The Use of Downside Risk Measures in Portfolio Construction and Evaluation

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Abstract

One of the challenges of using downside risk measures as an alternative constructor of portfolios and diagnostic devise is in their computational intensity. This paper outlines how to use downside risk measures to construct efficient portfolios and to evaluate portfolio performance in light of investor loss aversion. Further, this paper advocates the use of distributional scaling to forecast price movement distributions. This paper could be subtitled, “Strategic Asset Allocation is Dead,” in light of the simulation results.

What is so efficient about the "efficient frontier?" The standard method of constructing the set of efficient portfolios, from which investors are to choose from, is to use the Markowitz (1952) model which defines risk as the standard deviation of a portfolio. Markowitz (1952) recognized that there are many different ways to define risk, but the standard deviation (or variance) of a portfolio is easier to calculate than alternatives.

All portfolio optimization problems can be described as a sequence of mathematical programming problems. First, an analyst must construct a set of efficient frontiers, which are portfolios that maximize the expected return for any given level of risk over different investment horizons. Second, for each period, an investor's utility is to be maximized by picking the portfolio that the investor most prefers (in terms of risk-return combinations). This method allows for the efficient frontier to change with time, which suggests that an investor would modify the composition of their portfolio as time unfolds and as expectations are updated regarding the possible return distributions that the investor may choose from.

Under assumptions that are unrealistic (i.e., investors dislike upside as well as downside risk equally, or that security returns are multivariate normally distributed), it is best to define risk in terms of standard deviation (or variance). This method can be made more robust by defining a general risk function, R, for any given time horizon (the time subscripts will not be used as it clutters the notation without sufficient marginal value-added), as a function of the proportion of an investor's wealth put into each of n securities (with representative weight w_i), and a general probability distribution function of returns $f\left(r_1,...,r_n\right) \geq 0$ such that $\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f\left(r_1,...,r_n\right) dr_n ... dr_1 = 1$.

It simplifies things considerably if I assume there exists a nominally risk-free security over the investor's investment horizon. In this case, the proportion of the investor's wealth that is allocated amongst the risky securities need not equal one, as the allocation to the risk-free security will be endogenously determined (borrowing if it is negative, lending if it is positive). With this simplifying assumption, the applicable returns are risk premiums (excess of expected return over the appropriate risk-free rate). For purposes of presentation, for now on, it is to be assumed that the returns refer to risk-premiums.

To further simplify the notation, I adopt the following conventions:

1) $\mathbf{w}' = (w_1, ..., w_n)$ represents the fractional allocations of wealth to particular risky securities.

2) $\mathbf{r}' = (r_1, ..., r_n)$ represents the risk premiums of the risky securities.

3) $\mathbf{dr} = dr_n ... dr_1$ represents the variables of integration for the probability analysis.

4) The probability distribution can compactly be written as $\int_{\mathbf{a}'}^{\mathbf{b}'} f(\mathbf{r}')\mathbf{dr}$ where it is understood that the integral is an n-integral, with limits of integration given by $\mathbf{a}' = (a_1, ..., a_n)$ and $\mathbf{b}' = (b_1, ..., b_n)$ for each integral.

5) There exists a risk function, $R(\mathbf{w}', f(\mathbf{r}))$.

6) I will assume that the investor is a price-taker, so their portfolio allocation decision does not impact the probability distribution of returns.

Using the preceding conventions, the expected excess return of the portfolio, with allocation $\mathbf{w}'$, is

$$E\left(r_p\right)=\int_{-\infty}^{\infty}\mathbf{w}'\mathbf{r}f\left(\mathbf{r}'\right)\mathbf{dr} \tag{1}$$

The investor's problem, for each time period, is to construct an efficient frontier from the mathematical programming problem as follows

$$\begin{aligned}&\max_{w_1,\ldots,w_n} E\left(r_p\right)\\ &\text{s/t } R=R^*\end{aligned} \tag{2}$$

The investor chooses the weights of the component securities, having *R** vary from some lower bound to an upper bound, as relevant for the risk measure. The easiest way to formulate this problem is to set up the Lagrangian:

$$L=E\left(r_p\right)+\lambda_1\left(R^*-R\right) \tag{3}$$

Maximizing the Lagrangian yields the familiar first order conditions:

$$\frac{\left(\partial E\left(r_p\right)/\partial w_i\right)}{\left(\partial R/\partial w_i\right)}=\lambda_1 \;\; \forall i \tag{4}$$

λ_1 is often interpreted as the "market price of risk" and is equal across all securities within an efficient portfolio.

Since the expected excess-return is a convex combination of the expected excess-returns of the component securities, the differential of the expected excess-return of the portfolio with respect to any given weight is just the expected excess-return of that particular security:

$\partial E(r_p) / \partial w_i = E(r_i)$. The complicated part of this optimization problem is solving the differential of the risk measure with respect to the weights.

Different Efficient Frontiers

For the balance of the paper I examine the consequences on portfolio construction and evaluation of using three different measures of risk: variance, value-at-risk, and lower-partial-moments.

