Asset Pricing with Loss Aversion*

Lars Grüne[†] Willi Semmler[‡]

May 23, 2005

Abstract

Using standard preferences for asset pricing has not been very successful to match asset price characteristics such as the risk-free interest rate, equity premium and the Sharpe ratio to time series data. Behavioral finance has recently proposed more realistic preferences such as preferences with loss aversion to model asset pricing. Research has now started to explore the implications of behaviorally founded preferences for asset price characteristics. Encouraged by some studies of Benartzki and Thaler (1995) and Barberis et al. (2001) we study asset pricing with loss aversion in a production economy. We here employ a stochastic growth model and use a stochastic version of a dynamic programming method with adaptive grid scheme to compute the above mentioned asset price characteristics of a model with loss aversion in preferences. As our results show, a model with loss aversion performs considerably better than pure consumption based asset pricing models.

JEL classifications: C60, C61, C63, D90, G12

keywords: behavioral finance, loss aversion, stochastic growth models, asset pricing and stochastic dynamic programming

---

*We want to thank Blake LeBaron, David Backus and Richard Thaler for helpful communications.

[†]Mathematisches Institut, Fakultät für Mathematik und Physik, Universität Bayreuth, 95440 Bayreuth, Germany, email: lars.gruene@uni-bayreuth.de

[‡]Center for Empirical Macroeconomics, University of Bielefeld, 33501 Bielefeld, Germany; and Dept. of Economics, New School University, email: wsemmler@wiwi.uni-bielefeld.de

1 Introduction

Consumption based asset pricing models with time separable preferences, such as power utility or log utility have been shown to have serious difficulties to match financial market characteristics such as the risk-free interest rate, the equity premium and the Sharpe ratio to time series data. In those models, even if the coefficient of relative risk aversion in the power utility is significantly raised, neither the risk free rate nor the mean equity premium and Sharpe ratio fit the observed data. In particular, the latter two are much too low in the model as compared to the data.

One important concern has been that asset pricing models have often used models with exogenous dividend stream[1] and the difficulties to match stylized financial statistics may have come from the fact that consumption is not endogenized. There is a tradition of asset pricing models that is based on the stochastic growth model with production originating in Brock and Mirman (1972) and Brock (1979, 1982) which endogenizes consumption. The Brock approach extends the asset pricing strategy beyond endowment economies to economies that have endogenous state variables including capital stocks that are used in production. Authors, building on this tradition,[2] have argued that it is crucial how consumption is endogenized. In stochastic growth models the randomness occurs to the production function of firms and consumption and dividends are derived endogenously. Yet, models with production have turned out to be even less successful. Given a production shock, consumption can be smoothed through savings and thus asset market features are even harder to match.[3]

Recent development of asset pricing studies has turned to extensions of intertemporal models conjecturing that the difficulties to match real and financial time series characteristics may be related to the simple structure of the basic model. In order to match better asset price characteristics of the model to the data, economic research has extended the baseline stochastic growth model to include different utility functions, such as non-separable preferences represented for example by habit formation. Moreover, adjustment costs of investment have also been built into the model.[4]

[1]Those models originate in Lucas (1978) and Breeden (1979) for example.

[2]See Rouwenhorst (1995, Akdeniz and Dechert (1997), Jerman (1998), Boldrin, Christiano and Fisher (2001), Lettau and Uhlig (1999) and Hansen and Sargent (2002), the latter in a linear-quadratic economy. The Brock model has also been used to evaluate the effect of corporate income tax on asset prices, see McGrattan and Prescott (2001).

[3]For a recent account of the gap between such models and facts, see Boldrin, Christiano and Fisher (2001), Cochrane (2001, ch. 21), Lettau, Gong and Semmler (2001) and Semmler (2003, chs. 9-10).

[4]For further detailed studies of those extensions see, for example, Campbell and

An enormous effort has been invested into models with time non-separable preferences, such as habit formation models, which allow for adjacent complementarity in consumption. This type of habit specification gives rise to time non-separable preferences where risk aversion and intertemporal elasticity substitution are separated and a time varying risk aversion will arise. The risk aversion falls with rising surplus consumption and the reverse holds for falling surplus consumption. A high volatility of the surplus consumption will lead to a high volatility of the growth of marginal utility and thus to a high volatility of the stochastic discount factor.

Such habit persistence has been introduced in asset pricing models by Constantinides (1990) in order to account for high equity premia. Asset pricing models along this line have been further explored by Campbell and Cochrane (2000), Jerman (1998) and Boldrin et al. (2001). As the literature has demonstrated (Jerman 1998, and Boldrin et al. 2001) one needs not only habit formation but also adjustment costs of investment to reduce the elasticity of the supply of capital. It seems to be both habit persistence and adjustment costs for investment which are needed to generate higher equity premium and Sharpe ratio.

Yet, as it has been shown in Grüne and Semmler (2004b) habit formation models can only slightly improve the equity premium and Sharpe ratio. Even the above mentioned recently developed habit formation model do not generate enough co-variance of consumption growth with asset returns so as to match the data. Models with loss aversion in preferences where the loss or gain in financial wealth affects the agent's welfare do not have to increase the co-variance of consumption growth with asset returns to improve the aforementioned financial market characteristics. The time varying risk aversion, arising from gains and losses in financial wealth, will generate a higher volatility of asset prices, independently of dividend pay-offs, a higher equity premium and Sharpe ratio than the consumption based asset pricing models.

