Existence of Equilibrium for Integer Allocation Problems

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Abstract

In this paper we show that if all agents are equipped with well-behaved discrete concave production functions, then a feasible price allocation pair is a market equilibrium if and only if it solves a linear programming problem.
Using this result we are able to obtain a necessary and sufficient condition for existence that requires an equilibrium price vector to satisfy finitely many inequalities. A necessary and sufficient condition for the existence of market equilibrium when the maximum value function is Weakly Monotonic at the initial endowment that follows from our results is that the maximum value function is partially concave at the initial endowment. We also provide a discussion of the results and an alternative solution concept. The alternative solution concept is however, informationally and computationally inefficient.

1. Introduction: The equilibrium existence problem with indivisibilities has been investigated by Yang (2001) and more recently by Sun and Yang (2004). The model they consider is the model of a market game due to Shapley and Shubik (1969, 1976), with the goods being available for redistribution among the agents being available in integer units only. Yang (2004), shows that a constrained market equilibrium (i.e. a market clearing allocation, where each agent is constrained to maximize profit subject to its consumption not exceeding the initial endowment of the goods), exists if and only if there is a feasible price allocation pair that solves a certain linear programming problem. The result is obtained without any concavity assumption being imposed on the production function of the agents.
In this paper we show that if all agents are equipped with discrete concave production function, then a feasible price allocation pair is a market equilibrium (i.e. where agents solve an unconstrained profit maximization problem at given prices to obtain the resulting market clearing allocation) if and only if it solves a linear programming problem, similar to, but perhaps simpler than the one invoked in Yang (2001).
Using this result, but assuming discrete concave production function for the agents once again, we are able to show that the necessary and sufficient condition for the existence of market equilibrium available in Sun and Yang (2004), which involved obtaining a price vector that satisfied infinitely many inequalities, can be reduced to one where such a price vector satisfies finitely many such inequalities.

These results provide the necessary computational device for obtaining a market equilibrium for integer allocation problems.

Bikhchandani and Ostroy (1998), obtain existence results for what they call the package assignment model. Package assignment models turn out to have a structure similar to the one studied in Yang (2001).

In a final section we provide a discussion of the results and an alternative solution concept. However, the construction of this new solution for an integer allocation problem generally requires an infinite number of computations and hence an unrealistic data set. Thus, its practical usefulness as a solution for integer allocation problems is doubtful.

2. The Model: We now develop the general equilibrium model for the case where the inputs are available in integer amount only.

Let $Z = \aleph \cup \{0\}$, where $\aleph$ denotes the set of natural numbers. Let there be $H > 0$ agents and $L+1 > 1$ commodities. The first L commodities are used as inputs to produce the $L+1^{th}$ commodity, which is a numeraire consumption good. Let $w \in Z^L$ denote the aggregate initial endowment of the inputs which is available for distribution among the agents.

For $j = 1,\ldots,L$, let w_j denote the aggregate amount of commodity j that is initially available in the economy.

A function $f: Z^L \to \Re_+$(: the set of non-negative real numbers) is said to be discrete concave if there exists a continuous concave function $g: \Re_+^L \to \Re_+$ such that the restriction of g to Z^L coincides with f.

While discrete concavity could be defined for functions whose range may include negative real numbers or for that matter are unbounded below, for the purpose of the analysis reported here such a generalization would be superfluous, since we shall primarily be concerned with the discrete concavity of production functions.

Given functions $f: Z^L \to \Re_+$ and $g: \Re_+^L \to \Re_+$, let hypograph(f) $\equiv \{(x, \alpha) \in Z^L \times \Re / \alpha \leq f(x)\}$ and hypograph(g) $\equiv \{(x, \alpha) \in \Re_+^L \times \Re / \alpha \leq g(x)\}$.

Given a function $f: Z^L \to \Re_+$ its canonical extension is the function g^f: $\Re_+^L \to \Re_+$ such that the hypograph(g^f) = convex hull of hypograph(f). Clearly g^f is continuous and concave.

Let e denote the vector in $\Re^L$ all whose coordinates are equal to one and for $j = 1,\ldots,L$, let e^j denote the vector in $\Re^L$ whose j^{th} coordinate is equal to one and all other coordinates are equal to zero.

If f is discrete concave, then the restriction of its canonical extension g^f to Z^L coincides with f.

For $x \in Z^L$, let $C(x) = \{z \in Z^L / z \leq x\}$ and $C^*(x)$ = Convex hull of $C(x)$.

