

Option-Pricing in Incomplete Markets: The Hedging Portfolio plus a Risk Premium-Based Recursive Approach*

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Abstract

Consider a non-spanned security C_T in an incomplete market. We study the risk/return trade-offs generated if this security is sold for an arbitrage-free price \widehat{C}_0 and then hedged. We consider recursive “one-period optimal” self-financing hedging strategies, a simple but tractable criterion. For continuous trading, diffusion processes, the one-period minimum variance portfolio is optimal. Let $C_0(0)$ be its price. Self-financing implies that the residual risk is equal to the sum of the one-period orthogonal hedging errors, $\sum_{t \leq T} Y_t(0)e^{r(T-t)}$. To compensate the residual risk, a risk premium $y_t \Delta t$ is associated with every Y_t . Now let $C_0(y)$ be the price of the hedging portfolio, and $\sum_{t \leq T} (Y_t(y) + y_t \Delta t) e^{r(T-t)}$ is the total residual risk. Although not the same, the one-period hedging errors $Y_t(0)$ and $Y_t(y)$ are orthogonal to the trading assets, and are perfectly correlated. This implies that the spanned option payoff does not depend on y .

Let $\widehat{C}_0 = C_0(y)$. A main result follows. Any arbitrage-free price, \widehat{C}_0 , is just the price of a hedging portfolio (such as in a complete market), $C_0(0)$, plus a premium, $\widehat{C}_0 - C_0(0)$. That is, $C_0(0)$ is the price of the option’s payoff which can be spanned, and $\widehat{C}_0 - C_0(0)$ is the premium associated with the option’s payoff which cannot be spanned (and yields a contingent risk premium of $\sum y_t \Delta t e^{r(T-t)}$ at maturity). We study other applications of option-pricing theory as well.

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1 Introduction

In an incomplete market there is not a replicating portfolio for those non-spanned securities, and thus, one cannot apply the law of one price, and obtain a unique solution. On the contrary, there are an upper and a lower arbitrage bound, which contain the non arbitrage prices (see Merton (1973)). One must make further assumptions to select one of these prices, or to constrain the arbitrage bounds.

Consider a non-spanned option C_T with maturity T . Let C_0^- and C_0^+ be the arbitrage bounds. We study the risk/return trade-offs generated if this security is sold for $\widehat{C}_0 \in (C_0^-, C_0^+)$ and then hedged. We consider recursive “one-period optimal” self-financing hedging strategies, a simple but tractable criterion. For continuous trading, diffusion processes, the one-period hedging errors are Gaussian. This implies that the one-period minimum variance portfolio, \widehat{h}_{t+1} , is optimal, and, in this case, is the unique one-period hedging criterion. Let $X_t^{\widehat{h}_{t+1}}$ be its price at time t .

We now connect pricing with hedging. For every $t \in \{0, \Delta t, \dots, T - \Delta t\}$, we recursively define an option price process $C_t(y) = X_t^{\widehat{h}_{t+1}} + y_t \Delta t$, where y_t is a risk premium associated with the one-period hedging error $Y_{t+1}^{\widehat{h}_{t+1}}(y) = X_{t+1}^{\widehat{h}_{t+1}} - C_{t+1}(y)$. Note that $C_t(0)$ and $Y_{t+1}^{\widehat{h}_{t+1}}(0)$ are the option price and the hedging error, respectively, if all risk premiums are zero (i.e., $y = 0$).

Self-financing implies that the residual risk is equal to the sum (financed to the riskless rate r) of the one-period orthogonal hedging errors and their associated risk premiums, $\sum_{t=0}^{T-1} Y_{t+1}^{\widehat{h}_{t+1}}(y) e^{r(T-t)} + \sum_{t=0}^{T-1} y_t \Delta t e^{r(T-t)}$, which can be considered separately. Although the errors $Y_{t+1}^{\widehat{h}_{t+1}}(y)$ and $Y_{t+1}^{\widehat{h}_{t+1}}(0)$ are not the same if $y \neq 0$, they are orthogonal to the trading assets, and are perfectly correlated. This implies that the spanned option payoff does not depend on the process y .

Now, let $\widehat{C}_0 = C_0(y)$. Two main results follow.

1. Any arbitrage-free price, \widehat{C}_0 , is just the price of a hedging portfolio (such as in a complete market), $C_0(0)$, plus a premium, $\widehat{C}_0 - C_0(0)$. That is, $C_0(0)$ is the price of the option’s payoff which can be spanned, and $\widehat{C}_0 - C_0(0)$ is the premium associated with the option’s payoff which cannot be spanned (and yields a contingent risk premium of $\sum_{t=0}^{T-1} e^{r(T-t)} y_t \Delta t$ at maturity).
2. As we do not advocate a specific risk premium y , we do not provide a unique price \widehat{C}_0 . We derive (as \widehat{h}_t is one-period optimal) an optimal frontier in the “non arbitrage option prices/risk premiums” space (i.e., $y \rightarrow C_0(y)$), and it is the final user who provides the risk premium y .

In brief, our model reduces pricing in incomplete markets to the explicit valuation of a one-period orthogonal diffusion risk. One can constrain $C_0(y)$ by constraining y ; parametrize $y(\lambda)$, $\lambda \in \mathcal{R}$; compute the elasticity $\frac{1}{C_0(y)} \frac{dC_0(y(\lambda))}{d\lambda}$; define an upper (lower) bound if $y \geq 0$ ($y \leq 0$).

In addition, we explicitly obtain the latter price decomposition which can be applied to a complete market or to risk-neutral pricing. Moreover, y does not need to depend on a price of risk, differing from a complete market and risk-neutral pricing, a flexibility which can be used to fit volatility smiles.

Our model is tractable because it is based in recursive one-period optimal portfolios, and allows us to study other option-pricing applications such as risk management, American-style payoffs, portfolio constraints, etc. Our model is related to other approaches as will be demonstrated.

There are important pricing approaches in incomplete markets, including the equilibrium or utility maximization-based approach (see Rubinstein (1976)); to consider prices of risk associated with non-traded state variables (see Heston (1993)); to compute an optimal hedging portfolio, whose price is then the desired incomplete market price (Merton (1998)); to use a risk/reward criterion, such as the gains-to-losses ratio (Bernardo and Olivier (2000)) or the Sharpe ratio (Cochrane and Saá-Requejo (2000)). See also Carr et al. (2001), Cerný (2003), and references therein for other models.

Next, we describe the main results of our approach. First, recursive prices are consistent with recursive one-period optimal self-financing hedging strategies. A related approach is that based on fully optimal dynamic strategies.¹ This literature has indeed raised the issue of partially or fully optimal hedging, and focuses more on hedging than pricing. The main limitation of a fully optimal criterion is tractability. In our approach, e.g., multifactor models are feasible to analyze.

Second, the residual risk is equal to the sum of the one-period errors plus the risk premiums, and can be used for risk management (to compute moments, tails, etc.). In addition, it conveys a contingent risk premium. If we specify that larger one-period risk premiums are associated with more volatile or risky hedging errors, then a sample path of large (small) one-period hedging errors is associated, in probability, with a path of large (small) risk premiums. Therefore, these risk premiums are a risk management tool. Further, the residual risk can be used empirically to study market incompleteness, to extract prices of risk, or to study risk/return trade-offs in derivative markets.

Third, in continuous-time, diffusion processes, the hedging strategy is unique and thus recursive prices differ only in the one-period risk premiums. This result allows us to relate our model to option-pricing theory. In a complete market, the risk premium of every state variable depends on a price of risk to avoid arbitrage. Here, this constraint does not necessarily apply to the residual risk.

This pricing flexibility differentiates a complete market, and risk-neutral pricing, from an incomplete market, and allows us to better fit volatility smiles in option markets. (Certainly, we are pricing securities separately. If, later, these securities are traded, these risk premiums will depend on a price of risk.) In Merton (1998) and Cochrane and Saá-Requejo (2000), which are recursive methods also, the risk premium y_t is equal to zero and is proportional to the residual risk volatility, respectively.²

Fourth, recursive prices are easily characterized through PDE's equations or a risk-neutral dy-

¹See Duffie and Richardson (1991), Schweizer (1992), Heath et al. (2001), Bertsimas et al. (2001), among others; e.g., Duffie and Richardson (1991) study lognormal processes. Heath et al. (2001) focus on stochastic volatility models.

²Merton (1998, 333) argues that the one-period risk premium should be zero, since the residual risk is orthogonal with the traded assets and therefore with the equilibrium market portfolio. Equilibrium models, however, are not empirically supported in general. I thank Robert L. McDonald for pointing out this reference.

namic. Then, option-pricing applications are like those of continuous-time complete markets models, studied by the literature in detail. For example, both the hedging portfolio and the volatility of the residual risk depend linearly on the option's Deltas, which can be used for risk management.³

Fifth, recursive prices are equal to the price of a hedging portfolio, $C_0(0)$, plus a premium, $\widehat{C}_0 - C_0(0)$. We explicitly obtain this decomposition. $C_0(0)$ is given by a proper risk-neutral expectation of the discounted option payoff. This pricing measure is the so-called Minimal Martingale measure, Q^h . This result allows us to explicitly separate the pricing of the risk which can be hedged from the pricing of the risk which cannot be hedged in a dynamic context.

We derive further results from this decomposition. Let r be the risk-free interest rate. Then,

$$\widehat{C}_0 = E_0^{Q^h} [e^{-rT} C_T] + E_0^{Q^h} \left[\int_0^T e^{-rt} y_t dt \right].$$

The first part is the price of the hedging portfolio. The second is the premium which depends on y . Let y depend on $\lambda \in \mathcal{R}$. The option price elasticity is given by $\frac{dC_0}{d\lambda} = \frac{d}{d\lambda} E_0^{Q^h} \left[\int_0^T e^{-rt} y_t(\lambda) dt \right]$.

By using this decomposition, we can define an upper and a lower bound by

$$C_t = E_t^{Q^h} \left[e^{-r(T-t)} C_T \right] + a E_t^{Q^h} \left[\int_t^T e^{-r(s-t)} y_s ds \right],$$

where $a = +1$ ($a = -1$) for the upper (lower) bound. Assume that $y_s \geq 0$, $s \in [0, T]$. The upper bound is larger than the lower bound. Moreover, under technical conditions, under the Q^h probability measure, the discounted upper (lower) bound is a super-martingale (sub-martingale), and the discounted price of the hedging portfolio, $E_t^{Q^h} [e^{-rT} C_T]$, is the martingale component. This result is related to Ross (1978) and Harrison and Kreps (1979), but in incomplete markets.⁴

We are also interested in option-pricing from a portfolio perspective. In a complete market, the price of a portfolio of (any kind of) n securities is equal to the sum of the n individual prices, as linear pricing. In an incomplete market, the portfolio can be less expensive if there is some diversification. We apply the decomposition to a portfolio of n European securities, $C_T^p = (C_T^{(1)}, C_T^{(2)}, \dots, C_T^{(n)})$,

$$\begin{aligned} \sum_{i=1}^n C_0^{(i)} &= \sum_{i=1}^n E_0^{Q^h} [e^{-rT} C_T^{(i)}] + \sum_{i=1}^n E_0^{Q^h} \left[\int_0^{T^{(i)}} e^{-rt} y_t^{(i)} dt \right] \quad \text{and} \\ C_0^p &= \sum_{i=1}^n E_0^{Q^h} [e^{-rT} C_T^{(i)}] + E_0^{Q^h} \left[\int_0^T e^{-rt} y_t^p dt \right], \end{aligned}$$

where $y_t^{(i)}$ and y_t^p are the risk premium of every security and of portfolio p , respectively. Both prices differ only in the valuation of the residual risk. If the underlying securities are not (instantaneously) perfectly correlated, there is some diversification in the (instantaneous) residual risk of portfolio p , and consequently, $y_t^p \leq \sum_{i=1}^n y_t^{(i)}$ is a sensible specification for all t . That is, C_0^p is cheaper.

³See also Bertsimas et al. (2000) and Heath et al. (2001) for more on multiperiod residual risks.

⁴In a complete market, a decomposition result, in an equivalent European option plus an early exercise premium, also holds for American-style securities which are super-martingales under the pricing measure (see Carr et al. (1992)).

American-style securities and portfolio constraints are problems which have not been addressed in incomplete markets in general. For instance, one must rely on numerical methods to price American options even in a complete market. A recursive multiperiod model is a series of one-period models, easily to analyze. We adapt the one-period definition to American options. The one-period portfolio hedges the maximum of the option value and the exercise value in the next period. The same way, the one-period risk premium compensates the remaining one-period residual risk.⁵

Assume short-selling constraints. We can derive a volatility smirk for in-the-money put options in the standard Black-Scholes and Cox-Ross-Rubinstein binomial models. In-the-money put options cannot be perfectly hedged (especially short-term options) if there are short-selling constraints. It is an example of market incompleteness due to market frictions. Thus, it is intuitive to assume a positive risk premium associated with the residual risk. The deeper the option in-the-money, the larger the residual risks and consequently the larger the risk premiums and the volatility smirk.

The paper is organized as follows. Section 2 presents the one-period incomplete market model. Section 3 studies recursive bounds in multiperiod discrete-time incomplete markets. Section 4 studies recursive bounds in continuous-time incomplete markets for diffusion processes, and Section 5 solves a few examples under basis risk, stochastic volatility, and short-selling constraints. Section 6 concludes.

2 The One-period Model

Assume the standard one-period model of financial economics. Let t be the initial period and $t + 1$ be the final period. Let K be the number of states, $\Omega = \{\omega_1, \omega_2, \dots, \omega_K\}$ the state space, and P_t the true probability measure, with $P_t(\omega) > 0$ for all $\omega \in \Omega$. There exists a risk-free asset with price process $S_t^0 = 1$ and $S_{t+1}^0 = 1 + r > 0$, and N risky assets with initial prices $S_t = \{S_t^1, S_t^2, \dots, S_t^N\}$ and final prices $S_{t+1} = \{S_{t+1}^1, S_{t+1}^2, \dots, S_{t+1}^N\}$, which are defined on the space state Ω . Assume that there are no arbitrage opportunities. The objective is to determine the price C_t of a contingent claim, with a final-period payoff given by C_{t+1} .

Let H be the portfolio's or trading strategy's space. In particular, there are no constraints on this space (i.e., $H = \mathcal{R}^{N+1}$). Let $h_{t+1} = (h_{t+1}^0, h_{t+1}^1, \dots, h_{t+1}^N)$, for $h_{t+1} \in H$ and h_{t+1} chosen at time t , be a (hedging) portfolio with value process $X^h = \{X_t^h, X_{t+1}^h\}$; i.e., $X_t^h = h_{t+1}^0 + \sum_{n=1}^N h_{t+1}^n S_t^n$ and $X_{t+1}^h = h_{t+1}^0(1 + r) + \sum_{n=1}^N h_{t+1}^n S_{t+1}^n$. Assume that this market is incomplete (i.e., $K > N + 1$) and that the payoff C_{t+1} is not replicable. Equivalently, there does not exist a portfolio h_{t+1} such that $X_{t+1}^h(\omega) = C_{t+1}(\omega)$ for all $\omega \in \Omega$. Let C_t^- and C_t^+ be the two arbitrage bounds of this security, which solve two linear programs (see Ingersoll (1987) or Pliska (1997)). Then the price C_t must satisfy that $C_t^- < C_t < C_t^+$ to avoid arbitrage opportunities.⁶

⁵American options can be priced by simulation (e.g., Longstaff and Schwartz (2001)) using the risk-neutral dynamic.

⁶An example and source of market incompleteness are portfolio constraints and transaction costs. However, to

2.1 A Hedging Portfolio plus a Risk Premium-Based Approach

Let Y_{t+1}^h , the hedging error or residual risk produced by the portfolio h_{t+1} , be defined as

$$Y_{t+1}^h = X_{t+1}^h - C_{t+1}. \quad (1)$$

Let \hat{h}_{t+1} be an optimum portfolio associated with a hedging criterion $f(Y_{t+1}^h)$; i.e.,

$$\hat{h}_{t+1} = \arg \min_{\{h_{t+1}\}} f(Y_{t+1}^h), \quad (2)$$

and $X_t^{\hat{h}}$ its price. We do not specify the function $f(\cdot)$.

It is convenient to assume (we do it in the next subsection) that \hat{h}_{t+1} satisfies $C_t^- < X_t^{\hat{h}} < C_t^+$, which is equivalent to that the hedging error $Y_{t+1}^{\hat{h}}$ is positive for some states and negative for others if the model is arbitrage free, as we assume. For example, a hedging portfolio with a zero expected hedging error (i.e., $E_t^P [Y_{t+1}^h] = 0$) satisfies this constraint as it has both positive and negative errors.

Let y_t be a risk premium associated with $Y_{t+1}^{\hat{h}}$. Assume that y_t does not introduce arbitrage opportunities, i.e., $C_t^- - X_t^{\hat{h}} < y_t < C_t^+ - X_t^{\hat{h}}$. Equivalently, $C_t^- < X_t^{\hat{h}} + y_t < C_t^+$ and thus $X_t^{\hat{h}} + y_t$ is an arbitrage free price. In particular, y_t can be equal to zero if this risk is not priced.

