

# Distortionary taxation, debt, and the price level\*

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## Abstract

This paper considers the nominal and real determinacy of equilibria under an exogenously specified path of interest rates in an economy in which taxation is either lump-sum or distortionary. Under lump-sum taxation, we confirm the well-known finding that equilibria display nominal (in)determinacy if the primary surplus is exogenous (endogenous). Under distortionary taxation, this classification is no longer relevant. Nominal determinacy is always ensured since distortionary taxes establish a link between the allocation and the sequences of taxes and debt and, hence, the price level, regardless of whether the primary surplus is exogenous or endogenous. Distortionary taxation, however, increases the scope for real indeterminacy. As a general feature, the real (in)determinacy of equilibria depends on the interaction of fiscal and monetary policies, i.e. on the sequences of taxes, debt, and interest rates. If, for example, fiscal policy runs a balanced budget the central bank should set the nominal interest rate in a way consistent with long-run deflation in order to ensure real determinacy. This finding is different from a balanced-budget policy under lump-sum taxes where no such qualification with respect to the interest rate needs to be made.

*Keywords:* Monetary and fiscal policy, distortionary taxes, price level determination, balanced budget policy.

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# 1 Introduction

It is well established that a policy of an interest rate peg or, more generally, of an exogenously specified sequence of interest rates can be associated with an indeterminate price level (Patinkin, 1965, Sargent and Wallace, 1975). This property has been reconsidered in a number of studies with a particular focus on links between monetary and fiscal policy regimes (Leeper, 1991, Woodford, 1994, Sims, 1994, Kocherlakota and Phelan, 1999, Benhabib et al., 2001, Schmitt-Grohé and Uribe, 2000). Specifically, the ‘Fiscal Theory of the Price Level’ (*FTPL*) has claimed that the equilibrium under exogenous interest rates is consistent with a unique price level if fiscal policy is specified as an exogenous sequence of the primary surplus.<sup>1</sup> By contrast, indeterminacy of the price level prevails if the primary surplus reacts endogenously to the level of public debt such that government solvency is guaranteed for any price level sequence. A particular example of such a solvent fiscal policy regime is a balanced-budget policy, as discussed in Schmitt-Grohé and Uribe (2000).

A main assumption which underlies this classification of the nominal (in)determinacy of equilibria is that fiscal policy has access to lump-sum taxes. The primary contribution of this paper is to show that this classification breaks down if taxation is assumed to be distortionary. Intuitively, with distortionary taxes the ‘logic’ of the price level determination is different since such taxes establish a link between equilibrium allocations and the paths of taxes and debt and, hence, the price level.<sup>2</sup> This link is independent of whether the sequence of primary surpluses is exogenous or endogenous, implying that, in general, equilibria exhibit nominal determinacy under distortionary taxation. The determination of the price level is thus a straightforward implication of fiscal policy non-neutrality and does not rely on an equilibrium concept which has been applied to establish the *FTPL* and which has been criticized by some authors to be inconsistent (see Buiter, 2002, or Niepelt, 2004).

The paper also shows that, compared with lump-sum taxation, the non-neutrality of distortionary taxation increases the scope for the real indeterminacy of equilibria. This finding relates to the result of Schmitt-Grohé and Uribe (1997) that there might exist multiple equilibrium allocations under distortionary taxation. In our set-up, since not only fiscal policy but also monetary policy (due to transactions frictions) is non-neutral, the real (in)determinacy of equilibria depends, in general, on the interaction of fiscal and monetary policies, i.e. on the sequences of taxes, debt, and interest rates. As an illustration of this principle, it is shown that if fiscal policy runs a balanced budget the central bank should set the nominal interest rate in a way consistent with long-run deflation in order to ensure that equilibria are locally determinate. This finding is different from a balanced-budget policy under lump-sum taxes where no such qualification with respect to the interest rate needs to be made.

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<sup>1</sup>The term ‘surplus’ is used in a loose way as a short-cut for the primary budget balance, i.e. it includes the possibility of deficits.

<sup>2</sup>As pointed out by Bassetto and Kocherlakota (2004), this statement requires modifications if distortionary taxes are functions of past incomes. In our model, however, taxes are restricted to be a function of current income.

