Measuring the Effects of Real and Monetary Shocks in a Structural New-Keynesian Model[1]

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Abstract

We develop a technique for analyzing the dynamics of shocks in structural linear rational expectations models. Our work differs from standard SVARs since we allow expectations of future variables to enter structural equations. We show how to estimate the variance-covariance matrix of fundamental and non-fundamental shocks and we construct point estimates and confidence bounds for impulse response functions. Our technique can handle both determinate and indeterminate equilibria. We provide an application to U.S. monetary policy under pre and post Volcker monetary policy rules.

JEL-Classification: C39, C62, D51, E52, E58

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Non Technical Summary

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1 Introduction

This paper introduces a technique for analyzing the dynamic effects of structural shocks in linear rational expectations models. The method we propose is similar to the construction of impulse response functions in structural VARs. It differs from the SVAR literature because the models we study contain future expectations as explanatory variables. We apply our method to a New-Keynesian model and show how to trace the effects of fundamental and non-fundamental shocks on all of the variables of the system.

There is a well developed literature which evolved from the work of Sims [15] that traces the effects of different shocks on small systems of variables by estimating VARs. Sims identified alternative shocks by ordering the equations of an estimated VAR and imposing a Choleski decomposition on the variance covariance matrix of the residuals. An extension of this method combines sets of long-run and short run restrictions on a structural model. Although the models in this literature are known as structural VARs, (SVARs) this name is misleading. SVARs are not structural in the same sense as structural linear rational expectations models (SLREs) since they do not include forward looking elements to account for the effects of expectations on current behavior. The shocks, in an SVAR, are combinations of the fundamental shocks that confound direct effects of fundamental impulses to the structural equations with the indirect effect of these shocks on expectations.

The use of VARs and SVARs to identify monetary policy shocks is surveyed in Christiano Eichenbaum and Evans [5]. These authors present a range of methods that have been used in the literature to identify the effects of monetary shocks and they summarize a consensus opinion on the effects of a monetary shock, based on popular identification schemes.

> The nature of this [consensus opinion] is as follows: after a

> contractionary monetary policy shock, short term interest rates rise, aggregate output, employment, profits and various monetary aggregates fall, the aggregate price level responds very slowly, and various measures of wages fall, albeit by very modest amounts. In addition, there is agreement that monetary policy shocks acount for only a modest percentage of the volatility of aggregate output; they account for even less of the movements in the aggregate price level.
>
> Christiano Eichenbaum Evans, op. cit.

Given the existing consensus on the effects of monetary policy, one might wonder why we would write another paper on identifying monetary policy shocks. Our motivation is based on a concern that identification schemes that have been useful at describing the effects of past shocks may not be useful if the goal of the policy maker is to design a policy rule. In short, VARs and SVARs are subject to the Lucas critique of econometric policy evaluation. Our goal in this paper is to identify shocks in complete structural models that have been identified using restrictions suggested by economic theory.

In contrast to the SVAR literature, complete linear rational expectations models are identified by equality and exclusion restrictions applied to equations that contain expectations of future variables. In this paper we show that every structural linear rational expectations model has a representation as an SVAR; however, this representation is not necessarily unique. For some points in the parameter space, an SLRE has a unique determinate equilibrium that is driven solely by fundamental shocks such as those to preferences, endowments and technology. But for other points in the parameter space of the SLRE there may exist multiple indeterminate equilibria and in this case non-fundamental or 'sunspot' shocks may also play a role even though the remaining SVAR parameters are unchanged.

This issue is not simply a theoretical curiosity. Clarida Galí and Gertler [6] have argued that U.S. data in the period from 1960 through 1979 is well characterized by an indeterminate equilibrium. This possibility raises difficulties with the SVAR approach to identification of structural shocks since, if the nature of the shocks driving the economy can change when economic policy changes, fixed identification schemes in an SVAR cannot be considered structural.

We provide an alternative to standard SVAR identification schemes by deriving restrictions on a VAR that are implied by estimates of a structural rational expectations model. We give an example of our approach within a three equation New-Keynesian model of the U.S. economy and we show how to impose structural restrictions that are difficult or impossible to impose using standard methods. We estimate the model by GMM and replicate recent results on the determinacy properties of the system before and after 1979 as reported by Clarida-Galí-Gertler and substantiated by Lubik-Schorfheide [12] and Boivin-Giannoni [4]. We provide a method for estimation of the variance-covariance matrix of structural shocks for both the determinate and indeterminate cases and we illustrate our method by constructing impulse response functions for a set of fundamental and non-fundamental shocks estimated from U.S. data. Finally, we discuss the implications of our work for the analysis of monetary policy.

2 Estimating a Structural Linear Rational Expectations Model

In this section we discuss the use of GMM to obtain consistent estimates of the parameters of A, F, B and C in the structural model

$$
\begin{aligned}
AY_t + FE_t\left[Y_{t+1}\right] &= BY_{t-1} + C + V_t, &(1)\\
E_t\left[V_tV_s'\right] &= \begin{cases} \Omega_{vv}, & t = s, \\ 0, & \text{otherwise.} \end{cases} &(2)
\end{aligned}
$$

In this notation A, F, and B are $n \times n$ matrices of coefficients, C is an $n \times 1$ vector of constants, E_t is a conditional expectations operator, Y_t is an n-dimensional vector of endogenous variables, and $\{V_t\}$ is a weakly stationary i.i.d. stochastic process with covariance matrix Ω_{vv} and mean zero.[1] We maintain the convention that coefficients of endogenous variables appear on the left side of each equation with positive signs and explanatory variables appear on the right side of equations with positive signs. The first issue we face is that of identification. Each of the n equations in (1) contains $2n$ endogenous variables since the expectations terms $E_t\left(Y_{t+1}\right)$ are endogenous variables to be determined at date t. Application of order and rank conditions (as in Fisher [7]) should be checked for each equation, but identification does not pose additional complications over standard structural models.[2]

In order to estimate the parameters of model (1) in this paper we propose an estimator based on a systems GMM approach. Our method, originally suggested by McCallum [13], replaces unobserved expectations $E_t\left(Y_{t+1}\right)$ by their realizations Y_{t+1} and rewrites Equation (1) as a linear model that includes future values of the observed endogenous variables with moving average error terms

$$
AY_t + FY_{t+1} = BY_{t-1} + C + \Psi_v V_t + \Psi_w W_{t+1}. \tag{3}
$$

[1] We will focus on the case of one lag, but our method can easily be expanded to include additional lags or additional leads of expected future variables.

[2] For an application of the rank and order condition to a three equation New-Keynesian model see Beyer et. al. [3]

The vector W_{t+1} represents one-step-ahead forecast errors. Let the variance-covariance matrix between forecast errors and fundamental shocks be

$$\Omega = E\left[V_t, W_t\right]\left[V_t, W_t\right]'.$$

When the model has a unique rational expectations equilibrium these errors will be exact functions of the fundamental shocks V_{t+1}. In this case the $2n \times 2n$ covariance matrix

$$\Omega = \begin{bmatrix} \Omega_{vv} & \Omega_{vw} \\ \Omega_{wv} & \Omega_{ww} \end{bmatrix} \tag{4}$$

has rank n. When the model has an indeterminate equilibrium of degree r, the variance-covariance matrix Ω has rank $n + r > n$. In this case one can pick a particular rational expectations equilibrium by imposing the assumption that the elements of Ω_{ww} and Ω_{wv} are time invariant. In either case, estimation of Equation (3) must take account of the fact that the errors have an MA(1) structure.[3]

To estimate the parameters of (1), a number of alternative approaches have been suggested in the literature. One method, discussed in Anderson et. al [1] and implemented amongst other by Lindé [10], is full information maximum likelihood (FIML). We chose not to follow this route since it requires the econometrician to take a prior stand on the determinacy properties of the equilibrium.[4] To construct the likelihood function one must be prepared to specify the joint probability distribution of the errors and to make assumptions about the covariance matrix Ω. Since the rank of Ω can change

[3] In our empirical work we dealt with this issue by estimating (3) with GMM using a heteroskedastic-autocorrelation-consistent (HAC) estimator for the optimal weighting matrix.