Since accuracy of the solution method is an intricate issue for models with more complicated decision structure, one first have to have sufficient confidence in the accuracy of the solution method when solving such models. In Grüne and Semmler (2004a,b) a stochastic dynamic programming method with flexible grid size is used to solve such models. In that method an efficient and reliable local error estimation is undertaken and used as a basis for a local refinement of the grid in order to deal with regions of steep slopes or other non-smooth properties of the value function (such as non-differentiability). This procedure allows for a global dynamic analysis of deterministic as well

Cochrane (2000), Jerman (1998), Boldrin, Christiano and Fisher (2001) and Cochrane (2001, ch. 21).

as stochastic intertemporal decision problems. [5]

As has been shown in Grüne and Semmler (2004a) the errors, as compared to the analytical solutions, are small. This method has also been applied in Grüne and Semmler (2004b) where the habit formation model is numerically solved. Using this new method there it has been shown that models with habit formation are not sufficient to solve the above discussed financial market puzzles. A similar dynamic programming method is used in this paper and a model of loss aversion, as proposed by Benartzi and Thaler (1995) and Barberis et al. (2001), is reformulated for a production economy and numerically solved which comes much closer to solving the aforementioned financial market puzzles.

The paper is organized as follows. Section 2 discusses related literature. Section 3 presents the model. Section 4 introduce the stochastic dynamic programming algorithm. Section 5 studies our model of loss aversion and reports numerical results. Section 6 evaluates the results in the context of other recent studies. Section 7 concludes the paper.

2 Moving Beyond Consumption Based Asset Pricing Models

As above discussed, the basic problem in matching the asset market features to data using a consumption based model, is that empirically there is a lack of co-variance of consumption growth and asset returns. Consumption based asset pricing models have not been successful to capture the historical average return and volatility in stock returns.[6] Since even a power utility function with large coefficient of relative risk aversion fails to match the consumption based asset pricing model to the data researchers have used more sophisticated utility function.[7] As aforementioned one of the recent specification of a utility function is habit formation. One might think to improve on the equity premium and Sharpe ratio puzzles by building models that increase consumption volatility through increasing the parameter of risk aversion, as in power utility models, or through a time varying risk aversion as in habit formation models. Yet, since empirically the co-variance of consumption growth with asset returns is low this might be a misleading research strategy

[5]For deterministic versions of this paper, see Grüne (1997), Santos and Vigo–Aguiar (1998), and Grüne and Semmler (2004a).

[6]For an extensive account of this failure, see Campbell and Cochrane (2000). For an extensive exploration of the role different types of preferences for asset pricing, see Backus et al. (2004).

[7]For an elaborate overview, see Backus et al. (2004).

to improve the equity premium and Sharpe ratio.

The current research on loss aversion models moves away from consumption based models. The new strategy is to look for the impact of the fluctuation of wealth on the households' welfare, so that the decision on a stochastic portfolio is impacted by both preferences over a consumption stream as well as by changes in financial wealth. In the preferences there will be thus an extra term representing the change of wealth. Furthermore, as prospect theory has taught us, an investor may be much more sensitive to losses than to gains, known as loss aversion. This, in particular, seems to hold if there have been prior losses already. By extending the asset pricing model in this direction one does not need to raise the variance of consumption growth and increase the correlation of consumption growth with asset returns, a feature not to be found in the data anyway.[8]

A low variance of consumption growth but a higher mean and volatility of asset returns, might be achieved by a time varying risk aversion arising from the fluctuation of wealth. The idea is that after an asset price boom the agents may become less risk averse because the gains may dominate any fear of losses. On the other hand, after an asset price fall the agent become more cautions and more risk averse. This way the variation of risk aversion would allow the asset returns to be more volatile than the underlying pay-offs, the dividend payments, a property that Shiller (1991) has studied extensively. Generous dividend payments and an asset price boom makes the investor less risk averse and drives the asset price still higher. The reverse can be predicted to happen if large losses occur. This may give rise to some waves of optimism and pessimism and associated asset price movements.

Habit formation models attempt to increase the equity premium and Sharpe ratio by constructing a time varying risk aversion arising from the change of consumption. This occurs as current consumption moves closer to (or further away) from an (external) habit level for consumption. Risk aversion in models with loss aversion is varying not through surplus consumption, as in the habit formation model, but rather through the fluctuation in financial wealth. Hereby the risk aversion is affected by prior investment experiences. This is likely to produce a substantial equity premium and Sharpe ratio, high volatility of returns, yet it allows for a low co-variance of the growth rate of consumption and asset returns, actually to be found in the data.

Whereas the risk aversion in the habit formation model is finally driven by consumption, this is not so in the loss aversion model, where the changes of risk aversion are driven by changes of the value of assets. In the consumption

[8]See Semmler (2003, ch. 9)

based asset pricing model assets are only risky because they co-vary with consumption. In the loss aversion model changes of risk aversion arise from the fluctuation of asset prices regardless of whether those fluctuations are correlated with consumption growth.

Although the above is the most interesting feature of the loss aversion model, the feedback effect of asset value – and changes of wealth – on preferences on the one hand, and the choice of consumption path on asset value, on the other hand, creates a complicated stochastic dynamic optimization problem that we propose to be solved by a dynamic programming algorithm as presented in Grüne and Semmler (2004a).

Finally, we want to note that the basic idea of loss aversion has been developed in the so-called prospect theory which goes back to Kahneman and Tversky (1979) and Tversky and Kahneman (1992). It has been further developed for applications in asset pricing by Benartzi and Thaler (1995), although there in the context of a single period portfolio decision model, and Barberis et al. (2001) for an intertemporal model of an endowment economy.[9] Yet, without the asymmetry in gains and losses, with prior losses playing an important role, the risk aversion will be constant over time and the theory cannot contribute to the explanation of the equity premium and Sharpe ratio.