The analysis is conducted in a standard cash-in-advance model with exogenous government expenditures. At the outset, we allow for a distortionary income tax and a lump-sum tax. First, to reproduce the well-known reference case, we set the distortionary tax equal to zero and assume that all tax revenues are raised by lump-sum taxes. In line with the above cited literature, equilibria display nominal indeterminacy if the primary surplus responds endogenously to ensure government solvency, like under a balanced-budget regime (Schmitt-Grohé and Uribe, 2000). By contrast, equilibria display nominal determinacy under an exogenous sequence of the primary surplus, as shown by the *FTPL*. The intuition for this finding is well understood. Variations in the initial price level lead to revaluations of the outstanding stock of nominal government liabilities. In equilibrium, where government solvency has to be satisfied, these revaluations have to be offset by appropriate adjustments of primary surpluses. Under lump-sum taxation these adjustments do not affect the equilibrium allocation. Hence, as long as the sequence of the primary surplus itself is not exogenously specified, equilibria display nominal indeterminacy, i.e. any particular equilibrium allocation is consistent with multiple price level sequences. In the main, second part of our analysis we reconsider this classification under the assumption that all taxation is distortionary. As a consequence, there exists a link between equilibrium allocations and the primary surplus. To see the implications of this link one may, again, consider a variation in the initial price level which leads to a revaluation of outstanding government liabilities. Under a solvent fiscal policy ‘offsetting’ variations in the sequence of primary surpluses can, again, be found which respect all equilibrium conditions. Yet, in contrast to the reference case of lump-sum taxation, these variations are now associated with different equilibrium allocations. Put differently, any particular equilibrium allocation is now associated with exactly one price level sequence. As long as there are no other restrictions on fiscal policy beyond the constraint that government solvency needs to hold for arbitrary sequences of the price level, there exists an infinite number of such variations consistent with equilibrium. This implies that, compared with lump-sum taxation, the nominal determinacy of equilibria is ‘achieved’ at the expense of real indeterminacy, unless further restrictions are considered.<sup>3</sup>

Specifically, the equilibrium can exhibit real determinacy if fiscal policy is not only constrained to be solvent, but is additionally characterized by a particular financing pattern that restricts the sequences of taxes and debt. As a specific example of such a financing restriction we follow Schmitt-Grohé and Uribe (2000) and consider a balanced-budget regime. We show that in this case a sequence of nominal interest rates consistent with long-run deflation leads to steady-state uniqueness and real determinacy. Intuitively, under deflation the government tends to receive negative seigniorage revenues because of falling prices and falling nominal balances. Under the balanced-budget regime these losses need to be offset by the issuance of additional debt, leading over time to potentially unstable debt dynamics. For this process to be consistent with an equilibrium, the interest payments on debt, which are also potentially unstable, would need to be refinanced en-

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<sup>3</sup>To establish clear benchmarks, our economy has the feature that in the first experiment (i.e. under lump-sum taxation) real equilibrium determinacy is always ensured, irrespective of whether the sequence of the primary surplus is exogenous or endogenous.

tirely out of distortionary tax revenues. Such revenues, however, cannot grow without bounds for feasible equilibrium allocations. Yet, there exists a unique initial price level (associated with uniquely defined tax sequences and equilibrium allocations) that devalues the initial nominal liabilities in a manner consistent with the real determinacy of the equilibrium. By contrast, this mechanism does not work under a sequence of nominal interest rates consistent with long-run inflation, i.e. such a sequence fails to establish the real determinacy of the equilibrium.

Finally, as regards a regime of exogenous primary surpluses, we show for our benchmark specification that the shift from lump-sum to distortionary taxation may leave the real determinacy of the equilibrium unaffected. However, we indicate that this result is not generic, i.e. we show that also in this case distortionary taxes increase the scope for real indeterminacy.