[4] See Benhabib-Farmer [2] for a discussion of determinacy and sunspots in macroeconomics.

across regions of the parameter space, depending on the degree of indeterminacy, this approach requires the econometrician to estimate a different theoretical model for every such region.[5]

3 Accounting for Shocks

In this section we discuss the problem of disentangling the dynamic effects of different kinds of shocks. This problem involves first, estimating Ω, the variance-covariance matrix of the fundamental and non-fundamental shocks and second, attributing the effects of these shocks to the reduced form equations. A complication arises when equilibria are indeterminate since the impact effects of alternative shocks must be attributed to fundamental and non-fundamental sources.

We begin by constructing an estimator for Ω using a two-step approach. First, we obtain estimates of the structural parameters $\hat{A}, \hat{F}, \hat{B}, \hat{C}, \hat{\Psi}_v$ and $\tilde{\Psi}_w$ in (3) by GMM; second, we use these parameter estimates to construct reduced form residuals from which we estimate Ω. This two-step procedure involves a complication which we discuss in the following section. It arises from the fact that the reduced form of the model obtained by standard solution algorithms will not generally be free of unobserved expectations.[6]

[5] For an application of Maximum Likelihood from a Bayesian perspective, see the paper by Lubik and Schorfheide [12].

[6] We compute the reduced form of the model with a QZ decomposition. Our algorithm, `SysSolve`, is described in Appendix A It is based on code by Sims [16] and amended by Lubik and Schoefheide [11] to account for the possibility that there may be multiple indeterminate solutions.

3.1 Finding an Observable Reduced Form

To compute the reduced form of our structural model we define a vector

$$X_t = \begin{bmatrix} Y_t \\ E_t [Y_{t+1}] \end{bmatrix}$$

which consists of observable variables Y_t and (possibly unobserved) expectations variables $E_t [Y_{t+1}]$. Our procedure for computing the reduced form of the model uses an algorithm, `SysSolve,`which returns a VAR(1) in this augmented state vector. When the equilibrium is determinate, this system can be broken down into two separate subsystems. One is a VAR(1) in the observable variables Y_t and the other is a static function that determines $E_t [Y_{t+1}]$ as a function of Y_t. When the equilibrium is indeterminate, however, it is not generally possible to carry out this decomposition. The following example illustrates this problem in practise and proposes a solution that can be generalized.

Consider the single equation model

$$p_t = \frac{1}{\alpha} E_t [p_{t+1}] + v_t,$$

where p_t is observable and v_t is a fundamental error. This model can be written as follows,

$$\begin{bmatrix} 1 & -\frac{1}{\alpha} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ E_t [p_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ E_{t-1} [p_t] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t.$$

When $|\alpha| < 1$, the solution is indeterminate and `SysSolve` returns the solution

$$p_t = E_{t-1} [p_t] + w_t, \tag{5}$$

$$E_t [p_{t+1}] = \alpha E_{t-1} [p_{t-1}] - \alpha v_t + \alpha w_t. \tag{6}$$

Although this solution is valid, both equations contain unobservable expectations and for some purposes it might be helpful to have an alternative dynamic representation that involved a single stochastic difference equation in the observable variable p_t. In this example one can find such a representation by rearranging Equation (5) to find $E_{t-1}[p_t]$ as a function of p_t and w_t and substituting this solution at dates $t-1$ and $t-2$ into Equation (6). This process leads to the expression

$$p_t = \alpha p_{t-1} - \alpha v_{t-1} + w_t. \tag{7}$$

Equation (7) is a VARMA(1,1) in the observable variable p_t and the vector of shocks $(v_t, w_t)'$.

The coexistence of VAR and VARMA representations, exhibited in this example, carries over to more general SLRE models when the solution is indeterminate. The QZ solution method suggested by Sims and implemented in `SysSolve`, leads to a reduced form expression of the form

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_V^* V_t + \Psi_W^* W_t.$$

Since Γ^* is generally singular, there will be more than one way to partition X_t into two subsets (X_t^1, X_t^2) such that X_t^1 forms an autonomous VARMA(1,1) model that is independent of X_t^2. One or more of these representations will be in terms of the observable variables Y_t and Y_{t-1}, but these observable representations will not generally reduce to a VAR(1). An exception, is the case of a determinate equilibrium when the solution is unique. In Appendix B, we provide an algorithm to generalize the above example to the case of an $n-$dimensional SLRE and we provide MATLAB code, `Arrange`, that implements our algorithm by rearranging the output from the QZ decomposition provided by `SysSolve`.

3.2 Computing an Estimate of Ω

In this section we provide a method to recover consistent estimates of the population variance-covariance matrix Ω. First, we write the reduced form as a VARMA(1,1) in the observable variables Y_t and the unobserved shocks $\eta_t = (V_t, W_t^1)'$,

$$Y_t = \Gamma_a^* Y_{t-1} + C_a^* + \Psi_a^* \eta_t + \Psi_b^* \eta_{t-1}. \tag{8}$$

We assume that the econometrician can obtain consistent estimates of the population parameters $\Gamma_a^*, C_a^*, \Psi_a^*$ and Ψ_b^* which we refer to as $\hat{\Gamma}_{aT}^*, \hat{C}_{aT}^*, \hat{\Psi}_{aT}^*$ and $\hat{\Psi}_{bT}^*$ where T is the sample size.

Let e_t be a vector of sample residuals defined as follows;

$$e_t = Y_t - \hat{\Gamma}_{aT}^* Y_{t-1} - \hat{C}_{aT}^*, \tag{9}$$

where Y_t are observable variables and $\hat{\Gamma}_{aT}^*$ and $\hat{C}_{aT}^*$ are consistent estimates of the VARMA(1,1) representation of the reduced form. Now define the sample autocorrelations $\hat{S}_{0T}$ and $\hat{S}_{1T}$ as follows;

$$\hat{S}_{0T} = \frac{1}{T} \sum_{t=1}^{T} \left(Y_t - \hat{\Gamma}_{aT}^* Y_{t-1} - \hat{C}_{aT}^* \right) \left(Y_t - \hat{\Gamma}_{aT}^* Y_{t-1} - \hat{C}_{aT}^* \right)', \tag{10}$$

$$\hat{S}_{1T} = \frac{1}{T} \sum_{t=1}^{T} \left(Y_t - \hat{\Gamma}_{aT}^* Y_{t-1} - \hat{C}_{aT}^* \right) \left(Y_{t-1} - \hat{\Gamma}_{aT}^* Y_{t-2} - \hat{C}_{aT}^* \right)'. \tag{11}$$

In Appendix C, we show that one can obtain consistent estimates of the elements of Ω by finding a solution to the equations

$$\underset{n \times n}{\hat{S}_T} = \underset{n \times (n+r)}{\hat{\Psi}_T} \; \hat{\Omega}_T \; \underset{(n+r) \times n}{\hat{\Psi}_T'}, \tag{12}$$

where

$$\hat{S}_T = \hat{S}_{0T} + \hat{S}_{1T} + \hat{S}_{1T}', \tag{13}$$

$$\hat{\Psi}_T = \hat{\Psi}_{aT}^* + \hat{\Psi}_{bT}^*, \tag{14}$$

and $\hat{\Psi}_{aT}^*$ and $\hat{\Psi}_{bT}^*$ are consistent estimates of Ψ_a^* and Ψ_b^*.

It is important to notice that Equation (12) cannot be solved uniquely for the elements of $\hat{\Omega}_T$ since it consists of $n(n+1)/2$ independent equations in $(n+r)(n+r+1)/2$ unknowns.[7] The non-uniqueness of the solution to Equation (12) means that, when equilibrium is indeterminate, the econometrician cannot distinguish between fundamental and non-fundamental disturbances to the economy.

