MULTI-STEP PERTURBATION SOLUTION OF NONLINEAR RATIONAL EXPECTATIONS MODELS

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January 30, 2005

EXTENDED ABSTRACT

Perturbation methods have recently received much attention as methods for accurately and quickly computing numerical solutions of dynamic stochastic economic equilibrium models, both single-agent or rational-expectations models and multi-agent or game-theoretic models. A perturbation method is based on the following three aspects. (1) The goal is to solve a set of nonlinear algebraic equations in terms of the same number of current endogenous variables (dated t), for given values of predetermined endogenous variables (dated t-1 or earlier), predetermined exogenous variables (dated any period), and parameters (constant, hence, undated). The current endogenous variables, predetermined endogenous and exogenous variables, and parameters are, respectively, collected in the decision vector y, the state vector x, and the parameter vector θ. (2) The equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters. (3) The approximate solution function to be computed, $y = \hat{f}(x)$, is a kth-order Taylor series approximation of the exact, unknown, true solution function, $y = f(x)$, and is centered around a point in the state space, x_0, which is usually a steady state of a nonstochastic version of the algebraic equations. The parameters, θ, are implicit in $f(x)$ and $\hat{f}(x)$.

The computed approximate solution is evaluated in a region, R_0, centered at x_0 and can usually achieve maximal numerical accuracy at $x = x_0$, which usually means that the approximation error, $\varepsilon = |f(x)-\hat{f}(x)|$, is about double precision or $\varepsilon \cong 10^{-16}$. If $\hat{f}(x)$ is a kth-order approximation of $f(x)$ in R_0, then, Taylor-series theory says that $\varepsilon = c|x-x_0|^{k+1}$, for $x \in R_0$ and some constant $c > 0$, or, equivalently, in orders of magnitude, $0(\varepsilon) = |x-x_0|^{k+1}$. We call this standard

approach single-step perturbation or SSP. In this paper, we develop and illustrate the multi-step generalization of SSP or MSP.

Think of a data sample over periods t = 1, ..., T. Each (discrete) period t is viewed as a continuous unit-duration interval, starting at the (continuous) moment t-1. At the starting moment, for given predetermined state variables in x_0 and given parameter values, the goal every period is to solve the nonlinear algebraic equations for the period t decision variables in y. Then, the given x_0 and the computed y determine the end-of-period-t predicted value of x. We could take this x as the initial value, x_0, for next period t+1's computations or we could start anew and take the observed period t+1 sample value of x as x_0. We are interested in solving algebraic equations that represent Euler equations and dynamic state equations of nonlinear rational expectations (NLRE) models. Thus, every period t, we wish to apply MSP to compute y, hence, to compute the predicted x, for given x_0.

In SSP, we can think of evaluating at x the computed approximate solution based on x_0 as moving from x_0 to x in "one big step" along the straight-line vector $x-x_0$. By contrast, in MSP we move from x_0 to x along any chosen, continuous, curved-line or connected-straight-line, path in h steps of equal length h^{-1}, where h is a positive integer. We could consider steps of unequal length (Allgower and Georg, 1997), but do not. If at each step we apply SSP, Taylor-series theory says that the approximation error per step is $0(\varepsilon) = h^{-k-1}$, so that the total approximation error in moving away from x_0 to x in h steps is $0(\varepsilon) = h^{-k}$. Thus, MSP has two major advantages over SSP.

First, both SSP and MSP accuracy declines as the approximation point, x, deviates from the initial point, x_0, although only in MSP can the decline be countered by increasing h. Increasing k is much more costly than increasing h, because increasing k requires new derivations of derivatives, more computer programming, more computer storage, and more computer run time. By contrast, increasing h generally requires only more computer run time and often only slightly more, as our experience shows (Chen and Zadrozny, 2004). Once MSP has been programmed, h can be set to any numbers of steps. Zadrozny and Chen (2004) explain the details of implementing MSP, for k = 4, to the static consumer allocation model and Chen and Zadrozny (2004) apply the derived MSP formulas, for k = 4 and h = 100, to an econometric analysis of the mathematically-equivalent producer allocation model with a data sample of 50 periods and obtained about double-precision MSP solution accuracy for every sample period.

Second, in SSP the initial point is usually a nonstochastic steady state but can sometimes also be set up in function space as the known exact solution of a close but simpler model. This "closeness" of a related, simpler, and known solution can be exploited much more explicitly in MSP, when moving away from x_0 toward x. Specifically, in MSP the state space could include parameters, so that the initial point, x_0, would represent the simpler model with the known solution, and the final point, x, would continue to represent the model of

interest. Then, as we would move from the initial x_0 to the final x in h steps, the state variables and parameters would move together from their initial to final values and the model being solved would vary continuously from the simple model to the model of interest. This is exactly what numerical path-following methods are designed to do (Allgower and Georg, 1997).

Both advantages of MSP facilitate repeatedly, accurately, and quickly solving a NLRE model in an econometric analysis, over a range of data values, which could differ enough from nonstochastic steady states of the model of interest to render computed SSP solutions, for a given k, inadequately accurate.

In the present paper, we extend the derivation of SSP for k = 4 in Chen and Zadrozny (2000) to MSP, also for k = 4. As before, we use a mixture of gradient and differential form differentiations (Chen and Zadrozny, 2003, appendix A) to derive the MSP computational equations in conventional linear-algebraic form and illustrate the derived computational equations with a version of the stochastic optimal one-sector growth model. Although in this extension nonstochastic and static models are extended to stochastic and dynamic models, it is the extension from static to dynamic models which is especially challenging. In particular, whereas all computations for static allocation models are linear, dynamic NLRE models, whether stochastic or nonstochastic, also require solving a nonlinear matrix quadratic equation. An accurate and quick method for solving this equation has been developed (Zadrozny, 1998) and has been programmed and successfully used to compute solutions with about double-precision accuracy (Chen and Zadrozny, 2000).

REFERENCES

Allgower, E.L. and K. Georg (1997), "Numerical Path Following," pp. 3-207 in Handbook of Numerical Analysis, Vol. 5, P.G. Ciarlet and J.L. Lions (eds.), Amsterdam, Netherlands: North Holland Publishing Co.

Chen, B. and P.A. Zadrozny (2000), "Perturbation Solution of Nonlinear Rational Expectations Models," paper presented August 2000 at the Stanford Institute for Theoretical Economics, Stanford University, Stanford, CA.

Chen, B. and P.A. Zadrozny (2003), "Higher Moments in Perturbation Solution of the Linear-Quadratic Exponential Gaussian Optimal Control Problem," Computational Economics 21: 45-64.

Chen, B. and P.A. Zadrozny (2004), "Testing Substitution Bias of the Solow-Residual Measure of Total Factor Productivity Using CES-Class Production Functions," paper presented July 2003 at the Conference on Index Number Theory and Measurement of Productivity, Vancouver, Canada.

Zadrozny, P.A. (1998), "An Eigenvalue Method of Undetermined Coefficients for Solving Linear Rational Expectations Models," Journal of Economic Dynamics and Control 22: 1353-1373.

Zadrozny, P.A. and B. Chen (2004), "Multi-Step Perturbation Method for Accurately Computing and Econometrically Evaluating the Cost of Living," working

paper, Division of Price Index Number Research, Bureau of Labor Statistics, Washington, DC.