,n} = f_n(\mathbb{I}_{t-1}) \quad . \tag{2.4}$$

The function f_n in the right hand-side of (2.4) gives a complete description of the investment decision of n-th agent and in what follows we will refer to it as the "investment function" of agent n. Notice that Assumption 1 implies that the demand function of any agent at time t can be written as $x_{t,n}\, W_{t,n}/P_t$, with $x_{t,n}$ independent from his present wealth $W_{t,n}$ and price level P_t. This assumption is consistent with a framework in which agents investment decisions are obtained from maximization of an expected utility characterized by constant relative risk aversion (CRRA)[3].

However, the simultaneous determination of price and wealth in (2.3) entails some restriction on the possible market positions available to agents[4]. We derive this restriction below, but before let us introduce a notation that will prove useful to present the dynamics in a more compact form.

Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents, i.e.

$$\langle a \rangle_t = \frac{\sum_{n=1}^{N} \, a_n\, w_{t,n}}{w_t} \quad , \quad \text{where} \qquad w_t = \sum_{n=1}^{N} w_{t,n} \quad . \tag{2.5}$$

The next result gives the condition for which the dynamical system implicitly defined in (2.3) can be made explicit without violating the requirement of positiveness of prices

Proposition 2.1. *From equations* (2.3) *it is possible to derive a map* $\mathbb{R}^{+N} \to \mathbb{R}^{+N}$ *that describes the evolution of traders wealth* $w_{t,n}$ $\forall n \in \{1, \dots, N\}$ *so that prices* $p_t \in \mathbb{R}^+$ $\forall t$ *remain positive provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t\, x_{t+1} \rangle_t\Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_t)\langle x_t\, x_{t+1} \rangle_t\Big) > 0 \qquad \forall t \quad . \tag{2.6}$$

[3]Assumption 1 is not satisfied, however, if agents are assumed constant absolute risk aversion (CARA) expected utility maximizers.

[4]The simplest way to understand this is to consider the case of a single agent. With a little bit of algebra it is easy to show that (2.1) and (2.2) imply that at time t the price should satisfy the equation $P_t = x_t\,(P_t + B_t)$, where B_t denotes the amount of the numéraire available to the agent before the trade. Therefore, a positive price requires $x_t < 1$. In the dynamical setting with many agents the matter is more complicated, since both current and previous investment choices of all agents are involved in the determination of price.

If this is the case, the price growth rate $r_{t+1} = p_{t+1}/p_t - 1$ *reads*

$$r_{t+1} = \frac{\left\langle x_{t+1} - x_t \right\rangle_t + e_t \left\langle x_t\, x_{t+1} \right\rangle_t}{\left\langle x_t \left(1 - x_{t+1}\right) \right\rangle_t} \tag{2.7}$$

and the evolution of wealth, described by the wealth growth rates $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$*, is given by*

$$\rho_{t+1,n} = x_{t,n} \left(r_{t+1} + e_t\right) = x_{t,n} \frac{\left\langle x_{t+1} - x_t \right\rangle_t + e_t \left\langle x_t \right\rangle_t}{\left\langle x_t \left(1 - x_{t+1}\right) \right\rangle_t} \qquad \forall n \in 1, \ldots, N \quad . \tag{2.8}$$

Proof. See appendix A. □

The explicit price dynamics can be obtained from (2.7) in a trivial way but price will be positive only if condition (2.6) is satisfied. In general, it may be quite difficult to check the validity of this condition at each time step. We do not deal with this issue here, but let us notice that sometimes, for practical purposes, it is better to replace this condition with a more binding but simpler one. For instance in Anufriev et al. (2004) it is shown that if investment choices $x_{t,n}$ are *uniformly* bounded inside interval $(0, 1)$ with respect to t and n, then condition (2.6) is satisfied and, therefore, equations (2.7) and (2.8) give well-defined dynamics in terms of price and wealths.

Having obtained the explicit dynamics for the evolution of price and wealth one is interested in the asymptotic behavior of the system. Unluckily, the dynamics defined by (2.7) and (2.8) does not possess any interesting fixed point. Indeed, if the price and the wealth are constant, one would have $r_{t+1} = \rho_{t+1,n} = 0$ for any t and n. This would imply, in periods when a positive dividend e_t is paid, that $x_{t,n} = 0$ for any n. That is, the only possible fixed point is the one in which there is no demand for the risky asset. The reasons behind this is that, after removing an exogenous expansion due to the risk-free interest rate, we still have an expansion due to the dividend payments. The presence of such an expansion would suggest to look for possible asymptotic states of *steady growth*. Notice, however, that if the dividend yield e_t depends on price, it is impossible to rewrite the dynamics in Proposition 2.1 in terms of the sole price and wealth returns. This issue is solved as soon as one makes the following

Assumption 2. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support.

This assumption is common to several works in the literature, for instance Chiarella and He (2001), so that a further reason to have this assumption introduced is to maintain comparability with previous investigations. Under Assumption 2, direct dependence on price disappears from (2.7) and (2.8). Consequently, the dynamics of the economy is fully specified in terms of r_t and $\rho_{t,n}$.

2.3 Agent Investment Function

In the present work we are mainly concerned with the effect of speculative behaviors on the market aggregate performance. In order to eliminate effects due to asymmetric evaluation and/or knowledge of the underlying fundamental process we model the agents' investment choice as depending on the sole realized price returns, assuming, for any agent, a perfect knowledge of the dividend process. This amounts to replace Assumption 1 with

Assumption 3. Agents possess complete knowledge about the dividend yield process $\{e_t\}$ so that (2.4) reduces to

$$x_{t+1,n} = f_n(r_t, r_{t-1}, \dots) \quad . \tag{2.9}$$

We call the investment behavior in Assumption 3 *"technical trading"* stressing the fact that the agents' decision is not affected by the past realization of the fundamental e_t. At the same time, also agent-specific variables, like past investment choices or investment choices of other traders, do not affect agent's demand.

In the majority of models discussed in the literature the investment choice described by (2.9) is obtained as the result of two distinct steps. In the first step agent n, using certain estimator g_n, forms his expectation at time t about certain future price statistics, for instance expected price return $\mathrm{E}_n[r_{t+1}] = g_n(\mathbb{I}_{t-1})$. With this expectations, using a choice function h_n, he computes the fraction of wealth invested in the risky asset $x_{t+1,n} = h_n(\mathrm{E}_n[r_{t+1}])$. Investment function f_n is the result of the composition of estimator g_n and of choice function h_n.

Even if such interpretation is intuitive and common in the economic literature, and compatible with (2.9), we would like to stress the fact that it is not required by our framework. In our model agents are not forced to use some specific predictors, rather they are allowed to map the past return history into the future investment choice, with whatever smooth function they like.

Function f_n, as it is written in (2.9), can be infinite dimensional. In this case, a complete dynamical system for the return dynamics should be, generally speaking, infinite-dimensional as well. In some cases, however, it is possible to reduce the effective dimension of the dynamical system. For instance in Anufriev et al. (2004) agents use exponentially weighted moving average (EWMA) estimations based on past returns in order to forecast both average return value and its variance. In this case the dynamics can be described with the use of a low dimensional system even in presence of an infinite information set. This dimensional reduction is possible any time agents forecasting activity admits a recursive definition.

In what follows, however, we want to consider a more generic situation. We assume that each agent n has his own, so to speak, "memory time span" L_n, so that at each time step his new investment choice is determined as a function of the last L_n return realizations. For the following discussion, L_n must be finite, but can be arbitrary large. In this case, without loss of generality, we can assume that their spans are all the same and equal to the largest span $L = \max\{L_1, \dots L_N\}$, so that each investment function can be thought as having exactly L arguments

$$x_{t+1,n} = f_n(r_t, r_{t-1}, \dots, r_{t-L+1}) \quad . \tag{2.10}$$

To our knowledge, there are no published analyses that deal with such a generic expression for the investment decision of agents. Indeed only a quite small class of functions is, usually, taken in to consideration. For instance, Chiarella and He (2001) makes rigorous analysis only for a specific mean-variance investment function and only considering $L = 1$. As we will show below, however, some important results can be obtained under our, more general, assumptions.

3 Single Agent Case

We start with the analysis of the very special situation in which a single agent operates on the market. The main reason to perform this analysis rests in its relevance for the multi-agent case. Indeed, in the setting with N heterogeneous traders each generic multi-agent equilibrium requires, as necessary condition for stability, the stability of a suitably defined single agent equilibrium.

This Section starts laying down the dynamics of the single agent economy as a multidimensional dynamical system of difference equations of the first order. All possible equilibria of this system are identified and the associated characteristic polynomial, which can be used to analyze their stability, derived.

3.1 Dynamical System

In the case of one single agent the dynamical system describing the market evolution can be considerably simplified since the explicit evolution of wealth shares in (2.8) is not needed. As a consequence, the whole system can be described with only $L + 1$ variables: one variable represents the current investment choice x_t and the other variables the L past returns.

The current return can be defined by means of the function in the right hand-side of (2.7):

$$R(x', x, e) = \frac{x' - x + e\, x'\, x}{(1 - x')\, x} \quad , \tag{3.1}$$

where the first variable x' denotes the current (contemporaneous with return) investment choice, while the variables x and e stands for the *previous period* investment choice and dividend yield, respectively.

With such definitions the dynamical system governing the evolution of the economy with a single agent reads

$$\begin{cases} x_{t+1} & = \; f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big) \\ r_{t+1,0} & = \; R\Big(f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big),\ x_t,\ e_t\Big) \\ r_{t+1,1} & = \; r_{t,0} \\ \quad\vdots & \\ r_{t+1,L-1} & = \; r_{t,L-2} \end{cases} , \tag{3.2}$$

where $r_{t,l}$ stands for the price return at time $t - l$.