For the purposes of examining the dynamic properties of the model, the inability to distinguish between determinate and indeterminate shocks is not a problem as long as the variance-covariance matrix Ω remains time invariant - it is simply a question of how we choose to name the observed disturbances to each equation.[8] For the purposes of constructing impulse response functions in the New-Keynesian model that we describe below, we chose to ascribe all shocks in the indeterminate regime to fundamentals by setting the elements of Ω_w and Ω_{wv} to zero.

4 Application to the New-Keynesian Model

In this section we describe a New-Keynesian model that puts together simplified versions of specifications of the representative agent's Euler equation by Fuhrer and Rudebusch [8], the Phillips curve by Galí-Gertler [9], and the Central Bank reaction function by Clarida-Galí-Gertler [6].

[7] As an example, consider the case when there are two equations and one degree of indeterminacy. In this case $\hat{S}_T$ is a known symmetric 2×2 matrix and $\hat{\Psi}_T$ is a known 2×3 matrix both of which are functions of the data. For this example, Equation (12) consists of 4 equations in 9 unknowns. Since $\hat{S}_T$ is symmetric only 3 of these equations are independent and since $\hat{\Omega}$ is symmetric only 6 elements of Ω need to be independently calculated. Only three linear combinations of the variance-covariance parameters Ω are identified from the data.

[8] The question becomes more interesting if we observe data from different regimes since then one might ascribe a change in the observed variance of the data to the additional contribution of sunspots as suggested by Clarida-Galí-Gertler [6].

4.1 A Description of the Model

The model we estimate consists of the following three equations.

$$y_t = \alpha_0 + \alpha_1 E_t[y_{t+1}] + \alpha_2(i_t - E_t[\pi_{t+1}]) + \alpha_3 y_{t-1} + v_t^1, \tag{15}$$

$$\pi_t = \beta_0 + \beta_1 E_t[\pi_{t+1}] + \beta_2 y_t + \beta_3 \pi_{t-1} + v_t^2, \tag{16}$$

$$i_t = \gamma_0 + \gamma_1(1-\gamma_3)(E_t[\pi_{t+1}] - \pi^*) + \gamma_2(1-\gamma_3)(y_t - y^*) + \gamma_3 i_{t-1} + v_t^3. \tag{17}$$

The variable y_t is a measure of the output gap (we used the same one-sided HP-filtered series as in [3], BFHM), π_t is the GDP deflator, i_t is the Federal Funds rate and E_t is again a conditional expectations operator. Equation (15) is an output equation derived from the representative agent's Euler equation, Equation (16) is a hybrid New-Keynesian Phillips curve, and Equation (17) is a Central Bank reaction function, (also referred to as a Taylor rule after the work of Taylor [17]).

4.2 Unrestricted Parameter Estimates

In this section we report the results of estimating Equations (15)–(17) by GMM on the full system. Table 1 reports these estimates using three lags of the endogenous variables as instruments.[9] Since there is evidence of parameter instability across the full sample, particularly in the policy rule, we split the data in 1979. This follows the lead of Clarida, Galí and Gertler [6], who suggest that the rule followed in the pre-Volcker period (1960:4–1979:3), has very different properties from that during the Volcker-Greenspan years. We

[9] Beyer et. al. [3] estimate this model on the same data set that we use here. They report results from a number of alternative estimation methods and show how to obtain more efficient parameter estimates using factors as instruments. The reader is referred to their work for a more complete description of the robustness properties of the system GMM estimator and for a discussion of parameter stability across different subsamples.

discarded the quarters 1979:4–1982:4 since this was a period of considerable instability in which the Fed followed a money targeting rule that was quickly abandoned. Our second sub-sample consists of the years 1983:1–1999:3.[10]

TABLE 1: UNRESTRICTED GMM PARAMETER ESTIMATES

		Sample 60:4 79:3				Sample 83:1 99:3			
Eqn	**Param**	**Coeff**	**Std Err**	**t-stat**	**p-val**	**Coeff**	**Std Err**	**t-stat**	**p-val**
Euler Eq.	$gap_{t+1}(\alpha 1)$	0.480***	0.032	14.85	0.00	0.394***	0.048	8.230	0.00
	$ri_t(\alpha 2)$	-0.073*	0.038	-1.91	0.06	0.001	0.012	0.070	0.95
	$gap_{t-1}(\alpha 3)$	0.549***	0.034	16.21	0.00	0.595***	0.040	14.920	0.00
Phil. Curve	$\pi_{t+1}(\beta 1)$	0.739***	0.083	8.95	0.00	0.539***	0.087	6.170	0.00
	$gap_t(\beta 2)$	-0.020	0.024	-0.85	0.40	-0.068***	0.019	-3.660	0.00
	$\pi_{t-1}(\beta 3)$	0.268***	0.073	3.7	0.00	0.276***	0.063	4.380	0.00
Pol. Rule	$\pi_{t+1}(\gamma 1)$	0.876***	0.190	4.62	0.00	1.660***	0.427	3.890	0.00
	$gap_t(\gamma 2)$	0.644**	0.244	2.64	0.01	0.257**	0.124	2.060	0.04
	$i_{t-1}(\gamma 3)$	0.843***	0.051	16.64	0.00	0.761***	0.058	13.150	0.00
		J-stat = 10.0290 **p-val** = 0.9310				**J-stat** = 9.8014 **p-val** = 0.9382			

* (**) (***) denotes significance at 10% (5%) (1%) level

Table 1 is divided into three sections, one for each equation of the New-Keynesian model. The table is further divided into two halves reporting estimates, in the left panel, for the sub-sample from 1960:4–1979:3 and in the right panel, for the sub-sample 1983:1–1999:3. For each sub-sample we were able to fit a tightly parameterized model; the equality and exclusion restrictions that we imposed to achieve identification passed Hansen's $J-$test with $p-$values of 93% and 94% for the two samples. Further, as reported in Beyer et. al. [3], the residuals for this model are consistent with the model assumptions. After removing an MA(1) component, predicted by theory, the residuals passed a range of misspecification tests including absence of ARCH effects, absence of additional serial correlation and the Jarque Bera test for normality.

As in BFHM, we find that detrended output and inflation are well de-

[10] We chose to end at this date for comparability with other studies.

scribed by their own future and lagged values. Coefficients on future and lagged output in the Euler equation are tightly estimated and qualitatively similar across sub-periods. Our point estimate for α_1, (the estimated coefficient on future output), is equal to 0.48 in the first sub-period and 0.39 in the second and both coefficients are significant at the 1% level using HAC standard errors. The coefficient on lagged output, a_3, is estimated as 0.55 and 0.59 in the two sub-samples.

Theory predicts that the real interest rate coefficient in the Euler equation, α_2, should be negative and of the same order of magnitude as the average value of the real interest rate. In practice this coefficient is small and imprecisely estimated but, at least for the first sub-sample, the point estimate lies within two standard errors of 0.02 which we take to be a ball park figure for the quarterly real return to capital. Similar results hold for our estimates of the hybrid Phillips curve. The coefficients on future and lagged inflation, β_1 and β_3, are significant and qualitively similar across sub-samples although the coefficient on the output gap, β_2, is small and insignificant in the first sub-sample and has the wrong sign in the second.

Our estimates of the policy rule are similar to those reported by Clarida-Galí-Gertler (CGG). Like CGG, we find that the estimated coefficient on future inflation in the policy rule, γ_1, switches from 0.87 in the pre-Volcker period to 1.66 in the Volcker-Greenspan years.[11] This is an important coefficient since, when the parameters of the Phillips curve and the Euler equation are calibrated to values suggested by economic theory, γ_1 regulates the determinacy of equilibrium. If γ_1 is less than one, the Fed responds to expected future inflation by lowering the real rate of interest; a policy of this kind

[11] CGG used GMM in a single equation framework and used a larger instrument set. Our findings for the policy rule are, however, qualitatively the same as theirs for the unrestricted model.

is called *passive*. If γ_1 is greater than one, the Fed responds to expected inflation by raising the real interest rate; a policy of this kind is called *active*.