Let $C_t^- < X_t^{\hat{h}} + y_t < C_t^+$. This paper defines the incomplete market price of C_t as the price of the hedging portfolio, $X_t^{\hat{h}}$, plus the risk premium, y_t ; i.e.,

$$C_t = X_t^{\hat{h}} + y_t. \quad (3)$$

Note that the price C_t in equation (3) is computed in two steps. First, $X_t^{\hat{h}}$ corresponds formally with the application of the law of one price nonarbitrage condition if we assume that the residual risk is zero. Second, we add the risk premium y_t to compensate the residual risk $Y_{t+1}^{\hat{h}}$.

Moreover, y_t should be invested in riskless bonds to not change the properties of the optimal portfolio \hat{h}_{t+1} . Again, we define the hedging strategy as $\hat{h}_0 + y_t$ bonds and \hat{h}_n risky assets for $n = 1, 2, \dots, N$. Therefore, the total risk assumed by the writer of this security is

$$X_{t+1}^{\hat{h}} - C_{t+1} + y_t(1+r) = Y_{t+1}^{\hat{h}} + y_t(1+r). \quad (4)$$

In an incomplete market, one is interested in defining two bounds, an upper (lower) bound, C_t^s (C_t^l), obtained when hedging the short (long) position; i.e., $-C_{t+1}$ (C_{t+1}). Moreover, these bounds should satisfy $C_t^s \geq C_t^l$ to make economic sense. Consider two optimal hedging portfolios, $\hat{h}_{t+1}(s)$ and $\hat{h}_{t+1}(l)$, and two risk premiums, y_t^s and y_t^l , associated with the hedging errors $Y_{t+1}^{\hat{h}(s)} = (X_{t+1}^{\hat{h}(s)} - C_{t+1})$ and $Y_{t+1}^{\hat{h}(l)} = -(X_{t+1}^{\hat{h}(l)} - C_{t+1})$ for the short and the long position's, respectively. Then these two bounds can be defined as in equation (3); i.e.,

$$C_t^s = X_t^{\hat{h}(s)} + y_t^s \text{ and } C_t^l = X_t^{\hat{h}(l)} - y_t^l. \quad (5)$$

formalize these cases (in H), we have to include new assumptions and definitions, and we lose the simplicity of the actual formulation. We prefer to deal with these problems on a case-by-case basis as we show in the examples below.

In particular, $X_t^{\hat{h}(s)} \geq X_t^{\hat{h}(l)}$ and $y_t^s \geq -y_t^l$ are sufficient conditions for $C_t^s \geq C_t^l$. For example, $\hat{h}_{t+1} = \hat{h}_{t+1}(s) = \hat{h}_{t+1}(l)$ and $y_t = y_t^s = y_t^l \geq 0$, i.e., the same portfolio and the same nonnegative risk premium, imply that $C_t^s = X_t^{\hat{h}} + y_t \geq C_t^l = X_t^{\hat{h}} - y_t$.

2.2 A Risk-Neutral Pricing Formulation

In standard frictionless markets, for both complete and incomplete markets, an arbitrage free price can be expressed as an expectation under a risk-neutral probability measure (henceforth, RNP measure). By using RNP measures also, we are going to derive a related but novel result, for which its importance is evident in the multi-period model.

Let Q_t be a RNP measure, and $E_t^Q[\cdot]$ be the conditional expectation operator. Q_t satisfies

$$S_t^n = E_t^Q \left[\frac{S_{t+1}^n}{1+r} \right], \quad n = 1, 2, \dots, N. \quad (6)$$

Recall the implications of a RNP measure $Q_t > 0$ (see, e.g., Pliska (1997)). First, the existence of Q_t is equivalent to nonarbitrage. Second, the uniqueness of Q_t is equivalent to market completeness. Third, let $C = \{C_t, C_{t+1}\}$ be the value process of an arbitrary security. Then, if $C_t = E_t^Q \left[\frac{C_{t+1}}{1+r} \right]$, C_t is an arbitrage-free price. In particular, if Q_t is unique, then C_t is the unique arbitrage-free price. Therefore, Q_t is a tool that allows us to compute the price C_t as a simple risk-neutral expectation. This last point is the one that is important in the present nonarbitrage, incomplete market context.

Recall that portfolio \hat{h}_{t+1} satisfies $C_t^- < X_t^{\hat{h}} < C_t^+$, then there is a RNP measure $Q_t^{\hat{h}}$ such that

$$X_t^{\hat{h}} = E_t^{Q_t^{\hat{h}}} \left[\frac{C_{t+1}}{1+r} \right], \quad (7)$$

where the notation $Q_t^{\hat{h}}$ highlights the dependence on portfolio \hat{h} . That is, $Q_t^{\hat{h}}$ allows us to compute the price of the hedging portfolio $X_t^{\hat{h}}$ by the risk-neutral expectation of the discounted payoff C_{t+1} . Consequently, from equation (3), the incomplete market price of C_t can also be expressed as

$$C_t = E_t^{Q_t^{\hat{h}}} \left[\frac{C_{t+1}}{1+r} \right] + y_t. \quad (8)$$

In sum, from equations (5) and (7),

$$C_t = X_t^{\hat{h}} + ay_t = E_t^{Q_t^{\hat{h}}} \left[\frac{C_{t+1}}{1+r} \right] + ay_t, \quad (9)$$

where $a = +1$ (-1) for the short (long) position and upper (lower) bound.

On the other hand, if $y_t \neq 0$, there exists a different RNP measure $Q_t^{\hat{h},y}$ such that

$$C_t = X_t^{\hat{h}} + y_t = E_t^{Q_t^{\hat{h},y}} \left[\frac{C_{t+1}}{1+r} \right], \quad (10)$$

which can be used for pricing purposes, or to prove that C_t is arbitrage free. Note that $Q_t^{\hat{h}}$ and $Q_t^{\hat{h},y}$ depend on the payoff C_{t+1} . Therefore, different from a complete market where the unique RNP measure prices any security, in our model $Q_t^{\hat{h}}$ and $Q_t^{\hat{h},y}$ may only price the security C_{t+1} .

Remark 1. In addition to the definition of the price of an arbitrary security in incomplete markets in equation (3), equation (8) is the main result of the one-period model. The measure $Q_t^{\hat{h}}$ will allow us to derive an important result on the decomposition of the price C_t in a proper hedging portfolio plus a multiperiod risk premium in multiperiod markets.

Remark 2. Although we have assumed a frictionless market, the definition of equation (3) is independent of market frictions such as portfolio constraints or transaction costs. For equations (7) (and (8)) to hold, it is necessary to find a probability measure which allows us to compute the price $X_t^{\hat{h}}$ as the discounted expectation of C_{t+1} . For a frictionless market, we have shown (since $C_t^- < X_t^{\hat{h}} < C_t^+$) that $Q_t^{\hat{h}}$ is a RNP measure. For a friction market, we prefer to study this problem on a case-by-case basis.

Remark 3. The results of the one-period model do not depend on a specific portfolio, h_{t+1} , and on a specific risk premium, y_t . We only assume that C_t is arbitrage free. In an incomplete market and, in practice, there can be different functions for \hat{h} and y , which depend of the problem of interest. They can depend on the model's statistical properties (such as jumps, skewness or kurtosis) or on economic factors (such as default, initial wealth or regulation issues). Note that y_t can also be studied in an equilibrium or portfolio context (e.g., if the hedging error can be diversified; see Merton (1976)). This is relevant in practice, because C_t can be just a derivative from a larger portfolio of securities.⁷

Examples of option pricing in incomplete markets are left for the continuous-time model.

3 The Multiperiod Model

Consider a discrete-time multiperiod model with initial time 0, final time T , and M trading dates such that $t = 0, 1, \dots, M - 1$, and $\Delta t = \frac{T}{M}$. This multiperiod model is defined over a probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, with $\Omega = \{\omega_1, \omega_2, \dots, \omega_K\}$ finite, where the stochastic processes S_t^n are adapted and the hedging strategies h_{t+1}^n are predictable with regard to the filtration \mathcal{F}_t , respectively, for $t = 0, 1, \dots, M$ and $n = 0, 1, \dots, N$. The risk-free asset corresponds with a “bank account” with value process $S^0 = \{S_0^0, S_1^0, \dots, S_M^0\} = \{1, e^{r\Delta t}, \dots, e^{rT}\}$.⁸

Assume that the model is arbitrage free and incomplete. The objective is to price a contingent claim C_0 , whose payoff C_M occurs in the last period and is not replicable.

Let $h = \{h_1, h_2, \dots, h_M\}$ be a self-financing dynamic portfolio with value process $X^h = \{X_0^h, X_1^h, \dots, X_M^h\}$, where $X_0^h = \sum_{n=0}^N h_1^n S_0^n$ and $X_t^h = \sum_{n=0}^N h_t^n S_t^n$ for $t = 1, 2, \dots, M$. The asterisk denotes

⁷See Hansen and Jaganathan (1991), Cerný and Hodges (2001), Jaschke and Kücheler (2001), and Longarela (2001) for other models of incomplete markets. See Artzner et al. (1999) for risk measures.

⁸See Pliska (1997, chapter 3) for details. In particular, the same results hold if the interest rate r is a predictable stochastic proces, i.e., the one-period short-term interest rate r_{t+1} is \mathcal{F}_t -measurable.

discounted values. It is well known that a portfolio h is self-financing if it holds that

$$X_M^{h*} = X_0^{h*} + \sum_{t=0}^{M-1} \Delta X_{t+1}^{h*}, \quad (11)$$

where $X_t^{h*} = e^{-rt\Delta t} X_t^h$, $\Delta X_{t+1}^{h*} = \sum_{n=1}^N h_{t+1}^n \Delta S_{t+1}^{n*}$, and $\Delta S_{t+1}^{n*} = e^{-r(t+1)\Delta t} (S_{t+1}^n - e^{r\Delta t} S_t^n)$ is the discounted gain process for every risky asset $n = 1, 2, \dots, N$.

3.1 A Hedging Portfolio plus a Risk Premium-Based Recursive Approach

For a self-financing portfolio h , the hedging error is defined by $Y_T^h = a(X_M^h - C_M)$, where $a = +1$ ($a = -1$) is the short (long) position. Let us rewrite this hedging error, which is important for deriving the optimal hedging portfolio that follows. That is, let $Y_T^{h*} = e^{-rT} Y_T^h$ be written as

$$Y_T^{h*} = a(X_M^{h*} - C_M^*) = a(X_0^{h*} - C_0^*) + \sum_{t=0}^{M-1} a(\Delta X_{t+1}^{h*} - \Delta C_{t+1}^*) = \sum_{t=0}^{M-1} \Delta Y_t^{h*}, \quad (12)$$

where $C = \{C_0, C_1, \dots, C_M\}$ is a \mathcal{F}_t -adapted stochastic process (to be specified below, except for the maturity payoff C_M). $\Delta C_t^* = C_t^* - C_{t-1}^*$, $C_t^* = e^{-rt\Delta t} C_t$, and $\Delta Y_{t+1}^{h*} = e^{-r(t+1)\Delta t} \Delta Y_{t+1}^h = a(\Delta X_{t+1}^{h*} - \Delta C_{t+1}^*)$ for $t = 0, 1, \dots, M-1$ and $\Delta Y_0^h = a(X_0^h - C_0)$. Consequently, the total hedging error, Y_T^{h*} , can be understood as the sum of one-period replication errors ΔY_t^{h*} , $t = 0, 1, \dots, M$.

In practice, to find an optimum dynamic portfolio $\hat{h} = \{\hat{h}_1, \hat{h}_2, \dots, \hat{h}_M\}$ which minimizes a proper hedging criterion $f(Y_T^h)$ could be difficult. Moreover, the risk of the hedging error and its associated risk premium, could also be difficult to quantify. Therefore, because of tractability (see footnote 1) we consider only recursive bounds, which allows us to study general problems in incomplete markets.

The optimal hedging criterion

$$\min_{\{h_1, h_2, \dots, h_M\}} f(Y_T^h) = \min_{\{h_1, h_2, \dots, h_M\}} f\left(\left(\Delta Y_0^{h*} + \Delta Y_1^{h*} + \dots + \Delta Y_M^{h*}\right) e^{rT}\right) \quad (13)$$

is changed as follows. First, consider a more simple recursive series of one-period hedging problems⁹

$$\min_{\{h_{t+1}\}} f(\Delta Y_{t+1}^h) = \min_{\{h_{t+1}\}} f\left(a(X_{t+1}^h - C_{t+1}) - a(X_t^h - C_t) e^{r\Delta t}\right) \text{ for } t = M-1, M-2, \dots, 0, -1, \quad (14)$$

where we define $X_{-1}^h = C_{-1} = 0$. Second, for every t define that $X_t^h = C_t$ from the application of the law of one price (since otherwise (14) produces an additional hedging error, which can be hedged

⁹Note that

$$\begin{aligned} \Delta Y_{t+1}^h &= a(\Delta X_{t+1}^{h*} - \Delta C_{t+1}^*) e^{r(t+1)\Delta t} = a\left(\sum_{n=1}^N h_{t+1}^n \Delta S_{t+1}^{n*} - \Delta C_{t+1}^*\right) e^{r(t+1)\Delta t} \\ &= a\left(\sum_{n=0}^N h_{t+1}^n S_{t+1}^n - e^{r\Delta t} \sum_{n=0}^N h_{t+1}^n S_t^n\right) - a(C_{t+1} - e^{r\Delta t} C_t) = a(X_{t+1}^h - C_{t+1}) - a(X_t^h - C_t) e^{r\Delta t}. \end{aligned}$$

with the bank account). In particular, for $t = -1$, $X_0^h = C_0$. Then, the problems to solve are

$$\min_{\{h_{t+1}\}} f\left(\Delta Y_{t+1}^h\right) = \min_{\{h_{t+1}\}} f\left(a\left(X_{t+1}^h - C_{t+1}\right)\right) \text{ for } t = M-1, M-2, \dots, 0, \quad (15)$$

which are similar to the previous one-period problem, wherein \hat{h} is computed.

Therefore, this problem is solved recursively from $t = M-1$ until $t = 0$. For every period t , taking as given the previously solved C_{t+1} , two items are computed: the portfolio \hat{h}_{t+1} , which solves equation (15), and the option price $C_t = X_t^{\hat{h}_{t+1}} + ay_t\Delta t$, where $y_t\Delta t$ is the risk premium per period associated with $\Delta Y_{t+1}^{\hat{h}_{t+1}}$. Note that this recursion is well-defined since C_M is known at maturity, and that C_t is \mathcal{F}_t -adapted. Note also that these recursive prices (i.e., the value process) $C = \{C_0, C_1, \dots, C_{M-1}\}$ are arbitrage free if and only if each price determined in each one-period model is arbitrage free (equivalently it does exist a RNP measure, see Pliska (1997)).

However, this recursive portfolio \hat{h} derived from equation (15) is not self-financing if the risk premium $y_{t+1} \neq 0$, or if the hedging error $\Delta Y_{t+1}^{\hat{h}} \neq 0$, since \hat{h}_{t+1} and \hat{h}_{t+2} are chosen in two independent steps. The new notation $X_t^{\hat{h}_{t+1}}$ (instead of $X_t^{\hat{h}} = \sum_{n=0}^N \hat{h}_t^n S_t^n$) is to distinguish between $X_{t+1}^{\hat{h}_{t+1}} = \sum_{n=0}^N \hat{h}_{t+1}^n S_{t+1}^n$ and $X_{t+1}^{\hat{h}_{t+2}} = \sum_{n=0}^N \hat{h}_{t+2}^n S_{t+1}^n$, and applies only to this non-self-financing portfolio. Consequently, the non-self-financing portfolio \hat{h} must be changed to a self-financing portfolio denoted by \tilde{h} and with value process $X^{\tilde{h}}$. Recall the definitions $C_t = X_t^{\hat{h}_{t+1}} + ay_t\Delta t$ and $\Delta Y_{t+1}^{\hat{h}_{t+1}} = a\left(X_{t+1}^{\hat{h}_{t+1}} - C_{t+1}\right)$ for $t = 0, 1, \dots, M-1$, and note that $a\Delta Y_{t+1}^{\hat{h}_{t+1}} = \left(X_{t+1}^{\hat{h}_{t+1}} - C_{t+1}\right)$ since $a^2 = 1$. Recall that $X_t^{h*} = e^{-rt\Delta t} X_t^h$ is the discounted value.