In the rest of this Section we are interested in analyzing the so-called *deterministic skeleton* of this $L+1$-dimensional system. That is, we substitute the yield by its mean value $\bar{e}$ in order to obtain the deterministic dynamical system which give, in a sense, "average" representation of the stochastic dynamics.

3.2 Determination of Equilibria

The following Proposition provides the characterization of the fixed points of the deterministic skeleton of (3.2).

Let x^* denote the agent's wealth share invested in the risky asset at equilibrium and let r^* be the the equilibrium return. In any fixed point the realized returns are constant, so that $r_0 = r_1 = \cdots = r_{L-1} = r^*$. One has

Proposition 3.1. *Let $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ be a fixed point of the system (3.2). The equilibrium return r^* satisfies*

$$\frac{r^*}{\bar{e} + r^*} = f(\underbrace{r^*, \ldots, r^*}_{L}) \quad , \tag{3.3}$$

and the equilibrium investment share x^* is defined accordingly to

$$x^* = f(r^*, \ldots, r^*) \quad . \tag{3.4}$$

Proof. Equation (3.4) directly follows from the first equation of (3.2) computed in the fixed point. From the second equation of (3.2) one has

$$r^* = R(x^*, x^*, \bar{e}) = \bar{e}\,\frac{x^*}{1 - x^*} \quad ,$$

inverting this relation to obtain x^* as a function of r^* and using (3.4) one obtains (3.3). $\square$

Notice that (3.3) provides a simple geometric characterization of all possible fixed points $\boldsymbol{x^*}$ of (3.2). To see it, start from the graph of the function f of L variables and consider its full symmetrization $f(r) = f(r, \ldots, r)$, i.e. its restriction to the one dimensional line defined by the $L-1$ equations $r_0 = r_1 = \cdots = r_L$. The intersections of this function with the function defined on this line by the left hand side of (3.3) are the equilibria of the system. Since the latter function will play a major role in our analysis we introduce the following

Definition 3.1. The *Equilibria Market Line* (EML) is the function $l(r)$ defined according to

$$l(r) = \frac{r}{\bar{e} + r} \quad . \tag{3.5}$$

As an example, consider the case of investment functions which are one-dimensional functions of the sole last return ($L = 1$). In Fig. 1 we show how the equilibrium return can be found for two different functions of this type, drawn as thick lines. The thin line represents the hyperbolic curve of the EML defined in (3.5). Notice that this line is made of two branches and that it is not defined in the point[5] $-\bar{e}$. The intersection of the investment function with the EML are the possible equilibria of the system. The nonlinear function has two equilibria: S_1 with small positive return and U_1 with high positive return. The linear function has also

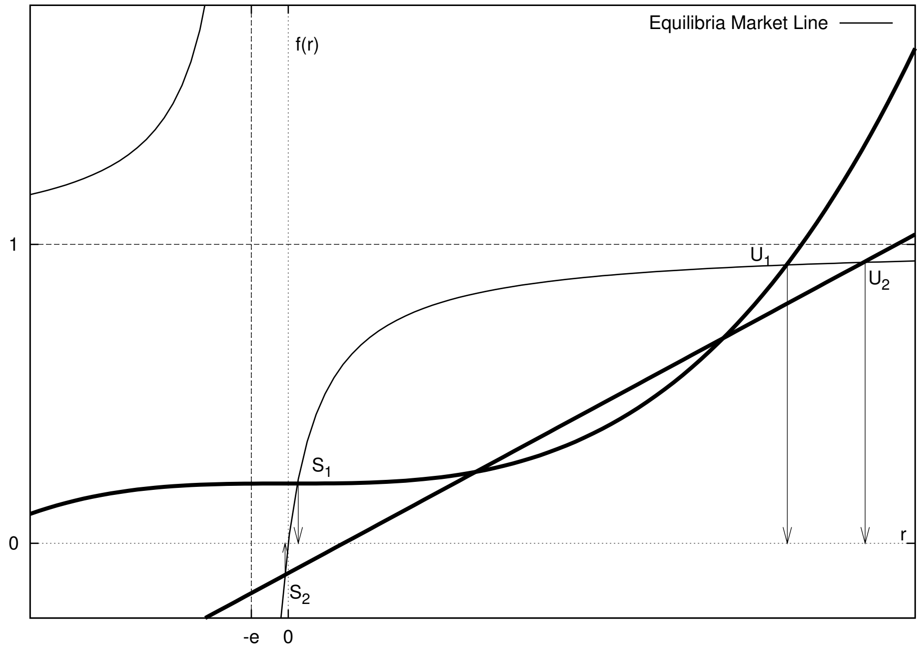


Figure 1: Equilibria for two different investment functions based on the last realized return.

two equilibria, S_2 and U_2. Even if in S_2 the equilibrium price return is negative, it is greater than $\bar{e}$, so that the return in terms of unscaled price is still positive.

A second "geometrical" example is presented in Fig. 2. Here the agent's memory lag is equal to 2 periods, and the investment function is given by $x_{t+1} = |r_t| \left(r_t + 0.4 \left(r_t - r_{t-1} \right) \right)$. Look at the two-dimensional surface representing the investment function. The thick line on this surface is the intersection of the function with the plane defined by the condition $r_t = r_{t-1}$. Now consider, on this plane, the EML function $l(r)$. The intersection of the two curves on this plane define all possible equilibria. In this case we have one trivial equilibrium with zero return and another equilibrium with positive price return r^* and investment share $x^* = f(r^*) = |r^*| \, r^*$.

3.3 Stability Conditions of Equilibria

As the next natural step we move to discuss the stability conditions for the equilibria which has been identified in the previous Section. We will do it by deriving the characteristic polynomial associated to the Jacobian matrix computed in the fixed point $\boldsymbol{x^*}$. The following applies

[5]It is general property of the system with N agents defined in (2.7) and (2.8) that the the equilibrium return cannot be equal to $-\bar{e}$. Indeed, if the price return would offset the positive dividend yield in the equilibrium, each agent would have constant wealth over time. But it immediately leads to constant price, and therefore to zero return.

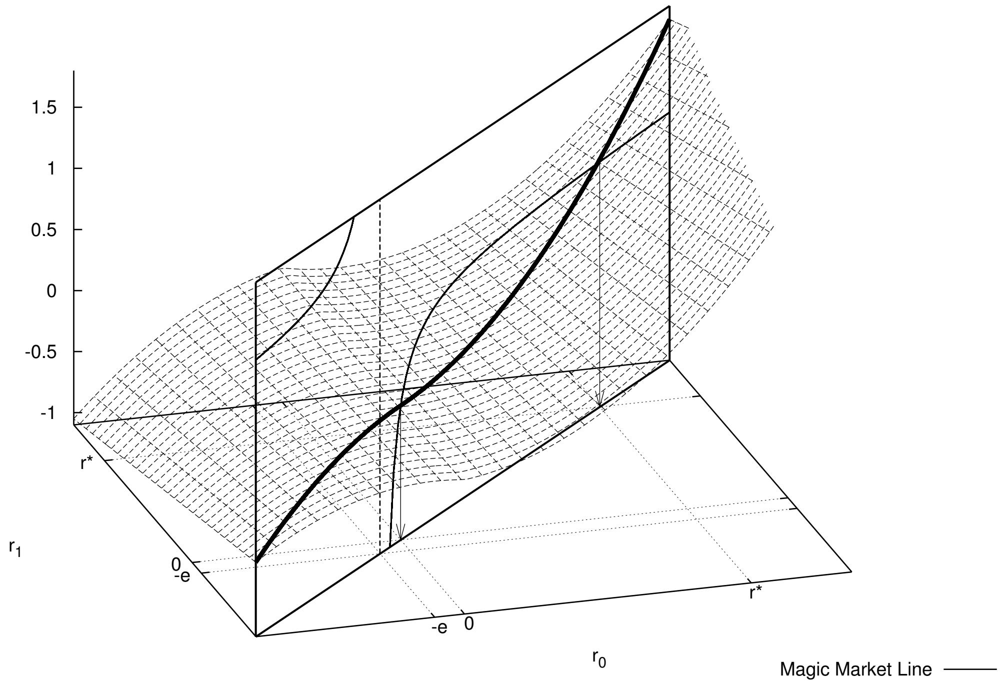


Figure 2: Equilibria for an investment function based on the last two realized returns.

Lemma 3.1. *The characteristic polynomial of system* (3.2) *in fixed point* $\boldsymbol{x^*}$ *reads:*

$$Q(\mu) = (-1)^{L+1} \left(\mu^{L+1} - \left(R'_x + \mu\, R'_f\right) \left(f'_0\, \mu^{L-1} + f'_1\, \mu^{L-2} + \cdots + f'_{L-2}\, \mu + f'_{L-1}\right)\right), \quad (3.6)$$

where

$$R'_x = \frac{\partial R(x^*, x^*)}{\partial x}\,, \qquad R'_f = \frac{\partial R(x^*, x^*)}{\partial x'}\,, \qquad f'_j = \frac{\partial f(r^*, \ldots, r^*)}{\partial r_j}\,. \quad (3.7)$$

Proof. See appendix B. □

The general result of the theory of dynamical system states that if all *multipliers*, i.e. the roots of polynomial $Q(\mu)$, lie inside the complex unit circle of the complex space, then the equilibrium $\boldsymbol{x^*}$ is *locally asymptotically stable.* On the contrary, if there exist at least one multiplier outside the unit circle, then this equilibrium is *unstable.* Finally, in the situation when there are multipliers on the unit circle and others inside, the equilibrium may or may not be stable. In this case the stability depends on the higher order terms of the Taylor expansion of the right hand-side of system (3.2).