3 The Asset Pricing Model with Loss Aversion

In order to formalize the new idea on asset pricing we may follow Barberis et al. (2001) and specify the following preference

$$E\left[\sum_{t=0}^{\infty}\left(\rho^t\frac{C_t^{1-\gamma}}{1-\gamma}+b_t\rho^{t+1}\nu(X_{t+1},S_t,z_t)\right)\right] \qquad (1)$$

The first term in equ. (1) represents, as usual, the utility over consumption, using power utility, ρ is the discount factor and γ, the parameter of relative risk aversion. For $\gamma = 1$ we replace $\frac{C_t^{1-\gamma}}{1-\gamma}$ by the log–utility $\ln C_t$. The second term captures the effect of the change of wealth on the agent's welfare. Hereby X_{t+1} is the change of wealth, S_t, the value of the agent's risky assets. Finally, we want to note that z_t is a variable, measurng the agent's gains or losses prior to period t as fraction of S_t. The variables S_t

[9]Further important literature along this kine is Thaler et al. (1997), Barberis and Huang (2003), Barberis et al. (2004a,b), Barberis and Thaler (2003).

and z_t express the way of how the agent has experiences gains or losses in the past affecting his or her willingness to take risk.

In particularly, it is presumed that

$$X_{t+1} = S_t R_t - S_t R_f \tag{2}$$

which means that the gain or loss $S_t R_t$, with R_t the risky return, R_f the risk free return, is measured relative to a return $S_t R_f$ from a risk-free asset. The difference $R_t - R_f$ can be positive, zero or negative and the variable z_t can be greater, equal or smaller than one, with

$$\nu(X_{t+1}, S_t, 1) = \begin{cases} X_{t+1} & \text{for} \quad X_{t+1} \geq 0 \\ \lambda X_{t+1} & \text{for} \quad X_{t+1} < 0 \end{cases} \tag{3}$$

and $\lambda > 1$ as defined by

$$\lambda(z_t) = \lambda + k(z_t - 1) \tag{4}$$

expressing the fact that a loss is more severe than a gain with $k > 0$, and

$$z_{t+1} = \eta z_t \frac{\overline{R}}{R_{t+1}} + (1 - \eta) \tag{5}$$

with $\eta \in [0, 1]$ and $\overline{R}$ a fixed parameter which is chosen to be the long time average of the risk free interest rate. Moreover, it is presumed that

$$b_t = b_0 \tilde{C}_t^{-\gamma} \tag{6}$$

with b_0, a scaling factor, and $\tilde{C}_t$ some aggregate consumption which will be specified below, so that the price-dividend ratio and the risky asset premium remain stationary. Hereby b_0 is an important parameter indicating the relevance that financial wealth has in utility gains or losses relative to consumption. In case $b_0 = 0$, we recover the consumption based asset pricing model with power utility.

Barberis et al. (2001) employ such a model of loss aversion and asset pricing to two stochastic variants of an endowment economy without production. In the first model variant there is only one stochastic pay-off for the asset holder, a stochastic dividend, whereby dividend pay-offs are always equal to consumption. In the other model variant dividends and consumption follow different stochastic processes.

From the agent's Euler equation for optimality of the equilibrium Barberis et al. (2001) obtain a characterization of a stochastic discount factor for the risk-free rate

$$1 = \rho R_f E_t \left[(\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma} \right] \tag{7}$$

and for the risky asset

$$1 = \rho E_t \left[R_{t+1} (\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma} \right] + b_0 \rho E_t \left[\hat{\nu}(R_{t+1}, z_t) \right] \tag{8}$$

with

$$\hat{\nu}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_{f,t}, & R_{t+1} \geq z_t R_{f,t} \text{ and } z_t \leq 1 \\ (z_t - 1) R_{f,t} + \lambda (R_{t+1} - z_t R_{f,t}), & R_{t+1} < z_t R_{f,t} \text{ and } z_t \leq 1 \\ R_{t+1} - R_{f,t}, & R_{t+1} \geq R_{f,t} \text{ and } z_t > 1 \\ \lambda(z_t)(R_{t+1} - R_{f,t}), & R_{t+1} < R_{f,t} \text{ and } z_t > 1 \end{cases} \tag{9}$$

From (7) we obtain the stochastic discount factor for the risk-free rate, R_f

$$m_{f,t+1} = (\tilde{C}_{t+1}/\tilde{C}_t)^{-\gamma},$$

which coincides with the stochastic discount factor for the consumption based model, see Cochrane (2001, sect. 1.2)

As compared to (7), equ. (8) has two terms. The first term represents the usual one also found in (7), obtained from consumption based asset pricing. The second term expresses the fact that if the agent consumes less today and invests in risky assets the agent is exposed to the risk of greater losses, a risk that is represented by the state variable z_t.