That is, at the initial time $t = 0$,

$$\begin{aligned} X_0^{\tilde{h}*} &= C_0^* = X_0^{\hat{h}_1^*} + ay_0^*\Delta t, \text{ and} \\ \tilde{h}_1^n &= \hat{h}_1^n \text{ for } n = 1, 2, \dots, N \text{ and } \tilde{h}_1^0 = \hat{h}_1^0 + ay_0^*\Delta t. \end{aligned}$$

At time $t = 1$,

$$\begin{aligned} X_1^{\tilde{h}*} &= X_1^{\hat{h}_1^*} + ay_0^*\Delta t = X_1^{\hat{h}_1^*} + ay_0^*\Delta t - C_1^* + \left(X_1^{\hat{h}_2^*} + ay_1^*\Delta t\right) \\ &= X_1^{\hat{h}_2^*} + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + y_1^*\Delta t\right), \text{ and} \\ \tilde{h}_2^n &= \hat{h}_2^n \text{ for } n = 1, 2, \dots, N \text{ and } \tilde{h}_2^0 = \hat{h}_2^0 + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + y_1^*\Delta t\right). \end{aligned}$$

At time $t = 2$,

$$\begin{aligned} X_2^{\tilde{h}*} &= X_2^{\hat{h}_2^*} + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + y_1^*\Delta t\right) = X_2^{\hat{h}_2^*} + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + y_1^*\Delta t\right) - C_2^* + \left(X_2^{\hat{h}_3^*} + ay_2^*\Delta t\right) \\ &= X_2^{\hat{h}_3^*} + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + \Delta Y_2^{\hat{h}_2^*} + y_1^*\Delta t + y_2^*\Delta t\right), \text{ and} \\ \tilde{h}_3^n &= \hat{h}_3^n \text{ for } n = 1, 2, \dots, N \text{ and } \tilde{h}_3^0 = \hat{h}_3^0 + a\left(\Delta Y_1^{\hat{h}_1^*} + y_0^*\Delta t + \Delta Y_2^{\hat{h}_2^*} + y_1^*\Delta t + y_2^*\Delta t\right). \end{aligned}$$

In general, for any time $t = 0, 1, \dots, M - 1$,

$$X_t^{\tilde{h}^*} = X_t^{\hat{h}_{t+1}^*} + \sum_{i=0}^{t-1} a \left(\Delta Y_{i+1}^{\hat{h}^*} + y_i^* \Delta t \right) + ay_t^* \Delta t, \text{ and} \quad (16)$$

$$\tilde{h}_{t+1}^n = \hat{h}_{t+1}^n \text{ for } n = 1, 2, \dots, N \text{ and } \tilde{h}_{t+1}^0 = \hat{h}_{t+1}^0 + \sum_{i=0}^{t-1} a \left(\Delta Y_{i+1}^{\hat{h}^*} + y_i^* \Delta t \right) + ay_t^* \Delta t \quad (17)$$

and

$$X_M^{\tilde{h}^*} = X_M^{\hat{h}_{M+1}^*} + \sum_{t=0}^{M-2} a \left(\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t \right) + ay_{M-1}^* \Delta t - C_M^* + C_M^* = C_M^* + \sum_{t=0}^{M-1} a \left(\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t \right). \quad (18)$$

The following proposition summarizes these results and gives the multiperiod hedging error.

Proposition 1 *Recursive prices based on “every recursive price is defined as the price of a one-period hedging portfolio plus a risk premium associated with the one-period hedging error” are consistent with “recursive one-period optimal self-financing hedging strategies,” which depend on the previously computed recursive prices, and where the one-period hedging errors, and risk premiums, are financed or invested at the riskless rate. Thus, the multiperiod hedging error $Y_T^{\tilde{h}}$, is the sum of the one-period hedging errors, plus the associated risk premiums, financed or invested at the riskless rate r ,*

$$Y_T^{\tilde{h}} = a \left(X_M^{\tilde{h}} - C_M \right) = a \sum_{t=0}^{M-1} a \left(\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t \right) e^{rT} = \sum_{t=0}^{M-1} \left(\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t \right) e^{rT}. \quad (19)$$

That is, $aC_M = aX_M^{\tilde{h}} - Y_T^{\tilde{h}}$, where $aX_M^{\tilde{h}}$ is the risk that can be hedged and $-Y_T^{\tilde{h}}$ is the (residual) risk that cannot be hedged. ■

Recursive prices are defined as a generalization of equations (3) and (8) in the one-period model, where \hat{h} is the previous non-self-financing portfolio solving (15). That is,

$$C_t = X_t^{\hat{h}} + ay_t \Delta t, \text{ and} \quad (20)$$

$$C_t = E_t^{Q^{\hat{h}}} \left[\frac{C_{t+1}}{e^{r\Delta t}} \right] + ay_t \Delta t, \quad (21)$$

$t = 0, 1, \dots, M - 1$, and $a = +1$ ($a = -1$) defines the upper (lower) price bound. As the one-period model, we assume that $C_t^- < X_t^{\hat{h}} < C_t^+$ and that $C_t^- < X_t^{\hat{h}} + ay_t < C_t^+$, $t = 0, 1, \dots, M - 1$.

The next results show important properties of these multiperiod recursive prices.

At maturity, the price is equal to C_M . One period before maturity $M - 1$, the price is equal to

$$\begin{aligned} C_{M-1} &= E_{M-1}^{Q^{\hat{h}}} \left[\frac{C_M}{e^{r\Delta t}} \right] + ay_{M-1} \Delta t, \text{ or equivalently,} \\ &= X_{M-1}^{\hat{h}} + ay_{M-1} \Delta t. \end{aligned}$$

Two periods before maturity $M - 2$,

$$\begin{aligned} C_{M-2} &= E_{M-2}^{Q^{\hat{h}}} \left[\frac{C_{M-1}}{e^{r\Delta t}} \right] + ay_{M-2}\Delta t \\ &= E_{M-2}^{Q^{\hat{h}}} \left[\frac{C_M}{e^{r2\Delta t}} \right] + a \left(y_{M-2}\Delta t + E_{M-2}^{Q^{\hat{h}}} \left[\frac{y_{M-1}}{e^{r\Delta t}} \Delta t \right] \right), \end{aligned}$$

by using the law of the iterated expectation $E_{M-2}^{Q^{\hat{h}}} [C_M] = E_{M-2}^{Q^{\hat{h}}} \left[E_{M-1}^{Q^{\hat{h}}} [C_M] \right]$, or equivalently,

$$\begin{aligned} C_{M-2} &= X_{M-2}^{\hat{h}} + ay_{M-2}\Delta t \\ &= X_{M-2}^{\hat{h}} - aE_{M-2}^{Q^{\hat{h}}} \left[\frac{y_{M-1}}{e^{r\Delta t}} \Delta t \right] + a \left(y_{M-2}\Delta t + E_{M-2}^{Q^{\hat{h}}} \left[\frac{y_{M-1}}{e^{r\Delta t}} \Delta t \right] \right). \end{aligned}$$

And, recursively, at the initial period 0, we have the following result.

Theorem 2 *If multiperiod prices are derived recursively and if one-period prices are equal to the price of a hedging portfolio plus a risk premium, as in equations (20) and (21), then C_0 is as follows.*

$$C_0 = E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] + aE_0^{Q^{\hat{h}}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right], \text{ or equivalently,} \quad (22)$$

$$= X_0^{\hat{h}} - aE_0^{Q^{\hat{h}}} \left[\sum_{t=1}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right] + aE_0^{Q^{\hat{h}}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right]. \quad (23)$$

Similar to the one-period model, C_0 can be divided in two parts. First,

$$E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] = X_0^{\hat{h}} - aE_0^{Q^{\hat{h}}} \left[\sum_{t=1}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right], \quad (24)$$

and second,

$$aE_0^{Q^{\hat{h}}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right], \quad (25)$$

which depends on the risk premium y . ■

To understand the term $E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right]$ we need an additional assumption. We advance that $E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right]$ is related to the price of a hedging portfolio. We require that every $Q_t^{\hat{h}}$ is independent of all the previous risk premiums $y_{t+1}, y_{t+2}, \dots, y_{M-1}$. Note that C_{t+1} , and therefore \hat{h}_{t+1} , depend on all these risk premiums because recursive pricing. Then, denote by $\hat{h}_{t+1}(y=0)$ the optimal portfolio when all risk premiums are zero, i.e., $y_{t+1} = y_{t+2} = \dots = y_{M-1} = 0$, and by $Q_t^{\hat{h}(y=0)}$ the associated RNP measure. The $Q_t^{\hat{h}(y=0)}$ is the appropriate RNP measure and we have the desired result.

Note that $E_0^{Q^{\hat{h}(y=0)}} \left[\frac{C_M}{e^{rT}} \right] = X_0^{\hat{h}(y=0)}$ from equation (24). Therefore, the assumption $Q_t^{\hat{h}} = Q_t^{\hat{h}(y=0)}$ implies that $E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] = E_0^{Q^{\hat{h}(y=0)}} \left[\frac{C_M}{e^{rT}} \right]$, and from equation (24), $E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] = X_0^{\hat{h}(y=0)}$ is unique and well-defined. That is, the hedging portfolio corresponds with the initial recursive one-period hedging portfolio computed when all risk premiums are zero, $\hat{h}(y=0)$. If there are multiple RNP

measures $Q^{\hat{h}}$ and $Q^{\hat{h}(y=0)}$, we require that $\{Q^{\hat{h}}\} = \{Q^{\hat{h}(y=0)}\}$, where $\{Q^{\hat{h}}\}$ means the set of RNP measures which verify equation (20) and (21).

Consequently, by assuming that $\{Q^{\hat{h}}\} = \{Q^{\hat{h}(y=0)}\}$, then

$$C_0 = X_0^{\hat{h}(y=0)} + aE_0^{Q^{\hat{h}(y=0)}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right] = E_0^{Q^{\hat{h}(y=0)}} \left[\frac{C_M}{e^{rT}} \right] + aE_0^{Q^{\hat{h}(y=0)}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right], \quad (26)$$

which is the multiperiod extension of equation (9) in the one-period model. The condition that $\{Q^{\hat{h}}\} = \{Q^{\hat{h}(y=0)}\}$ depends on the one-period hedging criterion and on the risk premiums specification. We show that this condition holds in the continuous-time model for diffusion processes.¹⁰

Remark 4. Theorem 2, and equation (26), give a novel decomposition of multiperiod recursive prices in incomplete markets. If $\{Q^{\hat{h}}\} = \{Q^{\hat{h}(y=0)}\}$, the recursive price of a non American-style security, C_0 , is equal to a risk-neutral expectation of the discounted payoff at maturity (i.e., the price of a hedging portfolio) plus a risk-neutral expectation of the discounted one-period risk premiums (i.e., the multiperiod risk premium). In the next section, we show that the spanned option payoff does not depend of y , and thus (26) is not only a mathematical decomposition but economic meaningful.

We can derive an alternative expression for C_0 . Specify the risk premium as proportional to the price C_t , i.e., $y_t = \alpha_t C_t$. Now, instead of discounting the risk premiums, we reinvest them in the option C_t . Then, by again using equation (21), i.e., $C_t = E_t^{Q^{\hat{h}}} \left[(1 - a\alpha_t \Delta t)^{-1} \frac{C_{t+1}}{e^{r\Delta t}} \right]$, $t = 0, 1, \dots, M-1$,

$$\begin{aligned} C_0 &= E_0^{Q^{\hat{h}}} \left[(1 - a\alpha_0 \Delta t)^{-1} \frac{C_1}{e^{r\Delta t}} \right] \\ &= E_0^{Q^{\hat{h}}} \left[(1 - a\alpha_0 \Delta t)^{-1} \frac{E_1^{Q^{\hat{h}}} \left[(1 - a\alpha_1 \Delta t)^{-1} \frac{C_2}{e^{r\Delta t}} \right]}{e^{r\Delta t}} \right] = E_0^{Q^{\hat{h}}} \left[(1 - a\alpha_0 \Delta t)^{-1} (1 - a\alpha_1 \Delta t)^{-1} \frac{C_2}{e^{r2\Delta t}} \right] \\ &= \dots = E_0^{Q^{\hat{h}}} \left[\left(\prod_{t=0}^{M-1} (1 - a\alpha_t \Delta t) \right)^{-1} \frac{C_M}{e^{rT}} \right]. \end{aligned} \quad (27)$$

The one-period risk premium $\alpha_t C_t \Delta t$ is similar to a stochastic dividend flow paid by C_t , and from (27), C_t is equal also to the risk-neutral expectation of the discounted payoff at maturity adjusted by these reinvested risk premiums. Note that $\left(\prod_{t=0}^{M-1} (1 - a\alpha_t \Delta t) \right)^{-1}$ can be approximated as $\exp\{a \sum_{t=0}^{M-1} \alpha_t \Delta t\}$. For example, if α_t is constant ($\alpha_t = \alpha$), C_0 can be approximated as

$$C_0 \approx e^{a\alpha T} E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right]. \quad (28)$$

¹⁰Nevertheless, assume that the density ratio is finite, i.e., $\frac{dQ^{\hat{h}}}{dQ^{\hat{h}(y=0)}} < \infty$ for all $\omega \in \Omega$,

$$\begin{aligned} E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] &= E_0^{Q^{\hat{h}(y=0)}} \left[\frac{dQ^{\hat{h}}}{dQ^{\hat{h}(y=0)}} \frac{C_M}{e^{rT}} \right] = E_0^{Q^{\hat{h}(y=0)}} \left[\frac{dQ^{\hat{h}}}{dQ^{\hat{h}(y=0)}} \right] E_0^{Q^{\hat{h}(y=0)}} \left[\frac{C_M}{e^{rT}} \right] + \text{cov}^{Q^{\hat{h}(y=0)}} \left(\frac{dQ^{\hat{h}}}{dQ^{\hat{h}(y=0)}}, \frac{C_M}{e^{rT}} \right) \\ &= X_0^{\hat{h}(y=0)} + E_0^{Q^{\hat{h}(y=0)}} \left[\left(\frac{dQ^{\hat{h}}}{dQ^{\hat{h}(y=0)}} - 1 \right) \left(\frac{C_M}{e^{rT}} - X_0^{\hat{h}(y=0)} \right) \right]. \end{aligned}$$

As equation (10) in the one-period model, there exists a different RNP measure $Q^{\hat{h},y}$ such that $C_t = X_t^{\hat{h}} + ay_t = E_t^{Q^{\hat{h},y}} \left[\frac{C_{t+1}}{e^{r\Delta t}} \right]$ for $t = 0, 1, \dots, M-1$ (since $C_t^- < X_t^{\hat{h}} + ay_t < C_t^+$). Therefore, from the law of the iterated expectation,

$$C_0 = E_0^{Q^{\hat{h},y}} \left[\frac{C_M}{e^{rT}} \right]. \quad (29)$$

Consequently, the multiperiod risk premium also satisfies that

$$aE_0^{Q^{\hat{h}}} \left[\sum_{t=0}^{M-1} \frac{y_t}{e^{rt\Delta t}} \Delta t \right] = E_0^{Q^{\hat{h},y}} \left[\frac{C_M}{e^{rT}} \right] - E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right], \quad (30)$$

from equations (22) and (29). Note that $Q^h = Q^{h,y}$ if $y = 0$; i.e., $Q^h = Q^{h,0}$.

Finally, if all the risk premiums are zero because, for example, the market is complete (and \hat{h} is the replicating portfolio), then in all the equations above we obtain the very well-known result,

$$\text{if } y_0 = y_1 = \dots = y_{M-1} = 0, \text{ then } C_0 = E_0^{Q^{\hat{h}}} \left[\frac{C_M}{e^{rT}} \right] = X_0^{\hat{h}}. \quad (31)$$

In sum, Proposition 1 and Theorem 2 are two important results, which distinguish our paper from the extant literature. For example, Cochrane and Saá-Requejo (2000) bounds verify equation (20) and are recursive (see their Propositions 1 and 2, respectively), and therefore, verify these properties.

3.1.1 Recursive Prices in Continuous Time

Since $e^{r\Delta t} = 1 + r\Delta t + O(\Delta t^2)$, equation (21) can be rewritten as

$$\frac{1}{\Delta t} E_t^{Q^{\hat{h}}} [C_{t+1} - C_t] + O(\Delta t) = rC_t - ay_t, \quad t = 0, 1, \dots, M-1. \quad (32)$$

Assume that the N risky assets follow diffusion processes. Therefore, with the help of Itô's Lemma and other stochastic calculus tools, when $\lim \Delta t \rightarrow 0$, from (32), recursive bounds can be characterized through PDE's, once the stochastic processes for the state variables, the hedging strategy, and the risk premiums are specified. This PDE is derived in two simple steps: first, the risk-neutral drift, $\frac{1}{\Delta t} E_t^{Q^{\hat{h}}} [C_{t+1} - C_t]$, is equal to rC_t , and second, the term ay_t is subtracted from this riskless return. This is formally proved in the next section. Note that equation (32) can also be extended to study problems where the N risky assets follow stochastic processes more complex than diffusions.¹¹

3.1.2 American-style Securities in Incomplete Markets

The valuation of American-style securities in incomplete markets is a problem that has not been addressed in the literature in general. The joint determination of an optimal dynamic self-financing portfolio \tilde{h} and an optimal stopping-time (or exercise policy) is a complex problem. However, the

¹¹The study of more general process is an important subject of future research given the evidence of non-normality in returns. Beyond diffusion models, see affine jump-diffusion models in Duffie et al. (2000), or Carr and Wu (2002).

recursive approach allows us to price American securities easily, in a manner similar to other recursive numerical methods in a complete market. Let $I(S_t, E)$ be the intrinsic payoff with $C_M = I(S_M, E)$ and E the strike price. It is enough to substitute the term C_{t+1} in the recursive hedging equation (15) and then in the recursive pricing equations (20) and (21) with $\max\{I(S_{t+1}, E), C_{t+1}\}$.