The variance of the portfolio is given by:

$$\sigma_p^2 = \int_{-\infty}^{\infty} \left(\mathbf{w}'\mathbf{r} - E(r_p)\right)^2 f(\mathbf{r}')\mathbf{dr} \qquad (5)$$

The value-at-risk (VaR) (Jorion, 2001)—in terms of excess-return—of a portfolio is defined as $-V^*$, such that for a given level of significance (α):

$$\alpha = \int_{-\infty}^{v_1,\ldots,v_n} f(\mathbf{r}')\mathbf{dr}, \text{ such that } V^* = \sum_{i=1}^{n} w_i v_i \qquad (6)$$

The lower partial moment (with moment *a* and threshold return *T*) (Sortino and Price, 1994) is given by:

$$LPM(a,T) = \int_{-\infty}^{T_1,\ldots,T_n} (T - \mathbf{w}'\mathbf{r})^a f(\mathbf{r}')\mathbf{dr}, \text{ such that } T = \sum_{i=1}^{n} w_i T_i \qquad (7)$$

In light of the maximization problem posed in (2), how do these different risk measures affect the efficient frontiers?

The basic programming problem of the investor using each risk measure can be summarized by the following three Lagrangians:

For the mean-variance framework, where the target variance is allowed to vary from an appropriate lower bound to an appropriate upper bound:

$$\max_{\mathbf{w}'} L_{\sigma_2} = \int_{-\infty}^{\infty} (\mathbf{w}'\mathbf{r}) f(\mathbf{r}')\mathbf{dr} + \lambda_1^{\sigma^2}\left(\sigma_p^2 * - \int_{-\infty}^{\infty} (\mathbf{w}'\mathbf{r} - E(\mathbf{w}'\mathbf{r}))^2 f(\mathbf{r}')\mathbf{dr}\right) \tag{8}$$

For the VaR framework, where the value-at-risk is allowed to vary from some lower bound to some upper bound:

$$\max_{\mathbf{w}'} L_{VaR} = \int_{-\infty}^{\infty} (\mathbf{w}'\mathbf{r}) f(\mathbf{r}')\mathbf{dr} + \lambda_1^{VaR}\left(\alpha - \int_{-\infty}^{v_1,\ldots,v_n} f(\mathbf{r}')\mathbf{dr}\right) + \lambda_2^{VaR}(V * - \mathbf{w}'\mathbf{v}) \tag{9}$$

where $\mathbf{v}' = (v_1, \ldots, v_n)$

Provided that the weight of a risky security is non-zero (say it is security 1—if it is not, just reorder the securities so that it is), the second constraint can be incorporated into the limits of integration (a similar method will be used for the LPM framework):

$$\max_{\mathbf{w}'} L_{VaR} = \int_{-\infty}^{\infty} (\mathbf{w}'\mathbf{r}) f(\mathbf{r}')\mathbf{dr} + \lambda_1^{VaR}\left(\alpha - \int_{-\infty}^{\frac{V*-w_2v_2-\ldots-w_nv_n}{w_1}, v_2\ldots, v_n} f(\mathbf{r}')\mathbf{dr}\right) \tag{10}$$

For the LPM framework, where the lower partial moment is allowed to vary from some lower bound to some upper bound:

$$\max_{\mathbf{w}'} L_{LPM} = \int_{-\infty}^{\infty} (\mathbf{w}'\mathbf{r}) f(\mathbf{r}')\mathbf{dr} + \lambda_1^{LPM}\left(LPM * - \int_{-\infty}^{\frac{T-w_2T_2-\ldots-w_nT_n}{w_1}, T_2,\ldots, T_n} (T - \mathbf{w}'\mathbf{r})^a f(\mathbf{r}')\mathbf{dr}\right) \tag{11}$$

The issue at this point is to characterize the efficient portfolios in each context. This is most easily done by seeing what the solutions to the preceding Lagrangians tell us:

For the mean-variance efficient portfolios:

$$\frac{E\left(r_i\right)}{\int_{-\infty}^{\infty}\left(\mathbf{w}'\mathbf{r}-E\left(r_p\right)\right)\left(r_i-E\left(r_i\right)\right)f\left(\mathbf{r}\right)\mathbf{dr}}=2\lambda_1^{\sigma^2} \tag{12}$$

For value-at-risk efficient portfolios:

$$\frac{E\left(r_i\right)}{\dfrac{\partial\left(\displaystyle\int_{-\infty}^{\frac{V^*-w_2v_2-\ldots-w_nv_n}{w_1},v_2\ldots,v_n} f\left(\mathbf{r}'\right)\mathbf{dr}\right)}{\partial w_i}}=\lambda_1^{VaR}\ \forall i \tag{13}$$

For LPM efficient portfolios:

$$\frac{E\left(r_i\right)}{\dfrac{\partial\left(\displaystyle\int_{-\infty}^{\frac{T-w_2T_2-\ldots-w_nT_n}{w_1},v_2\ldots,v_n} \left(T-\mathbf{w}'\mathbf{r}\right)^a f\left(\mathbf{r}'\right)\mathbf{dr}\right)}{\partial w_i}}=\lambda_1^{LPM}\ \forall i \tag{14}$$

The difference between the portfolios is in terms of the market price of risk. Instead of normalizing the weights by security 1, I can normalize by security *i*, and then change the order of integration such that the outside integral is for security *i*. Leibniz's rule can then be applied to derive the partial derivatives:

For value-at risk:

If $F(r_i) = \int_{-\infty}^{v_1} .. \int_{-\infty}^{v_{i-1}} \int_{-\infty}^{v_{i+1}} . \int_{-\infty}^{v_n} f(r_1, ..., r_n)\, dr_n ... dr_{i+1} dr_{i-1} ... dr_1$

then $G(w_i) = \int_{-\infty}^{\frac{V^* - w_1 v_1 - ... - w_{i-1} v_{i-1} - w_{i+1} v_{i+1} ... - w_n v_n}{w_i}} F(r_i, w_i)\, dr_i$

and $\frac{dG(w_i)}{dw_i} = -F(r_i, w_i) \frac{V^* - w_1 v_1 - ... - w_{i-1} v_{i-1} - w_{i+1} v_{i+1} ... - w_n v_n}{w_i^2} + \int_{-\infty}^{\frac{V^* - w_1 v_1 - ... - w_{i-1} v_{i-1} - w_{i+1} v_{i+1} ... - w_n v_n}{w_i}} \frac{\partial F(r_i, w_i)}{\partial w_i} dr_i$

$$\text{thus } \frac{E(r_i)}{-F(r_i, w_i) \frac{V^* - w_1 v_1 - ... - w_{i-1} v_{i-1} - w_{i+1} v_{i+1} ... - w_n v_n}{w_i^2} + \int_{-\infty}^{\frac{V^* - w_1 v_1 - ... - w_{i-1} v_{i-1} - w_{i+1} v_{i+1} ... - w_n v_n}{w_i}} \frac{\partial F(r_i, w_i)}{\partial w_i} dr_i} = \lambda_1^{VaR} \quad (15)$$

A similar result holds for the lower partial moment measure of risk.

By assumption, the distribution of returns is not affected by the reallocation. Also, these equalities should hold for a given security, or even for the entire portfolio. What this means is there can be three variants of the capital asset pricing model, with different risk measures, with the most familiar being the one from the mean-variance framework:

$$E(r_i) = \frac{\int_{-\infty}^{\infty} \left(\mathbf{w'r} - E(r_p)\right)\left(r_i - E(r_i)\right) f(\mathbf{r})\, \mathbf{dr}}{\int_{-\infty}^{\infty} \left(\mathbf{w'r} - E(r_p)\right)\left(r_i - E(r_p)\right) f(\mathbf{r})\, \mathbf{dr}} E(r_p)$$

$$\rightarrow E(r_i) = \frac{\sigma_{p,i}}{\sigma_p^2} E(r_p) \quad (16)$$

$$\rightarrow E(r_i) = \beta_i E(r_p)$$

The resulting investment opportunity sets (with mean excess return plotted versus standard deviation, value-at-risk, lower partial moment(1,0) and lower partial moment(2,0)), using daily data on the Standard and Poor's 500 and the Dow Jones 2 Year Corporate Bond Index from December 31, 1996 to March 18, 2005, are shown in Figures 1 through 4.

Figure 1 Investment Opportunity Set Defined According to Mean (Excess Return)-Standard Deviation Paradigm