Given $z^* \in Z^L$, a discrete concave function $f: Z^L \to \Re_+$ is said to be well behaved up to z^* if for all $x \in C(z^*)$ and $y \in C^*(x)$: $g^f(y) = \mathrm{Max}\ \{ \sum_{z \in C(x)} t(z) f(z)$ / $t(z) \geq 0$ for all $z \in C^*(x)$, $\sum_{z \in C(x)} t(z) f(z) = y$ and $\sum_{z \in C(x)} t(z) = 1\}$.

Each agent i has preferences defined over Z^L which is represented by a discrete concave production function f^i.

The pair <{f^i/i= 1,…,H}, w> where for all i∈{1,…,H}, f^i is well behaved up to w +e, is called an integer allocation problem and is assumed to be a given for the rest of the analysis.

An integer allocation problem <{f^i/i= 1,…,H}, w> is said to be a multi-unit auction if for all i = 1,…,H and $x \in Z^L$, $f^i(x)$ = Max{f(z)/ z ∈C(y)∩C(w)}= Max{f(z)/ $z^j \leq \min\{x^j, w^j\}$ for all j = 1,…,L}.
An input consumption vector of agent i is denoted by a vector $X^i \in Z^L$.
A price vector p is an element of $\Re^L_+\backslash\{0\}$, where for j = 1,…,L, p_j denotes the price of input j.
At a price vector p, the objective of agent i is to maximize profits:
Maximize $[f^i(X^i) - p^T X^i]$
An allocation is an array X = <X^i/ i = 1,…,H> such that $X^i \in Z^L$ for all i = 1,…,H.
Given $x \in Z^L$, let F(x) = { X = <X^i/ i = 1,…,H>/ X is an allocation satisfying $\sum_{i=1}^{H} X^i$ = x}.
An allocation X is said to be feasible if X∈F(w).
A market equilibrium is a pair <p*,X*> where p* is a price vector, X* is a feasible allocation and for all i = 1,…,H, X^{*i} maximizes profits for agent i.
The function $V: Z^L \rightarrow \Re_+$ such that for all $x \in Z^L$: V(x) = Max{ $\sum_{i=1}^{H} f^i(X^i)$ / X = <X^i/ i = 1,…,H>∈F(x)}, is called the maximum value function.
Since under our assumptions for all i = 1,…,H, f^i is non-decreasing (i.e. for all $x,y \in Z^L$:[x ≥ y] implies $[f^i(x) \geq f^i(y)]$), it must be the case that V is non-decreasing as well (i.e. for all $x,y \in Z^L$:[x ≥ y] implies [V(x) ≥ V(y)]).
A feasible allocation X* = <X^{*i}/i = 1,…,H> is said to be efficient if $\sum_{i=1}^{H} f^i(X^{*i})$ = V(w).

The following result is available in Lahiri (2005) and will be used in the sequel.

Proposition 1: Let <p*,X*> be a market equilibrium. Then X* is an efficient allocation.

A constrained market equilibrium is a pair <p*,X*> where p* is a price vector, X* is a feasible allocation such that for all i = 1,…,H, X^{*i} solves:
Maximize $[f^i(X^i) - p^T X^i]$
Subject to $X^i \leq w$.
Clearly a market equilibrium is a constrained market equilibrium. The following example shows that a constrained market equilibrium need not be a market equilibrium.

Example 1: Let H = 1, L = 1, $f^1(x)$ = x for all x∈Z and w = 1. Let X^{*1} = 1 and X* = <X^{*1}>. For all p∈(0, 1], <p, X*> is a constrained market equilibrium, though for p∈(0,1), <p, X*> is never a market equilibrium. <1,X*> is the unique market equilibrium for this integer allocation problem.

However, if $<\{f^i / i = 1,\ldots,H\}, w>$ is a multi-unit auction then $<p^*, X^*>$ is a market equilibrium if and only if it is a constrained market equilibrium.
For $x, y \in Z^L$ let $m(x,y)$ be the L-vector whose j^{th} coordinate is $\min\{x^j, y^j\}$.
Thus, if $<\{f^i / i = 1,\ldots,H\}, w>$ is a multi-unit auction then for all $i = 1,\ldots,H$ and $x \in Z^L$, $f^i(x) = f^i(m(x,w))$.

Proposition 2: Let $<\{f^i / i = 1,\ldots,H\}, w>$ be a multi-unit auction. Then $<p^*, X^*>$ is a market equilibrium if and only if it is a constrained market equilibrium.

Proof: We only need to show that if $<p^*,X^*>$ is a constrained market equilibrium for the multi unit auction $<\{f^i / i = 1,\ldots,H\}, w>$, then it is also a market equilibrium.
Hence suppose $<p^*,X^*>$ is a constrained market equilibrium. Let $x \in Z^L$.
If $x \le w$, then $x = m(x,w)$ and clearly for all $i = 1,\ldots,H$: $f^i(X^{*i}) - p^{*T}X^{*i} \ge f^i(x) - p^{*T}x$.
Thus suppose, $x^j > \min\{x^j, w^j\}$ for some $j = 1,\ldots,L$. Hence for all $i = 1,\ldots,H$: $f^i(X^{*i}) - p^{*T}X^{*i} \ge f^i(m(x,w)) - p^{*T}m(x,w) \ge f^i(x) - p^{*T}x$. Q.E.D.