If we consider the cases in (9) seperately, one sees that for each single case the right hand side of (8) is affinely linear in R_{t+1}. More precisely, we can rewrite (8) as

$$1 + \rho b_0 \alpha_2 R_{f,t} = E_t \left[\left(\rho \left(\tilde{C}_{t+1}/\tilde{C}_t \right)^{-\gamma} + b_0 \alpha_1 \right) R_{t+1} \right] \tag{10}$$

with α_1 and α_2 given by

$$\begin{array}{llll} \alpha_1 = 1, & \alpha_2 = 1 & \text{for } R_{t+1} \geq z_t R_{f,t} & \text{and } z_t \leq 1 \\ \alpha_1 = \lambda, & \alpha_2 = (\lambda - 1) z_t + 1 & \text{for } R_{t+1} < z_t R_{f,t} & \text{and } z_t \leq 1 \\ \alpha_1 = 1, & \alpha_2 = 1 & \text{for } R_{t+1} \geq R_{f,t} & \text{and } z_t > 1 \\ \alpha_1 = \lambda(z_t), & \alpha_2 = \lambda(z_t) & \text{for } R_{t+1} < R_{f,t} & \text{and } z_t > 1 \end{array} \tag{11}$$

Using the equation

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

for the risky return with P_t denoting the asset price and D_t the dividend, which we chose equal to $\tilde{C}_t$ in our model and plugging this equation into (10) and using (9) we obtain

$$P_t = E_t \left[\underbrace{\frac{\rho \left(\tilde{C}_{t+1}/\tilde{C}_t\right)^{-\gamma} + \rho b_0 \alpha_1}{1 + \rho b_0 \alpha_2 R_{f,t}}}_{=m_{t+1}} (\tilde{C}_{t+1} + P_{t+1}) \right] \tag{12}$$

Note, again, that for $b_0 = 0$ this equation coincides with the stochastic discount factor for the consumption based model, see Cochrane (2001, sect. 1.2) Note, however, that for $b_0 \neq 0$ in contrast to the consumption based case the stochastic discount factor depends on R_{t+1},[10] which in turn depends on P_{t+1}, thus the right hand side of (11) becomes nonlinear and even discontinuous in P_{t+1}.

In order to generate the consumption $\tilde{C}_t$, we use the basic growth model from Brock and Mirman (1972). This amounts to choosing $\tilde{C}_t$ to be the optimal control of the problem

$$\max_{\tilde{C}_t} E \left(\sum_{t=0}^{\infty} \rho^t \frac{\tilde{C}_t^{1-\gamma}}{1-\gamma} \right) \tag{13}$$

subject to the dynamics

$$k_{t+1} = y_t A k_t^{\alpha} - \tilde{C}_t \tag{14}$$

$$\ln y_{t+1} = \sigma \ln y_t + \varepsilon_t \tag{15}$$

with ε_t being i.i.d. random variables. Here γ is the same as in (1) and as there we replace the utility function by log–utility $\ln \tilde{C}_t$ for $\gamma = 1$. In this case, i.e. for log–utility, the optimal consumption policy is known and is given by

$$\tilde{C}(k_t, y_t) = (1 - \alpha\rho) A y_t k_t^{\alpha}.$$

For $\gamma \neq 1$ we compute $\tilde{C}_t$ numerically.

For this model we want to compute a number of financial measures: The risk free interest rate $R_{f,t}$, the equity return R_{t+1}, the stochastic discount factors m_{t+1} and $m_{f,t+1}$, all of which are specified above. In addition we will compute the Sharpe Ratio given by[11]

---

[10]See equ. (11) where it is visible that R_{t+1} relative to the risk free rate impacts the stochastic discount factor in equ. (12).

[11]See Cochrane (2001)

$$SR = \left| \frac{E(R_{t+1}) - R_{f,t}}{\sigma(R_{t+1})} \right| = \frac{-R_{f,t}\mathrm{Cov}\Big(m_{t+1}, R_{t+1}\Big)}{\sigma(R_{t+1})}. \tag{16}$$

4 Stochastic Dynamic Programming Approach

Our approach to solve the model described above is a stochastic dynamic programming method using the stochastic discount factors m_f and m from the previous section. More precisely, using the state vector

$$x_t = (k_t, \ln y_t, z_t),$$

the equations (14), (15) and (5) for k_{t+1}, $\ln y_{t+1}$ and z_{t+1} define dynamics for x_t which we can write shortly as

$$x_{t+1} = \varphi(x_t, \tilde{C}_t, \varepsilon_t).$$

Now using Bellman's optimality principle the optimal value function V of the problem (13) is characterized by

$$V(x) = \max_{\tilde{C}} E_t \left[\frac{\tilde{C}^{1-\gamma}}{1-\gamma} + \rho V(\varphi(x, \tilde{C}, \varepsilon)) \right] \tag{17}$$

which can be used as the basis of our algorithm.

In contrast to other stochastic dynamic programming methods, here the dynamic programming principle (17) is not sufficient in order to solve the problem, because the dynamics φ depend not only on the state vector x_t, the control $\tilde{C}_t$ and the random variable ε_t, but also on the the risk free interest rate $R_{f,t}$ and on the risky return R_{t+1}, i.e., the equations are *externally coupled*. Since $R_{f,t}$ and R_{t+1} are, in turn, obtained from the stochastic discount factors and from the asset price function P, now the crucial observation is that using m_f and m, the values P_t and $R_{f,t}$ are again characterized by the dynamic programming principles

$$R_{f,t} = E_t\left[m_{f,t+1}\right] \tag{18}$$

$$P_t = E_t\left[m_{t+1}(D_{t+1} + P_{t+1})\right] \tag{19}$$

Hence, we can solve the three equations (17), (18) and (19) in conjunction in order to obtain the solutions V, P and R_f simultaneously as functions of x.

In order to approximate these functions numerically, we chose an appropriate domain $\Omega \subset \mathbb{R}^3$ for our state vector and (in all our examples this was chosen as $\Omega = [1, 4] \times [-0.32, 0.32] \times [0.5, 2]$) and approximate the solutions V, R_f, $P : \Omega \to \mathbb{R}$ as continuous and piecewise multilinear functions on a cuboidal grid Γ on Ω. In order to make this approach efficient, we chose the grid adaptively using the a posteriori error estimation based grid generation technique described in Grüne and Semmler (2004a,b). For each set of parameters we have performed 3 adaptation steps resulting in a grid with ≈ 10000 cuboidal elements and an error of order 10^{-5} (measured accumulated along the optimal trajectories).