That is, $f(a(X_{t+1}^h - \max\{I(S_{t+1}, E), C_{t+1}\}))$, then $X_t^h = E_t^{Q^h} [e^{-r\Delta t} \max\{I(S_{t+1}, E), C_{t+1}\}]$ and $C_t = E_t^{Q^h} [e^{-r\Delta t} \max\{I(S_{t+1}, E), C_{t+1}\}] + ay_t\Delta t$, for $t = 0, 1, \dots, M-1$, respectively.

The total residual risk is given now by $\sum_{t=0}^{M-1} (\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t) 1_{\{t+1 \leq \hat{\tau}\}}$, where $\hat{\tau} \in \{1, 2, \dots, M\}$ is the optimal stopping-time defined by the first τ such that $I(S_\tau, E) \geq C_\tau$.

3.1.3 Empirical Applications of the Hedging Errors

We can devise empirical applications of the multiperiod hedging error $Y_T^{\hat{h}}$ in equation (19) as well. The multiperiod hedging error produces two main empirical testable implications. Note that $Y_T^{\hat{h}^*} = \sum_{t=0}^{M-1} (\Delta Y_{t+1}^{\hat{h}^*} + y_t^* \Delta t)$ has two terms: the one-period hedging errors and risk premiums. Assume that the conditional expectation $E_t^P [\Delta Y_{t+1}^{\hat{h}^*}] = 0$ for $t = 0, 1, \dots, M-1$. First, if the model being studied is complete then (there exists a portfolio \hat{h} such that) $\Delta Y_{t+1}^{\hat{h}^*} = 0$ and the risk premium $y_t = 0$ to avoid arbitrage opportunities. Consequently, a period-by-period hedged portfolio has a zero (expected) hedging error since $Y_T^{\hat{h}^*} = 0$. Second, if the model is incomplete and $y_t \neq 0$, then a period-by-period hedged portfolio has an expected hedging error $E_0^P [Y_T^{\hat{h}^*}] = \sum_{t=0}^{M-1} E_0^P [y_t^*] \Delta t$, which can be different from zero. We have the following result.

Proposition 3 *Assume, first, $E_t^P [\Delta Y_{t+1}^{\hat{h}^*}] = 0$ for $t = 0, 1, \dots, M-1$, second, $y_t = 0$ if and only if $\Delta Y_{t+1}^{\hat{h}^*} = 0$, and third y_t is a positive or a negative risk premium, but its sign does not change from time 0 to $M-1$. Then, given $N+1$ hedging assets S_t^n , $n = 0, 1, \dots, N$, the market is incomplete if and only if $E_0^P [Y_T^{\hat{h}^*}] = \sum_{t=0}^{M-1} E_0^P [y_t^*] \Delta t \neq 0$, i.e., a period-by-period hedged portfolio has an expected hedging error different from zero. ■*

This result is dependent upon the number of hedging assets S_t^n , $n = 0, 1, \dots, N$. For instance, assume a stochastic volatility model. If the hedging assets are only the bond and the stock, then the residual risk is the volatility risk (i.e., $\Delta Y_{t+1}^{\hat{h}^*} \neq 0$) and it can hold that $y_t \neq 0$, which is a volatility risk premium. On the other hand, if the hedging assets are the bond, the stock and a second option, then the residual risk is zero (i.e., $\Delta Y_{t+1}^{\hat{h}^*} = 0$) and $y_t = 0$ in order to avoid arbitrage opportunities.¹²

Finally, another interesting application is as follows. Assume that $E_t^P [\Delta Y_{t+1}^{\hat{h}^*}] = 0$ for $t = 0, 1, \dots, M-1$. Then every one-period hedging error $\Delta Y_{t+1}^{\hat{h}^*} + y_t \Delta t$ can be regressed on a series of variables to analyze if the risk premium y_t is related to such variables. See the basis risk example below.

¹²For instance, Fan, Gupta, and Ritchken (2003) study how many factors are necessary to price swaptions from the perspective of hedging effectiveness instead of the standard approach of pricing performance.

4 Recursive Prices in Continuous Time

Assume a vector of K state variables, $S(t) = (S_1(t), S_2(t), \dots, S_K(t))$, which follows the following diffusion process

$$dS(t) = \mu(t, S(t)) dt + \Sigma(t, S(t)) dz_t, \quad (33)$$

where μ is a $(K \times 1)$ vector, Σ is a $(K \times K)$ matrix, and z is a $(K \times 1)$ vector of independent Wiener processes. We assume that $\mu(t, S(t))$ and $\Sigma(t, S(t))$ satisfy growth and regularity conditions such that the process dS is well defined and has a unique solution (see Duffie (2001)). Let $r(t)$ be the instantaneous short interest rate, and $r(t) = r$ be constant to save notation.

We assume that only the first N state variables $S_1(t), S_2(t), \dots, S_N(t)$, with $0 \leq N \leq K$, are tradable and consequently the market is incomplete if $N < K$. For instance, the $S_{N+1}(t), S_{N+2}(t), \dots, S_K(t)$ correspond with illiquid assets, stochastic volatility, etc., (see the next section's examples). We consider the partition of the volatility matrix $\Sigma' = [A' B']$, where A and B contain the first N and the last $K - N$ rows of Σ , respectively. We assume that the rank of the matrix A is equal to N (almost sure), i.e., there are no redundant tradable assets. In particular, this implies that the model is arbitrage free, and equivalently, there exist multiple risk-neutral probability measures for the N tradable assets (under technical conditions, see Duffie (2001)).

We assume that there are no portfolio constraints, which are a real source of market incompleteness. Because the recursive approach is based on one-period optimization, portfolio constraints can be easily incorporated (e.g., we price a put option under short-selling constraints). Finally, we assume that the rank of Σ is equal to K (almost sure), every state variable represents a different risk.

Two special cases are $N = 0$ where the only hedging instrument is the risk-free asset (we define $\Sigma = B$), and $N = K$ where the market is complete ($\Sigma = A$ and A is invertible).

Although recursive prices can be derived in the limit from equation (32) when $\Delta t \rightarrow 0$, we find easier to derive them along the lines of Black and Scholes (1973) and Merton (1973).

4.1 The Hedging Strategy

Let $C(t, S(t))$ and $X_h(S(t))$ be the price of a derivative security and the price of a hedging portfolio h , respectively, where $X_h(S(t)) = \sum_{n=1}^N h_n(t) S_n(t)$, $h_0(t) = 0$, and $C(T, S(T))$ is the European option payoff at maturity (with the notation slightly changed). By Ito's lemma, dC and dX_h satisfy

$$dC = \mu_c dt + C'_S \Sigma dz \quad \text{and} \quad (34)$$

$$dX_h = \mu_h dt + h' A dz, \quad (35)$$

with

$$\mu_c = C_t + \frac{1}{2} \sum_{i=1}^K \sum_{j=1}^K C_{SS(i,j)} \left(\sum_{k=1}^K \Sigma_{i,k} \Sigma_{j,k} \right) + \mu' C_S \quad \text{and} \quad \mu_h = \mu'_{(1:N)} h, \quad (36)$$

and $h(t) = (h_1(t), h_2(t), \dots, h_N(t))$. Note that C_S is the $(K \times 1)$ vector of first derivatives and C_{SS} is the $(K \times K)$ matrix of second derivatives, and we have suppressed the dependence of all variables on t and $S(t)$. We assume that $C(T, S(T))$ depends on all state variables S . If C does not depend on some state variables, the corresponding partial derivatives C_S and C_{SS} are equal to zero.

Define the hedging error

$$dY_t^h = a(h'A - C'_S \Sigma) dz_t = a(h'A - C'_S [A' B']') dz_t = a((h - C_{S(1:N)})'A - C'_{S(N+1:K)}B) dz_t. \quad (37)$$

Because dC and dX follow diffusion processes and because of the fact that continuous trading is allowed, the infinitesimal one-period hedging errors are (conditionally) normally distributed and consequently the appropriate, and unique, hedging criterion is to minimize the variance. Therefore, the hedging criterion $f(dY_t)$ is given by minimizing

$$f(dY_t^h) = \frac{1}{dt} E_t^P [dY_t^h]^2 = \left\| (h - C_{S(1:N)})'A - C'_{S(N+1:K)}B \right\|^2, \quad (38)$$

where $\|\cdot\|^2$ is the Euclidean norm. Let denote $g = h - C_{S(1:N)}$. Then, the N orthogonality conditions (i.e., $E_t^P [dS_n dY_t^{\hat{h}}] = 0$, $n = 1, 2, \dots, N$) for this problem imply that

$$\hat{g} = (AA')^{-1} AB' C_{S(N+1:K)}, \text{ and } \hat{h} = C_{S(1:N)} + \hat{g} \quad (39)$$

is the optimal minimum variance portfolio, and the matrix AA' is invertible since the rank of A is equal to N . Then, $dX_{\hat{h}} = \mu_{\hat{h}} dt + \hat{h}' Adz$ is the dynamics of the optimal hedging portfolio, and

$$dY_t^{\hat{h}} = a(\hat{g}'A - C'_{S(N+1:K)}B) dz_t = aC'_{S(N+1:K)}B (A' (AA')^{-1} A - I) dz_t \quad (40)$$

is the remaining residual risk.

This residual risk $dY_t^{\hat{h}}$ has three components: $C_{S(N+1:K)}$ measures the non-traded option's risk, B are the volatilities of the non-traded variables, and $(A' (AA')^{-1} A - I)$ is related to the market incompleteness. Note that $B(A' (AA')^{-1} A - I) dz_t$ is the risk which is non-spanned by $S_{(1:N)}$, and note that the option's Deltas, $C_{S(N+1:K)}$, can be used for risk management.

4.2 The PDE Equation: the Law of One Price and the Risk Premium

First, if we forget the residual risk $dY_t^{\hat{h}}$, the law of one price implies that, similar to the Black-Scholes-Merton model, the return of a riskless portfolio must be equal to the riskless rate; i.e.,

$$a(\mu_{\hat{h}} - \mu_c) = a(X_{\hat{h}} - C)r, \quad (41)$$

Second, if $dY_t^{\hat{h}} \neq 0$, we add a risk premium y_t to compensate the residual risk; i.e.,

$$a(\mu_{\hat{h}} - \mu_c) = a(X_{\hat{h}} - C)r + y_t, \quad (42)$$

which is equation (20) but in continuous time. If $dY_t^{\hat{h}} = 0$, $y_t = 0$ and we have the standard application of non arbitrage arguments with complete markets.

In other words, the risk-return trade-off of “ $y_t dt + dY_t^{\hat{h}}$,” i.e., $\mathcal{N}\left(y_t dt, \sqrt{\sum_{k=1}^K (\sigma_k^Y)^2} \sqrt{dt}\right)$, is attractive for the writer (or buyer) of the option, where $\sigma_k^Y = \left|C'_{S(N+1:K)} B \left(A' (AA')^{-1} A - I\right)\right|_{(k)}$, $k = 1, 2, \dots, K$, is the vector of volatilities of the residual risk. The investor in C obtains an extra premium $y_t dt$ for carrying extra risk on $dY_t^{\hat{h}}$. (Note that, under technical conditions, this premium is arbitrage-free due to the continuous support of $\mathcal{N}(0, \sqrt{dt})$ over all the real line \mathcal{R}).

Note further that the orthogonality conditions, $E_t^P \left[dS_n dY_t^{\hat{h}} \right] = 0$, imply that the risk premium y_t can be specified independently of the tradable assets dS_n , highlighting the importance of an optimal hedging portfolio. From the PDE equation (42), “ $-a(\mu_{\hat{h}} - rX_{\hat{h}}) + y_t$ ” is the option risk premium and y_t is the risk premium when the option is hedged by portfolio \hat{h} (and recall that $a = +1$ ($a = -1$) for a short (long) position). Moreover, “ $\mu_{\hat{h}} - rX_{\hat{h}}$ ” is an endogenous risk premium related to the traded assets, whereas y_t is the exogenous risk premium associated with the residual risk.

Equivalently, since $a^2 = 1$, the latter PDE equation can be rewritten as

$$\mu_c - (\mu_{\hat{h}} - rX_{\hat{h}}) = rC - ay_t, \quad (43)$$

which is similar to the risk-neutral equations (21) and (32) but in continuous time. Moreover, we are interested in two prices C^s and C^l , with $C^s \geq C^l$. Then, from the PDE equation (43), a sufficient condition is if that $-\infty < -y_t^s \leq y_t^l < +\infty$ (or equivalently, $+\infty > y_t^s \geq -y_t^l > -\infty$, and in particular, $+\infty > y_t^s = y_t^l \geq 0$), where y_t^s (y_t^l) is the risk premium associated with C^s (C^l). Intuitively, a lower (higher) term $-ay_t$ in the PDE equation (43) implies a higher (lower) option price. Note that it is the same condition on equation (5) in the one-period model.

Finally, substituting μ_c and \hat{h} in equation (43), and noting that

$$\begin{aligned} & \mu' C_S - (\mu_{\hat{h}} - rX_{\hat{h}}) \\ = & \mu' C_S - (\mu'_{(1:N)} \hat{h} - rX_{\hat{h}}) \\ = & \mu' C_S - (\mu - rS)'_{(1:N)} (\hat{g} + C_{S(1:N)}) \\ = & rS'_{(1:N)} C_{S(1:N)} + \mu'_{(N+1:K)} C_{S(N+1:K)} - (\mu - rS)'_{(1:N)} \hat{g} \\ = & rS'_{(1:N)} C_{S(1:N)} + \left(\mu'_{(N+1:K)} - (\mu - rS)'_{(1:N)} (AA')^{-1} AB' \right) C_{S(N+1:K)}, \end{aligned} \quad (44)$$

then equation (43) is given explicitly by

$$\begin{aligned} & C_t + \frac{1}{2} \sum_{i=1}^K \sum_{j=1}^K C_{SS(i,j)} \left(\sum_{k=1}^K \Sigma_{i,k} \Sigma_{j,k} \right) + \\ & rS'_{(1:N)} C_{S(1:N)} + \left(\mu'_{(N+1:K)} - (\mu - rS)'_{(1:N)} (AA')^{-1} AB' \right) C_{S(N+1:K)} = rC - ay_t. \end{aligned} \quad (45)$$

As noted before, the recursive approach can be generalized to include portfolio constraints or more general process than diffusions for the state variables; i.e., an equivalent equation to (43) can be derived along the same lines. For example, for jump-diffusion models, one needs a generalized version of Ito's Lemma (see Duffie (2001)) to derive equation (34). Then, the one-period hedging problem is not simply the standard minimum variance problem in equation (38).

By choosing different risk premiums y_t , we can connect several option-pricing models. Merton (1998) where $y_t = 0$. Recall that $\sigma_{(1:K)}^Y$ is the vector of volatilities of the residual risk.

In the risk-neutral approach based on prices of risk, where we define $y_t = \sum_{k=1}^K \sigma_k^Y \lambda_k$. We can distinguish several cases. First, $N = 0$ and there are not tradable assets. Then, we define $\hat{g} = 0$, $dY_t^{\hat{h}} = aC'_{S(N+1:K)} B dz_t$, and $\sigma_k^Y = \left| C'_{S(N+1:K)} B \right|_{(k)}$. In this case, the $\lambda_k(t, S(t))$ are naturally the market prices of risk associated with each orthogonal factor dz_k . Second, $0 < N < K$. In this case, it is possible that some of the λ_k correspond with tradable assets (and therefore, $\sigma_k^Y = 0$), whereas other are exogenous (e.g., the volatility price of risk in a stochastic volatility model). Third, complete markets, where $N = K$ and A invertible. Then, $dY_t^{\hat{h}} = 0$, $\sigma_k^Y = 0$ for all k , and consequently, $y_t = 0$.

Cochrane and Saá-Requejo (2000) bounds, where $y_t = \tilde{A} \sqrt{\sum_{k=1}^K (\sigma_k^Y)^2}$. Cochrane and Saá-Requejo show that \tilde{A} is a parameter related to a bound on the pricing kernel volatility, and equivalently, on the maximum Sharpe ratio. Indeed, from our equation (43), \tilde{A} is the Sharpe ratio of the hedged option, i.e., the market price of risk of the residual risk.

In the local risk minimization approach (see Heath, Platen and Schweizer (2001)), where $y_t = 0$.

4.3 The RNP Measures Q^h and $Q^{h,y}$

From equations (33) and (41) we extract the RNP measure Q^h . This is one of the innovations of our paper. The literature extracts the RNP measure $Q^{h,y}$ from equation (33) and (43). Q^h allows to separate the recursive price in the price of a hedging portfolio plus a premium, which has a natural interpretation in incomplete markets. $Q^{h,y}$ is best used for pricing purposes and to prove no arbitrage. Note that $Q^h = Q^{h,y}$ if $y = 0$; i.e., $Q^h = Q^{h,0}$.

Let Q be a RNP measure. Q can be characterized through the *Radon-Nikodym* derivative, i.e.,

$$\frac{dQ}{dP} = \xi_T,$$

where ξ_T is the state price density,

$$\xi_0 = 1 \text{ and } \frac{d\xi_t}{\xi_t} = -\lambda_t' dz_t \text{ for } t \in [0, T],$$

and λ_t is a vector of prices of risk (and the Novikov's condition holds, $E_0 \left[\exp \left(\frac{1}{2} \int_0^T \lambda_t' \lambda_t dt \right) \right] < \infty$).