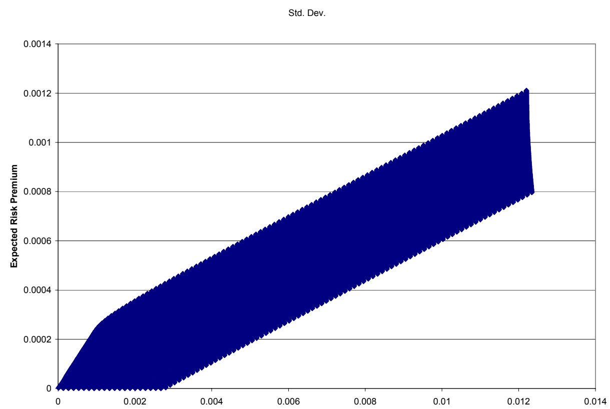


Figure 2

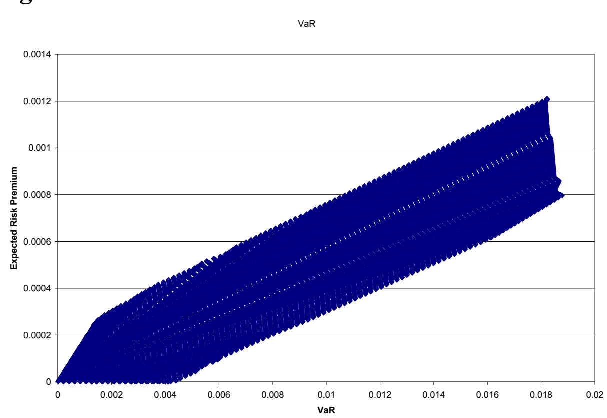


Figure 3

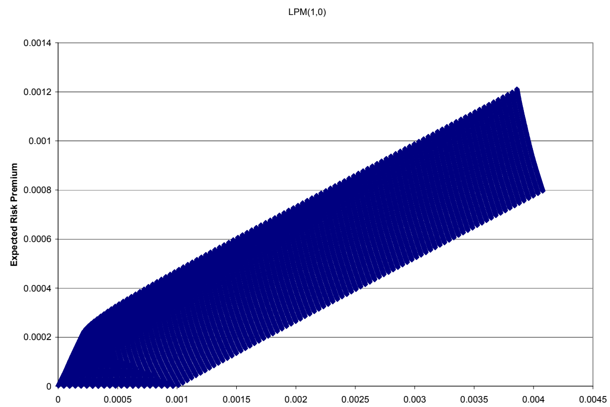
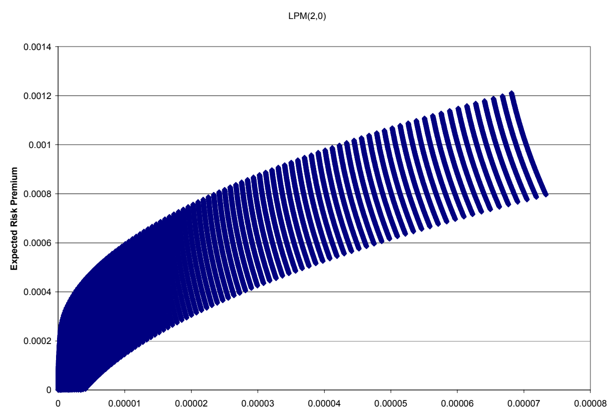
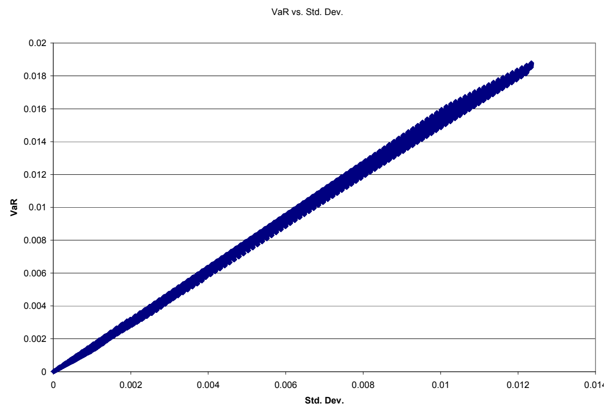


Figure 4



To get an idea about the relationship between these frontiers, Figure 5 plots the value-at-risk versus the standard deviation. As is clear, there is a strong, but not perfectly linear, relationship between these measures. The divergence between the measures is especially apparent in the extremes, which suggests that investment decisions based on either measure will be similar for "low" risk ventures, but could be markedly different for higher risk prospects.

Figure 5 VaR versus Standard Deviation from Efficient Frontiers



Which Measure is Best?

To answer the question about which measure is best requires a bit of qualification: is this a normative or a positive question? Normatively, I could argue which measure I think is the best, and then advocate the use of that pricing formula. Positively, I can look at the data and see which pricing formula best fits the data; inferring from that, which risk measure investors probably actually use in their portfolio construction decisions. Given that the purpose of this paper is to present "a better way" of measuring risk, I will use the normative approach, showing which method yields the highest investment returns.