The question addressed in this paper is directly related to Leeper (1991), Woodford (1994), Sims (1994), Benhabib et al. (2001) and, most importantly, Schmitt-Grohé and Uribe (2000). As in these studies, it is assumed that prices are fully flexible, but we relax the common assumption of lump-sum taxes.<sup>4</sup> Our findings correspond to Benassy (2000, 2005). These two papers depart from Ricardian equivalence not via taxation but instead by means of an overlapping generations structure and establish that an interest rate peg is consistent with nominal determinacy.<sup>5</sup> Canzoneri and Diba (2005) establish the possibility of nominal determinacy under an interest rate peg and endogenous primary surpluses if public debt is non-neutral due to transactions services of government bonds. In a careful overview paper, Leeper and Yun (2005) point out that the existence of asset revaluation effects induced by an exogenous primary surplus does not rely on the assumption of lump-sum taxes. In addition to this aspect, we examine fiscal policy regimes with entirely distortionary taxation and with an endogenous primary surplus. Benigno and Woodford (2003) emphasize the importance of distortionary taxes from the perspective of optimal Ramsey policies. Essentially, in their analysis it is the assumption of distortionary taxes which creates a considerable joint decision problem of monetary and fiscal policy. This corresponds to our key finding that under distortionary taxes equilibrium allocations cannot be established independently of the price level. Finally, Schmitt-Grohé and Uribe (1997) show in a real business cycle model that real indeterminacy can occur under empirically plausible labor income taxes.

The paper is structured as follows. Section 2 presents a model with a transaction friction and distortionary taxes, implying that both monetary and fiscal policy affect the equilibrium allocation and prices in a non-trivial way. Section 3 establishes the nominal (in)determinacy of equilibria under lump-sum and distortionary taxation. Similarly, Section 4 establishes the real (in)determinacy of equilibria under lump-sum and distortionary taxation. Section 5 concludes. The Appendix contains technical parts of the analysis.

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<sup>4</sup>Corresponding to the assumption of flexible prices, this paper does not consider interest rate feedback rules (e.g. Taylor-rules), which are typically designed to stabilize inflation in a sticky-price framework. For a comprehensive discussion, see Woodford (2003).

<sup>5</sup>Cushing (1999) stresses that within an overlapping generations structure departures from Ricardian equivalence are not a sufficient condition to ensure nominal determinacy under an interest rate peg.

## 2 The model

In this section we present a simple representative agent model with flexible prices. Money demand is introduced via a liquidity constraint in the goods market. Government purchases of the final good are financed by public debt, tax revenues, and seigniorage. Tax revenues are raised in a lump-sum way or by a proportional tax on labor income which is distortionary. Throughout the paper, small (large) letters denote real (nominal) variables.

### 2.1 Private sector

There exists a continuum of infinitely lived and identical households of mass one. Their utility increases in consumption  $c_t$  and decreases in working time  $l_t$ , the latter variable being bounded by some finite value  $\bar{l}$  such that  $l_t \in (0, \bar{l})$ . The objective of a representative household is given by

$$\max \sum_{t=0}^{\infty} \beta^t \left[ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\vartheta}}{1+\vartheta} \right], \quad \text{with } \sigma > 1, \quad \vartheta \geq 0, \quad \beta \in (0, 1), \quad (1)$$

where  $\beta$  denotes the discount factor,  $\sigma$  represents the inverse of the intertemporal elasticity of substitution in consumption and  $\vartheta$  denotes the inverse of the Frisch elasticity of labour supply. Households enter a representative period  $t$  with two types of nominal assets, money balances  $M_{t-1}$  and interest bearing government debt  $B_{t-1}$ . The latter is issued in the form of one-period riskless bonds, earning a net interest rate  $i_{t-1}$  in period  $t$ . Households pay a proportional tax on labour income  $\tau_t^d w_t l_t$  (where  $\tau_t^d$  and  $w_t$  denote the distortionary tax on labour income and the real wage, respectively) and a lump-sum tax  $\tau_t$ . Moreover, households face a liquidity constraint in the goods market

$$P_t c_t \leq M_t, \quad (2)$$

where  $P_t$  denotes the aggregate price level. The cash constraint (2) implies that households can in every period adjust their money holdings before they enter the goods market. Hence, private consumption in period  $t$  is not related to the predetermined stock of money carried over from the previous period,  $M_{t-1}$ . The budget constraint of households is given by

$$P_t c_t + B_t + M_t \leq (1 + i_{t-1})B_{t-1} + M_{t-1} + (1 - \tau_t^d) P_t w_t l_t - P_t \tau_t. \quad (3)$$