For completeness, we recall a standard result of pricing by arbitrage in frictionless markets (see Duffie (2001, 111-114) for technical details). Under technical conditions, the following three properties

are equivalent, (a) a well-defined market prices of risk process λ , (b) the existence of a risk-neutral probability measure Q , and (c) non arbitrage. It holds for both complete and incomplete markets.

Clearly, for the N tradable assets, the risk-neutral drift must be equal to the riskless rate r to avoid arbitrage opportunities, and therefore,

$$\mu_{(1:N)} - A\lambda = rS_{(1:N)}. \quad (46)$$

For the rest of nontradable assets, the risk-neutral drift is implicit in the PDE pricing equation (45). To extract it, we must look at the loading of the vector C_S . That is, they are obtained from the term

$$rS'_{(1:N)}C_{S(1:N)} + \left(\mu'_{(N+1:K)} - (\mu - rS)'_{(1:N)} (AA')^{-1} AB' \right) C_{S(N+1:K)} + ay_t. \quad (47)$$

Note that the risk-neutral drift of the N tradable assets is equal to r ; i.e., the PDE equation derived from the optimal minimum variance portfolio, \hat{h} , is consistent with equation (46).

Define the vector $D_{k-N} = -\frac{ay_t}{C_{S(k)}} \alpha_k 1_{\{C_{S(k)} \neq 0\}}$, $k = N+1, \dots, K$, and $\sum_{k=N+1}^K \alpha_k 1_{\{C_{S(k)} \neq 0\}} = 1$. Then, $ay_t = -D'C_{S(N+1:K)}$ (in particular, $D = 0$ if $y_t = 0$) and equation (47) can be rewritten as

$$rS'_{(1:N)}C_{S(1:N)} + \left(\mu'_{(N+1:K)} - (\mu - rS)'_{(1:N)} (AA')^{-1} AB' - D' \right) C_{S(N+1:K)}. \quad (48)$$

Therefore, the risk-neutral drift of the nontradable assets is given by the loadings of $C_{S(N+1:K)}$; i.e.,

$$\left(\mu_{(N+1:K)} - B\lambda \right) 1_{\{C_{S(N+1:K)} \neq 0\}} = \left(\mu_{(N+1:K)} - BA' (AA')^{-1} (\mu - rS)_{(1:N)} - D \right) 1_{\{C_{S(N+1:K)} \neq 0\}}, \quad (49)$$

with $1_{\{C_{S(N+1:K)} \neq 0\}}$ a vector of indicator functions. Equivalently,

$$\begin{pmatrix} A \\ B1_{\{C_{S(N+1:K)} \neq 0\}} \end{pmatrix} \lambda = \begin{pmatrix} I_{N \times N} \\ (BA' (AA')^{-1}) 1_{\{C_{S(N+1:K)} \neq 0\}} \end{pmatrix} (\mu - rS)_{(1:N)} + \begin{pmatrix} 0_{(1:N)} \\ D1_{\{C_{S(N+1:K)} \neq 0\}} \end{pmatrix}. \quad (50)$$

Since the rank of the matrix A and Σ is equal to N and K , respectively, then AA' and $\Sigma' = (A' B')$ are invertible and the system has well defined solutions. Note that the market prices of risk, λ , do not depend on the drift of the nontradable state variables, $\mu_{(N+1:K)}$. Let us show a few examples.

- If there are not hedging assets, $N = 0$, and $B1_{\{C_{S(N+1:K)} \neq 0\}} \lambda = D1_{\{C_{S(N+1:K)} \neq 0\}}$.
- If the market is complete, $K = N$, and $\lambda = A^{-1} (\mu - rS)$.
- If $K = N + 1$, λ and Q do not depend on the option being studied if $\frac{y_t}{C_{S(N+1)}}$ does not (e.g., if $y_t = 0$). If $K = N + 1$, λ and Q are unique. For example, if $K = N + 1$ and $C_{S(N+1)} \neq 0$, the system (50) simplifies to

$$A\lambda = (\mu - rS)_{(1:N)} \quad \text{and} \quad B\lambda = \left(BA' (AA')^{-1} \right) (\mu - rS)_{(1:N)} - a \frac{y_t}{C_{S(N+1)}}. \quad (51)$$

- However, if $K > N+1$, λ and Q depend on the option being studied from the term $1_{\{C_{S(N+1:K)} \neq 0\}}$. Moreover, if $K > N+1$, for $y_t = 0$, λ and Q are unique if the rank of $C_{S(N+1:K)}$ is equal to $K - N$. For $y_t \neq 0$, λ and Q are not unique even if the rank of $C_{S(N+1:K)}$ is equal to $K - N$ since the weights α in D are arbitrary.

For instance, assume that the N tradable assets only depend on the first N state variables, i.e., $A = [A_1 \ A_2]$ where A_1 contains the first N columns of A and A_2 is a matrix of zeros (i.e., $A_2 = 0_{N \times K-N}$). Partition the matrix $B = [B_1 \ B_2]$ where B_1 contains the first N columns of B . Then,¹³ the system (50) simplifies to

$$\lambda_{(1:N)} = A_1^{-1} (\mu - rS)_{(1:N)} \quad \text{and} \quad B_2 \lambda_{(N+1:K)} 1_{\{C_{S(N+1:K)} \neq 0\}} = D 1_{\{C_{S(N+1:K)} \neq 0\}}. \quad (52)$$

Moreover, if $1_{\{C_{S(N+1:K)} \neq 0\}} \neq 0$,

$$\lambda_{(1:N)} = A_1^{-1} (\mu - rS)_{(1:N)} \quad \text{and} \quad \lambda_{(N+1:K)} = B_2^{-1} D, \quad (53)$$

with $\lambda_{(N+1:K)} = 0_{(1:K-N)}$ if $y_t = 0$.

Finally, if $y_t = 0$ then $D = 0$ and equation (50) is given by

$$\begin{pmatrix} A \\ B 1_{\{C_{S(N+1:K)} \neq 0\}} \end{pmatrix} \lambda = \begin{pmatrix} I_{N \times N} \\ (BA' (AA')^{-1}) 1_{\{C_{S(N+1:K)} \neq 0\}} \end{pmatrix} (\mu - rS)_{(1:N)}, \quad (54)$$

which is independent of the risk premiums $y_s, t < s \leq T$, since this equation does not depend on $C(t, S(t))$. Consequently, the recursive price can be divided in the price of the initial one-period hedging portfolio with all zero risk premiums, plus a multiperiod risk premium, which is shown next.

4.4 Risk Management

First, the total hedging error is simply the sum of the one-period hedging errors plus the one-period risk premiums, financed or invested at the riskless rate r . The dynamic of a self-financing portfolio \hat{h} satisfies $X_h^*(T) = X_h^*(0) + \int_0^T dX_h^*(t)$, and $C^*(T, S(T)) = C^*(0, S(0)) + \int_0^T dC^*(t, S(t))$. We define $X_h^*(0) = C^*(0, S(0))$. Therefore,

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$$\begin{aligned} B \lambda 1_{\{C_{S(N+1:K)} \neq 0\}} &= BA' (AA')^{-1} (\mu - rS)_{(1:N)} 1_{\{C_{S(N+1:K)} \neq 0\}} + D 1_{\{C_{S(N+1:K)} \neq 0\}} \iff \\ (B_1 \lambda_{(1:N)} + B_2 \lambda_{(N+1:K)}) 1_{\{C_{S(N+1:K)} \neq 0\}} &= B_1 A_1^{-1} A_1 \lambda_{(1:N)} 1_{\{C_{S(N+1:K)} \neq 0\}} + D 1_{\{C_{S(N+1:K)} \neq 0\}} \iff \\ B_2 \lambda_{(N+1:K)} 1_{\{C_{S(N+1:K)} \neq 0\}} &= D 1_{\{C_{S(N+1:K)} \neq 0\}}. \end{aligned}$$

$$\begin{aligned}
Y_T^{\widehat{h}^*} &= a \left(X_{\widehat{h}}^*(T) - C^*(T, S(T)) \right) = a \int_0^T dX_{\widehat{h}}^*(t) - a \int_0^T dC^*(t, S(t)) \\
&= a \int_0^T e^{-rt} \left((\mu_{\widehat{h}} - rX_{\widehat{h}}^*)dt + \widehat{h}' Adz_t \right) - a \int_0^T e^{-rt} \left((\mu_c - rC) \right) dt + C'_S \Sigma dz_t \\
&= a \int_0^T e^{-rt} \left((\mu_{\widehat{h}} - rX_{\widehat{h}}^*) - (\mu_c - rC) \right) dt + a \int_0^T e^{-rt} dY_t^{\widehat{h}} \\
&= e^{-rT} \int_0^T e^{r(T-t)} y_t dt - a e^{-rT} \int_0^T e^{r(T-t)} C'_{S(N+1:K)} B \left(A' (AA')^{-1} A - I \right) dz_t, \quad (55)
\end{aligned}$$

where the third equality is from Itô's Lemma, and the fifth is from the pricing PDE equation (43) and from the hedging error in equation (40).

That is, $aX_{\widehat{h}}^*(T) = aC(T, S(T)) + Y_T^{\widehat{h}}$. The hedging portfolio replicates the option payoff except for a residual risk $Y_T^{\widehat{h}}$, which contains two parts. The first part depends on the risk premium y . The second part is orthogonal to the traded assets $\left(E_t^P \left[dS(t) dY_t^{\widehat{h}} \right] = 0 \right)$. Therefore, the spanned option payoff does not depend on y (except for the loadings $C_{S(N+1:K)}$), and it makes economic sense to decompose the option price in the price of a hedging portfolio plus a premium.

Second, under the risk-neutral dynamics $Q^{\widehat{h}}$ we have $(dz_t^{Q^{\widehat{h}}})$ are Wiener processes under $Q^{\widehat{h}}$,

$$C(T) = C(0) + \int_0^T (rC - ay_t) dt + \int_0^T C'_S \Sigma dz_t^{Q^{\widehat{h}}} \quad (56)$$

from equation (43), and in discounted prices

$$C^*(T) = C^*(0) - a \int_0^T e^{-rt} y_t dt + \int_0^T e^{-rt} C'_S \Sigma dz_t^{Q^{\widehat{h}}}. \quad (57)$$

Therefore, taking risk-neutral expectations under $Q^{\widehat{h}}$, and given $C(0) = C^*(0)$, we see that

$$E_0^{Q^{\widehat{h}}} [C^*(T)] = C(0) - a E_0^{Q^{\widehat{h}}} \left[\int_0^T y_t^* dt \right] \quad (58)$$

and consequently,

$$C(0) = E_0^{Q^{\widehat{h}}} \left[\frac{C(T)}{e^{rT}} \right] + a E_0^{Q^{\widehat{h}}} \left[\int_0^T \frac{y_t}{e^{rt}} dt \right]. \quad (59)$$

Denote by $\widehat{h}_t(y=0)$ (by $C_{\widehat{h}(y=0)}(0)$) the optimal portfolio (the option price) when all risk premiums are zero, i.e., $y_s = 0$ for $s \in (t, T]$. Equation (54) implies that $Q_t^h = Q_t^{h(y=0)}$. The portfolio $\widehat{h}_t(y=0)$ satisfies that $X_{\widehat{h}(y=0)}^*(0) = C_{\widehat{h}(y=0)}^*(0)$, and from equation (59), $X_{\widehat{h}(y=0)}^*(0) = E_0^{Q^{\widehat{h}}} \left[\frac{C(T)}{e^{rT}} \right]$. Consequently,

$$C(0) = X_{\widehat{h}(y=0)}^*(0) + a E_0^{Q^{\widehat{h}}} \left[\int_0^T \frac{y_t}{e^{rt}} dt \right], \quad (60)$$

and again, the option price is equal to the price of a hedging portfolio plus a multiperiod risk pre-

mium.¹⁴ Note that equations (55), (59) and (60) are just equations (19), (22) and (26), respectively, but in continuous time.

Assume that $y_t \geq 0$ (almost sure), for all t . Then, assuming technical conditions, under the $Q^{\hat{h}}$ probability measure, the discounted upper (lower) bound is a super-martingale (sub-martingale), and the price of the hedging portfolio, $e^{-rt} X_{\hat{h}(y=0)}(t) = E_t^{Q^{\hat{h}}} [e^{-rT} C_T]$, is the martingale component.

Finally, since

$$C(0) = E_0^{Q^{\hat{h},y}} \left[\frac{C(T)}{e^{rT}} \right], \quad (61)$$

then the multiperiod risk premium is also given by

$$aE_0^{Q^{\hat{h}}} \left[\int_0^T e^{-rt} y_t dt \right] = E_0^{Q^{\hat{h},y}} \left[\frac{C(T)}{e^{rT}} \right] - E_0^{Q^{\hat{h}}} \left[\frac{C(T)}{e^{rT}} \right] \quad (62)$$

For example, if $C(T)$ is an European call option, the premium is equal to the price difference of two call options, and if the two call options have a close form solution, the premium does too.

In the continuous-time framework for diffusion processes, Proposition 4 summarizes the results of the present section and distinguishes our paper from the extant literature. Moreover, that jump-diffusion processes, that American-style payoffs, and that portfolio constraints can be easily studied in incomplete markets through the recursive approach (i.e., in a series of one-period recursive and independent optimization problems) is also another contribution of our paper.

Proposition 4 *Assume a frictionless market and that dS_t satisfies equation (33). Then, the recursive optimal hedging portfolio $\hat{h} = C_{S(1:N)} + \hat{g}$ is given by equation (39), the recursive prices C are characterized by the PDE equation (45), or equations (59) or (61), subject to a boundary condition $C(T, S(T))$, the market prices of risk are given by equation (50), the total hedging error is given by equation (55), and the multiperiod risk premium is given by equation (62). ■*

5 Multiperiod Examples

We give three examples where we apply this methodology. We work in continuous time since it is easier to derive close-form solutions or characterize prices through partial differential equations, which

¹⁴The standard Black and Scholes (1973) formula can be analyzed from two angles, complete markets or (say) risk-neutral pricing. Let μ and σ be drift and volatility of S_t , respectively. With complete markets, the one-period residual risks and associated risk premiums are equal to zero and $C_0 = E_0^Q [e^{-rT} C_T]$, where Q is the unique risk-neutral probability measure. Then (in risk-neutral pricing), C_0 can be rewritten as

$$C_0 = E_0^{\mathcal{P}} [e^{-rT} C_T] - E_0^{\mathcal{P}} \left[\int_0^T e^{-rt} \frac{\mu - r}{\sigma} S_t \frac{\partial C_t}{\partial S_t} dt \right],$$

where the first part is the value of the hedging portfolio that invests only in the bond, which has a zero expected mean hedging error, and \mathcal{P} is the actual probability measure. A trivial result follows for call (the opposite for put) payoffs, $C_0 < E_0^{\mathcal{P}} [e^{-rT} C_T]$ if and only if $\mu > r$ since $\frac{\partial C_t}{\partial S_t} > 0$.

can be solved numerically. In particular, we assume that all the state variables follow a diffusion process, which is standard in the literature, and we can compare this with other results. First, we study the pricing of basis risk for lognormal processes, then the Heston (1993) stochastic volatility model, and finally, a put option under short-selling constraints in the standard Black-Scholes model.

5.1 Basis Risk

We price a real option, or an option subject to basis risk. We have an European call option C , which depends on an underlying asset V which is not traded or is illiquid. There are many examples of nontraded assets such as weather, electricity, or of illiquid assets such as options on real estate purchases, etc. However, there exists a second traded asset S that is correlated with the non-tradable one. For example, the option could be defined on an illiquid commodity (e.g., Mexican oil), but one could use a correlated and more liquid asset (e.g., the Texas oil future contract) as the hedging asset. Then, it is possible to hedge the option partially and to derive pricing implications.¹⁵

This problem is also studied by Cochrane and Saá-Requejo (2000), and here we use the same notation and continuous-time dynamics. The dynamics of both assets under the true (lognormal) probability measure P are given by

$$dS = \mu_s S dt + \sigma_s S dz_{1,t} \quad \text{and} \quad (63)$$

$$dV = \mu_v V dt + \sigma_v V (\rho dz_{1,t} + \sqrt{1 - \rho^2} dz_{2,t}), \quad (64)$$

where $dz_{1,t}$ and $dz_{2,t}$ are two standard orthogonal Brownian motions, the parameter ρ measures the correlation between the returns of V and S , and there exists a risk-free asset with return equal to r . Note that dz_2 is the residual risk and is orthogonal to dS , i.e., $E_t^P[dS_t dz_{2,t}] = 0$.

Let T be the option maturity and E the strike price, i.e., $C(V(T)) = \max\{V(T) - E, 0\}$. Because S is a tradable asset, from Merton (1973) we know that the no-arbitrage bounds of a call option $C(S)$ are $\max\{S - Ee^{-r(T-t)}, 0\} < C(S) < S$. However, because V is nontradable and if $|\rho| < 1$, one can show that the no-arbitrage bounds of $C(V)$ are much more unconstrained, i.e., $0 < C(V) < \infty$. Consequently, any non-negative price is feasible as it does not allow arbitrage opportunities, and the arbitrage bounds are unpractical.