Thus, in order to explore the stability condition for equilibria defined in Proposition 3.1, one has to compute the modulus of the roots of $Q(\mu)$ in (3.6). They clearly depend on the value of the derivatives of function f in equilibrium, and therefore, on function f itself. On the other hand, the derivatives of function R can be computed in general

Lemma 3.2. *In a fixed point* $\boldsymbol{x}^*$ *of the deterministic skeleton of* (3.2) *it is*

$$R'_x = \frac{\partial R}{\partial x} = -\frac{1}{x^*\,(1-x^*)} \qquad \text{and} \qquad R'_f \;= \frac{\partial R}{\partial x'} = \frac{1-x^*+\bar{e}\,x^*}{(1-x^*)^2\,x^*} = \frac{1+r^*}{x^*\,(1-x^*)} \tag{3.8}$$

Once investment function f is known, sing the two last lemmas it is possible to derive the stability conditions of equilibria and, consequently, discuss the role of the different parameters (like $\bar{e}$ or the slope of the function f) in stabilizing or destabilizing a given equilibrium.

For illustrative purposes we present here the explicit analysis in the simplest case, when $L = 1$. Some other interesting situations are considered in Anufriev et al. (2004).

3.3.1 Naïve forecast: $L = 1$.

The easiest situation is represented by an agent with a memory time span of a single lag ($L = 1$) so that his investment function reads $x_{t+1} = f(r_t)$. In this case (3.6) becomes a second degree polynomial, and the multipliers are the roots of the following quadratic equation

$$\mu^2 - \mu\, R'_f\, f' - R'_x\, f' = 0\ ,$$

where we used f' for f'_1. The conditions for both roots of this equation to be inside the unit circle are easily derived to be

$$-R'_x\, f' < 1\ , \qquad R'_f\, f' < 1 - R'_x\, f'\ , \qquad -1 + R'_x\, f' < R'_x\, f'\ .$$

When one of the previous inequalities is violated, the fixed point undertakes a first oder bifurcation. The nature of this bifurcation depends on the condition being violated as provided by the following

Proposition 3.2. *The fixed point of system* (3.2) *with* $L = 1$ *is (locally) asymptotically stable if*

$$\frac{f'}{x^*\,(1-x^*)} < 1\,, \qquad \frac{r^*\,f'}{x^*\,(1-x^*)} < 1 \qquad \text{and} \qquad \frac{(2+r^*)\,f'}{x^*\,(1-x^*)} > -1 \quad , \tag{3.9}$$

where $f' = df(r^*)/dr$*. The fixed point undertakes*

(i) Neimark-Sacker bifurcation, if $f' = x^*(1-x^*)$*;*

(ii) fold bifurcation, if $r^*\,f' = x^*\,(1-x^*)$*;*

(iii) flip bifurcation, if $(2+r^*)\,f' = -x^*\,(1-x^*)$*.*

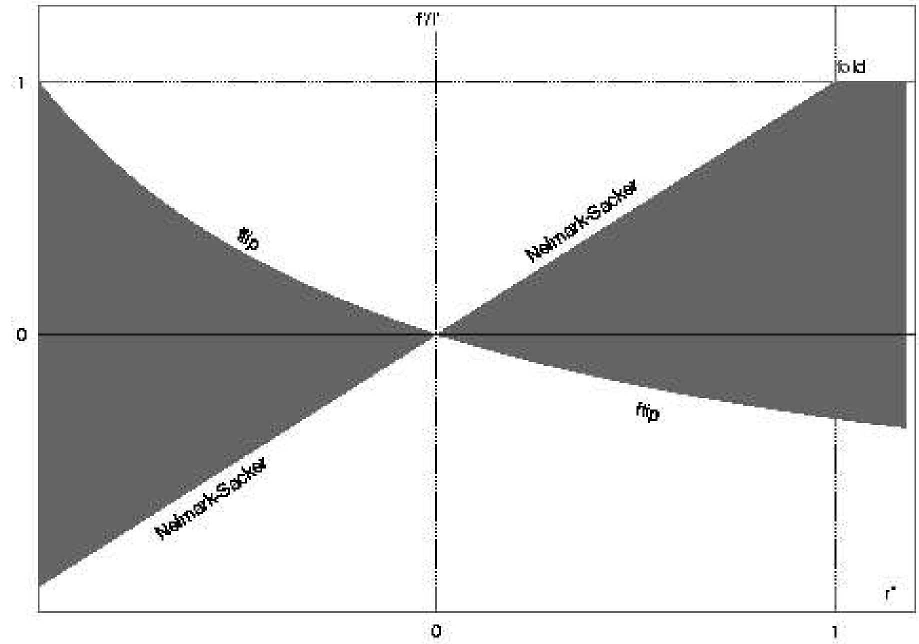


Figure 3: Stability region (gray) and fixed point bifurcations for system (3.2) with $L = 1$. The parameter space has coordinates r^* and $f'(r^*)/l'(r^*)$.

In order to visualize these conditions, let us notice that the slope of function $l(r)$ defined in (3.5) is equal to $l'(r) = x\,(1-x)/r$. Therefore, conditions (3.9) can be rewritten as

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < 1\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \text{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -1 \quad . \tag{3.10}$$

The stability region defined by the inequalities in (3.10) is shown in Fig. 3 in coordinates r^* and $f'(r^*)/l'(r^*)$. The last coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the Equilibrium Market Line $l(r)$. Notice that if the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, the second inequality in (3.10) requires the slope of investment function to be smaller than the slope of function $l(r)$.

As an example let us look at the equilibria in Fig. 1. One can immediately see that equilibrium U_1 of the nonlinear function is unstable: the slope of the function is higher than the slope of the EML. On the contrary, the slope of the investment function in S_1 is very small, so that, presumably, this equilibrium is stable. Analogously, for the linear function, equilibrium U_2 is clearly unstable while S_2 can result stable. As can be seen from Fig 3, if the slope of the first function in S_1 increases, this equilibrium would lose stability trough a Neimark-Sacker bifurcation. The increase of the slope of the second function in S_2 would instead lead to a flip bifurcation.

4 Economy with Many Agents

This Section extends the previous results to the case of finite, but arbitrarily large, number of heterogenous agents. It is organized as the previous one. It starts with the derivation of the $2N + L - 1$ dimensional stochastic dynamical system which describes the evolution of the economy and continues with the identification of all possible equilibria and their stability analysis.

4.1 Dynamical System

If there is more than one agent on the market, the evolution of agents wealth is not decoupled from the system and, consequently, all N equations in (2.8) are relevant for the dynamics. In this case it is convenient to rewrite the system using agents individual share of the total wealth $\varphi_{t,n} = w_{t,n}/w_t$. The dynamical system in terms of these new variables is provided by the following

Lemma 4.1. *Under the conditions of Proposition 2.1, the price growth rate* (2.7) *reads:*

$$r_{t+1} = \frac{\sum_{n=1}^{N} (x_{t+1,n} - x_{t,n} + e_t\, x_{t,n}\, x_{t+1,n})\, \varphi_{t,n}}{\sum_{n=1}^{N} x_{t,n}\, (1 - x_{t+1,n})\, \varphi_{t,n}} \quad , \tag{4.1}$$

while the agents' wealth shares evolve accordingly to

$$\varphi_{t+1,n} = \varphi_{t,n}\, \frac{1 + (r_{t+1} + e_t)\, x_{t,n}}{1 + (r_{t+1} + e_t)\, \sum_{m=1}^{N} x_{t,m}\, \varphi_{t,m}} \qquad \forall n \in \{1, \dots, N\} \quad . \tag{4.2}$$

Proof. See appendix C. ☐

The first-order dynamical system associated with (4.1) and (4.2) with investment function as in (2.10) has the following $2N + L - 1$ independent variables

$$x_{n,t} \quad \forall n \in \{1, \dots, N\}\,; \qquad \varphi_{n,t} \quad \forall n \in \{1, \dots, N-1\}\,; \qquad r_{l,t} \quad \forall l \in \{0, \dots, L-1\}\,, \tag{4.3}$$

where $r_{l,t}$ denotes the price return at time $t - l$. Notice that only $N - 1$ wealth shares are needed. Indeed, at any time step t, it is $\sum_{n=1}^{N} \varphi_{t,n} = 1$ so that $\varphi_{t,N} = 1 - \sum_{n=1}^{N-1} \varphi_{t,n}$.

In order to derive the dynamical system in the case of many agents we need the expression of the variables in (4.3) at time $t + 1$ as a function of the same set of variables at time t. One can proceed as follows. First, the investment choice in (2.10) can be written

$$x_{t+1,n} = f_n\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big) \qquad \forall n \in \{1, \dots, N\} \quad . \tag{4.4}$$

In order to find the expression of the return $r_{t+1,0}$ we introduce function R (which reduces to (3.1) in the case of a single agent)

$$\begin{aligned} R\Big(& f_1(r_0, r_1, \dots, r_{L-1}), f_2(r_0, r_1, \dots, r_{L-1}), \dots, f_N(r_0, r_1, \dots, r_{L-1});\\ & x_1, x_2, \dots, x_N; \varphi_1, \varphi_2, \dots, \varphi_{N-1}; e\Big) = \\ &= \frac{\sum_{n=1}^{N-1} \varphi_n\big(f_n\,(1 + e\, x_n) - x_n\big) + \Big(1 - \sum_{n=1}^{N-1} \varphi_n\Big)\,\big(f_N\,(1 + e\, x_N) - x_N\big)}{\sum_{n=1}^{N-1} \varphi_n\, x_n\,(1 - f_n) + (1 - \sum_{n=1}^{N-1} \varphi_n)\, x_N\,(1 - f_N)} \end{aligned} \tag{4.5}$$

so that one can write[6]

$$r_{t+1,0} = R\Big(f_1(r_0,\dots,r_{L-1}),\dots,f_N(r_0,\dots,r_{L-1});x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e\Big) \tag{4.6}$$

while the remaining returns variables are simply the outcome of a "lag" operation

$$r_{t+1,l} = r_{l-1} \qquad \forall l \in \{1,\dots,L-1\} \quad . \tag{4.7}$$

The evolution of wealth shares can be described with the help of functions Φ_n defined accordingly to

$$\begin{aligned}&\Phi_n\big(x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e;R\big) = \\ &\quad = \varphi_n \frac{1+x_n\,(R+e)}{1+(R+e)\left(\sum_{n=1}^{N-1}\varphi_n\,x_n + \big(1-\sum_{n=1}^{N-1}\varphi_n\big)\,x_N\right)} \qquad \forall n\in\{1,\dots,N-1\}\,.\end{aligned} \tag{4.8}$$