Since the equations (17)—(19) are nonlinear due to the external coupling, it is not a priori clear how they can be solved simultaneously. In our approach we have implemented the dynamic programming principles directly using a straightforward fixed point (or value) iteration. With this procedure we could achieve convergence for all considered parameter sets. We note, however, that more sophisticated Newton like techniques may perform better here, these methods are the topic of further research.

Once the asset price function P_t is computed, all financial measures can be directly obtained from this function using a second dynamic programming computation. Since all necessary values for the computation of the risky returns, the risk free interest rates, the stochastic discount factors etc. are stored as a result from the first computation this second step is numerically cheap and took less than 2 seconds in each example. We emphasize that here we can compute all expectations, variances etc. directly in terms of the formulas from section 3 and that all functions appearing in these formulas are x–dependent in our model, e.g., for the current state vector x_t we obtain $R_t = R(x_t)$ and $R_{t+1} = R(\varphi(x_t, \tilde{C}(x_t), \varepsilon_t))$. Thus, for instance, if we want to compute the equity return R_{t+1} for some state vector x_t, by direct numerical quadrature we can compute $E_t(R_{t+1}) = E_t[R(\varphi(x_t, \tilde{C}(x_t), \varepsilon_t))]$. In particular, we do not need to rely on (slowly converging) Monte Carlo simulations along optimal paths in order to compute these quantities; in our approach simulations are only used in order to average the obtained state dependent values along typical sample paths.

5 Presentation of the Results

Using the method described in the previous section we have determined several characteristic values for a number of parameters. Since our method derives these values as functions of k_t, y_t and z_t, in order to obtain representative values we have evaluated them along an optimal trajectory using 50 000 samples and the same random sequence ε_t for each set of parameters.

For the underlying Brock–Mirman model the parameters were chosen as

$$A = 5, \quad \alpha = 0.34, \quad \sigma = 0.9, \quad \rho = 0.95,$$

and ε_t was chosen as a Gaussian distributed random variable with standard deviation $\sigma = 0.008$, which we restricted to the interval $[-0.032, 0.032]$.

For the loss aversion asset pricing model our standard set of parameters was

$$\gamma = 1, \quad \lambda = 10, \quad \eta = 0.9, \quad b_0 = 1 \text{ and } k = 3.$$

In each of the tables below one of these parameters is varied in order to explore the variation of the data with respect to this parameter. The parameter in bold indicates the row containing the standard parameters specified above.

λ	0	1	2.25	**10**	20
R_f	1.05273	1.05273	1.05273	1.05273	1.05273
$\mathrm{Var}(R_f)$	0.00001	0.00001	0.00001	0.00001	0.00001
R_{t+1}	1.05279	1.05280	1.05333	1.05393	1.05405
$\mathrm{Var}(R_{t+1})$	0.00840	0.00843	0.00847	0.00085	0.00856
Sharpe Ratio	0.00798	0.00838	0.07187	0.14134	0.15480
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00006	-0.00003	-0.00003	-0.00002	-0.00002

Table 1: Results for varying λ

b_0	0	0.01	0.05	0.1
R_f	1.05273	1.05273	1.05273	1.05273
$\mathrm{Var}(R_f)$	0.00001	0.00001	0.00001	0.00001
R_{t+1}	1.05279	1.05282	1.05362	1.05396
$\mathrm{Var}(R_{t+1})$	0.00840	0.00840	0.00848	0.00855
Sharpe Ratio	0.00798	0.01114	0.10516	0.14453
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00006	-0.00006	-0.00005	-0.00005

Table 2: Results for varying b_0

b_0	0.5	**1**	3	10
R_f	1.05273	1.05273	1.05273	1.05273
$\mathrm{Var}(R_f)$	0.00001	0.00001	0.00001	0.00001
R_{t+1}	1.05419	1.05393	1.05338	1.05297
$\mathrm{Var}(R_{t+1})$	0.00858	0.00085	0.00879	0.00842
Sharpe Ratio	0.17072	0.14134	0.07438	0.02919
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00003	-0.00002	-0.00001	0.0

Table 3: Results for varying b_0

k	1	**3**	10	150
R_f	1.05273	1.05273	1.05273	1.05273
$\mathrm{Var}(R_f)$	0.00001	0.00001	0.00001	0.00001
R_{t+1}	1.05393	1.05393	1.05393	1.05396
$\mathrm{Var}(R_{t+1})$	0.00854	0.00085	0.00855	0.00856
Sharpe Ratio	0.14128	0.14134	0.14154	0.14463
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00002	-0.00002	-0.00002	-0.00002

Table 4: Results for varying k

γ	**1**	2	3
R_f	1.05273	1.05258	1.05174
$\mathrm{Var}(R_f)$	0.00001	0.00005	0.00005
R_{t+1}	1.05393	1.05505	1.05582
$\mathrm{Var}(R_{t+1})$	0.00085	0.01584	0.02196
Sharpe Ratio	0.14134	0.15985	0.18683
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00002	-0.00010	-0.00020

Table 5: Results for varying γ

ρ	**0.95**	0.98
R_f	1.05273	1.02049
$\mathrm{Var}(R_f)$	0.00001	0.00001
R_{t+1}	1.05393	1.02156
$\mathrm{Var}(R_{t+1})$	0.00085	0.00842
Sharpe Ratio	0.14134	0.12740
$\mathrm{Cov}(m_{f,t+1}, R_{t+1})$	-0.00002	-0.00002

Table 6: Results for varying ρ

As shown in the tables 1-6 the standard parameters generate already reasonably high Sharpe ratios and the $Cov(m_{f,t+1}, R_{t+1})$ is, as one would expect from the data, always very low. The risk-free interest rate has, as one also knows from the data, a low variance, but given that the subjective discount factor is low in our model (the subjective discount rate is high) the mean of the risk-free rate is high. The mean of the risk-free interest rate is, however, considerable reduced if we take $\rho = 0.98$, as in table 6.