CGG calibrated the parameters of the Phillips curve and Euler equation to values that are standard in the literature and showed that a model economy would display an indeterminate equilibrium with the pre-Volcker policy rule and a determinate equilibrium with a post-Volcker policy rule. They pointed out that output and inflation have been less volatile in the post-Volcker period and they raised the possibility that this reduction in volatility arose from the elimination of sunspot uncertainty associated with a switch from an indeterminate to a determinate policy rule.

4.3 Dynamics Implied by the Unrestricted Parameter Estimates

Our next step was to compute VARMA(1,1) representations of the reduced form for each regime using the `SysSolve` and `Arrange` algorithms described in Appendices A and B. In Table 2 we report the absolute values of the generalized eigenvalues of the companion forms for the first and second sub-samples. In the first sub-sample our point estimates suggest an indeterminate equilibrium with two unstable roots, and for the second sub-sample, a determinate equilibrium with three unstable roots. These findings are consistent with the reported results of CGG [6], LS [11] and BG [4], in spite of the fact that the point estimates of the interest coefficients in the Euler equation and the output gap coefficients in the Phillips curve are often insignificant and/or have the wrong signs.

TABLE 2:	UNRESTRICTED ESTIMATES OF GENERALIZED EIGENVALUES								
Sample	**Point estimates of roots**								
60:4—79:3	0.00	0.00	0.00	0.37	0.80	0.80	0.93	**2.44E+16***	**1.58**
83:1—99:3	0.00	0.00	0.00	0.34	0.76	0.95	**2.18E+17**	**1.55**	**1.55**

*Bold figures indicate unstable roots.

Estimated size of determinate, indeterminate and unstable regions of the parameter space

60:4—79:3	**83:1—99:3**
Point Estimates Imply **Indeterminate Equilibrium**	Point Estimates Imply **Determinate Equilibrium**
Percentage of Indeterminate Draws = 73	Percentage of Indeterminate Draws = 1.5
Percentage of Determinate Draws = 25.9	Percentage of Determinate Draws = 70.4
Percentage of Non-Existent Draws = 1.1	Percentage of Non-Existent Draws = 28.1
Percentage of Non-Positive Definite Draws = 0	Percentage of Non-Positive Definite Draws = 0
Percentage of Change in Dimension = 28.2	Percentage of Change in Dimension = 1.5

To check the robustness of our determinacy findings, for each sub-sample, we took $10,000$ parameter draws from a normal distribution centered on the point estimates of the parameters with a variance covariance matrix equal to the asymptotic estimate using HAC standard errors from the GMM estimation. For each draw, we calculated the number of stable generalized eigenvalues and calculated whether the implied equilibrium was determinate, indeterminate or non-existent. The results of this exercise are reported in Table 2. For the first sub-sample we found that 73% of our draws were consistent with the point estimate in the sense that they fell in the indeterminate region. A further 25.9% were in the determinate region and for 1.1% of the draws stationary equilibrium did not exist. For the second sub-sample 70.4% of the the draws were determinate, (consistent with the point estimates for this sub-sample), 1.5% were indeterminate and 28.1% implied non-existence. This exercise suggests a lower degree of confidence than that reported by Lubik and Schorfheide [12] who developed Bayesian techniques to determine the posterior odds ratio for the probability that any given model is associated with a determinate as opposed to an indeterminate region of the parameter space.[12]

Our next step was to study the dynamics of the economic response of the output gap, inflation and the Fed funds rate to alternative fundamental shocks to the system. First, we constructed an estimate of the variance-covariance matrix of the fundamental shocks using the methods described in Section 3. The results of this exercise are reported in Table 3.

TABLE 3: UNRESTRICTED ESTIMATES OF VCV MATRIX OF FUNDAMENTAL SHOCKS

	Sample 60:4—79:3					Sample 83:1—99:3		
$X\ 10^{-3}$	**Gap**	**Infl**	**Irate**	**Sunspot**	$X\ 10^{-3}$	**Gap**	**Infl**	**Irate**
Gap	10.3	0	0	0	**Gap**	4.50	0.00	0.00
Infl	7.3	7.9	0	0	**Infl**	-0.60	9.50	0.00
Irate	-32.4	15.5	30.8	0	**Irate**	6.10	1.90	8.30
Sunspot	0	0	0	0				

Since we found an indeterminate model in the first sub-period, we were forced to take a stand on how to attribute the residuals to three equations to four possible shocks. As described in Sub-section 3.2, there is no unique solution to this problem and we chose to identify the shocks by setting the variance and covariance terms of the sunspot shock equal to zero. The result of identifying shocks with this assumption is reported in Table 3 which reports the Choleski decomposition of the estimated variance-covariance matrices for each sub-sample. We find the estimated standard deviation of output to be roughly twice as high in the pre-Volcker period as in the Volcker-Greenspan years; the standard deviation of interest rate shocks is correspondingly four times bigger whereas the inflation shock is of the same order of magnitude.

[12] Lubik and Schorfheide used a slightly different sample and different identification assumptions. More importantly, the parameters α_2 and β_2, in their analysis, were strongly influenced by Bayesian priors. When we restrict these parameters we also obtain much tighter estimates of the determinacy and indetermionacy regions in line with the LS results.

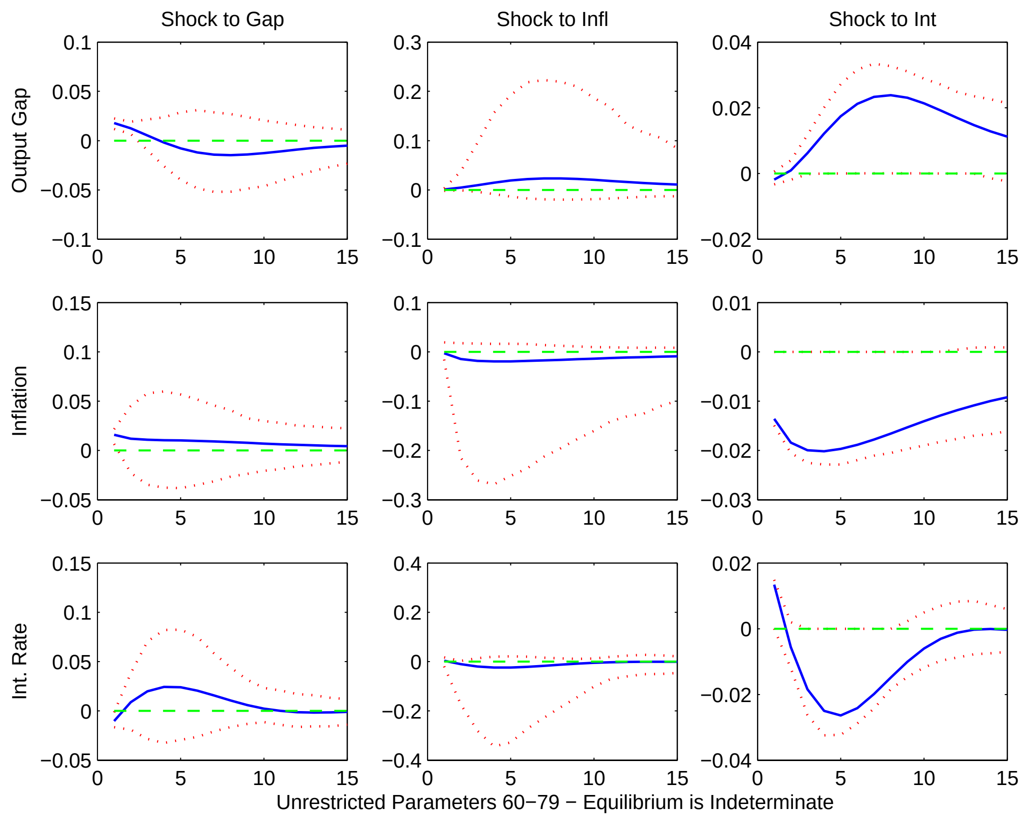

Figure 1: Unrestricted Impulse Responses: First Subsample

In Figures 1 and 2 we used our point estimates of the parameters to generate impulse response functions associated with the theoretical models for each sub-sample. The solid lines in each figure are impulse responses computed from the point estimates and the dashed lines are 90% confidence bounds. The upper and lower bounds were computed by simulating 10000 draws from the asymptotic distribution of the parameter estimates, ranking the responses for each quarter, and picking the values that delineate the $5th$ and $95th$ quantiles. In our simulations, we discarded draws for which the determinacy properties of the simulation were different from the point estimates. These confidence intervals should therefore be interpreted as conditional on the determinacy properties of the point estimates. For the pre-Volcker period our procedure resulted in discarding 28.8% of the draws.