Let  $\pi_t = P_t/P_{t-1}$  and  $R_{t-1} = 1 + i_{t-1}$  denote the gross inflation rate and the nominal gross interest rate, respectively. In the initial period  $t = 0$  households are endowed with nominal money balances  $M_{-1} > 0$  and (not necessarily positive) holdings of nominal bonds  $B_{-1}$  with  $R_{-1} > 1$ . Given these initial conditions, maximizing (1) subject to a no-Ponzi game condition  $\lim_{t \rightarrow \infty} (b_t + m_t) \prod_{i=1}^t \pi_i / R_{i-1} \geq 0$ , (2) and (3) leads to the first-order conditions

$$\frac{2R_t - 1}{R_t} c_t^\sigma l_t^\vartheta = (1 - \tau_t^d) w_t \quad (4)$$

$$\beta \frac{R_{t+1}}{2R_{t+1} - 1} c_{t+1}^{-\sigma} \pi_{t+1}^{-1} = \frac{1}{2R_t - 1} c_t^{-\sigma} \quad (5)$$

$$c_t \leq m_t. \quad (6)$$

where  $m_t = M_t/P_t$  and  $b_t = B_t/P_t$ . The first equation summarizes the first-order conditions associated with the labour supply and consumption decisions, the second equation describes the intertemporal Euler equation, and the third equation gives  $c_t = m_t$  if  $R_t > 1$ .<sup>6</sup> Further, the transversality condition

$$\lim_{t \rightarrow \infty} (b_t + m_t) \prod_{i=1}^t \pi_i / R_{i-1} = 0. \quad (7)$$

has to be satisfied. There is a continuum of perfectly competitive firms of mass one. Firms produce the consumption good with the linear technology  $y_t = l_t$ , i.e. labour is the only production factor supplied by the households. Assuming a competitive labour market, profit maximization leads to zero profits and  $w_t = 1$ . Total output  $y_t$  consists of private sector consumption  $c_t$  and government purchases of the consumption good  $g_t$ , i.e.  $y_t = c_t + g_t$ .

## 2.2 Public sector

Monetary policy is specified in terms of the nominal interest rate  $R_t$ . Throughout the paper we restrict our attention to the case where the central bank follows an exogenously specified path of the interest rate  $R_t > 1 \forall t \geq 0$ , implying that the cash-constraint is always binding. The fiscal authority collects taxes, issues risk-free one-period bonds, and purchases the amount  $g_t$  of the consumption good. The consolidated budget constraint of the public sector is given by  $B_t + M_t = (1 + i_{t-1})B_{t-1} + M_{t-1} - T_t + P_t g_t$ , where  $T_t = P_t \tau_t + P_t \tau_t^d w_t l_t$  summarizes total tax revenues. Government purchases  $g_t$  are assumed to be exogenously determined. In real terms the flow budget constraint of the government reads as

$$b_t + m_t + \tau_t + \tau_t^d w_t l_t = g_t + R_{t-1} b_{t-1} \pi_t^{-1} + m_{t-1} \pi_t^{-1}. \quad (8)$$

Integrating the flow government budget constraint (8) forward leads to the equation

$$\begin{aligned} \frac{S}{P_0} &= \sum_{t=1}^{\infty} \left( \prod_{i=1}^t \frac{\pi_i}{R_{i-1}} \right) \left( \tau_t + \tau_t^d w_t l_t - g_t + \frac{R_{t-1} - 1}{\pi_t} m_{t-1} \right) + \left( \tau_0 + \tau_0^d w_0 l_0 - g_0 \right) \\ &+ \lim_{t \rightarrow \infty} (b_t + m_t) \prod_{i=1}^t \frac{\pi_i}{R_{i-1}} \end{aligned} \quad (9)$$

in which  $S = R_{-1} B_{-1} + M_{-1}$  denotes the initial government liabilities in nominal terms, assumed to satisfy  $S > 0$ . We consider three fiscal policy regimes.