6 Interpretation of the Results

It is interesting to compare the numerical results that we have obtained, by using stochastic dynamic programming, to previous quantitative studies undertaken for consumption based asset pricing models (using power utility, habit formation or recursive preferences)[12], but employing other solution techniques. We in particular will restrict ourselves to a comparison with the results of habit formation models obtained by Boldrin et al. (2001) and Jerman (1998) and Grüne and Semmler (2004b).[13]

Whereas Boldrin et al. use a model with log utility for internal habit, but endogenous labor supply in the household's preferences, Jerman studies the asset price implication of a stochastic growth model, also with internal habit formation but, as in Grüne and Semmler (2004b), labor effort is not a choice variable. All three papers Boldrin et al. (2001), Jerman (1998) and Grüne and Semmler (2004b) use adjustment costs of investment in a model with habit formation.

Both, Boldrin et al. and Jerman claim that habit formation models with adjustment costs can match the financial characteristics of the data. Yet, both studies have chosen parameters that appear to be conducive to results which replicate better the financial characteristics such as risk free rate, equity premium and the Sharpe ratio. In comparison to their parameter choice Grüne and Semmler (2004b) have chosen parameters that have commonly been used for stochastic growth models[14] and that seem to describe the first and second moments of the data well. Table 7 reports the parameters and the results.

Both, the study by Boldrin et al. (2001) and Jerman (1998), have chosen a parameter, $\varphi = 4.05$, in the adjustment costs of investment, a very high value

[12]For a comparison of the relative performance of those three types of preferences, see Lettau and Uhlig (2002).

[13]The baseline stochastic growth model and its asset price implications is studied in Grüne and Semmler (2004c).

[14]See Santos and Vigo-Aguiar (1998).

which is at the very upper bound found in the data.[15] Since the parameter φ smoothes the fluctuation of the capital stock and makes the supply of capital very inelastic, we have rather worked with a $\varphi = 0.8$ in order to avoid such strong volatility of returns generated by high φ. Moreover, both papers use a higher parameter for past consumption, b, than Grüne and Semmler (2004b) have chosen. Both papers have also selected a higher standard deviation of the technology shock. Boldrin et al. take $\sigma = 0.018$, and Jerman takes a $\sigma = 0.01$, whereas Grüne and Semmler (2004b) use $\sigma = 0.008$ which has been employed in many models.[16] Those parameters increase the volatility of the stochastic discount factor, a crucial ingredient to raise the equity premium and the Sharpe ratio.

Boldrin et al.[a)]	Jerman[b)]	Grüne and Semmler[c)]	US Data[d)] (1954-1990)
b= 0.73-0.9 φ=4.15 σ= 0.018 $\overline{\rho}$= 0.9 ρ= 0.999 γ= 1	b= 0.83 φ=4.05 σ= 0.01 $\overline{\rho}$= 0.99 ρ= 0.99 γ= 5	b= -0.5 φ= 0.8 σ= 0.008 $\overline{\rho}$= 0.9 ρ= 0.95 γ= 1-3	
$R_f = 1.2$	$R_f = 0.81$	$R_f = 5.1 - 8.5$	$R_f = 0.8$
$E(R) - R_f$ =6.63	$E(R) - R_f$ =5.9	$E(R) - R_f$ =0.33	$E(R) - R_f$ =6.18
SR= 0.36	SR= 0.33	SR=0.057	SR=0.35

a) Boldrin et al.(2001) use a model with endogenous labor supply, log utility for habit formation and adjustment costs.

b) Jerman (1998) uses a model with exogenous labor supply, habit formation with coefficient of RRA of 5, and adjustment costs.

c) Grüne and Semmler (2004b). Note that here the risk-free rate is high, because the subjective discount factor, ρ, is low (which implies a high subjective discount rate).

d) The following financial characteristics of the data are reported in Jerman (1998).

Table 7: Habit formation models

[15] See for example, Kim (2002) for a summary of the empirical results reported on φ in empirical studies.

[16] This value of σ has also been used by Santos and Vigo-Aguiar (1998).

Barberis et al[e])	Grüne and Semmler[f])	US Data[g]) (1954-1990)
$\lambda = 2.25$ $b_0 = 2.0$ $k = 3.0$ $\gamma = 1.0$ $\rho = 0.98$	$\lambda = 10$ $b_0 = 1.0$ $k = 3.0$ $\gamma = 1.0$ $\rho = 0.95(0.98)$	
$R_f = 3.79$	$R_f = 5.3(2.1)$	$R_f = 0.8$
$E(R) - R_f$	$E(R) - R_f$	$E(R) - R_f = 6.18$
$SR = 0.17$	$SR = 0.14(0.13)$	$SR = 0.35$
–	$Cov(m_f, R) = -0.00002$	$Cov(\Delta c, R)$=0.0027

e) Barberis et al. (2001) use a loss aversion variant with exogenous dividends (equal consumption)

f) Note that Grüne and Semmler, this paper, use a model for a production economy with endogenous consumption.

g) Data sources, see table 7, $Cov(\Delta c, R)$ is from Campbell (1999).