Notice that due to the presence of function R in the last expression, all functions Φ_n depend on the same set of variables as R. Now, from (4.2) the agents' wealth shares $\forall n \in \{1,\dots,N-1\}$ reads

$$\begin{aligned}\varphi_{t+1,n} = \Phi_n\Big(&x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e;\\ &R\big(f_1(r_0,\dots,r_{L-1}),\dots,f_N(r_0,\dots,r_{L-1});x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e\big)\Big)\,.\end{aligned} \tag{4.9}$$

The set of equations (4.6), (4.7) and (4.9) defines the following $2N+L-1$ dimensional dynamical system (remember that all variables in the right-hand sides are from period t):

$$\begin{array}{l} \mathcal{X}: \left[\begin{array}{lcl} x_{t+1,1} & = & f_1\big(r_0,\dots,r_{L-1}\big) \\ \quad\vdots & \vdots & \vdots \\ x_{t+1,N} & = & f_N\big(r_0,\dots,r_{L-1}\big) \end{array}\right. \\ \mathcal{W}: \left[\begin{array}{lcl} \varphi_{t+1,1} & = & \Phi_1\Big(x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e; \\ & & \quad R\big(f_1(r_0,\dots,r_{L-1}),\dots,f_N(r_0,\dots,r_{L-1}); \\ & & \qquad x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e\big)\Big) \\ \quad\vdots & \vdots & \vdots \\ \varphi_{t+1,N-1} & = & \Phi_{N-1}\Big(x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e; \\ & & \quad R\big(f_1(r_0,\dots,r_{L-1}),\dots,f_N(r_0,\dots,r_{L-1}); \\ & & \qquad x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e\big)\Big) \end{array}\right. \\ \mathcal{R}: \left[\begin{array}{lcl} r_{t+1,0} & = & R\Big(f_1(r_0,\dots,r_{L-1}),\dots,f_N(r_0,\dots,r_{L-1}); \\ & & \qquad x_1,\dots,x_N;\varphi_1,\dots,\varphi_{N-1};e\Big) \\ r_{t+1,1} & = & r_0 \\ \quad\vdots & \vdots & \vdots \\ r_{t+1,L-1} & = & r_{L-2} \end{array}\right. \end{array} \tag{4.10}$$

[6]In order to avoid the overloading of the notation from now on we will leave out the indexes t from the variables in the right hand-sides of the equations.

We ordered the equations to obtain three separated blocks: $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{R}$. In block $\mathcal{X}$ there are N equations defining the investment choices of agents. Block $\mathcal{W}$ contains $N-1$ equations describing the evolution of the wealth shares. Finally, block $\mathcal{R}$ is composed by L equations which describe the evolution of the return. In the last block equations are in ascending order with respect to the time lag.

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (4.10): we replace yield realizations $\{e_t\}$ by their mean value $\bar{e}$ and analyze the equilibria of the resulting deterministic system.

4.2 Determination of Equilibria

The characterization of fixed points of system (4.10) is in many respect similar to the single agent case discussed above. Let $\boldsymbol{x}^* = (x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_{N-1}^*; r^*, \ldots, r^*)$ denotes a fixed point where r^* is the equilibrium return[7] and x_n^* and φ_n^* stay for the equilibrium value of the investment function and the equilibrium wealth share of agent n, respectively. Notice that in any fixed point it is $r_0 = r_1 = \cdots = r_{L-1} = r^*$. The following applies

Proposition 4.1. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (4.10)*. Two mutually exclusive cases are possible:*

(i) ***Single agent survival.*** *In $\boldsymbol{x}^*$ only one agent possesses a wealth share different from zero. Without loss of generality we can assume this agent to be agent* 1 *so that for the equilibrium wealth shares one has*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad . \tag{4.11}$$

Equilibrium return r^* is determined as the solution of

$$\frac{r^*}{\bar{e} + r^*} = f_1(\underbrace{r^*, \ldots, r^*}_{L_1}) \quad , \tag{4.12}$$

while the equilibrium investment shares are defined according to

$$x_n^* = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, N\}. \tag{4.13}$$

(ii) ***Many agents survival.*** *In $\boldsymbol{x}^*$ more than one agent possesses a wealth share different from zero. Without loss of generality one can assume that the agents with non-zero wealth shares are the first k agents (with $k > 1$) so that the equilibrium wealth shares satisfy*

$$\varphi_n^* = 0 \quad \text{if } n > k \; , \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \qquad . \tag{4.14}$$

[7] Remember that return in equilibrium r^* cannot be equal to $-\bar{e}$ as mentioned in Section 3.2.

The first k agents possess the same investment share $x^*_{1\diamond k}$ at equilibrium

$$x^*_1 = x^*_2 = \cdots = x^*_k = x_{1\diamond k} \quad .$$

and equilibrium return r^* must simultaneously satisfy the following set of k equations

$$\frac{r^*}{\bar{e}+r^*} = f_n(\underbrace{r^*,\ldots,r^*}_{L_n}) = x^*_{1\diamond k} \quad \forall n \in \{1,\ldots,k\} \quad . \tag{4.15}$$

The equilibrium investment shares of the last $N-k$ agents are defined according to

$$x^*_n = f_n(r^*,\ldots,r^*) \quad \forall n \in \{k,\ldots,N\} \quad . \tag{4.16}$$

Proof. From block $\mathcal{X}$ one immediately has (4.13) and (4.16). From block $\mathcal{W}$ using (4.8) and the condition $r^* + \bar{e} \neq 0$ one obtains

$$\varphi^*_n = 0 \quad \text{or} \quad \sum\nolimits_{m=1}^{N-1} \varphi^*_m\, x^*_m + \left(1 - \sum\nolimits_{m=1}^{N-1} \varphi^*_m\right) x^*_N = x^*_n \quad \forall n \in \{1,\ldots,N-1\} \quad . \tag{4.17}$$

Finally, from the first row of block $\mathcal{R}$ it is

$$r^* = \bar{e}\, \frac{\sum_{n=1}^{N-1} \varphi^*_n {x^*_n}^2 + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) {x^*_N}^2}{\sum_{n=1}^{N-1} \varphi^*_n\, x^*_n\,(1 - x^*_n) + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) x^*_N\,(1 - x^*_N)} \,. \tag{4.18}$$

The previous set of equations admits two types of solutions.

To derive the first type assume (4.11). In this case condition (4.17) is satisfied for all agents. From (4.18) one has $x^*_1 = r^*/(\bar{e}+r^*)$ which together with (4.13) leads to (4.12).

To derive the second type of solutions assume (4.14). In this case, the second equality from (4.17) must be satisfied for any $n \leq k$. Since its left-hand side does not depend on n, a $x^*_{1\diamond k}$ must exist such that $x^*_1 = \cdots = x^*_k = x^*_{1\diamond k}$. Substituting $x^*_n = 0$ for $n > k$ and $x^*_n = x^*_{1\diamond k}$ for $n \leq k$ in (4.18) one gets $x^*_{1\diamond k} = r^*/(\bar{e}+r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (4.16) for $n \leq k$.

Notice that in the second type of solutions all the investment shares $x^*_1,\ldots,x^*_k$ must *at the same time* be equal to a single value $x^*_{1\diamond k}$. Consequently, the equilibrium with $k > 1$ survivals exists only in the particular case in which k investment functions $f_1,\ldots,f_k$ satisfy this restriction. □

Strictly speaking, item *(i)* of the previous Proposition can be seen as a particular case of item *(ii)*. Nevertheless, the nature of the two situations is deeply different. In the first case, when a single agent survives, Proposition 4.1 defines a precise value for each component (x^*, φ^* and r^*) of the equilibrium $\boldsymbol{x}^*$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: while r^* and investment shares x^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (4.14). Consequently, item *(ii)* does not define a single equilibrium point but an equilibria hyperplane in the parameter space. The particular fixed point eventually chosen by the system will depend on the initial conditions. In the next Section we will see that the partially indeterminate nature of the equilibria in the case of many survivors will have a major effect also in the discussion of their stability.

The differences among the two cases of Proposition 4.1 does not only regard the geometrical nature of the *locus* of equilibria. Indeed, while in the first case no requirements are imposed on the behavior of the investment function of the different agents, the many-agent equilibria require that the values of the investment functions of the surviving agents are, at equilibrium, the same. Consequently, an economy composed by N agents having generic, so to speak "randomly defined", investment functions, has probability zero of displaying any multi-agent equilibrium. In other terms, the multi-agent equilibria are non-generic.

Notice that both types of multi-agent equilibria derived in Proposition 4.1 are strictly related to "special" single-agent equilibria. The determination of the equilibrium return level r^* for the multi-agent case in (4.12) or (4.15) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market. An useful consequence of this fact is that the geometrical interpretation of market equilibria presented in Section 3.2 can be extended to illustrate how equilibria with many agents are determined.