The first regime is characterized by the property that the sequence of tax receipts and, hence, of primary surpluses responds endogenously to the level of outstanding government liabilities such that the transversality condition (7) is satisfied for any paths of the price level and of the interest rate. Fiscal policies of this type are called solvent.

$$i) \ \{T_t(M_{t-1}, B_{t-1}, g_t, R_t)\}_{t=0}^{\infty} : \lim_{t \rightarrow \infty} (b_t + m_t) \prod_{i=1}^t \pi_i / R_{i-1} = 0 \quad (\text{Solvent fiscal policies})$$

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<sup>6</sup>We focus throughout the analysis on interior solutions consistent with  $l_t \in (0, \bar{l})$ . As will become clear below, this assumption will always be satisfied. The set of first-order conditions is derived in Appendix A.

The second regime uses a more specific description of fiscal policy. In particular, the fiscal authority is assumed to run a balanced budget, i.e. in every period tax revenues are equal to the sum of interest payments on outstanding debt and of government purchases of the consumption good.

$$ii) \quad T_t = i_{t-1}B_{t-1} + P_t g_t \quad (\text{Balanced Budget})$$

The third regime is characterized by the property that in every period tax revenues are equal to government purchases of the consumption good. This implies that interest payments on outstanding debt must be financed by issuance of government liabilities. In other words, this regime exhibits an exogenous primary surplus, for convenience assumed to be zero:

$$iii) \quad T_t = P_t g_t \quad (\text{Exogenous primary surplus})$$

In the equilibrium analysis below we will restrict our attention to two polar specifications of tax policy. In particular, we will consider the cases where taxes are either entirely lump-sum ( $T_t = P_t \tau_t$ ) or entirely distortionary ( $T_t = P_t \tau_t^d w_t l_t$ ).

### 2.3 Equilibrium

We now define an equilibrium of the economy and discuss some of its main properties.

**Definition 1** *A perfect foresight equilibrium is a set of sequences  $\{c_t > 0, l_t \in (0, \bar{l}), m_t > 0, \pi_{t+1} > 0, \tau_t^d, \tau_t, b_t\}_{t=0}^{\infty}$  and a price level  $P_0 > 0$  satisfying for all periods  $t \geq 0$*

$$m_t^\sigma (m_t + g_t)^\vartheta = (1 - \tau_t^d) \frac{R_t}{2R_t - 1} \quad (10)$$

$$\beta \frac{R_{t+1}}{2R_{t+1} - 1} m_{t+1}^{-\sigma} \pi_{t+1}^{-1} = \frac{1}{2R_t - 1} m_t^{-\sigma} \quad (11)$$

$$b_{t+1} + m_{t+1} + \tau_{t+1} + \tau_{t+1}^d (m_{t+1} + g_{t+1}) = g_{t+1} + R_t b_t \pi_{t+1}^{-1} + m_t \pi_{t+1}^{-1}, \quad (12)$$

$$\text{and } b_0 + m_0 + \tau_0 + \tau_0^d (m_0 + g_0) = g_0 + (R_{-1} B_{-1} + M_{-1}) / P_0, \quad (13)$$

$$c_t = m_t, \quad (13)$$

$$l_t = m_t + g_t, \quad (14)$$

the transversality condition (7), a fiscal policy of type *i*, *ii*) or *iii*), taking as given a sequence of government expenditures  $\{g_t > 0\}_{t=0}^{\infty}$ , nominal interest rates  $\{R_t > 1\}_{t=0}^{\infty}$ , and initial values  $M_{-1} > 0, R_{-1} > 1, B_{-1}$  and  $S = R_{-1} B_{-1} + M_{-1} > 0$ .

While the fiscal policy regime *i*) does not add any additional equilibrium requirement beyond the transversality constraint (7), this is different under regimes *ii*) and *iii*). The balanced-budget regime *ii*) requires

$$\tau_{t+1} + \tau_{t+1}^d (m_{t+1} + g_{t+1}) = (R_t - 1) b_t \pi_{t+1}^{-1} + g_{t+1} \quad (15)$$

and  $\tau_0 + \tau_0^d (m_0 + g_0) = (R_{-1} - 1) B_{-1} P_0^{-1} + g_0$ . This implies that the flow budget constraint (12) turns into  $b_{t+1} + m_{t+1} = (b_t + m_t) \pi_{t+1}^{-1}$ , and  $b_0 + m_0 = (B_{-1} + M_{-1}) / P_0$ . Hence,

under regime *ii*) the transversality condition (7) is always satisfied, since the sequence of nominal interest rates is assumed to satisfy  $R_t > 1$ . According to the solvency criterion the regime *ii*) is a particular example of a regime satisfying *i*).