25.9% of these discards were in the determinate region, 1.1% were in the non-existent region and 1.8% were indeterminate of a higher degree than the point estimates. For the Volcker-Greenspan period we rejected 29.6% of the draws (1.5% in the indeterminate region and 28.1% in the non-existent region of the space).

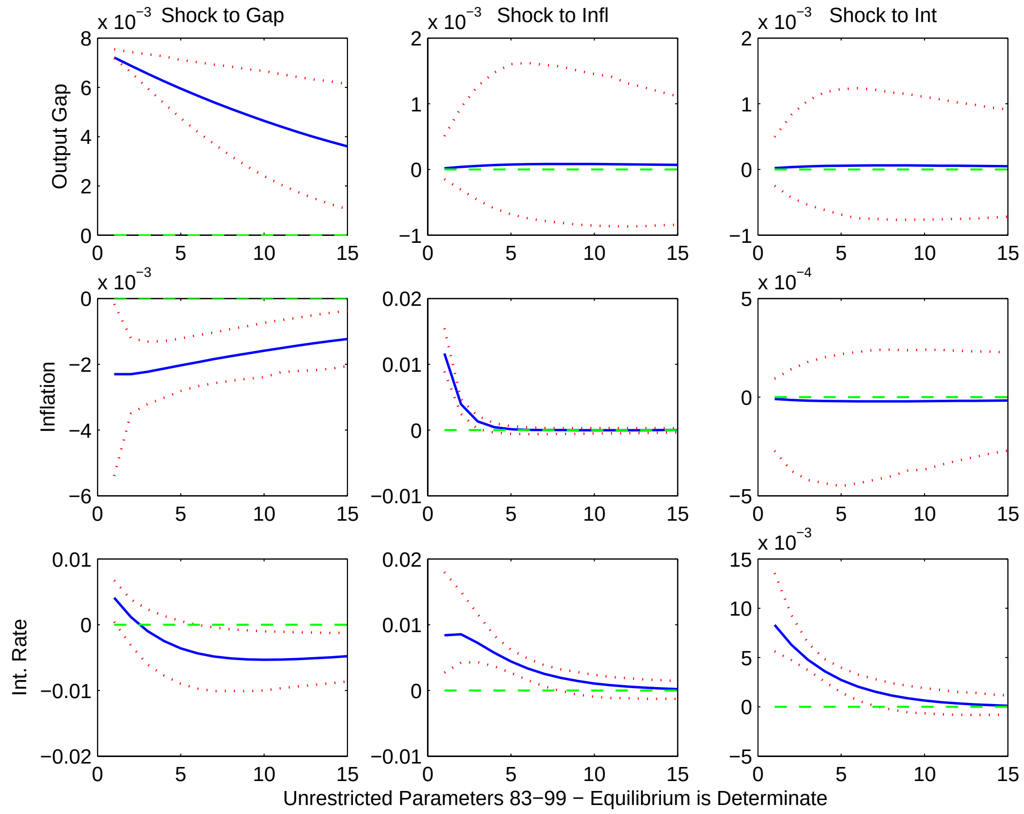

Figure 2: Unrestricted Impulse Responses: Second Subsample

Comparison of Figures 1 and 2 reveals wide error bands, very little consistency across sub-samples, and results that are difficult to interpret in terms of economic theory.[13] Inspection of these figures suggests that the exercise of

[13]The standard error bands are often asymmetric and in some examples (the top right panel of Figure 1 is an example) one or more of the confidence regions is coincident with the axis. This is because we are discarding draws for which the determinacy properties change and this may coincide with one or more of the variables changing sign and the

constructing impulse responses from structural shocks may have little use as a tool of analysis, and, the unrestricted estimates are in this sense a failure.

However, recall that the point estimates from the model do not incorporate restrictions from economic theory. Restrictions that have been successfully imposed by other authors include calibration of the magnitude and sign of the interest rate coefficient in the Euler equation and of the output gap coefficient in the Phillips curve. The following section repeats the estimation procedure after imposing these restrictions.

4.4 Restricted Parameter Estimates

Although our structural model is formally identified, when estimating models in this class they often suffer from problems of weak instruments. This arises when the instruments are only weakly correlated with their targets and it leads to possible inconsistency of parameter estimates even in large samples. A correlate of the weak instrument problem is that the data alone cannot distinguish between a range of alternative additional over-identifying assumptions and there is a possible role for economic theory to further restrict parameter estimates. An example of the ramifications of this issue related to our NKE model is the fact that in equations (15) and (**??**), the parameters α_2 and β_2 have important economic content. But they are estimated weakly in the data with large standard errors. To address this problem we added two pieces of identifying information from economic theory.

The parameter α_2 is the interest rate coefficient in the Euler equation and, theoretically, it is obtained from linearization of a representative agent's marginal utility of consumption. Theory suggests that α_2 should be in the range of 0.05 to 0.4, the same order of magnitude as a measure of the real interest rate. We experimented with a number of values in this range with

implied impulse function crossing the axis.

little qualitative difference from the results reported in In Table 4 which contains parameter estimates by GMM for the case when a_2 is equal to -0.02.

TABLE 4:		RESTRICTED GMM PARAMETER ESTIMATES							
		Sample 60:4 79:3				Sample 83:1 99:3			
Eqn	**Param**	**Coeff**	**Std Err**	**t-stat**	**p-val**	**Coeff**	**Std Err**	**t-stat**	**p-val**
Euler Eq.	$gap_{t+1}(\alpha 1)$	0.480***	0.026	18.270	0.00	0.506***	0.026	19.300	0.00
	$ri_t(\alpha 2)$	-0.020				-0.020			
	$gap_{t-1}(\alpha 3)$	0.542***	0.025	21.910	0.00	0.502**	0.022	22.680	0.00
Phil. Curve	$\pi_{t+1}(\beta 1)$	0.116*	0.069	1.670	0.09	0.807***	0.148	5.440	0.00
	$gap_t(\beta 2)$	0.250				0.250			
	$\pi_{t-1}(\beta 3)$	0.692***	0.064	10.860	0.00	0.353***	0.083	4.230	0.00
Pol. Rule	$\pi_{t+1}(\gamma 1)$	0.252	0.626	0.400	0.69	1.906**	0.729	2.620	0.01
	$gap_t(\gamma 2)$	2.290*	1.341	1.710	0.09	0.352*	0.201	1.760	0.08
	$i_{t-1}(\gamma 3)$	0.926***	0.042	21.900	0.00	0.894***	0.043	21.010	0.00
		J-stat = 11.8309 **p-val** = 0.9218				**J-stat** = 10.9645 **p-val** = 0.9471			

* (**) (***) denotes significance at 10% (5%) (1%) level

The parameter β_2 is the output coefficient in the Phillips curve and economic theory predicts that this parameter should equal the inverse of the representative agent's coefficient of relative risk aversion. Clarida-Galí-Gertler choose $\beta_2 = 1$ in their calibrated model and Lubik-Schorfheide set a prior mean of $\beta_2 = 0.5$. These values correspond to risk aversion coefficients of 1 and 2 respectively. We experimented with values of β_2 in the range 0.01 through 1 but our parameter estimates led to non-existence of stationary equilibrium for values much above 0.4. In Table 4 we report the results of GMM estimates in which we restrict $\beta_2 = 0.25$, which corresponds to a coefficient of relative risk aversion of 4.