3. Existence of Market Equilibrium: This section is devoted to obtaining results pertaining to the existence of market equilibrium for the given integer allocation problem. We now state and prove the main theorem of this paper.

Theorem 1: Let X^* be a feasible allocation and p^* a price vector. $<p^*,X^*>$ is a market equilibrium if and only if the pair $<p^*,m^*>$ solves:

Minimize $\sum_{i=1}^{H} m(i) + p^T w$

Subject to $\sum_{i=1}^{H} m(i) + p^T x \ge V(x)$ for all $x \in Z^L$ with $x \le w + e$, $p \in \Re_+^L$

where $m^* \in \Re^H$ with $m^* = <m^*(i) / i = 1,..H>$ satisfies $m^*(i) = f^i(X^{*i}) - p^{*T}X^{*i}$ for $i = 1,\ldots,H$.

Proof: Suppose $<p^*,X^*>$ is a market equilibrium. Let $x \in Z^L$ and $V(x) = \sum_{i=1}^{H} f^i(X^i)$ where $X = <X^i / i = 1,\ldots,H> \in F(x)$.
Thus, for all $i = 1,\ldots,H$: $f^i(X^{*i}) - p^{*T}X^{*i} \ge f^i(X^i) - p^{*T}X^i$.

Summing over i we get: $\sum_{i=1}^{H} m^*(i) + p^{*T}x \ge V(x)$ for all $x \in Z^L$.

Thus, $<p^*,m^*>$ satisfies the constraints.

Now, let $<p,m>$ satisfy the constraints. Thus, $\sum_{i=1}^{H} m(i) + p^T w \ge V(w)$.

However, by Proposition 1, X^* is efficient and hence $\sum_{i=1}^{H} m^*(i) + p^{*T}w = V(w)$.

Thus, $\sum_{i=1}^{H} m(i) + p^T w \ge \sum_{i=1}^{H} m^*(i) + p^{*T}w$.

Hence, $<p^*,m^*>$ solves the minimization problem.

Now, suppose <p*,m*> solves the given minimization problem. Towards a contradiction suppose <p*, X*> is not a market equilibrium. Thus, there exists $i \in H$ and $x \in Z^L$, such that $f^i(x) - p^{*T}x > f^i(X^{*i}) - p^{*T}X^{*i}$.

Suppose $x \leq X^{*i} + e$.

Since $X^* \in F(w)$, $w^* = x + \sum_{k \neq i} X^{*k} \leq X^{*i} + e + \sum_{k \neq i} X^{*k} = w + e$.

Thus, $V(w^*) \geq f^i(x) + \sum_{k \neq i} f^k(X^{*k})$

$$> f^i(X^{*i}) - p^{*T}X^{*i} + p^{*T}x + \sum_{k \neq i} f^k(X^{*k}) - p^{*T}\sum_{k \neq i} X^{*k} + p^{*T}\sum_{k \neq i} X^{*k}$$

$$= f^i(X^{*i}) - p^{*T}X^{*i} + \sum_{k \neq i} f^k(X^{*k}) - p^{*T}\sum_{k \neq i} X^{*k} + p^{*T}w^*$$

$$= \sum_{i=1}^{H} m^*(i) + p^{*T}w^*,$$

leading to a contradiction.

Thus it is not the case that $x \leq X^{*i} + e$.

In fact we have shown that for $i \in H$ and $x \in Z^L$, with $x \leq X^{*i} + e$, it is the case that $f^i(x) - p^{*T}x \leq f^i(X^{*i}) - p^{*T}X^{*i}$.

Since $X^{*i} << X^{*i} + e$, for $t \in (0,1)$, t sufficiently small, the real vector $X^{*i} + t(x - X^{*i}) << X^{*i} + e$.

Let $g^i = g^{f^i}$ denote the canonical extension of f^i.

Thus, $g^i(x) - p^{*T}x > g^i(X^{*i}) - p^{*T}X^{*i}$.

Since g^i is concave $g^i(X^{*i} + t(x - X^{*i})) \geq g^i(X^{*i}) + t(g^i(x) - g^i(X^{*i}))$.

Hence, $g^i(X^{*i} + t(x - X^{*i})) - p^{*T}(X^{*i} + t(x - X^{*i})) \geq g^i(X^{*i}) + t(g^i(x) - g^i(X^{*i})) - p^{*T}(X^{*i} + t(x - X^{*i}))$.