The third regime of a zero primary surplus requires

$$\tau_t + \tau_t^d(m_t + g_t) = g_t \quad (16)$$

implying that the flow budget constraint (12) turns into  $b_{t+1} + m_{t+1} = R_t b_t \pi_{t+1}^{-1} + m_t \pi_{t+1}^{-1}$ , and  $b_0 + m_0 = (R_{-1} B_{-1} + M_{-1}) / P_0$ . Evidently, regime *iii*) does not ensure solvency, i.e. under this regime the transversality condition remains a relevant equilibrium condition after (16) has been imposed.

For future reference, we summarize some noteworthy features of perfect foresight equilibria in this economy.

**Corollary 1** *For any equilibrium sequence  $\{m_t > 0\}_{t=0}^{\infty}$  the Euler equation (11) determines a unique equilibrium sequence  $\{\pi_{t+1} > 0\}_{t=0}^{\infty}$ , and (13)-(14) determine a unique set of equilibrium sequences  $\{c_t > 0, l_t > 0\}_{t=0}^{\infty}$ . Under distortionary taxation, equation (10) determines for any equilibrium sequence  $\{m_t > 0\}_{t=0}^{\infty}$  a unique equilibrium sequence of tax rates  $\{\tau_t^d\}_{t=0}^{\infty}$ .*

From now onwards the term ‘equilibrium sequence’ is meant to satisfy the sign restrictions listed in Definition 1, unless explicitly mentioned. Corollary 1 highlights the central role of the sequence  $\{m_t > 0\}_{t=0}^{\infty}$  for the characterization of equilibria in our economy. In particular, any particular equilibrium sequence  $\{m_t\}_{t=0}^{\infty}$  is associated with unique equilibrium sequences for consumption and working time  $\{c_t, l_t\}_{t=0}^{\infty}$ , and the latter two sequences fully determine social welfare. The following definitions of nominal and real (in)determinacy of equilibria correspond to those in Benhabib et al. (2001).

**Definition 2** *The equilibrium exhibits nominal (in)determinacy if for any sequence of equilibrium real allocations  $\{m_t, c_t, l_t\}_{t=0}^{\infty}$  there exists exactly one price level  $P_0 > 0$  (an infinite number of price levels  $P_0 > 0$ ) consistent with a perfect foresight equilibrium.*

Since the sequence of inflation rates  $\{\pi_{t+1}\}_{t=0}^{\infty}$  (starting with  $\pi_1 = P_1/P_0$ ) is uniquely determined for any given equilibrium allocation (see Corollary 1), the entire price level sequence  $\{P_t\}_{t=0}^{\infty}$  is also uniquely pinned down once the price level  $P_0$  is determined.

**Definition 3** *The equilibrium displays real (in)determinacy if there exists a unique sequence of equilibrium real allocations  $\{m_t, c_t, l_t\}_{t=0}^{\infty}$  (an infinite number of real allocations  $\{m_t, c_t, l_t\}_{t=0}^{\infty}$ ) consistent with a perfect foresight equilibrium.*

### 3 Price level determination

In this section we examine the nominal (in)determinacy of equilibria for exogenously given sequences of positive government expenditures  $\{g_t > 0\}_{t=0}^{\infty}$  and nominal interest rates  $\{R_t > 1\}_{t=0}^{\infty}$ . We consider two alternative assumptions regarding tax policy. In the first



scenario, all taxes are assumed to be lump-sum, i.e.  $T_t = P_t \tau_t$  and  $\tau_t^d = 0$ . The results of this first scenario are well-known from the literature (see e.g. Woodford, 1994, Sims, 1994, Schmitt-Grohé and Uribe, 2000, Benhabib et al. 2001) and serve as convenient benchmarks for the second scenario of distortionary taxation.