To test which measure is the best, I conducted a simulation study, pitting six investment strategies against each other. I constructed a portfolio allocation problem, using daily data from December 31, 1996 to March 18, 2005, on two indexes: the Standard and Poor's 500 and the Dow Jones 2 Year Corporate Bond Index. An investor is assumed to reallocate their portfolio between these two indexes daily, using data from the past 60 trading days to construct their empirical probability distribution for the next day. The investor picks the portfolio (with no short-selling allowed) that maximizes the market price of risk, recalibrating the portfolio every day. Six measures of risk are used: standard deviation, value-at-risk at the five percent significance level, and four lower partial moments (one through four with a minimal acceptable excess return of zero).

To pit the portfolio strategies against each other, I wrote a program that—for every day—calculated the market price of risk using portfolio allocation weights ranging from zero to one, increasing by increments of 0.01. All these portfolio allocations were compared, with the one that yielded the highest market price of risk being the allocation used for the next trading day. Cumulative excess returns (over the federal funds rate) were compared at the end of the simulation study.

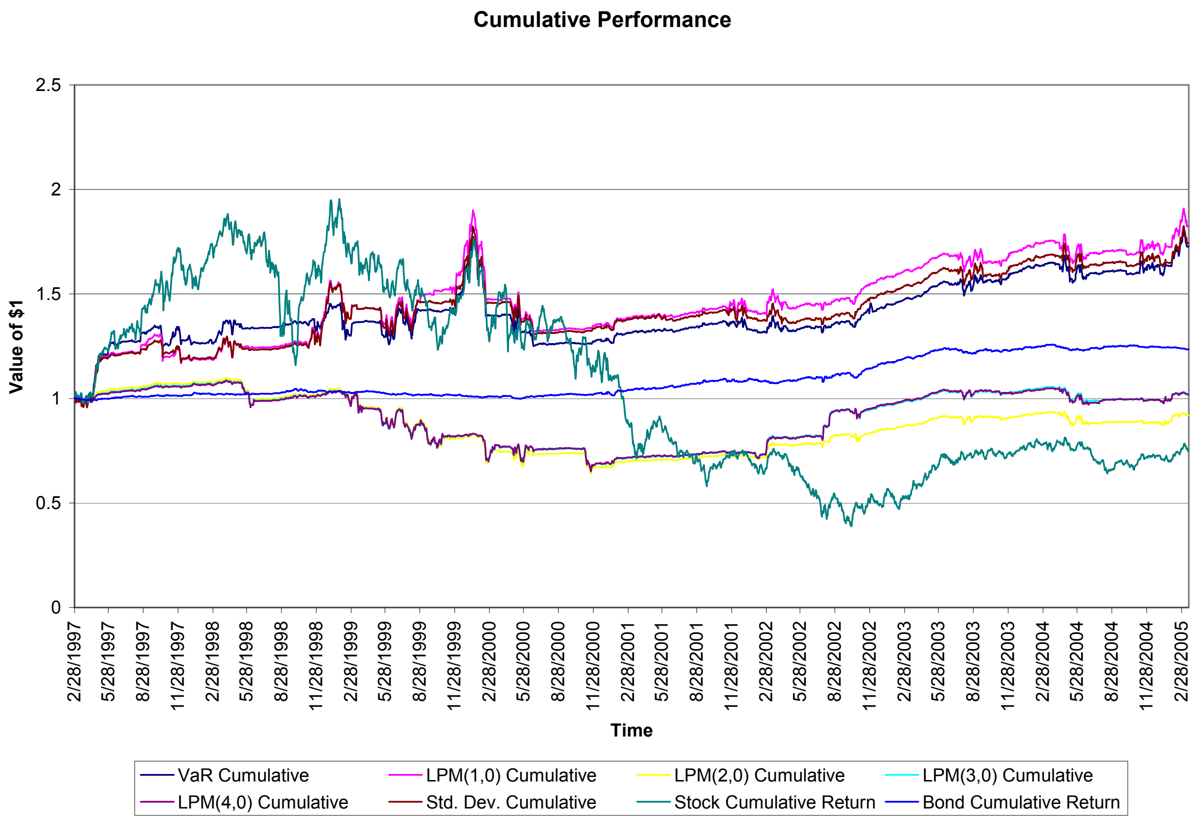
The active management definitely yielded higher returns than a passive investment in either index. Table 1 shows the amount that $1 invested on December 31, 1996 would have been worth, using each strategy, above the risk-free rate of return. The best strategy was maximizing the LPM(1,0) based market price of risk. These excess returns can be seen in Figure 6.