Table 4 also presents the parameter estimates for the remaining parameters of the model under the joint restrictions $\alpha_2 = -0.02$, $\beta_2 = 0.25$. The restricted estimates pass Hansen's $J-$test for overidentifying restrictions with $p-$values of 0.92 and 0.95 respectively for the pre-Volcker and post-Volcker samples. The restrictions do not have a big effect on the parameters of future and lagged inflation in the Phillips curve nor do they change qualitatively the coefficients on future and lagged output in the Euler equation. They do, however, have a significant effect on the estimates of the Taylor rule.

In the unrestricted case the coefficient on future inflation was estimated to be 0.87 in the first sub-sample and 1.67 in the second sub-sample; further, both estimated coefficients were significant at the 1% level. For the restricted estimates the pre-Volcker inflation response, γ_1 is insignificant whereas the post-Volcker response is higher and equal to 1.9.

TABLE 5: RESTRICTED ESTIMATES OF GENERALIZED EIGENVALUES

Sample	Point estimates of roots								
60:4—79:3	0.00	0.00	0.00	0.82	0.82	0.73	**2.66E+16***	**7.98**	**1.58**
83:1—99:3	0.00	0.00	0.00	0.66	0.58	0.66	**8.85E+15**	**1.24**	**1.24**

*Bold figures indicate unstable roots.

Estimated size of determinate, indeterminate and unstable regions of the parameter space

60:4—79:3	83:1—99:3
Point Estimates Imply **Determinate Equilibrium**	Point Estimates Imply **Determinate Equilibrium**
Percentage of Indeterminate Draws = 0.3	Percentage of Indeterminate Draws = 0.3
Percentage of Determinate Draws = 98.7	Percentage of Determinate Draws = 99.7
Percentage of Non-Existent Draws = 1	Percentage of Non-Existent Draws = 0
Percentage of Non-Positive Definite Draws = 0	Percentage of Non-Positive Definite Draws = 0
Percentage of Change in Dimension = 0.3	Percentage of Change in Dimension = 0.3

In Table 5 we report point estimates and frequency distributions for the determinacy properties of equilibrium. In contrast to Clarida-Galí-Gertler we find that for our restricted estimates equilibrium is determinate in both sub-samples. Further, this property is a robust feature of the estimates in the sense that in both sub-samples 99% or more of draws from the estimated asymptotic parameter distribution result in reduced form models with determinate equilibria.

TABLE 6:	RESTRICTED ESTIMATES OF VCV MATRIX OF FUNDAMENTAL SHOCKS						
	Sample 60:4—79:3				Sample 83:1—99:3		
X 10^{-3}	**Gap**	**Infl**	**Irate**	X 10^{-3}	**Gap**	**Infl**	**Irate**
Gap	8.8	0	0	**Gap**	5.9	0	0
Infl	-8.4	11.6	0	**Infl**	-18.3	23.5	0
Irate	-83.5	18.7	77.5	**Irate**	-31.2	41	18

Table 6 documents our estimates of a Choleski decomposition of the estimated variance-covariance matrix of the shocks using the restricted parameter estimates and Figures 3 and 4 compare the estimated impulse response functions for the two sets of restricted estimates.[14]

[14]For the pre-Volcker period, the estimated standard deviations of output shocks and inflation shocks are comparable but the interest rate standard deviation is two and a half times larger. For the Volcker-Greenspan period the standard deviation of the output shock is comparable to the unrestricted model, but the inflation standard deviation is twice as high and the interest rate shock-deviation increases by a factor of two. For the Pre-Volcker years the determinant of the matrix of standard deviations is a three times higher in the restricted versus unrestriced case and seven times higher in the Volcker-Greenspan period.

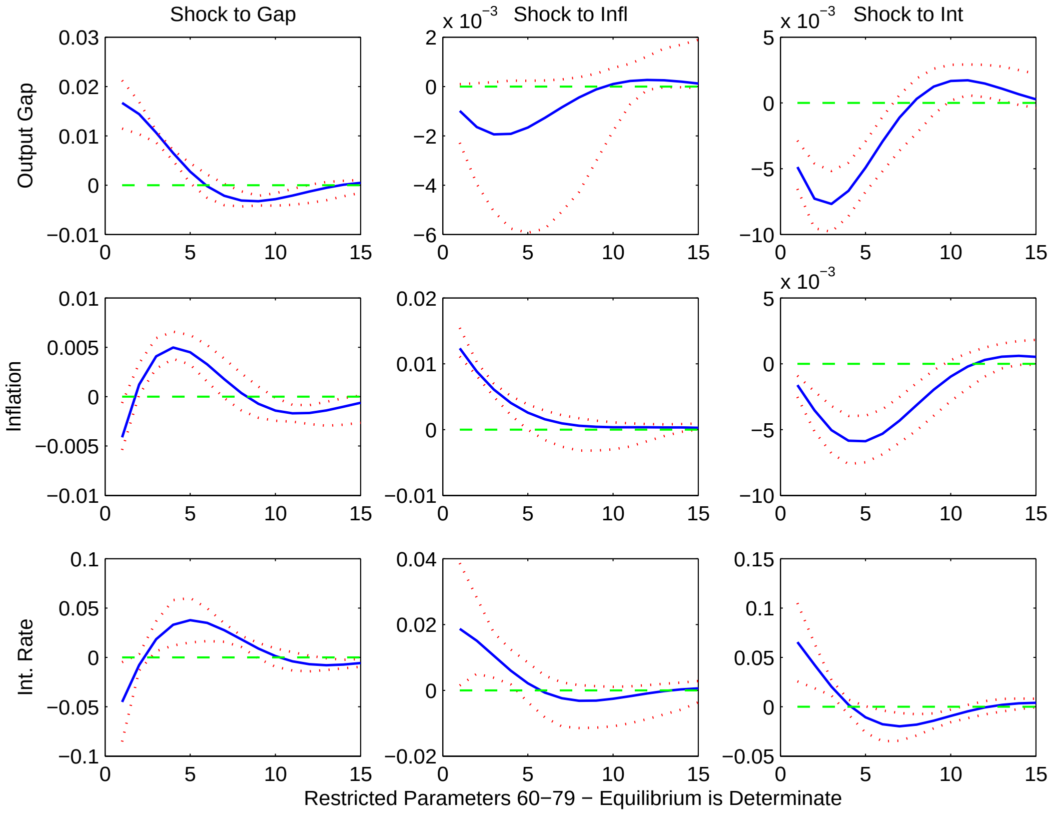

Figure 3: Restricted Impulse Responses: First Subsample

The main feature of these graphs is the consistency across sub-samples in the qualitative features of the impulse responses. The New-Keynesian model, estimated by GMM, provides a plausible economic interpretation of the effects of shocks and the way that the effects of these shocks trace themselves through the system. According to this interpretation, there are two fundamental shocks, one to the Phillips curve and one to the Euler equation. These shocks are negatively correlated and there is some evidence that the Phillips curve shock has increased in magnitude since 1979. In the following discussion we refer to the shock to the Euler equation as an output shock and the shock to the Phillips curve as an inflation shock.