Now, $g^i(X^{*i} + t(x - X^{*i})) - p^{*T}(X^{*i} + t(x - X^{*i})) \geq g^i(X^{*i}) - p^{*T}X^{*i} + t([g^i(x) - p^{*T}x] - [g^i(X^{*i})) - p^{*T}X^{*i}]) > g^i(X^{*i})) - p^{*T}X^{*i}$.

However, $X^{*i} + t(x - X^{*i}) << X^{*i} + e$.

Since f^i is well behaved up to w + e, $g^i(X^{*i} + t(x - X^{*i})) - p^{*T}(X^{*i} + t(x - X^{*i})) \leq \text{Max}\{f^i(y) - p^{*T}y / y \in Z^L \text{ such that } y \leq X^{*i} + e\} = f^i(X^{*i}) - p^{*T}X^{*i}$, leading to a contradiction.

This establishes the theorem. Q.E.D.

In the above proof, we used the well-behavedness of the production functions to conclude that local profit maximization implies global profit maximization. The following example (which emerged from a very illuminating correspondence with Katta Murty) shows that not all functions on Z^L which are restrictions of concave functions on $\Re^L_+$ are well-behaved. It also shows that Theorem 1 may not hold without the well-behavedness assumption and "local profit maximization" need not imply "global profit maximization" in the absence of this crucial assumption.

Example 2: Let $f: Z^2 \to \Re$ be defined thus:

$f(a,b) = 35a + 20b$ for $7a + 4b \leq 12$

$\quad\quad = 60 \quad\quad$ for $7a + 4b > 12$.

f is the restriction to Z^2 of a concave function on $\Re^2_+$.

f(0,3) = 60 and f attains a global maximum at (0,3).
Note, 55 = f(1,1).
Further, $g^f(\frac{1}{2}, 2) \geq \frac{1}{2} f(0,3) + \frac{1}{2} f(1,1) = 30 + \frac{55}{2} > 55$.
Since $(\frac{1}{2}, 2)$ belongs to C*((1,2)), and since if $(\frac{1}{2}, 2)$ is a convex combination of elements in C((1,2)) implies $(\frac{1}{2}, 2) = \frac{1}{2}(0, 2) + \frac{1}{2}(1, 2)$, it turns out that f is not well behaved.
For otherwise $g^f(\frac{1}{2}, 2) = 40\times\frac{1}{2} + 60\times\frac{1}{2} = 50$, contradicting $g^f(\frac{1}{2}, 2) > 55$.
Let $p = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$.
Then $h(a,b) = f(a,b) - p^T\begin{bmatrix} a \\ b \end{bmatrix} = 24a + 15b$ for $7a + 4b \leq 12$,

$= 60 - 11a - 5b$ for $7a + 4b > 12$.

h(2,2) = 28, h(2,1) = 33, h(2,0) = 38, h(1,2) = 39, h(1,1) = 39, h(1,0) = 24, h(0,2) = 30, h(0,1) = 15, h(0,0) = 0.
Thus, $f(1,1) - p^T\begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq f(a,b) - p^T\begin{bmatrix} a \\ b \end{bmatrix}$ for all $(a,b)\in C(2,2)$.
However, $f(0,3) - p^T\begin{bmatrix} 0 \\ 3 \end{bmatrix} = 60 - 15 = 45 > 39 = f(1,1) - p^T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Now, let us consider the integer allocation problem with H = 1, L = 2, $f^1 = f$, w = (1,1).
Had Theorem 1 been true without the well-behavedness assumption, it would reduce to the following in the present context:
<p*,w> is a market equilibrium where p* is a price vector if and only if <p*,m*> solves the following LP problem:
Min $m + p^Tw$
Subject to $m + p^Tx \geq f(x)$ for all $x\in C(2,2)$,
$p\in \Re^L_+\backslash\{0\}$,
with $m^* = f(w) - p^{*T}w$.
Let m* = 39 and $p^* = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. From our calculations above it follows that <p*,m*> satisfies the constraints. Since $m^* + p^{*T}w = f(w) \leq m + p^Tw$ for all <p,m> that is feasible for the LP problem, <p*,m*> actually solves the problem. Since $m^* = 39 = f(w) - p^{*T}w$, if Theorem 1 were to hold without the well-behavedness assumption, then <p*,w> would be a market equilibrium. However this is not the case since $f(0,3) - p^T\begin{bmatrix} 0 \\ 3 \end{bmatrix} > f(1,1) - p^T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Note: In the statement of Theorem 1 and in its proof, the constraint $p\in \Re^L_+$, which

appears for the linear, programming (minimization) problem, could be easily dispensed with without diluting the result in any way. The fact that the theorem concerns a price vector would then imply our version of Theorem 1.