Table 1 Returns, Average Allocations, and Standard Deviations of Allocations Using the Different Investment Strategies

Strategy	Value of $1, above the risk-free rate, Invested on 12/31/1996 as of 3/18/2005	Average Allocation to Stocks	Standard Deviation of Allocation
VaR	1.7265	0.2885	0.4057
LPM(1,0)	1.8231	0.3168	0.4357
LPM(2,0)	0.9196	0.2253	0.3519

LPM(3,0)	1.0216	0.22170	0.3529
LPM(4,0)	1.0195	0.2189	0.3520
Std. Dev.	1.7425	0.3088	0.4266

Figure 6 Excess Return Comparison of Investment Strategies



Clearly, the active management yields excess returns, but they also require *active* management of the portfolio. Each day, the investor must shift money between these two indexes, sometimes completely liquidating their position in one and shifting to the other.

The average allocation into the SP500 and the DJBond index for each method, along with the standard deviation of the allocations is presented in Table 1. The actual allocation histograms using each strategy are shown in Figures 7 through 12.

Figure 7 Std. Deviation histogram of allocations to stock index

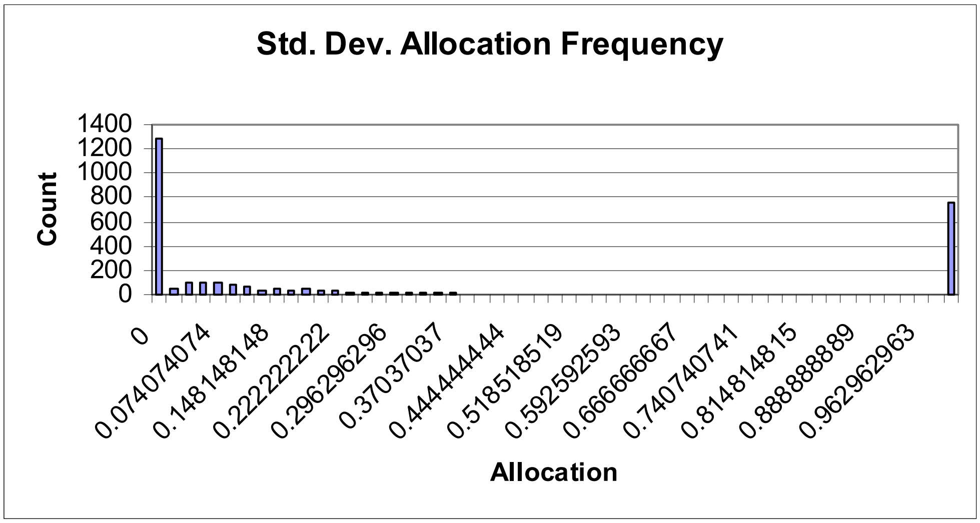


Figure 8

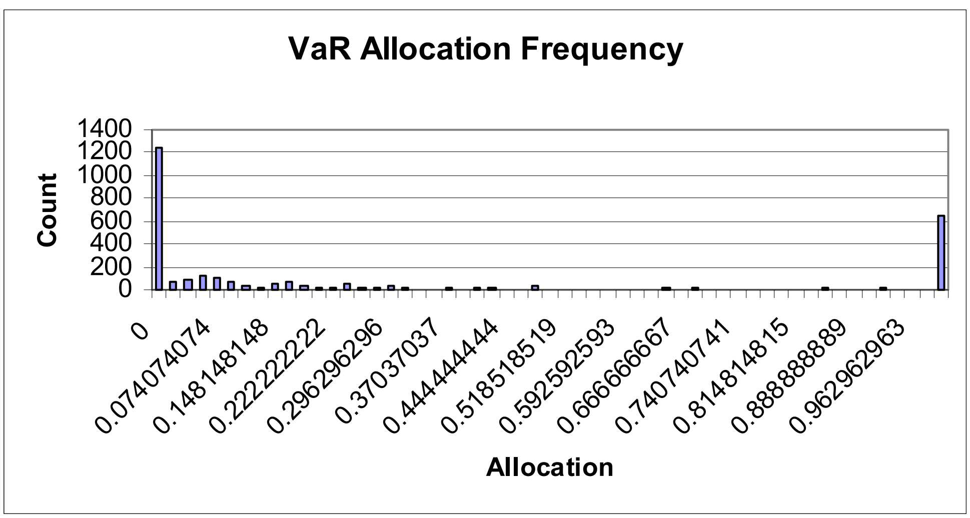


Figure 9

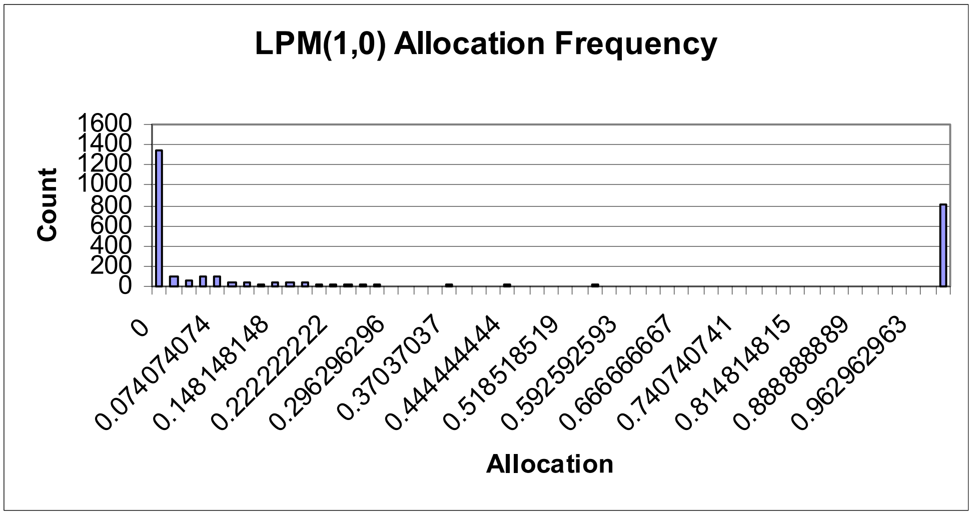


Figure 10

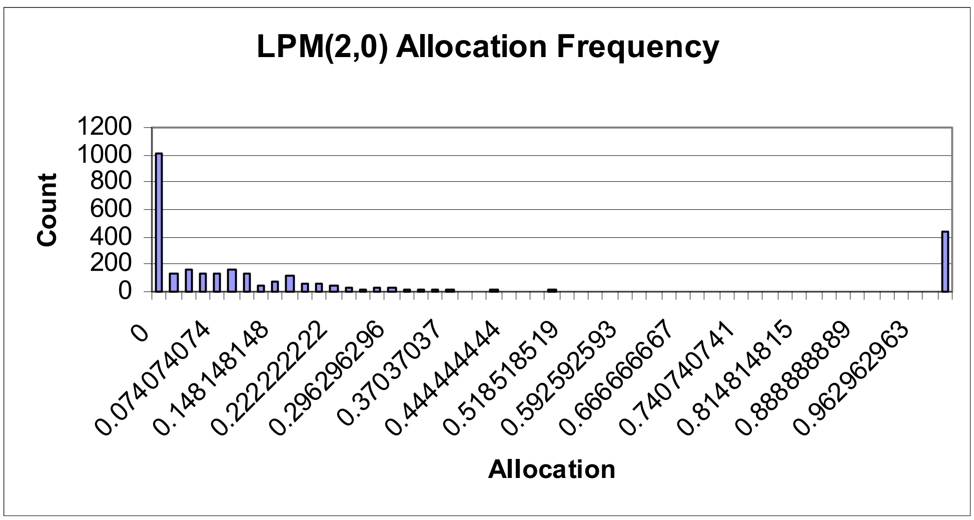


Figure 11

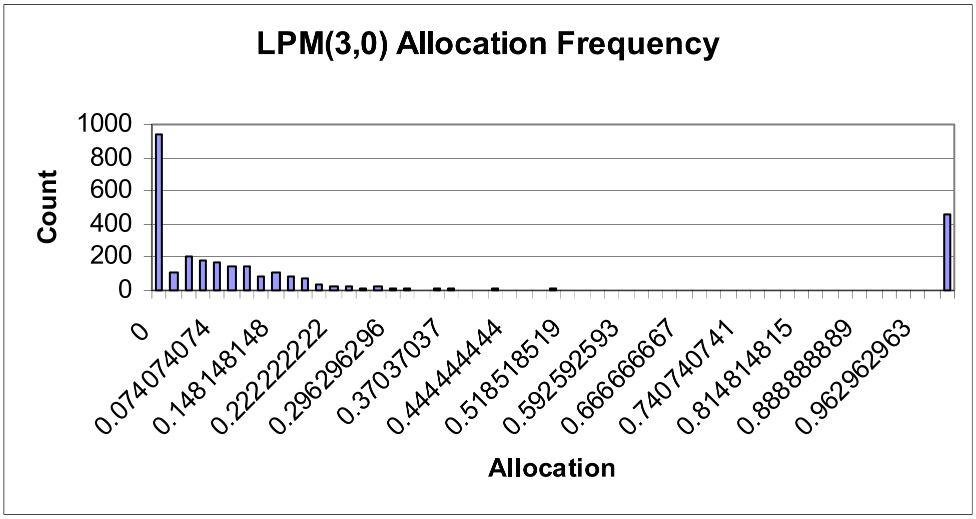
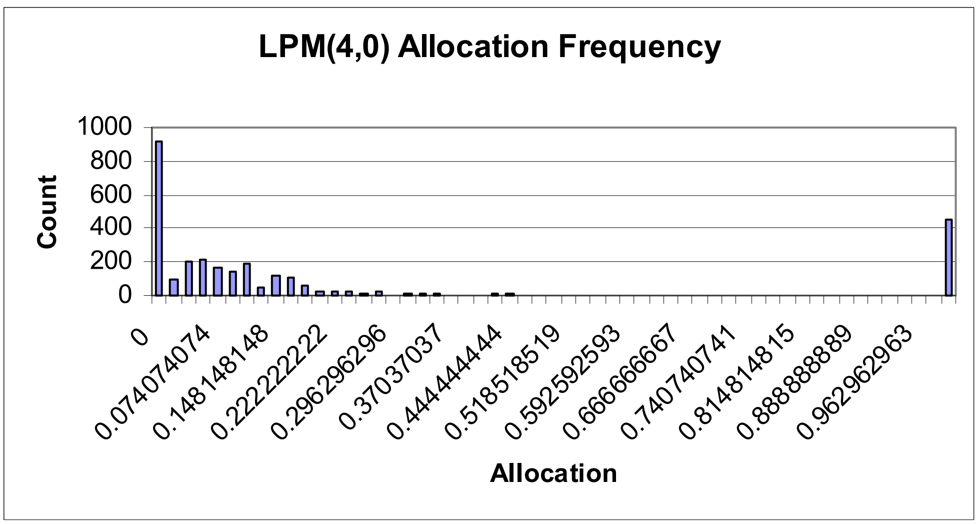


Figure 12



The allocation suggested by all the methods tends to not favor strategic diversification across asset classes; rather, the simulations are suggestive of a sort of extreme form of tactical asset allocation, exploiting opportunities while they arise. Philosophically, this makes sense to me as to get to the long-term (the realm of strategic allocation), we must pass through the short-term (tactical asset