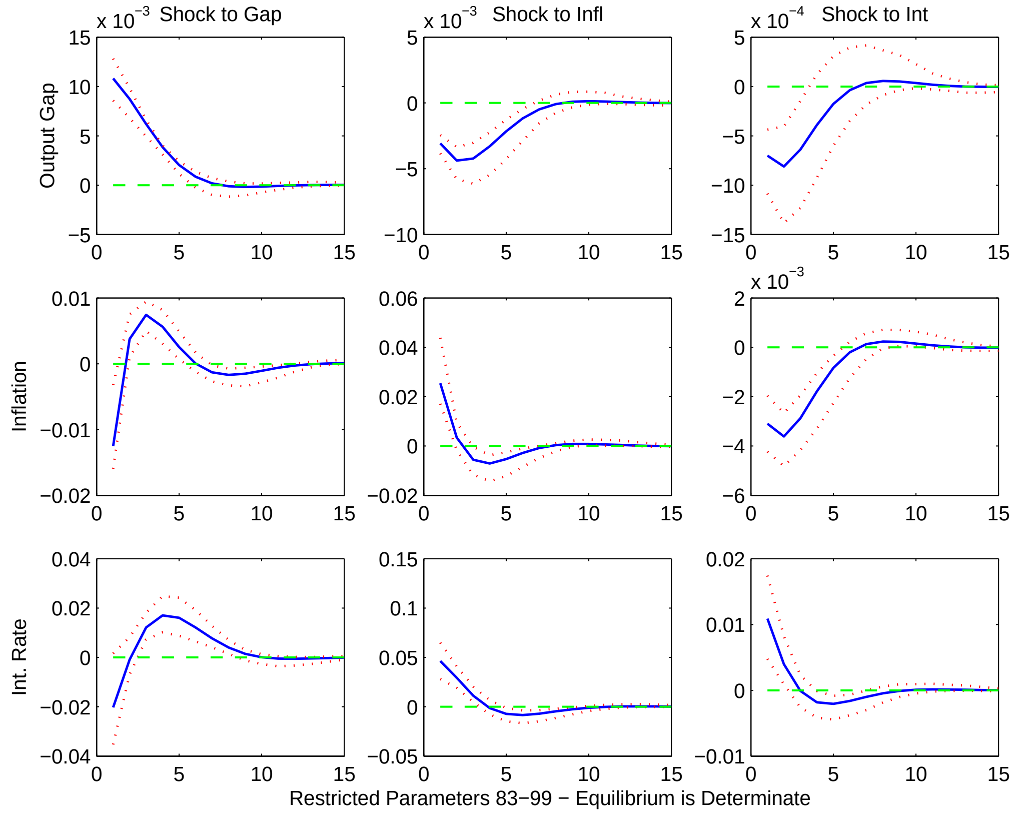

Figure 4: Restricted Impulse Responses: Second Subsample

A typical output shock causes the output gap to increase by 1% to 2% above trend and to return to trend in approximately six quarters. This response is graphed in the top left panels of Figures 3 and 4. Turning to the middle left panels, the output shock causes a small drop in inflation that quickly reverses itself and inflation remains above trend for the subsequent two to ten quarters after the impact. Under the pre-Volcker policy rule the inflation effect of an output shock was less than one half of a percentage point on impact and inflation remained within one half of a percentage point of its steady state level during the entire return path. In the Volcker-Greenspan period the response of inflation to an output shock is qualitively identical but the magnitude is roughly twice as big, possibly because the interest rate response to an output shock is lower in the Volcker Greenspan period. Notice from the lower left panels of these Figures that the time path of interest rates, following an output shock, is qualitively the same as that of inflation but, in the pre-Volcker period, the Fed's response was more aggressive. Within two quarters of an output shock, the Fed moved the nominal interest rate up by 5% above trend; in the Volcker-Greenspan period this response has been closer to 2% on average.

The middle column of Figures 3 and 4 illustrate estimated responses to an inflation shock. The typical inflation shock has been 1% in the pre-Volcker period and 2% under Volcker-Greenspan and our estimates suggest that inflation returned to trend in ten quarters in the first sub-sample and four quarters in the second. An inflation shock caused a 2% interest rate increase in the pre-Volcker period and a 5% increase in the post Volcker period; these interest rate increases were associated with small drops in output, varying from 0.2 to 0.5 basis points.

The final column of Tables 3 and 4 illustrates the effect of interest shocks to the economy. Interest rate policy is less volatile in the Volcker-Greenspan period. Interest rate increases cause falls in output and corresponding falls in inflation, both of which subsequently return to trend roughly ten quarters later.

[MAYBE A SHORT CONCLUSIVE PARAGRAPH ON THE MOST IMPORTANT LESSON FROM THE EMPIRICAL EXERCISE?][

5 Conclusion

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Appendix A

In this section we explain how our solution algorithm `SySolve` works in the case of an indeterminate equilibrium. The reader is referred to Sims [16] for a more detailed explanation of `GENSYS`, on which our algorithm is based. The structural model has the form

$$\tilde{A}_0 X_t = \tilde{A}_1 X_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \tag{A1}$$

Using a QZ decomposition, write this as

$$QSZX_t = QTZX_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \tag{A2}$$

where $QQ' = ZZ' = I$ and S and T are upper triangular and S and T are ordered such that all unstable generalized eigenvalues are in the bottom right corner. Recall that the generalized eigenvalues are defined as the ratios of the diagonal elements of T to the diagonal elements of S. Now define,

$$x_t = ZX_t \tag{A3}$$

and

$$e_t = S^{-1} Q' \left(\tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t \right), \tag{A4}$$

and partition x_t and e_t as follows;

$$x_t = \left(x_t^1, x_t^2 \right)', \quad e_t = \left(e_t^1, e_t^2 \right)' \tag{A5}$$

where $x_t^1 \in \mathbb{C}^{n_1}$, $x_t^2 \in \mathbb{C}^{n_2}$ are (possibly) complex vectors and n_1 and n_2 are the numbers of stable and unstable roots. Now partition the matrices S and T

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \tag{A6}$$

and the matrices S^{-1} and Q' as;

$$S^{-1} = \begin{bmatrix} S^{11} & S^{12} \\ 0 & S^{22} \end{bmatrix}, \; Q' = \begin{bmatrix} Q^{11} & Q^{12} \\ Q^{22} & Q^{22} \end{bmatrix}. \tag{A7}$$

Using this notation write (A2) as;

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} + \begin{bmatrix} e_t^1 \\ e_t^2 \end{bmatrix}. \tag{A8}$$

In Equation (A8) the lower block acts as an autonomous unstable subsystem in the transformed variables x_t^2. For the system to exhibit a non-explosive solution, one requires that $x_t^2 = e_t^2 = 0$ for all t. This restriction requires that that the non-fundamental errors W_t be chosen to remove the influence of the fundamental errors V_t. To this end, the solution algorithm sets

$$e_t^2 = S^{22} \begin{bmatrix} Q^{21} & Q^{22} \end{bmatrix} \left(\tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t \right) = 0. \tag{A9}$$

A necessary condition for these equations to have a solution is that there are at least as many elements of W_t as there are unstable roots (the number of rows in Equation system (A9)). In the case of r degrees of indeterminacy there are r more elements of W_t than one requires to eliminate unstable roots. In this case, our algorithm transfers the first r non-fundamental shocks to the vector V_t thereby treating the elements of $W_t^1 \in R^r$ as additional fundamentals. We refer to the expanded vector of fundamentals as (V_t, W_t^1). It might appear that this solution is arbitrary since a particular solution depends on the ordering of W_t. To see that this is not the case, let Ω represent the variance-covariance matrix of the expanded fundamentals

$$\Omega = E_t \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix} \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix}' = \begin{bmatrix} \Omega_{vv} & \Omega_{vw} \\ \Omega_{wv} & \Omega_{ww} \end{bmatrix}. \tag{A10}$$

Since we do not place any restrictions on Ω our algorithm is capable of generating the full range of sunspot solutions. Different solutions are captured by picking different values for the variance-covariance terms Ω_{ww} and Ω_{wv}.

Appendix B

This appendix explains how to generate a VARMA model in observable variables Y_t from the VAR in Y_t and $E_t[Y_{t+1}]$ that is generated by the solution algorithm `SySolve`. Consider the structural model

$$\tilde{A}_0 X_t = \tilde{A}_1 X_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t, \tag{B1}$$

where $X_t = (Y_t^1, Y_t^2)'$.[15] Our goal is to write Y_t^1 as a VARMA(1,1); that is, in the form,

$$Y_t^1 = \tilde{\Gamma} Y_{t-1}^1 + \Psi_a \eta_t + \Psi_b \eta_{t-1}, \tag{B2}$$

where Y_t^1 is a subset of X_t that are observable,

$$\eta_t = \begin{bmatrix} V_t \\ W_t^1 \end{bmatrix},$$

and W_t^1 is a subset of W_t.