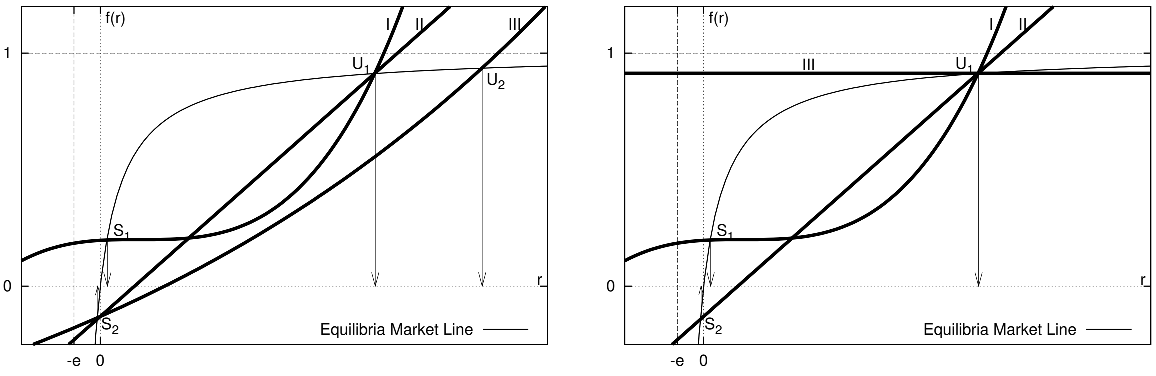
As an example consider again Fig. 1 and suppose that the two strategies shown there belong to two agents who are simultaneously operating on the market. According to Proposition 4.1 all possible equilibria can be found as intersections of one of the strategy with the Equilibria Market Line (c.f. (4.12) and (4.15)). In this example there are four possible equilibria. In two of them (S_1 and U_1) the first agent, with non-linear strategy, survives such that $\varphi_1^* = 1$ (and obviously $\varphi_2^* = 0$). In the other two equilibria (S_2 and U_2) is the second agent, with linear strategy, who survives so that, in these points, $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the Equilibria Market Line gives both equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the other agent can be found, accordingly to (4.16), as the intersection of its own investment function with the vertical line passing trough equilibrium return. Since the two investment functions shown in Fig. 1 do not possess common intersections with the EML, in this case equilibria with more than one survivors are impossible. An example of investment functions which allow for multiple survivors equilibria are reported in Fig. 4.

4.3 Stability Conditions of Equilibria

This Section presents two propositions relevant for the stability analysis of the equilibria defined in Proposition 4.1. The first Proposition provides the stability region in the parameters space for the generic case of one single survivor. The non-generic case of many survivors is partially addressed in the second Proposition, where the destabilizing effect of the existence of an entire hyperplane of equilibria is revealed. Since the proofs of these Propositions require quite cumbersome algebraic manipulations, we provide below only their statements and refer the reader to Appendix D for the intermediate Lemmas and the final proofs.

For the generic case of a single survivor equilibrium we have the following

Proposition 4.2. *Let* $\boldsymbol{x}^*$ *be a fixed point of* (4.10) *associated with a single survivor equilib-*



(a) (b)

Figure 4: The situation with 3 agents operating on the market. For the investment functions shown on the panel (a) there exist equilibria S_2 and U_1 with two survivals. For the investment strategies illustrated on the panel (b), in a fixed point U_1 all three agents survive.

rium. Without loss of generality we can assume that first agent survives, so that

$$\varphi_n^* = \begin{cases} 1 & \textit{if}\, n = 1 \\ 0 & \textit{if}\, n > 1 \end{cases} .$$

With the above hypothesis, point $\boldsymbol{x}^*$ is (locally) asymptotically stable if the two following conditions are met:

1) *the equilibrium investment behavior of the surviving agent x_1^* and the equilibrium return r^* defines a stable hyperbolic fixed point of the $L+1$ dimensional system in (3.2) with $f = f_1$, i.e. in the single-agent economy in which only agent 1 trades.*

2) *the equilibrium investment shares of the non-surviving agents are lower than the investment share of the survivor*

$$x_n^* < x_1^* \qquad \forall n \neq 1 . \tag{4.19}$$

The stability condition for a generic fixed point in the multi-agent economies is twofold. On the one hand, the stable equilibrium should be “self-consistent”, i.e. it should remain stable even if any non-surviving agent would be removed from the economy. This is however not enough. A further requirement is that the surviving agent must be the most aggressive among all agents in the economy, i.e. the one who invests the highest wealth share in the risky asset.

The EML “plot” can be used to obtain a geometrical illustration of the previous Proposition. Consider the two strategies in Fig. 1 and suppose they are both present on the market at the same time. In Section 4.2 we found four possible equilibria: S_1, S_2, U_1 and U_2. First notice that the dynamics cannot be attracted by U_1 or U_2. Since these equilibria were unstable in the respective single-agent cases, they cannot be stable when both agents are present in the market. Assume that S_1 is a stable equilibrium for the first agent alone and S_2 for the second.

In this case S_1 is the only stable equilibrium of the system with two agents. The point S_2 cannot be stable since the first agent possesses in this point a more aggressive strategy than the second: his investment function is above the investment function of the latter.

In the next Section we will discuss the economic interpretation of Proposition 4.2 and analyze its consequences for the aggregate behavior of the system. Before that, however, let us provide some hint on the local nature of the non-generic equilibria with several survivors. The following applies

Proposition 4.3. *A fixed point* $\boldsymbol{x^*}$ *of* (4.10) *belonging to a* $k-1$*-dimensional manifold of* k*-survivors equilibria defined by* (4.14),(4.15) *and* (4.16) *is never hyperbolic.*

Therefore, in the case of many survivors the non hyperbolic nature of the equilibria would require, for the determination of the stability region, the analysis of higher order expansions of the mapping around the fixed point. Due to the non-generic nature of these equilibria, however, this analysis would largely fall outside the intent of the present paper. We do not pursue this issue here and leave it for future investigations.

4.4 Market Selection and Asymptotic Dominance

In the following discussion we confine ourselves to the generic case in which only equilibria with a single survival trader are considered.

We want to exploit the geometric interpretation based on EML "plot" to understand some relevant implications of Proposition 4.2 about the asymptotic behavior of the model and its global properties.

The first implication concerns the aggregate dynamics of the economy. Let us consider a stable many-agent equilibrium $\boldsymbol{x^*}$. Let us suppose that r^* is the equilibrium return in $\boldsymbol{x^*}$ and that the first agent is dominating. According to (2.8) the wealth return of the survivor is equal to $\rho_1^* = x_1^*\,(r^* + \bar{e})$. Since the first agent asymptotically possesses the total wealth of the economy, ρ_1^* is also the asymptotic growth rate of the total wealth. Then, we can interpret the second requirement of Proposition 4.2 as saying that, in the dynamical competition, those agent survives who allows the economy to grow with the highest possible rate. Indeed, if any other agent n survived, the economy would have grown with a rate $x_n^*\,(r^* + \bar{e})$, which is lower than ρ_1^*, since $x_n^* < x_1^*$. To put the same statement in negative terms, the economy will never end up in equilibria where its growth rate is less then it would be if the survival would be substituted by some other agent. One could call this result an *optimal selection principle* since it characterizes the market endogenous selection toward the best aggregate outcome.

In a similar framework Chiarella and He (2001) introduce an analogous principle that has, however, in their less general case, much stronger consequence. They found that, when two agents have *linear* strategies, the dynamics endogenously tends to the equilibrium which has relatively higher return (among all "possible" equilibria). Indeed, with some additional care, they add the *quasi* specifier to the word *optimal* to stress that equilibrium is "possible" only if it is stable in the associated single agent case. Referring again to the Fig. 1: the selection is optimal since it chooses S_1 but not S_2. However, it is only quasi-optimal, since the dynamics does not end up in U_2 where the return is even higher.

Even if for the particular strategies considered in Chiarella and He (2001) such extended optimal selection principle worked, it will not work in the general setting due to the possibility of multiple stable equilibria. One can start, for instance, with an agent whose strategy is not linear and has two stable equilibria. Now, any market where such agent meets other agents who always invest less, will have more than one stable equilibrium strategy. Then the quasi-optimal principle in the sense of Chiarella and He (2001) is violated.

The existence of multiple equilibria also leads to a second interesting implication of Proposition 4.2, the fact that the dominance of one strategy on another depends on the market initial conditions. Consider the simple case with two agents on the market. Let suppose that their strategies are such that two stable equilibria exist. In the first equilibrium the first agent dominates while in the second equilibrium it is dominated by the second agent. Now assume that these two agents enter the market subsequently. It is immediate to see that it is *the order* in which these two agents enter the market that determines the final aggregate outcome. Ultimately, this proofs the impossibility (at least inside our framework) to build any dominance order relation on the space of trading strategies.

5 Conclusion

This paper extends the analysis presented in Chiarella and He (2001) and introduces novel results concerning the characterization and stability of equilibria in speculative pure exchange economies with heterogeneous traders.

Let us shortly review the assumptions we made and our achievements in order to sketch the possible future lines of research. We considered a simple analytical framework using a minimal number of assumptions (2 assets and Walrasian price formation). We modeled agents as speculative traders and we imposed the constraint that their participation to the trading activity is described by an individual demand function proportional to their wealth. Moreover, we assumed that agents form their individual demand decisions on the predictions about future price returns obtained from the publicly available past prices history. With prescribed but arbitrary specification about the agents' behavior, the feasible dynamics of the economy (i.e. the dynamics for which prices stay always positive) can then be described as a multi-dimensional dynamical system.

In such framework we started with a single agent case and presented the fixed point stability analysis of the corresponding system. Then we moved to the general framework with an arbitrary number of agents and showed that the conditions for the existence of fixed points and the conditions for their stability are related to the corresponding conditions in the situation with one single agent. We found that different scenarios are possible: in the generic case, the system possesses isolated equilibria where one single trader dominates the others and ultimately capture the entire market. Alternatively, in the non-generic case in which traders investment functions satisfy a special set of constraints, the system can possesses a continuous manifold of equilibria associated with non-hyperbolic fixed points. In these non-generic equilibria many agents possess a finite amount of the total wealth of the economy.

The present analysis can be extended in many directions. First of all, even if we proved that the existence of multiple equilibria is possible, the dynamics in this case remains to be unveiled. Probably numerical methods can be effectively applied to clarify the role of initial conditions, the determinants of the relative size of the attraction domains for different equilibria, etc. These methods can be also used to study the dynamics in the cases when there are no stable equilibria.