Table 8: Loss aversion models

Jerman, in addition, takes a very high parameter of relative risk aversion, a $\gamma = 5$, which also increases the volatility of the discount factor and increases the equity premium when used for the pricing of assets. Jerman also takes a much higher persistence parameter for the technology shocks, a $\overline{\rho} = 0.99$, from which one knows that it will make the stochastic discount factor more volatile too. All in all, both studies have chosen parameters which are known to bias the results toward the empirically found financial characteristics.[17] Grüne and Semmler (2004b) have chosen a model variant with no endogenous labor supply, which, as Lettau and Uhlig (2000) show, is the most favorable model for asset pricing in a production economy, since including labor supply as a choice variable, would even reduce the equity premium and the Sharpe

[17]We also want to remark that both, Jerman and Boldrin et al., do not provide any accuracy test for their procedure that they have chosen to solve the intertemporal decision problem. Boldrin et al. use the Lagrangian multiplier from the corresponding planner's problem to solve for asset prices with no accuracy test for the procedure. Jerman uses a log-linear approach to solve the model and an accuracy test of this procedure is also not provided in the paper. We also want to note that there is a crucial constraint in habit formation models, namely that the surplus consumption has to remain non-negative when the optimal solution, C_t, is computed. As shown in Grüne and Semmler (2004b) this constraint has to be treated properly in the numerical solution method.

ratio.

One is thus inclined to state that previous studies on consumption based asset pricing have not satisfactorily solved the dynamics of asset prices and the equity premium puzzle. There are still puzzles remaining for the consumption-based asset pricing model. At the heart of the consumptions based asset pricing model is the co-variance of consumption growth with asset return, which needs to be improved to get a higher equity premium and Sharpe ratio. Yet as the empirical data show, see table 8, column 3, this co-variance is very low.

On the other hand the models recently developed in behavioral finance using loss aversion, do not have to match consumption growth data with asset returns. Indeed, as table 8 shows, see column 3, the co-variance of consumption growth with asset returns is empirically very low and thus the (negative) co-variance with the growth rate of marginal utility would be low too. Consumption based models attempt to improve this co-variance by employing other preferences (such as power utility with a very large parameter of relative risk aversion, habit formation and recursive utility) but the co-variance does not need to be improved in our asset pricing model. In fact in our loss aversion model we have $Cov(m_f, R) = -0.00002$ which is very small. As can be seen from table 8 our proposed loss aversion model, where gains and losses of wealth also appear in the preferences, produce a time varying risk aversion, a low risk free rate (with low volatility), a high equity premium (with high volatility) and a reasonably high Sharpe ratio.

Moreover, for a discount factor of $\rho = 0.98$ one obtains a risk-free interest rate of approximately 2 percent, which is roughly half of the annual risk-free rate (see table 7, column 4). Thus, our risk-free rate corresponds roughly to a half of an annual risk free rate. If we use a conversion formula developed by Lo (2002)[18] with $SR(q) = \sqrt{q}SR$ with q the period return, then we have approximately an annual Sharpe ratio of 0.20 which is in the vicinity of the actual annual Sharpe ratio as reported in table 8, column 3. The habit formation model in Grüne and Semmler (2004b), see the above table 7, column 3, is the same, in terms of its basic structures and parameters, as the here solved model with loss aversion, see table 8, column 2. One can therefore be quite confident that the loss aversion model produces quantitatively important contributions to the equity premium and Sharpe ratio puzzles.

[18]This is developed for IID returns.

7 Conclusion

Extensive research effort has recently been devoted to study the asset price characteristics, such as the risk-free interest rate, the equity premium and the Sharpe ratio, arising from the stochastic growth model of the Brock type. The failure of the basic model to match the empirical characteristics of asset prices and returns has given rise to numerous attempts to extend the basic model by allowing for different preferences and technology shocks, adjustment costs of investment, the effect of leverage on asset prices and heterogenous households and firms.[19]

In this paper we have gone beyond the consumption based asset pricing model and have studied asset price characteristics when utility is not only obtained from a consumption stream but also the fluctuation of the agent's value of financial wealth affect the utility of the agent. We have presumed, along the recently proposed prospect theory, that agents become loss averse when they had prior experiences of large losses in wealth and they are again hit by a decline in their wealth in the current period. This gives rise, as we have shown in sect. 3 of the paper, to a new form of a stochastic discount factor pricing the income stream.

In the context of this model the agents do not have to experience large losses in current consumption in order to induce them to change asset holdings. In our model, as one finds in time series data, consumption growth is de-linked from asset prices booms and busts and the co-variance of consumption growth and asset returns can, as the empirical data show, indeed be weak. In future research, one thus might want to design empirical estimation strategies that excepts a de-linked relationship of consumption growth and asset returns.

[19]A model with heterogenous firms in the context of a Brock type stochastic growth model can be found in Akdeniz and Dechert (1997) who are able to match, to some extent, the equity premium by building on idiosynchratic stochastic shocks to firms.

References

[1] Akdeniz, L. and W.D. Dechert (1997), Do CAPM results hold in a dynamic economy? Journal of Economic Dynamics and Control 21: 981-1003.

[2] Backus, D.K., Routledge B.R. and S.E. Zin (2004), Exotic preferences for macroeocnomists, mimeo, New York University.

[3] Barberis, N., Huang, M. and T. Santos (2001), Prospect theory and asset prices, in: Quarterly Journal of Economics, vol. CXVI, no. 1: 1-53.