The Hedging Strategy. By applying Itô's Lemma we can decompose the return of dC into

$$dC = (C_t + \mu_v V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV}) dt + \sigma_v V C_V (\rho dz_{1,t} + \sqrt{1 - \rho^2} dz_{2,t}). \quad (65)$$

¹⁵There are many other applications of this basis risk model. V is a small stock, S is a correlated, but more liquid, stock. V is a basket of assets, S is an index. V is the short-term interest rate, S are the prices of liquid bonds. In emerging markets one finds at most one or two liquid bonds. V is inflation, S is a long-term bond. Executive stock options in the company V , where the executive can trade in any stock S except V . Option pricing with basis risk is studied in Davis (1998), Detemple and Sundaresan (1999), and Luenberger (2002), among others. Another incomplete market problem is that of hedging of long-term exposures by rolling over short-term futures contracts (see Ross (1997)).

Consider the following minimum variance hedging strategy, since dz_2 is orthogonal to dS ,

$$\hat{h}_1 = \frac{\sigma_v V C_V}{\sigma_s S} \rho. \quad (66)$$

Then, the return of the hedging portfolio, $a(\hat{h}_1 S - C)$, is equal to

$$a(\hat{h}_1 dS - dC) = -a \left(C_t + \left(\mu_v - \mu_s \frac{\rho \sigma_v}{\sigma_s} \right) V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV} \right) dt - a \sigma_v V C_V \sqrt{1 - \rho^2} dz_{2,t}. \quad (67)$$

The PDE equation. If we forget for a moment the residual risk, $dz_{2,t}$, then the return of this portfolio, $a(\hat{h}_1 S - C)$, is risk free. The law of one price no-arbitrage condition implies that

$$-a \left(C_t + \left(\mu_v - \mu_s \frac{\rho \sigma_v}{\sigma_s} \right) V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV} \right) = -a \left(C - \frac{\rho \sigma_v V C_V}{\sigma_s} \right) r. \quad (68)$$

Note that if $\rho = 1$, this is a standard complete markets problem, and therefore, we obtain the same no-arbitrage condition on the drift process.

However, we still have the residual risk, $dY_t^{\hat{h}} = \sigma_v V C_V \sqrt{1 - \rho^2} dz_{2,t}$, which cannot be hedged at all. Let $y_t dt$ (where $y_t = 0$ if $|\rho| = 1$) be this risk premium. The risk-return trade-off of $\mathcal{N}(y_t dt, \sigma_v V C_V \sqrt{1 - \rho^2} \sqrt{dt})$ is attractive for the writer (or buyer) of the option. If y_t is finite, under technical conditions, this premium is arbitrage-free. Then we have

$$-a \left(C_t + \left(\mu_v - \mu_s \frac{\rho \sigma_v}{\sigma_s} \right) V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV} \right) = -a \left(C - \frac{\rho \sigma_v V C_V}{\sigma_s} \right) r + y_t. \quad (69)$$

The investor in C obtains an extra premium $y_t dt$ for carrying extra risk on $dz_{2,t}$. Note that since $a^2 = 1$, the latter equation can be rewritten as

$$C_t + \mu_v V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV} - \frac{\mu_s - r}{\sigma_s} \rho \sigma_v V C_V = rC - ay_t. \quad (70)$$

Examples. If the risk premium is proportional to the option price, i.e., $y_t = \alpha \sigma_v \sqrt{1 - \rho^2} C$, similar to equation (28), then

$$C_t + \mu_v V C_V + \frac{1}{2} \sigma_v^2 V^2 C_{VV} - \frac{\mu_s - r}{\sigma_s} \rho \sigma_v V C_V = (r - \alpha \sigma_v \sqrt{1 - \rho^2}) C, \quad (71)$$

i.e., the risk-neutral return of the option is equal to $r - \alpha \sigma_v \sqrt{1 - \rho^2}$.

Another interesting example is to assume that the risk premium is proportional to the option Gamma, i.e., $y_t = \frac{1}{2} \alpha \sqrt{1 - \rho^2} \sigma_v^2 V^2 C_{VV}$, and $\alpha > 0$. We have

$$C_t + \left(\mu_v - \frac{\mu_s - r}{\sigma_s} \rho \sigma_v \right) V C_V + \frac{1}{2} \sigma_v^2 \left(1 + \alpha \sqrt{1 - \rho^2} \right) V^2 C_{VV} = rC. \quad (72)$$

Interestingly, the risk-neutral volatility is equal to $\sigma_v \sqrt{1 + \alpha \sqrt{1 - \rho^2}}$ ¹⁶ which is different from σ_v , the volatility under the actual probability measure, if $|\rho| \neq 1$. Whereas both volatilities must be

¹⁶Let $\alpha \geq 0$. We assume that $1 + \alpha \sqrt{1 - \rho^2} \geq 0$, and equivalently, $-\alpha \leq (\sqrt{1 - \rho^2})^{-1}$. Thus, for the upper bound ($a = +1$), this inequality holds for $\alpha \geq 0$. For the lower bound ($a = -1$), α is constrained, $0 \leq \alpha \leq (\sqrt{1 - \rho^2})^{-1}$, which makes sense to avoid negative option prices as the lower arbitrage bound is zero.

equal in a complete market model, this constraint does not necessarily hold in incomplete markets. If markets are incomplete, it is an empirical issue whether this risk premium specification is valid.

Assume now that $y_t = \tilde{A}\sigma_v VC_V \sqrt{1-\rho^2}$, i.e., y_t is proportional to the hedging error standard deviation, and note that $y_t > 0$ if and only if $\tilde{A} > 0$, since $C_V > 0$ for call payoffs.¹⁷ Then we can recover another Black-Scholes PDE type,

$$C_t + \mu_v VC_V + \frac{1}{2}\sigma_v^2 V^2 C_{VV} = rC + \left(\frac{\mu_s - r}{\sigma_s} \rho - a\tilde{A}\sqrt{1-\rho^2} \right) \sigma_v VC_V. \quad (73)$$

For this problem, equation (73) seems to be a very reasonable way of pricing this basis risk problem.

Let us remark that in continuous time and diffusion processes, the linearity of returns implies that the hedging criteria is unique and only the valuation of the residual risk makes the difference.

The Associated RNP Measure. Let us extract the RNP measure $Q^{\hat{h},y}$ and call $(S\mu_s^{\hat{h},y}, S\sigma_s^{\hat{h},y})$ and $(V\mu_v^{\hat{h},y}, V\sigma_v^{\hat{h},y})$ the risk neutral parameters under $Q^{\hat{h},y}$ for dS and dV , respectively. Clearly, $(\mu_s^{\hat{h},y}, \sigma_s^{\hat{h},y}) = (\mu_s - \frac{\mu_s - r}{\sigma_s} \sigma_s, \sigma_s) = (r, \sigma_s)$, and $(\mu_v^{\hat{h},y}, \sigma_v^{\hat{h},y}) = (\mu_v - \frac{\mu_s - r}{\sigma_s} \rho \sigma_v + a\frac{y_t}{VC_V}, \sigma_v)$ from equation (70). In other words, $\lambda_1 = \frac{\mu_s - r}{\sigma_s}$ and $\lambda_2 = -a\frac{y_t}{\sigma_v VC_V \sqrt{1-\rho^2}}$ are the market prices of risk associated with dz_1 and dz_2 , respectively. Note that the market prices of risk and therefore $Q^{\hat{h},y}$ are unique. For example, if $y_t = 0$, $\lambda_2 = 0$; if $y_t = \alpha\sigma_v \sqrt{1-\rho^2} C$, $\lambda_2 = -a\frac{\alpha C}{VC_V}$ (which is related to the inverse of the option price elasticity since $\lambda_2 = -\alpha\alpha (\frac{\partial C}{\partial V} \frac{V}{C})^{-1}$); and if $y_t = \tilde{A}\sigma_v VC_V \sqrt{1-\rho^2}$, $\lambda_2 = -a\tilde{A}$. On the other hand, if $y_t = \frac{1}{2}\alpha\sqrt{1-\rho^2}\sigma_v^2 V^2 C_{VV}$, the risk-neutral drift and volatility parameters can be interpreted differently, and $\lambda_2 = -a\frac{1}{2}\alpha\sigma_v \frac{VC_{VV}}{C_V}$, which is related to the curvature of the option price.

To check that these prices of risk λ_2 are well defined, one can simply check that lower option price bound is non-negative. Also, the risk premium y_t can be empirically estimated from the one-period errors. That is, the one-period hedging errors $y_t dt + dY_t^{\hat{h}} = y_t dt + \sigma_v VC_V \sqrt{1-\rho^2} dz_{2,t}$ can be regressed on C , VC_V , and $V^2 C_{VV}$ (or other variables) and test if they are statistically significant. This is related to the next point.

Risk Management. Assume that $y_t = \tilde{A}\sigma_v VC_V \sqrt{1-\rho^2}$. First, the total hedging error is simply the sum of the one-period hedging errors plus the one-period risk premiums, financed or invested at the riskless rate r , i.e.,

$$Y_T^{\hat{h}} = a(X_{\hat{h}}(T) - C(T, S(T))) = \sigma_v \sqrt{1-\rho^2} \int_0^T e^{r(T-t)} VC_V (\tilde{A} dt - a dz_{2,t}). \quad (74)$$

That is, $C(T, S(T)) = X_{\hat{h}}(T) - aY_T^{\hat{h}}$, where $X_{\hat{h}}(T) = \sigma_v \rho \int_0^T e^{r(T-t)} VC_V dz_{1,t}$ is the risk that can be hedged and $-aY_T^{\hat{h}} = \sigma_v \sqrt{1-\rho^2} \int_0^T e^{r(T-t)} VC_V dz_{2,t}$, besides the risk premium, is the risk that

¹⁷That $C_V > 0$ ($C_V < 0$) for call (put) payoffs can be proved following Bergman et al. (1996).

cannot be hedged. Second, the associated risk premium is given by

$$aE_0^{Q^{\hat{h}}} \left[\sigma_v \sqrt{1 - \rho^2} \tilde{A} \int_0^T e^{-rt} V C_V dt \right]. \quad (75)$$

The PDE equation (73) has a close form solution of the Black-Scholes type, this is the reason why we do not show this solution. The risk premium in equation (75) also has a close form solution given by the difference of two call options, which both satisfy equation (73); the first with $\tilde{A} > 0$ and the second with $\tilde{A} = 0$, which implies a positive (negative) premium if $a = 1$ ($a = -1$).

5.2 Stochastic Volatility

Let us assume a stochastic volatility model. Assume that the option market is illiquid and it is not possible to trade in a second option. Therefore, we face an incomplete market problem where the residual risk is going to be the volatility risk since we cannot *Vega* hedge. In other words, there is not a unique market price of risk associated with the volatility risk, since this risk cannot be hedged. This assumption is realistic since it could be very expensive to trade in illiquid options.

We consider the general mean-reverting model of Heston (1993),

$$dS = \mu S dt + \sqrt{v} S dz_{1,t}, \text{ and} \quad (76)$$

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v} \left(\rho dz_{1,t} + \sqrt{1 - \rho^2} dz_{2,t} \right) \quad (77)$$

where v is the stochastic variance, which is mean-reverting with parameters of mean reversion rate κ , long term level θ , and volatility σ , and $dz_{1,t}$ and $dz_{2,t}$ are two independent Wiener processes where ρ measures the correlation between dS and dv . Let C be the option price.

By Ito's Lemma,

$$dC = \left(C_t + \mu S C_S + \frac{1}{2} v S^2 C_{SS} + \kappa(\theta - v) C_v + \frac{1}{2} \sigma^2 v C_{vv} + \rho \sigma v S C_{vS} \right) dt \quad (78)$$

$$+ \sqrt{v} S C_S dz_{1,t} + \sigma \sqrt{v} C_v \left(\rho dz_{1,t} + \sqrt{1 - \rho^2} dz_{2,t} \right).$$

Consider the hedging strategy

$$\hat{h}_1 = \left(C_S + \rho \frac{\sigma C_v}{S} \right), \quad (79)$$

where we also hedge the correlated stochastic volatility, which depends on ρ . This strategy minimizes the hedging error variance. Then, by Ito's lemma the return of this hedging portfolio is given by

$$dC - \left(C_S + \rho \frac{\sigma C_v}{S} \right) dS \quad (80)$$

$$= \left(C_t - \mu S \rho \frac{\sigma C_v}{S} + \frac{1}{2} v S^2 C_{SS} + \kappa(\theta - v) C_v + \frac{1}{2} \sigma^2 v C_{vv} + \rho \sigma v S C_{vS} \right) dt + \sigma \sqrt{v} C_v \sqrt{1 - \rho^2} dz_{2,t},$$

where $\sigma \sqrt{v} C_v \sqrt{1 - \rho^2} dz_{2,t}$ is the residual risk.

Then, again, by applying the law of one price,

$$C_t - \mu S \rho \frac{\sigma C_v}{S} + \frac{1}{2} v S^2 C_{SS} + \kappa(\theta - v) C_v + \frac{1}{2} \sigma^2 v C_{vv} + \rho \sigma v S C_{vS} = r \left(C - \left(C_S + \rho \frac{\sigma C_v}{S} \right) S \right). \quad (81)$$

Let me further specify the model and assume that the risk premium is proportional to the residual risk volatility times volatility, i.e., $y_t = \tilde{A} \sigma \sqrt{1 - \rho^2} \sqrt{v} C_v \sqrt{v} = \tilde{A} \sigma \sqrt{1 - \rho^2} v C_v$. Note that $\tilde{A} > 0$ if we want to compute the two bounds of call (or put) option as $C_v > 0$. Then

$$\begin{aligned} & C_t - \mu S \rho \frac{\sigma C_v}{S} + \frac{1}{2} v S^2 C_{SS} + \kappa(\theta - v) C_v + \frac{1}{2} \sigma^2 v C_{vv} + \rho \sigma v S C_{vS} \\ &= r \left(C - \left(C_S + \rho \frac{\sigma C_v}{S} \right) S \right) - a \tilde{A} \sigma v C_v \sqrt{1 - \rho^2}, \end{aligned} \quad (82)$$

which can be rewritten as

$$\begin{aligned} & C_t + \mu S C_S + \frac{1}{2} v S^2 C_{SS} + \kappa(\theta - v) C_v + \frac{1}{2} \sigma^2 v C_{vv} + \rho \sigma v S C_{vS} \\ &= r C + \frac{\mu - r}{\sqrt{v}} \sqrt{v} S C_S + \left(\frac{\mu - r}{\sqrt{v}} \rho - a \tilde{A} \sqrt{v} \sqrt{1 - \rho^2} \right) \sigma \sqrt{v} C_v. \end{aligned} \quad (83)$$

This implies that $\lambda_1 = \frac{\mu - r}{\sqrt{v}}$ and $\lambda_2 = -a \tilde{A} \sqrt{v}$ (with $\lambda_2 = 0$ if $\tilde{A} = 0$) are the unique market prices of risk associated with dz_1 and dz_2 , respectively.

Equation (83) is related to the one derived in Heston (1993, 329, equation (6)). We refer to Heston for the economic assumptions behind this equation in complete markets or risk-neutral pricing and for the appropriate way to solve the equation by Fourier-transformed methods. Note that consistent with no-arbitrage option pricing theory, $\lambda_1 = \frac{\mu - r}{\sqrt{v}}$ is the endogenous market price of the risk that can be hedged, dz_1 , and $\lambda_2 = -a \tilde{A} \sqrt{v}$ is the exogenous price of the risk that cannot be hedged, dz_2 .

On the other hand, from Proposition 4, the hedging error is given by

$$Y_T^{\tilde{h}} = a \left(X_T^{\tilde{h}} - C_T \right) = \sigma \sqrt{1 - \rho^2} \int_0^T e^{r(T-t)} C_v \left(\tilde{A} v dt - a \sqrt{v} dz_{2,t} \right), \quad (84)$$

and the associated risk premium is given by

$$a E_0^{Q^{\tilde{h}}} \left[\sigma \sqrt{1 - \rho^2} \tilde{A} \int_0^T e^{-rt} v C_v dt \right]. \quad (85)$$

The recursive approach coincides also with local risk minimization in the literature that computes optimal hedging portfolios, and where the risk premium is chosen to be equal to zero. See Heath, Platen, and Schweizer (2001) for a theoretical and a numerical analysis of local risk minimization versus other fully optimal hedging strategies for a stochastic volatility model.¹⁸

¹⁸For instance, in Heath, Platen, and Schweizer (2001, 394-395), under the local risk minimization approach, equations (3.1) and (3.6), (3.3) and (3.4) give the PDE that satisfies the option price, the hedging strategy, and the residual risk, respectively. Note that these correspond to equations (81), (79), and to $\int_0^T \sigma \sqrt{v} C_v \sqrt{1 - \rho^2} dz_{2,t}$ in this paper, respectively, if we specialize Heath, Platen, and Schweizer to the Heston (1993) model and we call $v^Q = C$, $X = S$, $Y = \sqrt{v}$, and $b = \sigma \sqrt{v}$, and if $r = 0$ and $y_t = 0$. Moreover, the recursive RNP measure $Q^{\tilde{h}}$ in this paper is equal to the so-called Minimal Martingale Measure in this literature.