Second, one may ask what are the consequences of the optimal selection principle for a market in which the set of strategies is not "frozen", but instead is evolving in time, plausibly following some adaptive process. For instance, one can assume that agents imitate the behavior of the most successful traders (see e.g. Kirman (1991)). In such case, those situations which we referred as "non-generic" above may become, instead, typical. Proposition 4.3 can be considered only a first step in the analysis of such situations.

Third, inside our general framework, numerous different specifications of the traders strategies are possible, in addition to the ones analyzed here. They range from the evaluation of the "fundamental" value of the asset, maybe obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. The analysis of the consequences of the introduction of such strategies on the optimal selection principle may, ultimately, refute the statement about the impossibility of defining a dominance relation among strategies.

APPENDIX

A Proof of Proposition 2.1

Plugging the expression for $w_{t,n}$ from the second equation in (2.3) into the right hand-side of the first equation, assuming $p_{t-1} > 0$ and, consistently with (2.6), $p_{t-1} \neq \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}$, one obtains

$$p_t = \left(1 - \frac{1}{p_{t-1}} \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}\right)^{-1} \left(\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}\right) =$$

$$= p_{t-1} \frac{\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}}{\sum x_{t-1,n}\, w_{t-1,n} - \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}} =$$

$$= p_{t-1} \frac{\langle x_t \rangle_{t-1} - \langle x_{t-1}\, x_t \rangle_{t-1} + e_{t-1} \langle x_{t-1}\, x_t \rangle_{t-1}}{\langle x_{t-1} \rangle_{t-1} - \langle x_{t-1}\, x_t \rangle_{t-1}} \quad ,$$

where the second equality comes from the first equation in (2.3) rewritten for time $t-1$. Condition (2.6) is immediately obtained imposing $p_t > 0$. Then the price return and wealth return for each agent n at time t can be derived straightforwardly.

B Proof of Lemma 3.1

The $(L+1)\times(L+1)$ Jacobian $\boldsymbol{J}$ of system (3.2) reads

$$\boldsymbol{J} = \begin{bmatrix} 0 & f'_0 & f'_1 & \dots & f'_{L-2} & f'_{L-1} \\ R'_x & R'_f\, f'_0 & R'_f\, f'_1 & \dots & R'_f\, f'_{L-2} & R'_f\, f'_{L-1} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} . \tag{B.1}$$

Let $\boldsymbol{I}$ stands for the unit matrix and consider $\boldsymbol{J}-\mu\,\boldsymbol{I}$. Multiplying the first row by $-R'_f$ and adding it to the second row its determinant can be written

$$\det(\boldsymbol{J}-\mu\,\boldsymbol{I}) = \begin{vmatrix} -\mu & f'_0 & f'_1 & \dots & f'_{L-3} & f'_{L-2} & f'_{L-1} \\ R'_x+\mu\,R'_f & -\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\mu & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & -\mu \end{vmatrix} .$$

In order to obtain the characteristic polynomial expand this matrix by the minors of the upper row elements. The determinant of the first minor is, obviously, $(-\mu)^L$, while in all remaining minors the only non-zero element of the first column is $R'_x+\mu\,R'_f$. Therefore we get the following expression

$$(-\mu)^{L+1} + \left(R'_x+\mu\,R'_f\right)\Big(-f'_0\,(-\mu)^{L-1} + f'_1\,(-\mu)^{L-2} - f'_2\,(-\mu)^{L-3} \pm \dots \\ + (-1)^{L-1}\,f'_{L-2}\,(-\mu) + (-1)^L\,f'_{L-1}\Big) \tag{B.2}$$

which immediately leads to (3.6).

C Proof of Lemma 4.1

The dynamics of r_{t+1} in (4.1) is identical to the one given in (2.7) due to the definition of the wealth weighted average quantity introduced in (2.5). Indeed, $\langle a\rangle_t = \sum_{n=1}^N a_n\varphi_{t,n}$ for any quantity a. The evolution of agents wealth in (2.8) reads

$$w_{t+1,n} = w_{t,n}\,(1 + x_{t,n}\,(r_{t+1}+e_t)) \ .$$

Dividing both sides of this equation by the total wealth at time $t+1$ one gets

$$\begin{aligned} \varphi_{t+1,n} &= \frac{w_{t,n}}{\sum_m w_{t+1,m}}\,(1 + x_{t,m}\,(r_{t+1}+e_t)) = \\ &= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1}+e_t)\sum_m x_{t,m}w_{t,m}}\,(1 + x_{t,n}\,(r_{t+1}+e_t)) = \\ &= \frac{\varphi_{t,n}}{1+(r_{t+1}+e_t)\sum_m x_{t,m}\varphi_{t,m}}\,(1 + x_{t,n}\,(r_{t+1}+e_t)) \quad , \end{aligned}$$

where the last equality has been obtained dividing both numerator and denominator by $\sum_m w_{t,m}$. This proofs (4.2) for any agent n.

D Proofs of two propositions of Sec. 4.3

In order to prove Propositions 4.2 and 4.3 we have to explore the properties of the Jacobian matrix for the deterministic skeleton of system (4.10).

General Structure of the Jacobian Matrix

The Jacobian of system (4.10) is $\big(2N+L-1\big)\times\big(2N+L-1\big)$ matrix. In order to deal with such object we will write it in a following block structure where each of the small blocks is a matrix:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} & \frac{\partial \mathcal{X}}{\partial \mathcal{R}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} & \frac{\partial \mathcal{W}}{\partial \mathcal{R}} \\ \frac{\partial \mathcal{R}}{\partial \mathcal{X}} & \frac{\partial \mathcal{R}}{\partial \mathcal{W}} & \frac{\partial \mathcal{R}}{\partial \mathcal{R}} \end{bmatrix} \tag{D.1}$$

As it should be clear from the notation (remember also our notation of the blocks for system (4.10) introduced in Section 4.1), each of the blocks represents the matrix of the partial derivatives of either the agents' investment choices (if there is label $\mathcal{X}$ in numerator) or the agents' wealth shares (if there is label $\mathcal{W}$ in numerator) or the contemporaneous and lag return (if there is label $\mathcal{R}$ in numerator) with respect to either the agents' investment choices (if there is label $\mathcal{X}$ in denominator) or the agents' wealth shares (if there is label $\mathcal{W}$ in denominator) or the contemporaneous and lag return (if there is label $\mathcal{R}$ in denominator).

In order to write each of the small blocks we introduce the special notation for the derivatives of different functions used in our system. First, for the investment share functions f_n which depend, according to (2.10), on the last L returns, we define a gradient vector:

$$\boldsymbol{F}_n = \begin{bmatrix} \dfrac{\partial f_n}{\partial r_0} & \dfrac{\partial f_n}{\partial r_1} & \dots & \dfrac{\partial f_n}{\partial r_{L-1}} \end{bmatrix} \qquad \forall n \in \{1,\dots,N\} \quad ,$$

where instead of $\partial f_n/\partial r_{t-l}$ we simply write $\partial f_n/\partial r_l$. Second, function R defined in (4.5) provides new return $r(t+1)$. It depends on the past wealth shares and also on two investment choices, where the contemporaneous choice is captured by the function f_n. The following notation will be used:

R^{f_m}	$\forall m \in \{1,\dots,N\}$	for the derivative w.r. to the current choice given by f_m,
R^{x_m}	$\forall m \in \{1,\dots,N\}$	for the derivative w.r. to past choice x_m,
R^{φ_m}	$\forall m \in \{1,\dots,N-1\}$	for the derivative w.r. to wealth share φ_m.

Finally, all functions Φ_n defined in (4.8) depend on the investment choices, wealth shares and return R. We denote the derivatives in the following way:

$\Phi_n^{x_m}$	$\forall m \in \{1,\dots,N\}$	for the derivative w.r. to past choice x_m,
$\Phi_n^{\varphi_m}$	$\forall m \in \{1,\dots,N-1\}$	for the derivative w.r. to wealth share φ_m,
Φ_n^R		for the derivative w.r. to return given by R.

Let us now study the structure of the blocks constituting Jacobian (D.1). Short look at system (4.10) leads to the observation that there should be lot of zeros in Jacobian, because of the absence of many variables in the right hand-side of different equations. It concerns, especially, the first row. First block there $\frac{\partial \mathcal{X}}{\partial \mathcal{X}}$ is, clearly, $N \times N$ zero matrix, since the new investment choices of agents do not directly depend on the previous investment choices. Also the investment choices do not depend on the previous wealth shares, hence the next block in the first row $\frac{\partial \mathcal{X}}{\partial \mathcal{W}}$ of size $N \times (N-1)$ consists of only zeros. The last block in the first row expresses the dependence of the investment choices on the previous returns. Using our notation we write it as the following $N \times L$ matrix:

$$\frac{\partial \mathcal{X}}{\partial \mathcal{R}} = \left\| \begin{array}{c} \boxed{\boldsymbol{F}_1} \\ \vdots \\ \boxed{\boldsymbol{F}_N} \end{array} \right\|$$

The second row in (D.1) is relatively complicated. It contains blocks which, generally, do not have non-zero entries, as it can be seen from definition (4.8) of function $\varPhi_n$. The left matrix has size $(N-1) \times N$ and contains the derivatives of the wealth shares with respect to the investment choice. It is followed by the squared matrix of dimension $(N-1) \times (N-1)$ with the derivatives of the wealth shares with respect to the previous ones. These two blocks we will write as follows.

$$\frac{\partial \mathcal{W}}{\partial \mathcal{X}} = \left\| \begin{array}{ccc} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N-1,N} & \dots & A_{N-1,N} \end{array} \right\| , \qquad \frac{\partial \mathcal{W}}{\partial \mathcal{W}} = \left\| \begin{array}{ccc} B_{1,1} & \dots & B_{1,N-1} \\ \vdots & \ddots & \vdots \\ B_{N-1,N-1} & \dots & B_{N-1,N-1} \end{array} \right\| ,$$

where the coefficients are defined as

$$A_{n,m} = \varPhi_n^{x_m} + \varPhi_n^R \cdot R^{x_m} \qquad \forall n \in \{1, \dots, N-1\}, \quad \forall m \in \{1, \dots, N\}, \tag{D.2}$$

$$B_{n,m} = \varPhi_n^{\varphi_m} + \varPhi_n^R \cdot R^{\varphi_m} \qquad \forall n \in \{1, \dots, N-1\}, \quad \forall m \in \{1, \dots, N-1\}. \tag{D.3}$$

The third block in the middle row has $(N-1) \times L$ elements. Each element there is the derivatives of the wealth share of agent n with respect to lth lagged return. In our notation it is written as

$$\varPhi_n^R \cdot \left(R^{f_1} \cdot \frac{\partial f_1}{\partial r_l} + \dots + R^{f_N} \cdot \frac{\partial f_N}{\partial r_l} \right) \qquad \forall n \in \{1, \dots, N-1\}, \ \forall l \in \{0, \dots, L-1\}. \tag{D.4}$$

As we will see later, all these elements are zeros in any fixed point.