allocation). The traditional arguments in favor of strategic asset allocation—that trading costs will eat up any excess gains from short-term trading—are disintegrating as trading costs fall and data becomes cheaper to obtain and process. Strategic asset allocation has its place, but primarily for those who do not want to actively monitor their portfolios. With the extent of the market expanding, there are now specialists who will actively monitor your portfolio for you, so this active management can easily and cheaply be outsourced. So, in my opinion, tactical asset allocation based on using the LPM(1,0) measure of risk is a rational alternative to naïve strategic asset allocation. Now, the strategic decision is more an issue of selecting the specialist who will monitor your portfolio than selecting a static asset mix to hold onto regardless of what happens.

Forecasting Distributions (Distributional Scaling)

Allocation tools should be prospective, not retrospective, but usually the past is where our forecasts for the future distributions of returns come from. Mandelbrot (2001 a, b, c, d, 2005) advocates the recognition of scaling in financial markets. To this end, I have enriched the Mandelbrot model with economic fundamentals to identify the invariants across time scales of return distributions.

Figures 13 through 14 show that a modified version of a time elasticity of price has a characteristic distribution for an asset across time scales. Define $P(t)$ as the price process of an asset. The ratio $\beta_{MJ} = \dfrac{\ln\left(P(t+\Delta) / P(t) \right)}{\ln\left((t+\Delta) / t \right)}$ has a similar distributional form across different time steps, Δ.

Figure 13 MJ-Beta Distribution for Tick-by-Tick Prices of Crude Oil for March 24, 2005, from Reuters-Bridge Station