[4] Barberis, N., Huang, M. and R.H. Thaler (2003), Individual preferences, monetary gambles and the equity premium, mimeo, Stanford Unviersity.

[5] Barberis, N. and R.H. Thaler (2003), A survey of behavioral finance, in: Handbook of the Economics of Finance, ed. by G.M. Constantinides, M. Harris and R. Stulz, Elsevier Science B.V

[6] Barberis, N. and M. Huang (2004a), Preferences with frames: a new utility specification that allows for the framing of risks, mimeo, Stanford University.

[7] Barberis, N. and M. Huang (2004b) The loss aversion / narrow framing approach to stock market pricing and participation puzzles, mimeo, Stanford University.

[8] Benartzi, S. and R.H. Thaler (1995), Myopic loss aversion and the equity Premium puzzle, in: The Quarterly Journal of Economics, vol. 110, no. 1: 73-92.

[9] Breeden, D.T. (1979), An intertemporal asset pricing model with stochastic consumption and investment opportunities. Journal of Financial Economics 7: 231-262.

[10] Boldrin, M., L.J. Christiano and J.D.M. Fisher (2001), Habit persistence, asset returns and the business cycle. American Economic Review, vol. 91, 1: 149-166.

[11] Brock, W. (1979) An integration of stochastic growth theory and theory of finance, part I: the growth model, in: J. Green and J. Schenkman (eds.), New York, Academic Press: 165-190.

[12] Brock, W. (1982) Asset pricing in a production economy, in: The Economics of Information and Uncertainty, ed. by J.J. McCall, Chicgo, University pf Chicago Press: 165-192.

[13] Brock, W. and L. Mirman (1972), Optimal economic growth and uncertainty: the discounted case, Journal of Economic Theory 4: 479-513.

[14] Campbell, J.Y. (1999), Asset prices, consumption, and the business cycle, in: Handbook of Macroeconomics, vol. 1C, ed. by J.B. Taylor and M. Woodford, Amsterdam: Elsevier.

[15] Campbell, J.Y. and J.H. Cochrane (2000), Explaining the poor performance of consumption-based asset pricing models, in: The Journal of Finance, vol. LV. no. 6: 2863-2878.

[16] Cochrane, J. (2001), Asset Pricing, Princeton University Press, Princeton.

[17] Constantinides, G.M. (1990), Habit formation: a resolution of the equity premium puzzle, in: Journal of Political Economy 98: 519-543.

[18] Grüne, L. (1997), An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation, Numer. Math., 75: 1288-1314.

[19] Grüne, L. and W. Semmler (2004a), Using dynamic programming with adaptive grid scheme for optimal control problems in economics. Journal of Economic Dynamics and Control, vol 28: 2427-2456.

[20] Grüne, L. and W. Semmler (2004b), Asset pricing - constrained by past consumption decisions, CEM working paper no. 77, Bielefeld University.

[21] Grüne, L. and W. Semmler (2004c), Solving asset pricing models with stochastic dynamic programming, CEM working paper no. 54, Bielefeld university

[22] Hansen, L.P. and T. Sargent (2002) Robust control, book manuscript, Stanford Unversity.

[23] Jerman, U.J. (1998), Asset pricing in production economies, Journal of Monetary Economies 41: 257-275.

[24] Kahneman, D. and A. Tversky (1979), Prospect theory: an analysis of decision under risk. Econometrica 47: 263-291.

[25] Kim, J. (2002), Indeterminacy and investment adjustment costs: an analytical result. Dept. of Economics, Unviersity of Virginia, mimeo.

[26] Lettau, M. and H. Uhlig (1999), Volatility bounds and preferences: an analytical approach, revised from CEPR Discussion Paper No. 1678.

[27] Lettau, M. and H. Uhlig (2000), Can habit formation be reconciled with business cycle facts? Review of Economic Dynamics 3: 79-99.

[28] Lettau, M. and H. Uhlig (2002), The Sharpe ratio and preferences a parametric approach, in: Macroeconomic Dynamics, vol. 6: 242-265.

[29] Lettau, M. G. Gong and W. Semmler (2001), Statistical estimation and moment evaluation of a stochastic growth model with asset market restrictions, Journal of Economic Organization and Behavior, Vol. 44: 85-103.

[30] Lo, A.W. (2002), The statistics of Sharpe ratios, Financial Analysts Journal, July/August: 36-52.

[31] Lucas, R. Jr. (1978), Asset prices in an exchange economy. Econometrica 46: 1429-1446.

[32] McGrattan, E.R. and E.C. Prescott (2001) Taxes, regulation and asset prices. Working paper, Federal Reserve Bank of Minneapolis.

[33] Rouwenhorst, K.G. (1995), Asset pricing implications of equilibrium business cycle models, in: T. Cooley: Frontiers of Business Cycle Research. Princeton, Princeton University Press: 295-330.

[34] Santos, M.S. and J. Vigo–Aguiar (1998), Analysis of a numerical dynamic programming algorithm applied to economic models, Econometrica 66: 409-426.

[35] Semmler, W. (2003), Asset prices, booms and recessions, Springer Publishing House, Heidelberg and New York.

[36] Shiller, R.J. (1991), Market volatility. Cambridge: MIT Press.

[37] Thaler, R.H., Tversky, A., Kahnemann, D. and A. Schwartz (1997), The effect of myopic and loss aversion on risk taking: an experimental test, in: The Quarterly Journal of Economics, May: 647-660.

[38] Tversky, A. and D. Kahneman (1992), Advances in prospect theory: cumulative representation of uncertainty. Journal of Risk and Uncertainty 5: 297-323.