Finally, let us analyze three bottom blocks. Since the set of equations $\mathcal{R}$ of system (4.10) contains lagged variables, these blocks have relatively simple structure. Namely, in all the rows except the first there will be a lot of zeros. So, we have:

$$\frac{\partial \mathcal{R}}{\partial \mathcal{X}} = \left\| \begin{array}{cccc} R^{x_1} & R^{x_2} & \dots & R^{x_N} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array} \right\| , \qquad \frac{\partial \mathcal{R}}{\partial \mathcal{W}} = \left\| \begin{array}{cccc} R^{\varphi_1} & R^{\varphi_2} & \dots & R^{\varphi_{N-1}} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array} \right\| ,$$

Only in the last block, i.e. where the derivatives of return are taken with respect to the same variable, we will have not only zeros in the last rows:

$$\frac{\partial \mathcal{R}}{\partial \mathcal{R}} = \left\| \begin{array}{cccc} C_0 & C_1 & \dots & C_{L-1} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{array} \right\| ,$$

where the coefficients are defined as follows:

$$C_l = R^{f_1} \cdot \frac{\partial f_1}{\partial r_l} + \cdots + R^{f_N} \cdot \frac{\partial f_N}{\partial r_l} \qquad \forall l \in \{0, \ldots, L-1\}\,. \tag{D.5}$$

Jacobian Matrix in the Fixed Points

Let us now focus on equilibria of system (4.10). In these point the Jacobian can be further simplified. First of all, we can compute the derivatives of functions R and Φ_n in fixed points.

Lemma D.1. *In any fixed point* $\boldsymbol{x}^*$ *of system* (4.10) *for all* $m \in \{1, \ldots, N\}$ *it is:*

$$\begin{aligned} R^{f_m} &= \varphi^*_m \, \frac{1+r^*}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})} \quad , \\ R^{x_m} &= -\,\varphi^*_m \, \frac{1}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})} \quad , \\ R^{\varphi_m} &= x^*_m \,(r^*+\bar{e})\, \frac{x^*_m - x^*_{1\diamond k}}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})} \quad . \end{aligned}$$

Lemma D.2. *In any fixed point* $\boldsymbol{x}^*$ *of system* (4.10) *for all* $n \in \{1, \ldots, N-1\}$ *it is:*

$$\begin{aligned} \Phi^{x_m}_n &= \varphi^*_n \,(\delta_{n,m} - \varphi^*_m)\, \frac{\bar{e}+r^*}{1+r^*}\,, \qquad \forall m \in \{1, \ldots, N\} \quad , \\ \Phi^{\varphi_m}_n &= \frac{1}{1+r^*}\Big(\delta_{n,m}\big(1 + x^*_n(r^*+\bar{e})\big) - \varphi^*_n(r^*+\bar{e})(x^*_m - x^*_N)\Big), \qquad \forall m \in \{1, \ldots, N-1\}\,, \\ \Phi^{R}_n &= \varphi^*_n \, \frac{x^*_n - x^*_{1\diamond k}}{1+r^*}\,, \end{aligned}$$

where $\delta_{n,m}$ *is the Kronecker delta.*

Both these lemmas can be checked by the direct differentiation. Additional simplifications come from two results established in Section 4.2. First, in any fixed point for any agent n it is either $\varphi^*_n = 0$ or $x^*_n = x^*_{1\diamond k}$. Second, from (4.15) it is $x^*_{1\diamond k}(r^*+\bar{e}) = r^*$.

These two lemmas show that in any particular fixed point Jacobian $\boldsymbol{J}$ can be simplified. For instance, in any equilibrium $R^{\varphi_m} = 0$ for all survivals, while $R^{f_m} = R^{x_m} = 0$ for all other agents. Also notice that $\Phi^R_n = 0$, since if agent n survives in the equilibrium, then $x^*_n = x^*_{1\diamond k}$, and if he does not survive there, then $\varphi^*_n = 0$. We summarize all the information about the blocks forming the Jacobian for the fixed point in the following

Lemma D.3. *Consider equilibrium* $\boldsymbol{x}^*$ *with* k *survived agents and number them from* 1 *to* k. *Then the blocks of Jacobian* $\boldsymbol{J}$ *given in* (D.1) *in this fixed point have the following structure:*

1. Block $\partial\mathcal{X}/\partial\mathcal{X}$**.** *This is zero block.*

2. Block $\partial\mathcal{X}/\partial\mathcal{W}$**.** *This is zero block.*

3. Block $\partial\mathcal{X}/\partial\mathcal{R}$**.** *This is block where the rows are gradient vectors* $\boldsymbol{F}_n$*.*

4. Block $\partial\mathcal{W}/\partial\mathcal{X}$. *Here, element $A_{n,m}$ defined in* (D.2) *is reduced to $\Phi_n^{x_m}$. In particular, if $n > k$ and if $m > k$ but $m \neq n$, then $A_{n,m} = 0$. Therefore, the rows and the columns of those agents who did not survive contain only zero elements. The only non-zero part in this block is $k \times k$ upper-left square.*

5. Block $\partial\mathcal{W}/\partial\mathcal{W}$. *Here, $B_{n,m}$ defined in* (D.3) *is reduced to $\Phi_n^{\varphi_m}$. If $n > k$ and $n \neq m$ then $B_{n,m} = 0$. Therefore, all rows for those agents who did not survive contain only zeros, except the diagonal elements.*

6. Block $\partial\mathcal{W}/\partial\mathcal{R}$. *This is zero block, as it follows from* (D.4)*.*

7. Block $\partial\mathcal{R}/\partial\mathcal{X}$. *In the first row of this block the elements are R^{x_m}. If $m > k$, i.e. for all agents who did not survive, they are all zeros. All other rows contain only zeros.*

8. Block $\partial\mathcal{R}/\partial\mathcal{W}$. *In the first row of this block the elements are R^{φ_m}. If $m < k$, i.e. for all survivals, these elements are zeros. All other rows contain only zeros.*

9. Block $\partial\mathcal{R}/\partial\mathcal{R}$. *The first row of this block contains elements C_l defined in* (D.5)*. In the fixed point they can be written as follows:*

$$C_l = \frac{1 + r^*}{x^*_{1\diamond k}\,(1 - x^*_{1\diamond k})} \left(\varphi_1^* \frac{\partial f_1}{\partial r_l} + \cdots + \varphi_k^* \frac{\partial f_k}{\partial r_l}\right) \qquad \forall l \in \{0, \ldots, L-1\} \quad . \tag{D.6}$$

Our next step consists in looking for multipliers of Jacobian matrix (D.1) in different equilibria. First, we consider the equilibrium with one survival described in the item *(i)* of Proposition 4.1. The statement of Proposition 4.2 will then follow immediately. Second, we consider non-generic equilibrium as it is described in the item *(ii)* of Proposition 4.1 and complete the proof of Proposition 4.3.

Case of one survival: multipliers of the Jacobian matrix

Let us suppose that in fixed point $\boldsymbol{x^*}$ only first agent survives, i.e. $\varphi_1^* = 1$ while $\varphi_n^* = 0$ for all $n \in \{2, \ldots, N\}$. Consider characteristic polynomial $P(\mu) = \det(\boldsymbol{J} - \mu \boldsymbol{I})$ of system (4.10), where $\boldsymbol{J}$ stands for Jacobian matrix (D.1) and $\boldsymbol{I}$ denotes an identity matrix of size $2N + L - 1$. We have the following

Proposition D.1. *The characteristic polynomial $P(\mu)$ computed in fixed point $\boldsymbol{x^*}$ has $N-1$ zero roots. Another $L+1$ roots are the roots of the polynomial:*

$$\tilde{Q}(\mu) = (-1)^{L+1} \left(\mu^{L+1} - \left(R^{x_1} + \mu\, R^{f_1}\right) \left(\mu^{L-1} \frac{\partial f_1}{\partial r_0} + \mu^{L-2} \frac{\partial f_1}{\partial r_1} + \cdots + \frac{\partial f_1}{\partial r_{L-1}}\right)\right) . \tag{D.7}$$

Finally, the remaining $N-1$ roots are given as follows:

$$B_{1,1} = \frac{1 + (\bar{e} + r^*)\, x_N^*}{1 + (\bar{e} + r^*)\, x_1^*}\,, \qquad\qquad B_{n,n} = \frac{1 + (\bar{e} + r^*)\, x_n^*}{1 + (\bar{e} + r^*)\, x_1^*} \quad \forall n \in \{2, \ldots, N-1\}\,. \tag{D.8}$$

Proof. The proof will have a recursive structure. In the process of finding new multipliers (which will be equal either to 0, or to $B_{n,n}$) we will subsequently reduce the size of the matrix from which all the remaining multipliers come. At the end, we will reach the point when the reduced matrix will have characteristic polynomial $\tilde{Q}(\mu)$.