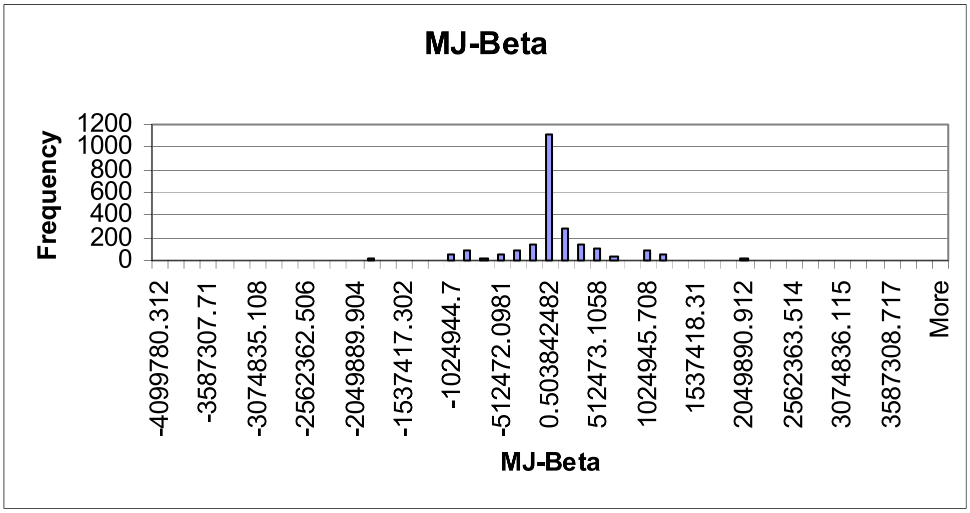
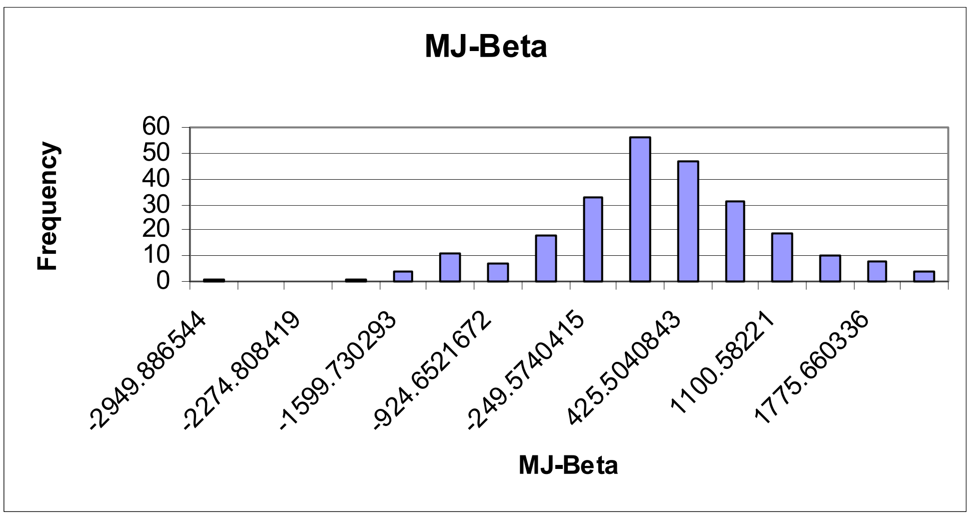
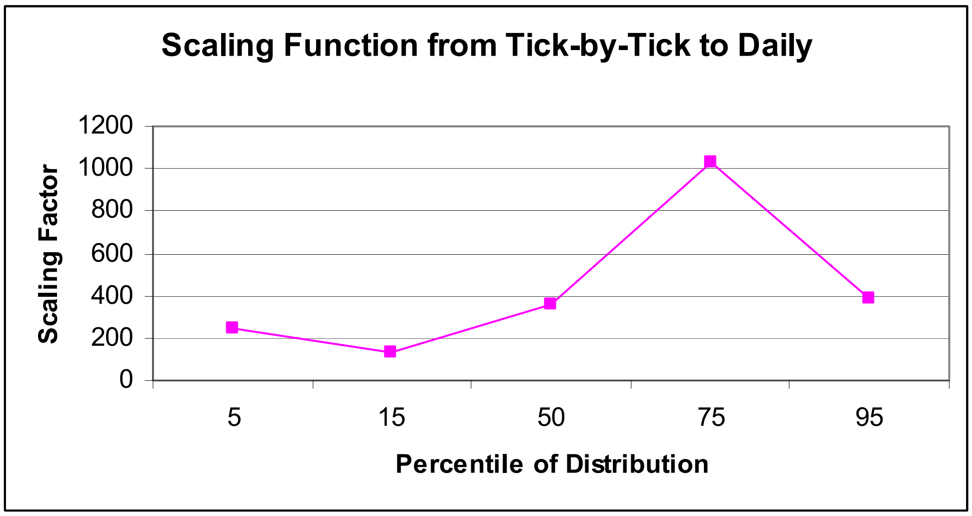


Figure 14 MJ-Beta Distribution for Daily Close Prices of Crude Oil from March 24, 2004 to March 25, 2005, from Reuters-Bridge Station



This implies that daily observations are distributed in a manner similar to tick-by-tick observations, or year-by-year observations, by a multifractal process (Mandelbrot, Fisher, and Calvet, 1997). A multifractal process can be thought of as a stochastic scaling factor from one time scale to another. When it comes to forecasting distributions, or determining which distributional assumptions to make about returns, the daily close distribution (transformed) can be used with the appropriate multifractal applied. The scaling function of the distribution must be appropriately applied to go from one time scale to another, which can be derived from the self-similarity of the distributions. What this means is that one time scale is a transformation of the distribution from another time-scale, which may be idiosyncratic to that particular asset. For example, Figure 15 shows the scaling function from the tick-by-tick distribution of crude oil β_{MJ} to the daily close distribution. This function can be applied to go from one time scale to another for inferring prospective distributions of returns.

Figure 15 Scaling Function from Tick-by-Tick Distribution of Crude Oil MJ-Beta to Daily Close MJ-Beta



Conclusion

The standard mean-variance efficient frontier has mathematical properties that make it easy to work with. However, the mathematical ease comes at a high price: it does not acceptably capture investors' preferences; nor does it generate superior investment returns. A modified efficient frontier must be optimized to determine the best mix of securities to add to a portfolio (essentially, measuring the marginal VaR or LPM through simulation), but the cost is offset by the benefit of higher returns with lower risk…whatever "risk" means to each individual.

Prospectively, distributions can be derived from the self-similarity of return distributions—properly transformed.

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