Since the case of a determinate equilibrium is well understood, we concentrate in this appendix on the case of indeterminacy. Using a QZ decomposition (see Sims **??**) to eliminate the influence of unstable generalized eigenvalues, write the reduced form of (B1) as follows,

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_V^* V_t + \Psi_W^* W_t^1 \tag{B3}$$

where $X_t = (Y_t^1, Y_t^2)$. By construction, all of the roots of Γ^* are inside the unit circle and W_t^1 has dimension equal to the degree of indeterminacy. For each unstable generalized eigenvalue of $\left\{\tilde{A}_0, \tilde{A}_1\right\}$, Γ^* has one zero root. It follows that Γ^* has rank $k \leq m + n$.

[15] For the model in Section ??, $Y_t^1 = Y_t$ and $Y_t^2 = E_t[Y_{t+1}]$ but more generally, the dimensions of the vectors Y_t^1 and Y_t^2 need not be equal. For example, if the model has additional lags then Y_t^1 might consist of the vector $[Y_t, Y_{t-1}]'$.

Without loss of generality we assume that

$$rank\left(\Gamma^{*}\right)=k\leq n.$$

If $k>n$, Equation (B1) can be expanded by increasing the space of predetermined observable variables from $Y_t^1 = Y_t$, to $Y_t^1 = (Y_t, Y_{t-1})$ and adding the identity row $Y_{t-1} = Y_{t-1}$to Equation (B1). This expansion leaves k unchanged, (since Γ^* is augmented by n columns of zeros) and m unchanged (since the dimension of Y_t^2 remains the same) but it increases the dimension of Y_t^1 by n.

Using the decomposition

$$\Gamma^{*}=Q\Lambda Q^{-1}$$

where Λ is a diagonal matrix of eigenvalues ordered with the $n-k$ zero eigenvalues of Γ^* in the lower right position, rewrite (B3) as

$$x_t=\Lambda x_{t-1}+d_t \tag{B4}$$

where

$$x_t=Q^{-1}X_t,\ X_t=Qx_t \tag{B5}$$

$$d_t=Q^{-1}\left(C^{*}+\Psi_V^{*}V_t+\Psi_W^{*}W_t\right). \tag{B6}$$

Now partition $x_t = \left(x_t^a, x_t^b\right)'$ $\Lambda = (\Lambda_a, 0)'$and $d_t = (d_{1t}, d_{2t})'$ such that the variables x_t^b correspond to the zero roots of Γ^* and partition Q conformably as

$$Q=\begin{bmatrix} Q_{aa} & Q_{ab} \\ Q_{ba} & Q_{bb}\end{bmatrix}.$$

Consider the following four equations:

$$x_t^a=\Lambda_a x_{t-1}^a+d_t^a, \tag{B7}$$

$$x_t^b=d_t^b, \tag{B8}$$

$$Y_t^a = Q_{aa}x_t^a + Q_{ab}x_t^a, \tag{B9}$$

$$Y_{t-1}^a = Q_{aa}x_{t-1}^a + Q_{ab}x_{t-1}^b. \tag{B10}$$

Equation (B7) contains the rows of (B4) associated with the k non-zero eigenvalues and (B8) the rows associated with the $n-k$ zero eigenvalues. Equations (B8) and (B10) follow from the first k rows of Equation (B5). Solving these equations for Y_t^a in terms of Y_{t-1}^a, d_t^a, d_t^b and d_{t-1}^b gives the following expression

$$Y_t^a = \left[Q_{aa}^{-1}\Lambda_a Q_{aa}\right] Y_{t-1}^a + Q_{aa}da + Q_{ab}d_t^b + \left[Q_{aa}^{-1}\Lambda_a Q_{aa}\right] Q_{ab}d_{t-1}^b.$$

From Equation (B6) one can find an expression for d_t^a and d_t^b as functions of C^*, Ψ_V^*, Ψ_W^*, V_t and W_t^1. This leads to an expression in the form of (B2), which is what we set out to accomplish.

Appendix C

This Appendix shows that the estimator of Ω proposed in Section 3.2 is consistent. Taking probability limits of (10) and making use of Equation (8), the consistency of $\hat{\Gamma}_a^*$ and $\hat{C}_a^*$ and the assumption (Equation (**??**)) that η_t is uncorrelated with its own lags leads to;

$$p\lim_{T\to\infty} \hat{S}_{0T} = \sum_{t=1}^{T} \frac{\left(\Psi_a^*\eta_t + \Psi_b^*\eta_{t-1}\right)\left(\Psi_a^*\eta_t + \Psi_b^*\eta_{t-1}\right)}{T} = \Psi_a^*\Omega\Psi_a^{*\prime} + \Psi_b^*\Omega\Psi_b^{*\prime}, \tag{C1}$$

$$p\lim_{T\to\infty} \hat{S}_{1T} = \sum_{t=1}^{T} \frac{\left(\Psi_a^*\eta_t + \Psi_b^*\eta_{t-1}\right)\left(\Psi_a^*\eta_t + \Psi_b^*\eta_{t-1}\right)}{T} = \Psi_a^*\Omega\Psi_b^{*\prime}. \tag{C2}$$

Now form the sum

$$\hat{S}_T = \hat{S}_{0T} + \hat{S}_{1T} + \hat{S}_{1T}'. \tag{C3}$$

Taking probability limits of (C1), using (C2) and (C3) gives

$$p\lim_{T\to\infty} \hat{S}_T = \left(\Psi_{aT}^* + \Psi_{bT}^*\right)\Omega\left(\Psi_{aT}^* + \Psi_{bT}^*\right)'. \tag{C4}$$

Now replace Ψ_a^* and Ψ_b^* by consistent estimates $\hat{\Psi}_{aT}^*$ and $\hat{\Psi}_{bT}^*$ obtained from passing the GMM estimates of the structural parameters through the QZ solution algorithms and `Arrange` to obtain the following system of n^2 equations in the $(n+r)^2$ unknown elements of the variance-covariance matrix Ω.

$$\underset{n\times n}{\hat{S}_T} = \left(\underset{n\times(n+r)}{\hat{\Psi}_{aT}^*} + \underset{n\times(n+r)}{\hat{\Psi}_{bT}^*}\right)\underset{(n+r)\times(n+r)}{\hat{\Omega}_T}\left(\underset{n\times(n+r)}{\hat{\Psi}_{aT}^*} + \underset{n\times(n+r)}{\hat{\Psi}_{bT}^*}\right)'. \tag{C5}$$

Now define

$$\hat{\Psi}_T = \hat{\Psi}_a^* + \hat{\Psi}_b^*. \tag{C6}$$

Since $\hat{S}_T$ and $\hat{\Omega}_T$ are symmetric this system reduces to $n\,(n+1)/2$ equations in $(n+r)\,(n+r+1)/2$ unknowns which we write as

$$\text{vech}\left(\hat{S}_T\right) = B\left(\hat{\Psi}\right)\text{vech}\left(\hat{\Omega}_T\right), \tag{C7}$$

where vech is the operator that stacks the lower triangular elements of a symmetric matrix into a row vector. For $r > 1$ Equation (C7) system will have multiple solutions and we are free to choose $r(n+r+1)/2$ linear combinations. We identify a solution by adding an arbitrary $[r(n+r+1)/2] \times [(n+r)(n+r+1)/2]$ matrix R such that.

$$\text{vech}\left(\hat{S}_T\right) = R\text{vech}\left(\hat{\Omega}_T\right). \tag{C8}$$

Our estimator of $\hat{\Omega}_T$ is given by

$$\text{vech}\left(\hat{\Omega}_T\right) = \begin{bmatrix} B\left(\hat{\Psi}_T\right) \\ R \end{bmatrix}^{-1} \text{vech}\left(\hat{S}_T\right). \tag{C9}$$

Consistency follows for arbitrary R from the properties of probability limits and the fact that

$$\text{vech}\left(S_T\right) = \begin{bmatrix} B\left(\Psi\right) \\ R \end{bmatrix} \text{vech}\left(\Omega\right). \tag{C10}$$