5.3 Portfolio Constraints

We assume the standard Black-Scholes-Merton model. The stock S follows the lognormal process

$$dS = \mu S dt + \sigma S dz_t. \quad (86)$$

We are interested in pricing a put option with maturity T and strike price E , i.e., $C(T, S(T)) = \{E - S(T)\}^+$. We have to sell the stock S to hedge the option risk. In practice, in many markets there are short-sales constraints.¹⁹ Assume that $h_1 \geq \delta_m$, where $-1 \leq \delta_m < 0$. We can study a discrete-time Cox-Ross-Rubinstein binomial model in the same way.

By applying Ito's Lemma, we can decompose the risk-return of dC into

$$dC = \left(C_t + \mu S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} \right) dt + \sigma S C_S dz_t. \quad (87)$$

Define the hedging strategy h_1 as related to the well-known delta hedge, $C_S = \frac{\partial C_t(S)}{\partial S}$. That is,

$$h_1 = C_S 1_{\{C_S \geq \delta_m\}} + \delta_m 1_{\{C_S < \delta_m\}}, \quad (88)$$

and clearly h_1 is the minimum variance portfolio for this problem with short-selling constrains, with $1_{\{\cdot\}}$ the indicator function. Then, the return of the hedging portfolio, $h_1 S - C$, is given by

$$h_1 dS - dC = - \left(C_t + \mu S (C_S - h_1) + \frac{1}{2} \sigma^2 S^2 C_{SS} \right) dt - \sigma S (C_S - \delta_m) 1_{\{C_S < \delta_m\}} dz_t. \quad (89)$$

As in the previous examples, if we forget the residual risk, $\sigma S (C_S - \delta_m) 1_{\{C_S < \delta_m\}} dz_t$, then the return of this portfolio is risk free. Therefore, the law of one price implies that

$$- \left(C_t + \mu S (C_S - h_1) + \frac{1}{2} \sigma^2 S^2 C_{SS} \right) = -r (C - h_1 S). \quad (90)$$

Note that if $\delta_m = -1$, the market is complete. Then, $h_1 = C_S$ and we obtain the no-arbitrage condition on the drift process. This equation can be rewritten as

$$C_t + \mu S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} = rC + \frac{\mu - r}{\sigma} \left(1_{\{C_S \geq \delta_m\}} + \frac{\delta_m}{C_S} 1_{\{C_S < \delta_m\}} \right) \sigma C_S S, \quad (91)$$

from where the market price of risk associated with dz is given by $\lambda = \frac{\mu - r}{\sigma} \left(1_{\{C_S \geq \delta_m\}} + \frac{\delta_m}{C_S} 1_{\{C_S < \delta_m\}} \right)$, which is nonlinear and stochastic. Note that if $C_S \geq \delta_m$ then $\lambda = \frac{\mu - r}{\sigma}$, and if $C_S < \delta_m$ then $|\lambda| = \left| \frac{\mu - r}{\sigma} \right| \frac{\delta_m}{C_S} < \left| \frac{\mu - r}{\sigma} \right|$.

However, we still have the residual risk, $\sigma S (\delta_m - C_S) 1_{\{C_S < \delta_m\}} dz_t$, which cannot be hedged at all. Let $y_t 1_{\{C_S < \delta_m\}} dt$ (i.e., $y_t 1_{\{C_S < \delta_m\}} = 0$, if $C_S \geq \delta_m$) be this risk premium. Again, the risk-return trade-off of $\mathcal{N}(y_t dt, \sigma S (\delta_m - C_S) \sqrt{dt}) 1_{\{C_S < \delta_m\}}$ is attractive for the writer of the option. Then,

$$C_t + \mu S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} = rC + \frac{\mu - r}{\sigma} \left(1_{\{C_S \geq \delta_m\}} + \frac{\delta_m}{C_S} 1_{\{C_S < \delta_m\}} \right) \sigma C_S S - y_t 1_{\{C_S < \delta_m\}}. \quad (92)$$

¹⁹Short-sale constraints appear to avoid insider trading, default issues, etc., or in executive stock options, where executives are legally subject to portfolio constraints.

Specialize the model further and assume that y_t is proportional to the hedging error standard deviation, i.e., $y_t = \tilde{A}\sigma S(\delta_m - C_S)1_{\{C_S < \delta_m\}}$, and note that $y_t > 0$ if and only if $\tilde{A} > 0$, since $(\delta_m - C_S)1_{\{C_S < \delta_m\}} > 0$. Then we have

$$C_t + \mu SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} = rC + \left(\frac{\mu - r}{\sigma} + \left(\frac{\mu - r}{\sigma} - \tilde{A} \right) \frac{\delta_m - C_S}{C_S} 1_{\{C_S < \delta_m\}} \right) \sigma C_S S. \quad (93)$$

From this equation, $\lambda = \frac{\mu - r}{\sigma} + \left(\frac{\mu - r}{\sigma} - \tilde{A} \right) \frac{\delta_m - C_S}{C_S} 1_{\{C_S < \delta_m\}}$ is the market price of risk. Therefore, the upper bound C^s verifies this equation following our methodology. Note that it is a nonlinear PDE, but can be solved through classic finite difference or binomial trees methods.

Because the short-selling constraint does not affect the hedging of the long position, then the lower bound C^l is equal to the Black-Scholes-Merton price, which solves

$$C_t + \mu SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} = rC + \frac{\mu - r}{\sigma} \sigma SC_S. \quad (94)$$

The condition $C^s \geq C^l$ implies that $\left(\frac{\mu - r}{\sigma} - \tilde{A} \right) \frac{\delta_m - C_S}{C_S} 1_{\{C_S < \delta_m\}} \sigma C_S S \leq 0$ from equations (93) and (94). Equivalently, $\tilde{A} \geq \frac{\mu - r}{\sigma}$. The empirical implication of this result is that bid (ask) prices should be greater than (closer to) the standard no-arbitrage price.

A regular empirical anomaly of option markets is a volatility smile or smirk inconsistent with the Black-Scholes formula. The Black-Scholes model is based on the normality of returns, and the main way to explain this abnormal pattern is to consider a more general process for the underlying as with jumps or stochastic volatility. However, the assumption of frictionless markets has not been removed in spite of the fact that it is not completely realistic. If $\tilde{A} > \frac{\mu - r}{\sigma}$, the short-selling constraint studied in this paper implies a positive risk premium, which increases with the option's moneyness.

Therefore, this model produces a volatility smirk for put options, which is empirically plausible especially for short-term options. Given the evidence of transaction costs in short-selling positions in equity markets (see Ofek et al. (2003)), and the interest in put options, as a way of providing portfolio insurance in downward markets, this pricing formula is relevant.

6 Summary and Extensions

Markets can be incomplete from jumps and stochastic volatility in stock and bond returns of well-developed and liquid financial option markets to more illiquid real options and other projects which have different embedded options. Market frictions (e.g., non-continuous trading, transaction costs, portfolio constraints, illiquidity, etc.), nontradable assets (e.g., stochastic volatility or basis risk), real options, or emerging markets imply that markets can be incomplete.²⁰

In a model of incomplete markets, which assumes recursive one-period optimal portfolios, a simple but tractable criterion, and that hedging errors and their risk premiums are financed to the riskless

²⁰Figlewski and Green (1999) show that even the most liquid and developed option markets bear many residual risks.

rate, we have derived two main results. First, that the residual risk does not depend on the risk premium process (for continuous-time and diffusion processes). Therefore, any arbitrage-free price is just the price of a hedging portfolio (such as in a complete market) plus a premium associated to the residual risk (which produces a contingent risk premium at maturity). Second, we derive an optimal frontier in the non-arbitrage option prices/risk premiums space, and thus, we reduce pricing in incomplete markets to the explicit valuation of a one-period orthogonal diffusion risk.

First, further research could apply the present technology to other problems such as real options or other investment or corporate projects (see, e.g., Merton (1998)). Second, the study of different hedging strategies and risk premiums in discrete-time models, which can depend on the model's statistical properties (such as skewness or kurtosis) or on economic factors (such as default issues, initial wealth, etc.), deserves future research. Third, research could investigate the comparison of one-period recursive strategies with fully optimal hedging strategies, which produce a less risky residual risk. Fourth, it is worth studying an extension of the recursive formulation to general jump-diffusion and other stochastic processes in continuous time. Fifth, note that the risk premium associated with the hedging errors is not constrained to depend on a price of risk. Therefore, a more flexible estimation of this premium in empirical work is consistent with non arbitrage. Sixth, in an incomplete market the risk premium of each security can be determined with the view of total portfolio risk.

A natural and relevant example, in which illiquidity and other frictions are observed daily is that of emerging markets. Many other examples appear on the very near horizon, as well.

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7 Numerical results

Here we solve the “pde” equations associated with two of the three problems studied in Section 5. The objective is to develop further intuition for these problems. We show below that our incomplete markets pricing model yields sensible and intuitive results.

For Basis Risk, asset prices are lognormally distributed and the pde can be solved in close form solution. We present several Figures where the different parameters are analyzed in detail. For Short-Sale Constraints, we do not have a close form solution, so we use binomial trees or finite-different methods. This problem is similar to the free-boundary problem seen in American-style securities. As the option put payoff is known at maturity, both methods can easily manage the free-boundary with one-state variable by backward recursion. We explain both methods, and note that the binomial method does not work well. And for Stochastic Volatility, we do not provide results since option prices can be solved following Heston (1993).

7.0.1 Basis Risk

In the examples that follow the risk premium associated with the residual risk is proportional to the residual risk volatility, which is equivalent to defining a market price of risk associated with the residual risk, $\tilde{A} = \lambda_V$. Let λ_S be the price of risk associated with the traded asset, S . Therefore (see equation (73)), the risk premium associated with the call option is given by $(\lambda_S \rho - a \lambda_V \sqrt{1 - \rho^2}) \sigma_V V C_V$.

For example, even if $\lambda_V = \lambda_S$, this risk premium does not simplify to $\pm \sigma_V V C_V \lambda_S$, but is given by $\lambda_S (\rho - a \sqrt{1 - \rho^2}) \sigma_V V C_V$ (except for market completeness, i.e., $|\rho| = 1$). As base case parameters, we take $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and therefore, $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. For simplicity, we take $\lambda_V = \lambda_S > 0$ for the upper bound, $\lambda_V = -\lambda_S < 0$ for the lower bound, and also give the zero-premium case where $\lambda_V = 0$. The correlation is $\rho = 0.9$, and the drift and the volatility of V and S are the same, $\mu_V = \mu_S$ and $\sigma_V = \sigma_S$, respectively.

As “ $\ln S_t$ ” is normally distributed, a close-form solution for the pde is given by

$$C(0, V_0) = V_0 e^{(\mu_V^* - r)T} \mathcal{N}(d_1) - E e^{-rT} \mathcal{N}(d_2), \quad (95)$$

$$\mu_V^* = \mu_V - (\lambda_S \rho - a \lambda_V \sqrt{1 - \rho^2}) \sigma_V, \quad d_1 = \frac{\ln \frac{V_0}{E} + (\mu_V^* + \frac{1}{2} \sigma_V^2) T}{\sigma_V \sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{T}, \quad (96)$$

where $a = +1$ ($a = -1$) for the upper (lower) bound, and $\lambda_V = 0$ if the residual risk is not priced.

Figure 1 contains option prices for different values of the non-traded underlying asset V . The intuition is clear. First, the three bounds are increasing and convex functions of V , similar to the Black-Scholes-Merton formula. Second, for far out-the-money payoffs, the probability for which the option can end in-the-money is very low. This implies that the residual risk is also very low, and therefore, the three bounds are very close. Third, the spread between the upper and the lower bound,

like the option risk, go up with the option moneyness. Fourth, in the same Figure 1, at-the-money options are approximately 10% more expensive (cheaper) for the upper (lower) bound than those if the residual risk is not priced, though this difference lowers with the option moneyness.

Figure 2 reports the volatility smile associated with the option prices of Figure 1 (we use the same axes in both Figures). This implicit volatility is computed assuming that $\mu_V^* = r$ under the risk-neutral Q -measure (as if V is a traded security). One can argue that a more reasonable drift under Q is given by $\mu_V^* = \mu_V - \rho\lambda_S\sigma_V$ if $\lambda_V = 0$. However, in this second case, option prices are not always increasing functions of the volatility σ_V , and thus, we do not have a one-to-one relationship. The spread between the upper and the lower bound implicit volatilities goes up with the option moneyness. The zero implicit volatility for the lower bound is simply because this price is less than $V_0 - Ee^{-rT}$ (i.e., a lower arbitrage bound, which does not apply here). And the implicit volatility for the upper bound increases unboundedly.

Figure 3 shows option prices as a function of volatility, σ_V , and proves that option prices are not necessarily increasing functions of volatility in an incomplete markets framework. This result is due to the fact that the risk-neutral drift, which is not simply equal to r , depends on σ_V (i.e., $\mu_V^* = \mu_V - \lambda_S \left(\rho - a\sqrt{1 - \rho^2} \right) \sigma_V$). This can be clearly shown for the lower bound ($a = -1$). If volatility raises, there is an unambiguous option price depression from the negative residual-risk premium, which can dominate any other effect (e.g., from $\sigma_V = 0.02$ to $\sigma_V = 0.03$).

Figure 4 shows one of the innovations of this paper, an optimal frontier between option prices and the risk premium (i.e., the price of risk, λ_V) of the residual risk. The frontier is optimal in the sense that the residual risk is one-period minimized, and hence, its associated risk premium is also the minimum (for a given valuation of this residual risk). The upper (lower) curve are associated with the upper (lower) bound. Figure 4 also shows the option price as a convex function of the variable λ_V , similar results for an at-the-money call option.

Because options are securities in zero-net supply, our model tells us that the option writer and the option buyer will agree in a price which belongs to this frontier, but does not produce a unique point. This parallels the Markowitz mean-variance framework, where investors demand portfolios of the “mean-variance efficient frontier,” but the chosen portfolio depends on the investor risk-aversion. Certainly, one can derive the CAPM if markets clearing and equilibrium is imposed, which is not considered here.

Figure 5 shows the effect of the correlation, ρ . For $|\rho| = 1$, the three prices are the same as the market is complete. The result of ρ is non-linear and non-monotonic for the upper and the lower bound where $\lambda_V \neq 0$. As the market risk premium is positive ($\lambda_S > 0$), increasing ρ depress call option prices (except for a small part) and raises the expected return of option holders. For $\rho = 0$, the residual risk is the largest, but as λ_V and λ_S are equal, option prices are not necessarily larger.

7.0.2 Short-Sale Constraints

Now consider the problem of short-selling constraints, a clear example of market incompleteness. In what follows, we price a put with $T = 0.25$ and $E = 100$, and study how the put price depends on the price of risk of the residual risk, $\tilde{A} = \lambda_r$, and depends on the short-selling constraint, δ_m . The other parameters are $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$.

Let $\delta_m = -0.5$. Figure 6 shows the put price as a function of the stock price and for $\tilde{A} = \{0.0, 1.0, 2.0, 4.0\} \times \lambda_S$. The put price goes up with \tilde{A} , and implies an inverted volatility smirk in Figure 7 (if $\tilde{A} > \lambda_S$), which has been empirically documented for very short-term options. The volatility smirk raises with the moneyness of the put option (if $\tilde{A} > \lambda_S$), being almost negligible for out-the-money options. If $\tilde{A} = \lambda_S$, one obtains the Black-Scholes-Merton price.

Let $\tilde{A} = 2\lambda_S$. Figure 8 shows the put price as a function of the stock price and for $\delta_m = \{0.0, -0.4, -0.7, -1.0\}$. For $\delta_m = -1.0$, the market is complete, and for $\delta_m = 0.0$, short-selling is forbidden. The put price raises with δ_m since $\tilde{A} > \lambda_S$, and implies an inverted volatility smirk in Figure 9, which goes up with the option moneyness since now the put is more difficult to hedge.

In Figure 10, we show the implicit volatility for different maturities. For in-the-money (at- and out-the-money) options, the implicit volatility decreases (raises) with the maturity.

We solve equation (93) by a finite-different method, which requires to discretize this PDE. At each node of the $S \times T$ grid, we simply check if the constraint $C_S < \delta_m$ holds and then, appropriately, solve equation (93). Finite-difference seems to produce convergent results in the limit. We check that the results converge to the complete markets, or Black-Scholes-Merton, price either if $\delta_m = -1.0$ or if $\tilde{A} = \lambda_S$.