The main result (Theorem 4.1) in Yang (2001) can be strengthened without requiring the production functions to be discrete concave, as follows:

Theorem 2: Let X* be a feasible allocation and p* a price vector. <p*,X*> is a constrained market equilibrium if and only if the pair <p*,m*> solves:

Minimize $\sum_{i=1}^{H} m(i) + p^T w$

Subject to $\sum_{i=1}^{H} m(i) + p^T x \geq V(x)$ for all $x \in Z^L$ with $x \leq w$, $p \in \Re_+^L$,

$m(i) + p^T x \geq f^i(x)$, for all $x \in Z^L$ with $x \leq w$ and $i = 1,\ldots,H$.

where $m^* \in \Re^H$ with $m^* = \langle m^*(i) / i = 1,..H \rangle$ satisfies $m^*(i) = f^i(X^{*i}) - p^{*T} X^{*i}$ for $i = 1,\ldots,H$.

Proof: The proof is similar to the proof of Theorem 1, but is being provided here for completeness.

Suppose <p*,X*> is a constrained market equilibrium. Let $x \in Z^L$ with $x \leq w$ and $V(x) = \sum_{i=1}^{H} f^i(X^i)$ where $X = \langle X^i / i = 1,\ldots,H \rangle \in F(x)$.

Clearly, $X^i \leq x \leq w$ for all $i = 1,\ldots,H$.

Thus, for all $i = 1,\ldots,H$: $f^i(X^{*i}) - p^{*T} X^{*i} \geq f^i(X^i) - p^{*T} X^i$.

Summing over i we get: $\sum_{i=1}^{H} m^*(i) + p^{*T} x \geq V(x)$ for all $x \in Z^L$.

Thus, <p*,m*> satisfies the constraints.

Now, let <p,m> satisfy the constraints. Thus, $\sum_{i=1}^{H} m(i) + p^T w \geq V(w)$.

However, by Proposition 1, X* is efficient and hence $\sum_{i=1}^{H} m^*(i) + p^{*T} w = V(w)$.

Thus, $\sum_{i=1}^{H} m(i) + p^T w \geq \sum_{i=1}^{H} m^*(i) + p^{*T} w$.

Thus, <p*,m*> solves the minimization problem.

Now, suppose <p*,m*> solves the given minimization problem. Towards a contradiction suppose <p*, X*> is not a constrained market equilibrium. Thus, there exists $i \in H$ and $x \in Z^L$ with $x \leq w$, such that $f^i(x) - p^{*T} x > f^i(X^{*i}) - p^{*T} X^{*i}$.

Thus, $f^i(x) > m^*(i) + p^{*T} x$, which leads to a violation of a constraint of the minimization problem and consequently a contradiction.

Thus, <p*, X*> is a constrained market equilibrium. Q.E.D.

Note: Sun and Yang (2004) establish the existence of market equilibrium allocations without assuming that the production functions are discrete concave. They show that a market equilibrium exists if and only if **there exists a price vector p* such that $V(w) - p^{*T}w \geq V(x) - p^{*T}x$ for all $x \in Z^L$.**

However, we are able to show that under the assumption of concave production functions the following is true:

Theorem 3: There exists a market equilibrium if and only if there exists a price vector p* such that **$V(w) - p^{*T}w \geq V(x) - p^{*T}x$ for all $x \in Z^L$ with $x \leq w + e$.**

Proof: Let <p*, X*> be a market equilibrium. By Proposition 1, $\sum_{i=1}^{H} f^i(X^{*i}) = V(w)$.

Let $x \in Z^L$ with $x \leq w + e$. By Theorem 1, $V(w) - p^*w \geq V(x) - p^{*T}x$, since $\sum_{i=1}^{H} X^{*i} = w$.

Now suppose there exists a price vector p* such that $V(w) - p^{*T}w \geq V(x) - p^{*T}x$ for all $x \in Z^L$ with $x \leq w + e$. Let X* be an efficient allocation. Since the feasible set is finite such an allocation always exists. Thus, $\sum_{i=1}^{H} f^i(X^{*i}) = V(w)$. Let, $m^* \in \Re^H$ with m* = <m*(i)/ i = 1,..H> satisfying $m^*(i) = f^i(X^{*i}) - p^{*T}X^{*i}$ for i = 1,…,H. Since , $\sum_{i=1}^{H} m^*(i) + p^{*T}x = V(w) - p^{*T}w + p^{*T}x \geq V(x)$ for all $x \in Z^L$ with $x \leq w + e$, <p*,m*> satisfies the constraints of the linear programming problem in Theorem 1.