With the help of Lemma D.3 we get the following structure of the Jacobian in a fixed point with one survival:

$$\begin{bmatrix} \left|\begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right| & \left|\begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right| & \left|\begin{matrix} \boxed{\boldsymbol{F}_1} \\ \vdots \\ \boxed{\boldsymbol{F}_N} \end{matrix}\right| \\ \left|\begin{matrix} A_{1,1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right| & \left|\begin{matrix} B_{1,1} & B_{1,2} & \dots & B_{1,N-1} \\ 0 & B_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N-1,N-1} \end{matrix}\right| & \left|\begin{matrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{matrix}\right| \\ \left|\begin{matrix} R^{x_1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right| & \left|\begin{matrix} 0 & R^{\varphi_2} & \dots & R^{\varphi_{N-1}} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix}\right| & \left|\begin{matrix} C_0 & \dots & C_{L-2} & C_{L-1} \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{matrix}\right| \end{bmatrix}$$

Let us, first, consider the three left blocks of the last matrix. As one can see, all the columns there contain only zero elements except the first column. Therefore, we have $N-1$ zero columns and, consequently $N-1$ zero multipliers. It also implies that all the other multipliers can be founded as the multipliers of the reduced matrix which can be obtain from $\boldsymbol{J}$ by means of deleting all columns and rows corresponding to that zero multipliers.

Now, let us consider the rows in the middle blocks. Obviously, each of that rows has only one non-zero element, moreover, it belongs to the main diagonal of matrix $\boldsymbol{J}$ (and therefore, also of the matrix which we got on the previous step.) As before, it implies, first, that the Jacobian has eigenvalues $B_{2,2}, B_{3,3}, \dots, B_{N-1,N-1}$, and, second that the matrix can be further reduced if we delete rows and columns corresponding to that elements. But in this, reduced matrix, element $B_{1,1}$ is the only one in his row. It is so, because, all elements $B_{1,2}, \dots, B_{1,N-1}$ were deleted on the previous steps. Therefore, also $B_{1,1}$ is a multiplier of the system and its row and column can be deleted. The direct substitution $\varphi_1^* = 1$ and $\varphi_n^* = 0$ for all other n into the expression for $\Phi_n^{\varphi_m}$ provided in Lemma D.2 leads to (D.8).

After all those reductions, the matrix which remains has size $L \times L$. The only non-zero element from first two columns is R^{x_1}. In the row it is followed by the elements $C_0, C_1, \dots, C_n$ computed as a sum in (D.6). In an equilibrium with one survival this sum contains only one term, however, since $R^{f_n} = 0$ for all $n > 1$. Thus, the following matrix remains:

$$\begin{bmatrix} 0 & \frac{\partial f_1}{\partial r_0} & \frac{\partial f_1}{\partial r_1} & \dots & \frac{\partial f_1}{\partial r_{L-2}} & \frac{\partial f_1}{\partial r_{L-1}} \\ R^{x_1} & R^{f_1}\frac{\partial f_1}{\partial r_0} & R^{f_1}\frac{\partial f_1}{\partial r_1} & \dots & R^{f_1}\frac{\partial f_1}{\partial r_{L-2}} & R^{f_1}\frac{\partial f_1}{\partial r_{L-1}} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

This matrix coincides with matrix (B.1) and, therefore, following the same lines as in the proof of Lemma 3.1, we can conclude that expression (D.7) is indeed the characteristic polynomial of this matrix. □

Case of one survival: Proof of Proposition 4.2

Let us compare characteristic polynomial $\tilde{Q}(\mu)$ from Proposition D.1 with characteristic polynomial $Q(\mu)$ which we derived in Lemma 3.1 for the single agent case. Apart from a slight difference in the notation, they are the same, since derivatives R^{x_1} and R^{f_1} computed in Lemma D.1 coincide in equilibrium $\boldsymbol{x^*}$ with derivatives R'_x and R'_f computed in Lemma 3.2.

Statement of Proposition 4.2 immediately follows now. For stability of fixed point $\boldsymbol{x^*}$ it is enough to have all multipliers inside of the unit circle. $L+1$ zeros are, obviously, there. The roots of $\tilde{Q}(\mu)$ will be there as soon as the survival has stable hyperbolic equilibrium. And conditions $|B_{n,n}| < 1$ for all $n \in \{1, \dots, N\}$ are equivalent to (4.19).

Case of many survivals: Proof of Proposition 4.3

Let us now move to the non-generic case when more than one agent survives. In order to proof Proposition 4.3 we have to show that Jacobian matrix (D.1) has at least one multiplier equal to 1. We start from matrix $\boldsymbol{J}$ in (D.1), study its structure with the help of Lemma D.3 and after proceed in the similar recursive way as we did while we proved Proposition D.1.

The simplification of the Jacobian comes from two sources. First, in three left blocks, there are $N-k$ zero columns. These are the last columns of that blocks. Indeed, block $\partial\mathcal{X}/\partial\mathcal{X}$ consists of only zero elements, while blocks $\partial\mathcal{W}/\partial\mathcal{X}$ and $\partial\mathcal{R}/\partial\mathcal{X}$ contain zero elements in the columns corresponding to those agents who did not survive, according to items **4** and **7** of Lemma D.3, respectively. Therefore, we have found $N-k$ zero multipliers and can eliminate the rows and columns corresponding to them.

Second, in each of the last $N-k$ rows in the "middle" part of the Jacobian there is only one non-zero element. Indeed, from item **6** of Lemma D.3, block $\partial\mathcal{W}/\partial\mathcal{R}$ is zero block. Those rows in $\partial\mathcal{W}/\partial\mathcal{X}$ which correspond to those agents who did not survive are zeros because of item **4**. And, finally, item **5** tells that also in block $\partial\mathcal{W}/\partial\mathcal{W}$ only diagonal elements of the last $N-k$ rows are non-zero. Since non-zero elements $B_{k+1,k+1}, \dots, B_{N,N}$ belong to the main diagonal in the Jacobian, we can eliminate corresponding rows and columns.

The Jacobian matrix is reduced to

$$
\begin{bmatrix}
\begin{vmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix} &
\begin{vmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix} &
\begin{Vmatrix} \boxed{\quad \boldsymbol{F}_1 \quad} \\ \vdots \\ \boxed{\quad \boldsymbol{F}_k \quad} \end{Vmatrix} \\
\begin{vmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,k} \\ A_{2,1} & A_{2,2} & \dots & A_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k,1} & A_{k,2} & \dots & A_{k,k} \end{vmatrix} &
\begin{vmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,k} \\ B_{2,1} & B_{2,2} & \dots & B_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k,1} & B_{k,2} & \dots & B_{k,k} \end{vmatrix} &
\begin{vmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{vmatrix} \\
\begin{vmatrix} R^{x_1} & R^{x_2} & \dots & R^{x_k} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix} &
\begin{vmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix} &
\begin{vmatrix} C_0 & \dots & C_{L-2} & C_{L-1} \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{vmatrix}
\end{bmatrix} \tag{D.9}
$$

Let us prove that matrix (D.9) has at least one multiplier equal to 1. (Therefore, our initial Jacobian $\boldsymbol{J}$ also has such unit multiplier.) Coefficients $B_{n,m}$ are equal to $\Phi_n^{\varphi_m}$ and were computed in Lemma D.2. We can simplify those expressions in the fixed point because the investment choices of the first k agents are equal to $x^*_{1\diamond k}$. Moreover, $x^*_{1\diamond k}\,(\bar{e}+r^*) = r^*$. Therefore, we have:

$$B_{n,m} = \delta_{n,m} - \varphi^*_n\,(x^*_{1\diamond k} - x^*_N)\,\frac{\bar{e}+r^*}{1+r^*} \qquad \forall n,m \in \{1,\ldots,k\} \quad . \tag{D.10}$$

Notice that difference $B_{n,m} - \delta_{n,m}$ does not depend on m. It implies that if one subtracts 1 from any diagonal element of matrix (D.9), in a new matrix the k columns in the middle will be identical. Such matrix has zero determinant. It proves the existence of the unit multiplier of matrix (D.9) and also of Jacobian $\boldsymbol{J}$.

References

Anufriev, M., G. Bottazzi and F. Pancotto, (2004) "Price and Wealth Asymptotic Dynamics with CRRA Technical Trading Strategies", *L.E.M. Working Paper Series*, 2004/23, Sant'Anna School for Advanced Studies, Pisa.

Campbell, J.Y. and L.M.Viceira, (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press, Oxford.

Chiarella, C. and X. He, (2001). "Asset Price and Wealth Dynamics under Heterogeneous Expectations", *Quantitative Finance*, 1, 509-526.

Chiarella, C. and T. He, (2002). "Heterogeneous Beliefs, Risk and Learning in a Simple Asset Pricing Model", *Computational Economics*, 19 (1), 95-132.

Kirman, A. (1991). "Epidemics of opinion and speculative bubbles in financial markets", ch.17 in M.Taylor (ed.), *Money and Financial Markets*, Blackwell, Cambridge.

LeBaron B. (2000). "Agent-based Computational Finance: Suggested Readings and Early Research", *Journal of Economic Dynamics and Control*, 24, 679-702.

Levy M., H. Levy and S. Solomon (1994). "A Microscopic Model of the Stock Market: Cycles, Booms, and Crashes", *Economics Letters* 45, 1, 103-111.

Levy M., H. Levy and S. Solomon, (2000). *Microscopic Simulation of Financial Markets*, Academic Press, London.

Zschischang, E. and T. Lux, "Some new results on the Levy, Levy and Solomon microscopic stock market model", *Physica A*, v.291, 563-573.