On the other hand, we solve equation (93) by a binomial method as well. However, it does not converge as we can check either if $\delta_m \rightarrow -1.0$ or if $\tilde{A} \rightarrow \lambda_S$. Fix a node (t, S_t) of the binomial tree. We price the put option as $C(t, S_t) = H_t^0 e^{-r\Delta t} + \max\{C_S, \delta_m\} S_t$, where $C_S = \frac{C(t+\Delta t)^{up} - C(t+\Delta t)^{down}}{S_{t+\Delta t}^{up} - S_{t+\Delta t}^{down}}$ and where H_t^0 is such that the expected hedging error at time $t + \Delta t$ is zero. Next, if $C_S < \delta_m$, we add a risk premium proportional to the residual risk volatility times $\sqrt{\Delta t}$, which is order Δt . The model, however, underprices the option when we check the two previous cases if $\Delta t \rightarrow 0$. A more detailed analysis of this issue is left for future research.

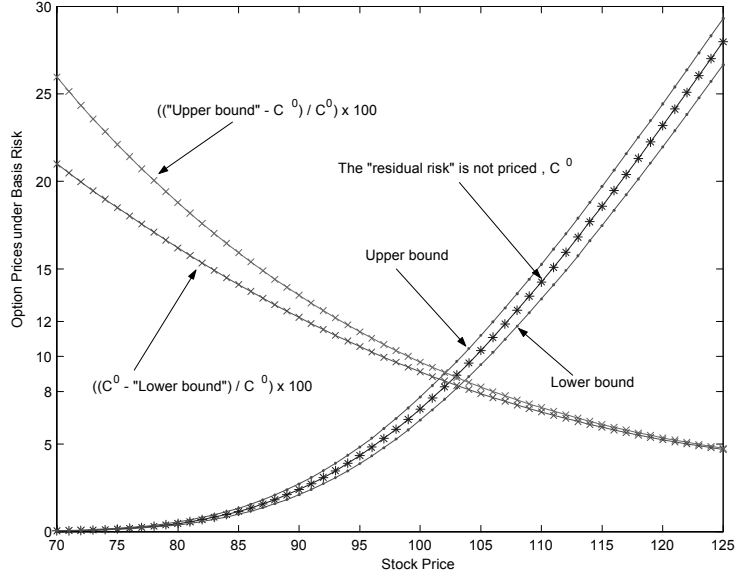


Figure 1: Option call prices under Basis Risk, where $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. We take $\lambda_V = \lambda_S > 0$ for the upper bound, $\lambda_V = -\lambda_S < 0$ for the lower bound, and $\lambda_V = 0$ for the zero premium. The correlation is $\rho = 0.9$, and drift and volatility of V and S are the same, $\mu_V = \mu_S$ and $\sigma_V = \sigma_S$, respectively.

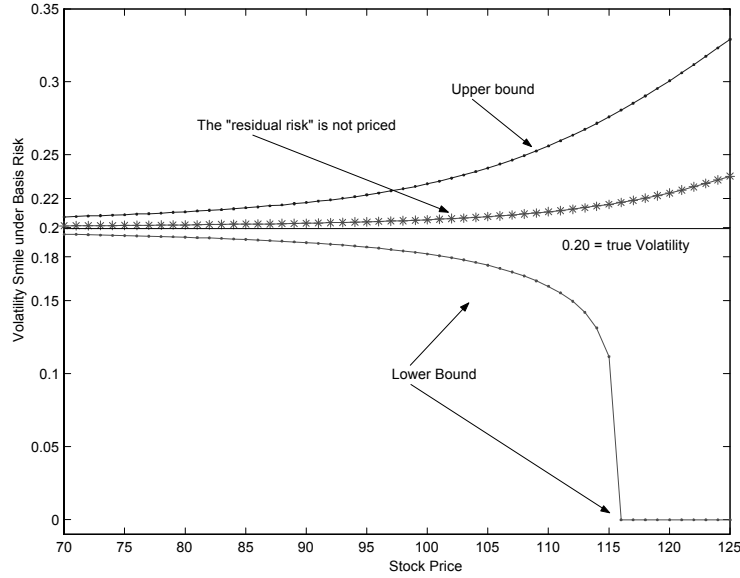


Figure 2: Volatility smile derived from Figure 1. The parameters are $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. Then, $\lambda_V = \lambda_S > 0$ for the upper bound, $\lambda_V = -\lambda_S < 0$ for the lower bound, and $\lambda_V = 0$ for the zero premium. The correlation is $\rho = 0.9$, and $\mu_V = \mu_S$ and $\sigma_V = \sigma_S$. The true price is derived assuming that $\mu_V^* = r$.

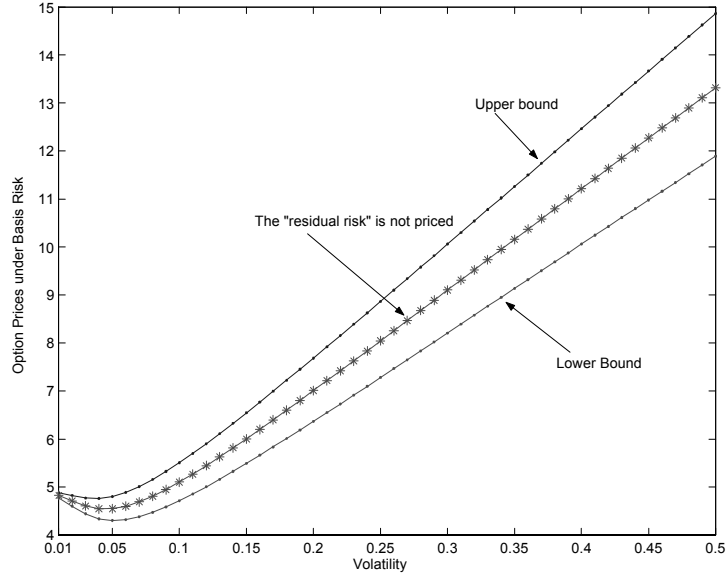


Figure 3: Option call prices under Basis Risk, where $S_0 = 100$, $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. We take $\lambda_V = \lambda_S > 0$ for the upper bound, $\lambda_V = -\lambda_S < 0$ for the lower bound, and $\lambda_V = 0$ for the zero premium. The correlation is $\rho = 0.9$, and the drift of V and S are the same, $\mu_V = \mu_S$.

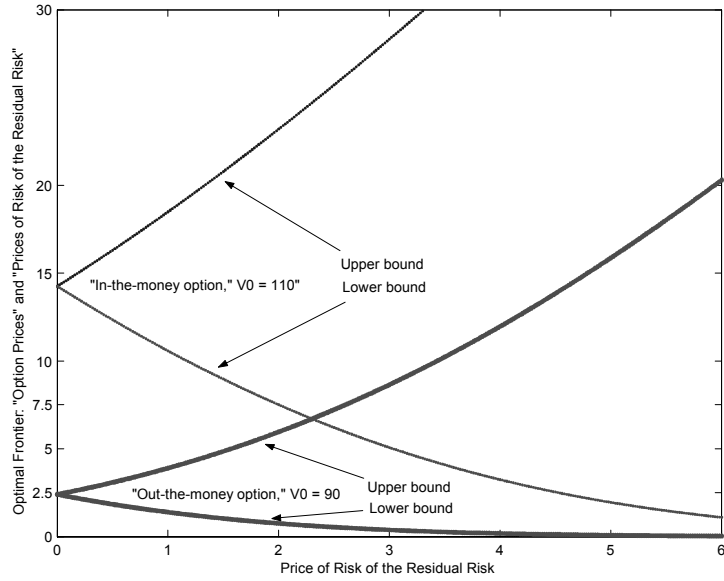


Figure 4: Optimal Frontier, option call prices and price of risk of the residual risk. $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. We take $\lambda_V > 0$ for the upper bound and $\lambda_V < 0$ for the lower bound. The correlation is $\rho = 0.9$, and drift and volatility of V and S are the same, $\mu_V = \mu_S$ and $\sigma_V = \sigma_S$, respectively.

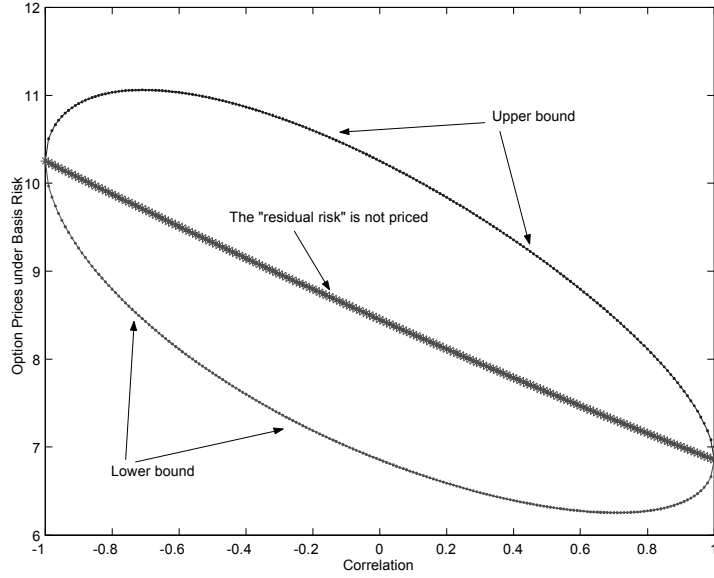


Figure 5: Option call prices under Basis Risk, where $E = 100$ and $T = 0.5$, and $r = 0.0488$, $\mu_S = r + 0.05$, $\sigma_S = 0.20$, and $\lambda_S = \frac{\mu_S - r}{\sigma_S} = 0.25$. We take $\lambda_V = \lambda_S > 0$ for the upper bound, $\lambda_V = -\lambda_S < 0$ for the lower bound, and $\lambda_V = 0$ for the zero premium. The drift and volatility of V and S are the same, $\mu_V = \mu_S$ and $\sigma_V = \sigma_S$, respectively.

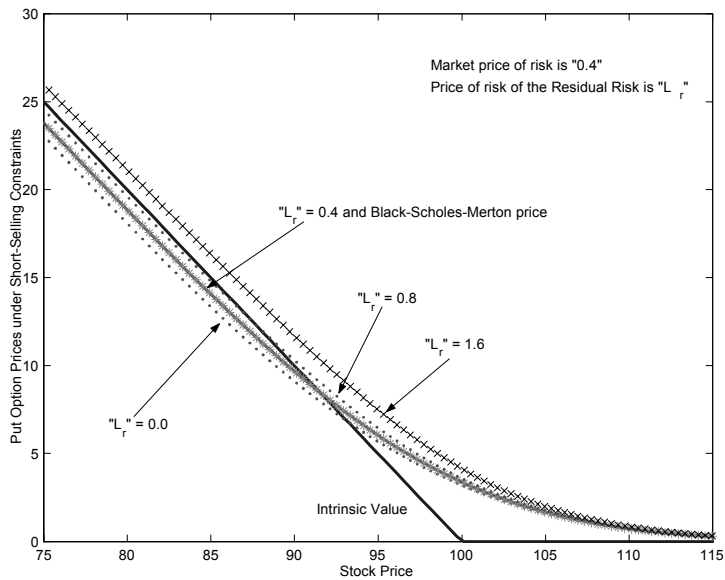


Figure 6: Put option price under short-selling constraints. The parameters are $T = 0.25$ and $E = 100$, and $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$. The short-selling constraint is $\delta_m = -0.5$.

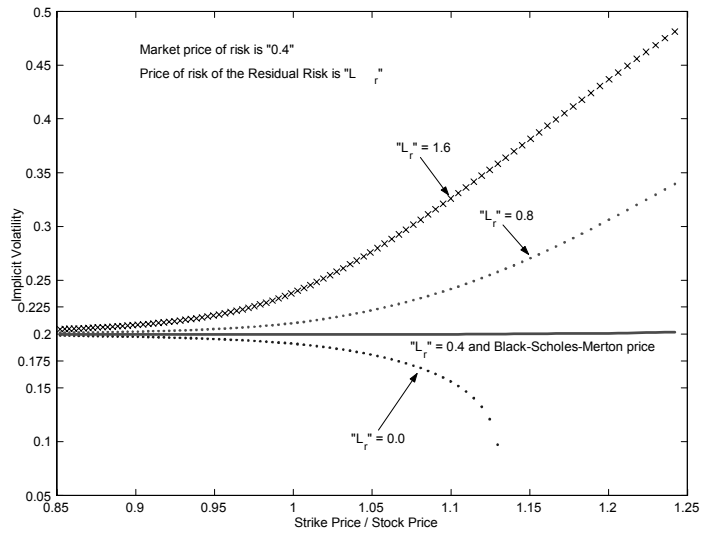


Figure 7: Volatility smile derived from Figure 6. The parameters are $T = 0.25$ and $E = 100$, and $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$. The short-selling constraint is $\delta_m = -0.5$.

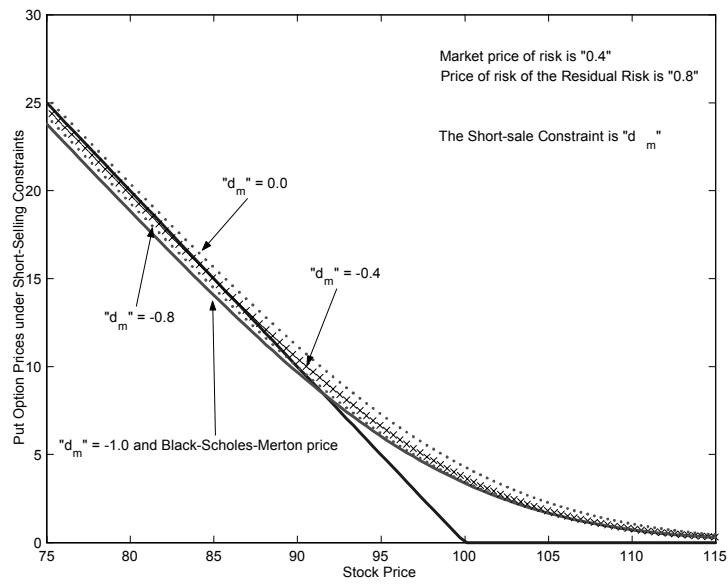


Figure 8: Put option price under short-selling constraints. The parameters are $T = 0.25$ and $E = 100$, and $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$. The price of risk of the residual risk is $\lambda_r = 0.80$.

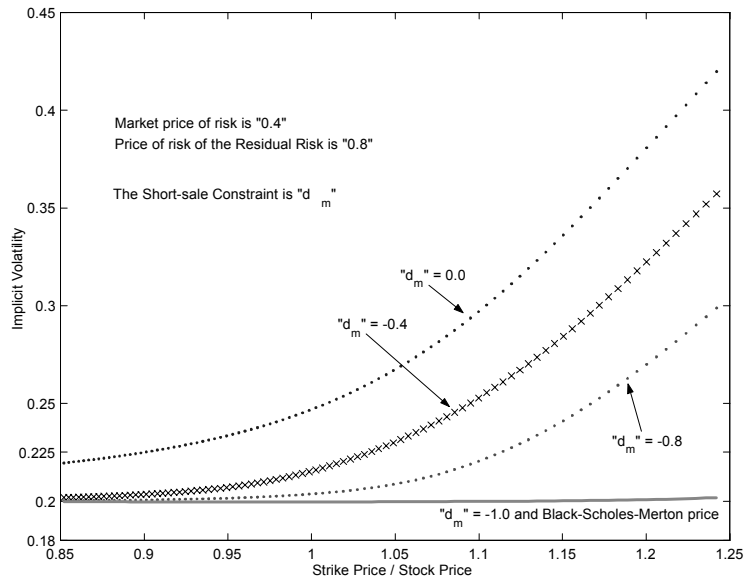


Figure 9: Volatility smile derived from Figure 7. The parameters are $T = 0.25$ and $E = 100$, and $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$. The price of risk of the residual risk is $\lambda_r = 0.80$.

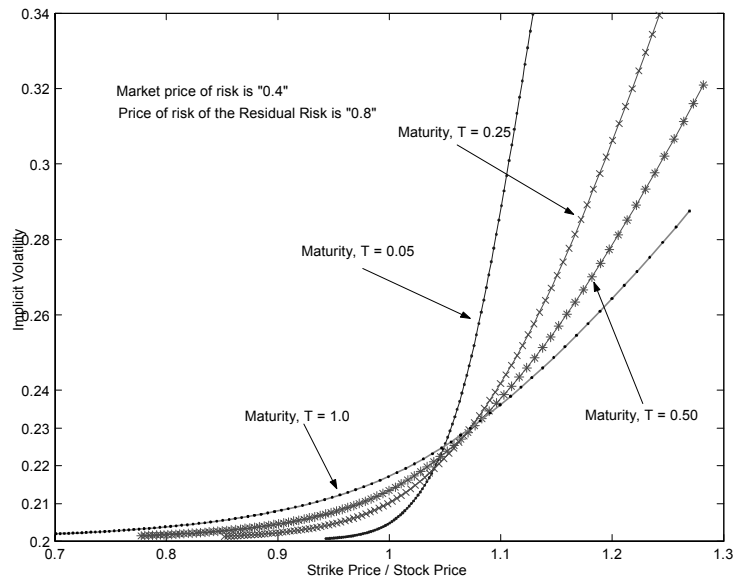


Figure 10: Put option price under short-selling constraints. The parameters are $E = 100$, and $r = 0.0488$, $\sigma = 0.20$, $\mu = r + 0.08$, and $\lambda_S = 0.40$. The price of risk of the residual risk is $\tilde{\Lambda} = 0.80$ with $\delta_m = 0.50$.