Let $m \in \Re^H$ with m = <m(i)/ i = 1,..H> satisfy $\sum_{i=1}^{H} m(i) + p^Tx \geq V(x)$ for all $x \in Z^L$ with $x \leq w + e$.

Thus, $\sum_{i=1}^{H} m(i) + p^Tw \geq V(w) = \sum_{i=1}^{H} m^*(i) + p^{*T}w$.

Thus, <p*, m*> solves the linear programming problem in Theorem 1. By Theorem 1, <p*, X*> is a market equilibrium. Q.E.D.

4. Properties of the maximum value function for existence of equilibrium: We now investigate properties, which when satisfied by the maximum value function, guarantees the existence of a market equilibrium. In this section we assume that $w \in Z^L \cap \Re^L_{++}$.

The cardinality of C(w+e) = $\prod_{j=1}^{L}(w^j + 2)$. Let M denote the integer $\prod_{j=1}^{L}(w^j + 2) - 1$.

A function f: $Z^L \rightarrow \Re$ is said to be partially concave at w if for any positive integer K and arrays $<x^k/ k = 1,\ldots,K>$, $<t^k/ k = 1,\ldots,K>$ with $x^k \in C(w+e)$ and $t^k \geq 0$ for k = 1,…,K: [$w = \sum_{k=1}^{K} t^k x^k$, $\sum_{k=1}^{K} t^k = 1$] implies [$f(w) \geq \sum_{k=1}^{K} t^k f(x^k)$].

Lemma 1: A function f: $Z^L \rightarrow \Re$ is partially concave at w if and only if there exists $p \in \Re^L$ such that $f(w) - p^Tw \geq f(x) - p^Tx$ for all $x \in C(w+e)$.

Proof: Let C(w+e)\{w} be equal to the set $\{x^k/ k = 1,\ldots,M\}$.

Suppose f is partially concave at w. Towards a contradiction suppose there does not exist $p \in \Re^L$ such that $f(w) - p^T w \geq f(x) - p^T x$ for all $x \in C(w + e)$.
Hence, there does not exist $\alpha, \beta \in \Re^L_+$ and $\gamma \in \Re^M_+$: $\alpha^T(x^k - w) - \beta^T(x^k - w) - \gamma^k \geq f(x^k) - f(w)$ for all k = 1,…,M.
By Farka's Theorem there exists $t \in \Re^M_+$ such that $\sum_{k=1}^{M} t^k(x^k - w) \leq 0$, $-\sum_{k=1}^{M} t^k(x^k - w) \leq 0$ and $\sum_{k=1}^{M} t^k[f(x^k) - f(w)] > 0$.

Thus, $\sum_{k=1}^{K} t^k > 0$.

Dividing the three inequalities above by $\sum_{k=1}^{K} t^k$, we get there exists $s \in \Re^M_+$ such that $\sum_{k=1}^{M} s^k x^k = w$, $\sum_{k=1}^{M} s^k = 1$ and $\sum_{k=1}^{M} s^k f(x^k) > f(w)$, contradicting that f is partially concave at w.
Hence, there exists $p \in \Re^L$ such that $f(w) - p^T w \geq f(x) - p^T x$ for all $x \in C(w + e)$.
Now suppose that there exists $p \in \Re^L$ such that $f(w) - p^T w \geq f(x) - p^T x$ for all $x \in C(w + e)$.
Hence there exists $\alpha, \beta \in \Re^L_+$ and $\gamma \in \Re^M_+$: $\alpha^T(x^k - w) - \beta^T(x^k - w) - \gamma^k \geq f(x^k) - f(w)$ for all k = 1,…,M.
By Farka's Theorem there does not exist $t \in \Re^M_+$ such that $\sum_{k=1}^{M} t^k(x^k - w) = 0$ and $\sum_{k=1}^{M} t^k[f(x^k) - f(w)] > 0$.

Thus, [$t \in \Re^M_+$, $t^k \geq 0$ for k = 1,…,M, $w = \sum_{k=1}^{M} t^k x^k$, $\sum_{k=1}^{M} t^k = 1$] implies [$f(w) \geq \sum_{k=1}^{M} t^k f(x^k)$].
Thus, f is partially concave at w. Q.E.D.

A function $f: Z^L \rightarrow \Re$ is said to be Weakly Monotonic at w if:
(1) For all j = 1,…,L: $f(w + e^j) \geq f(w)$;
(2) $f(w + e) > f(w)$.

It is easy to see that if for some i, and weakly increasing (i.e. for all $x, y \in Z^L$:[$x \gg y$] implies [$f^i(x) > f^i(y)$], then V is Weakly Monotonic at w.

Lemma 2: Suppose $f: Z^L \rightarrow \Re$ is Weakly Monotonic at w. Then, f is partially concave at w if and only if there exists $p \in \Re^L_+ \backslash \{0\}$ such that $f(w) - p^T w \geq f(x) - p^T x$ for all $x \in C(w + e)$.

Proof: By Lemma 1, f is partially concave at w if and only if there exists $p \in \Re^L$ such that $f(w) - p^T w \geq f(x) - p^T x$ for all $x \in C(w + e)$.
Suppose towards a contradiction f is partially concave but $p_j < 0$, for some j. Then, $0 \leq f(w + e^j) - f(w)$ by Weak Monotonicity of f at w and $f(w + e^j) - f(w) \leq p^T(w + e^j - w) = p_j < 0$, leads to a contradiction.
Thus $p \in \Re^L_+$.
If p = 0, then $0 < f(w + e) - f(w)$ by Weak Monotonicity and $f(w + e) - f(w) \leq p^T e = 0$, again leads to a contradiction.
Thus, $p \in \Re^L_+ \backslash \{0\}$. Q.E.D.

In view of Lemmas 1 and 2 and Theorem 3, we can state the following result:

Theorem 4: Suppose V is Weakly Monotonic at w. A market equilibrium exists if and only if V is partially concave at w.

Without the Weak Monotonicity of V at w, we are not able to ensure that the vector of prices is non-zero. We can only guarantee that the vector is non-negative, which is not enough for it to be a price vector.

5. An Illustrative Example: Consider the following two agent (H = 2), three input (L = 3) integer allocation problem with w = e. Let $f: \Re^L_+ \to \Re_+$ be defined as follows: $f(0) = 0 = f(e^j)$ for j = 1,2,3; $f(e^j + e^k) = 3$ for $j,k \in \{1,2,3\}$ with $j \neq k$; $f(e) = 4$; for all $x \in Z^3 \backslash \{y \in Z^3 / y \leq e\}$, let $f(x) = f(\sum_{\{j / x_j > 0\}} e^j)$. f is discrete concave. Let $f^i = f$ for i = 1, 2. The integer allocation problem $<\{f^1, f^2\}, e>$ is an example of a bundle auction.
For this problem, if X is an efficient allocation then either $X^1 = e$ and $X^2 = 0$ or $X^1 = 0$ and $X^2 = e$.
Suppose $<p,X>$ is a market equilibrium. Then by Proposition 1, X is efficient. Without loss of generality, suppose $X^1 = e$ and $X^2 = 0$. In order that X^2 maximize profits for agent 2 at price vector p, it must be that $p_j + p_k \geq 3$ for all $j,k \in \{1,2,3\}$ with $j \neq k$. Thus, $2(p_1 + p_2 + p_3) \geq 9$ or $p_1 + p_2 + p_3 \geq 4.5 > 4 = f(e) = f^1(e)$. Thus, X^1 does not maximize profits for agent 1 at price vector p.
Thus, this integer allocation problem does not have a market equilibrium.
It is easy to verify that the maximum value function V is not partially concave at w = e.
Note that $V(2e) = 8$, $V(e) = 4$, $V(e^j + e^k) = 3$ for $j,k \in \{1,2,3\}$ with $j \neq k$. Now $e = \frac{1}{4}(2e) + \frac{1}{4}(e^1 + e^2) + \frac{1}{4}(e^1 + e^3) + \frac{1}{4}(e^2 + e^3)$.

$$\frac{1}{4}V(2e) + \frac{1}{4}V(e^1 + e^2) + \frac{1}{4}V(e^1 + e^3) + \frac{1}{4}V(e^2 + e^3) = 2 + \frac{9}{4} = 4.25 > 4 = V(e).$$

Thus, V is not partially concave at w.

6. Discussion of the results and an alternative solution concept: The partial concavity of V at w can be equivalently expressed as follows:

V(w) is the optimal value of the linear programming maximization problem:

Maximize $\sum_{x \in C(w+e)} \alpha(x)V(x)$

Subject to $\sum_{x \in C(w+e)} \alpha(x)x = w$,

$\sum_{x \in C(w+e)} \alpha(x) = 1$,

$\alpha(x) \geq 0$ for all $x \in C(w+e)$.

Having made this observation one may be tempted to conclude that a constrained market equilibrium exists if for all $y \in C(w)$, V(y) is the optimal value of the following linear programming maximization problem:

Maximize $\sum_{x \in C(w)} \alpha(x)V(x)$

Subject to $\sum_{x \in C(w)} \alpha(x)x = y$,

$\sum_{x \in C(w)} \alpha(x) = 1$,

$\alpha(x) \geq 0$ for all $x \in C(w)$.

That such a conclusion would be incorrect is once again illustrated by the illustrative example in section 5. That example does not admit a constrained market equilibrium, although it satisfies the requirements spelt out in the latter group of linear programming problems.

It is precisely this problem with the existence of market equilibrium, constrained or otherwise, that has lead to alternative solution concepts, primarily for the class of bundle auction problems. One of them is due to Wurman and Wellman (1999), which can be easily extended to the class of all integer allocation problems.

Towards that end let us define a payment function to be a function $\pi: Z^L \rightarrow \Re_+$, where for $x \in Z^L$, $\pi(x)$ is the payment that is required to be made to purchase the bundle x.

A payment equilibrium is an ordered pair $<\pi^*, X^*>$, where π^* is a payment function and X^* is a feasible integer allocation that together satisfy the following condition:

For all i = 1,…,H: $f^i(X^{*i}) - \pi^*(X^{*i}) \geq f^i(x) - \pi^*(x)$ for all $x \in Z^L$.

Let X* be an efficient allocation.

Consider the following two linear programming minimization problems:

Problem 1

Minimize $\sum_{i=1}^{H} q_i$

Subject to $s_i + q_k \geq f^i(X^{*k})$ for all i, k = 1,…,H,

$\sum_{i=1}^{H} [s_i + q_i] = V(w)$,

$s_i, q_i \geq 0$ for all i = 1,…,H.

Problem 2

Minimize $\sum_{i=1}^{H} s_i$

Subject to $s_i + q_k \geq f^i(X^{*k})$ for all i, k = 1,…,H,

$\sum_{i=1}^{H} [s_i + q_i] =$ V(w),

$s_i, q_i \geq 0$ for all i = 1,…,H.

Let (s^1, q^1) be a solution for Problem 1 and (s^2, q^2) be a solution for Problem 2.
For j = 1,2 and i = 1,…,H, let s_i^j denote the i[th] coordinate of s^j and q_i^j denote the i[th] coordinate of q^j.

Note: (a) $[s_i + q_i \geq f^i(X^{*i})$ for all i = 1,…,H, and $\sum_{i=1}^{H} [s_i + q_i] = V(w)]$ implies $[s_i + q_i = f^i(X^{*i})$ for all i = 1,…,H].
(b) Suppose $X^{*i} = X^{*k}$ for some i, k $\in$ {1,…,H}. Then, $s_i + q_k \geq f^i(X^{*k}) = f^i(X^{*i}) = s_i + q_i$ implies $q_k \geq q_i$. Similarly $q_i \geq q_k$. Thus, $q_i = q_k$.
For j = 1,2, let π^j be the function fro Z^L to $\Re$ defined as follows: $\pi^j(X^{*i}) = q_i^j$ for i=1,…,H; $\pi^j(x) = \max\{\max\{f^i(x) - s_i^j / i = 1,\ldots,H\},0\}$ for x $\in Z^L \setminus \{X^{*i}/ i = 1,\ldots,H\}$.
For t$\in$[0,1], let $\pi^t: Z^L \rightarrow \Re_+$ be defined as follows: $\pi^t(x) = t\pi^2(x) + (1-t)\pi^1(x)$ for all $x \in Z^L$.
It is easily verified that π^t is a payment function for all t $\in$[0,1].

The following result extends Lemma 1 of Wurman and Wellman (1999) to all integer allocation problems.

Theorem 5: For all t$\in$[0,1], $<\pi^t, X^*>$ is a payment equilibrium.

Proof: For all i, k $\in$ {1,…,H}and j = 1,2:
(a) $f^i(X^{*i}) - \pi^j(X^{*i}) = f^i(X^{*i}) - q_i^j = s_i^j \geq f^i(X^{*k}) - q_k^j = f^i(X^{*k}) - \pi^j(X^{*k})$;
(b) For $x \in Z^L \setminus \{X^{*i}/ i = 1,\ldots,H\}$, $[\pi^j(x) = \max\{\max\{f^i(x) - s_i^j / i = 1,\ldots,H\},0\}]$ implies $[f^i(X^{*i}) - \pi^j(X^{*i}) = s_i^j \geq f^i(x) - \pi^j(x)]$.
Thus, for j = 1,2 and i = 1,…,H: $f^i(X^{*i}) - \pi^j(X^{*i}) = s_i^j \geq f^i(x) - \pi^j(x)$ for all $x \in Z^L$.
Hence, for all t$\in$[0,1] and i = 1,…,H: $f^i(X^{*i}) - \pi^t(X^{*i}) = s_i^j \geq f^i(x) - \pi^j(x)$ for all $x \in Z^L$.
Q.E.D.

However, the construction of the payment function for an integer allocation problem, generally requires an infinite number of computations and hence an unrealistic data set. Thus, its practical usefulness as a solution for integer allocation problems is quite limited.

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