### 3.1 Lump-sum taxation

Assume that all taxes are lump-sum, i.e.  $T_t = P_t \tau_t$  and  $\tau_t^d = 0$ . The set of equilibrium sequences  $\{m_t, \pi_{t+1}, \tau_t, b_t\}_{t=0}^{\infty}$  and the price level  $P_0$  have to satisfy (10)-(12), i.e.

$$m_t^\sigma (m_t + g_t)^\vartheta = \frac{R_t}{2R_t - 1} \quad (17)$$

$$\beta \frac{R_{t+1}}{2R_{t+1} - 1} m_{t+1}^{-\sigma} \pi_{t+1}^{-1} = \frac{1}{2R_t - 1} m_t^{-\sigma} \quad (18)$$

$$b_{t+1} + m_{t+1} + \tau_{t+1} = g_{t+1} + R_t b_t \pi_{t+1}^{-1} + m_t \pi_{t+1}^{-1}, \quad (19)$$

$$\text{and } b_0 + m_0 + \tau_0 = g_0 + (R_{-1} B_{-1} + M_{-1}) / P_0,$$

and

$$\text{for regime } ii) \quad \tau_{t+1} = (R_t - 1) b_t \pi_{t+1}^{-1} + g_{t+1}, \quad (20)$$

$$\text{and } \tau_0 = (R_{-1} - 1) B_{-1} P_0^{-1} + g_0$$

$$\text{for regime } iii) \quad \tau_t = g_t, \quad (21)$$

as well as

$$\frac{S}{P_0} = \sum_{t=1}^{\infty} \left( \prod_{i=1}^t \frac{\pi_i}{R_{i-1}} \right) \left( \tau_t - g_t + \frac{R_{t-1} - 1}{\pi_t} m_{t-1} \right) + (\tau_0 - g_0). \quad (22)$$

Note that (22) needs to hold under all three fiscal regimes  $i$ ),  $ii$ ), and  $iii$ ), since it follows from (9) in combination with the transversality condition (7). Equation (22) says that the real value of the outstanding nominal liabilities  $S$  of the public sector has to be equal to the discounted streams of future primary surpluses and seigniorage revenues.

**Solvent fiscal policies** We start the analysis with regime  $i$ ). Consider a sequence  $\{m_t\}_{t=0}^{\infty}$  satisfying (17). According to Corollary 1 there exists a unique associated sequence  $\{\pi_{t+1}\}_{t=0}^{\infty}$ . Then, for arbitrary price levels  $P_0 > 0$  one finds sequences  $\{\tau_t, b_t\}_{t=0}^{\infty}$  which satisfy (19) and (22). Intuitively, variations in  $P_0 > 0$  lead within the left-hand side of (22) to revaluations of outstanding government liabilities in real terms. These revaluations can be offset within the right-hand side of (22) by variations in the sequences  $\{\tau_t\}_{t=0}^{\infty}$  and, hence,  $\{b_t\}_{t=0}^{\infty}$ , and under lump-sum taxation these variations do not affect the other equilibrium conditions. This is the well-known nominal indeterminacy result under an exogenously specified path of interest rates and solvent (or ‘Ricardian’) fiscal policies under lump-sum taxation.

**Balanced Budget** Evidently, the balanced-budget regime *ii*), which is a special case of a solvent fiscal policy, also exhibits nominal indeterminacy. To see why consider again a sequence  $\{m_t\}_{t=0}^{\infty}$  with a unique associated sequence  $\{\pi_{t+1}\}_{t=0}^{\infty}$ . Then, using (20) in (19) it is still possible to find for arbitrary values of  $P_0 > 0$  sequences  $\{\tau_t, b_t\}_{t=0}^{\infty}$  which satisfy (19) and (22). Specifically, lump-sum taxes in period 0 satisfy  $\tau_0 - g_0 = (R_{-1} - 1)B_{-1}P_0^{-1}$ . Corollary 1 establishes a unique sequence  $\{\pi_{t+1}\}_{t=0}^{\infty}$ , but the initial inflation rate  $\pi_0$  and, hence,  $P_0$  is not pinned down. Because of this degree of freedom the right-hand side of (22) is not uniquely determined in equilibrium. This exactly restates the nominal indeterminacy result derived by Schmitt-Grohé and Uribe (2000).

Note that in contrast to the first fiscal regime for any  $P_0 > 0$  the sequences  $\{\tau_t, b_t\}_{t=0}^{\infty}$  are now uniquely determined. While consistent with nominal indeterminacy, the second fiscal regime of balanced budgets is evidently more restrictive than regime *i*), a feature which will become important under distortionary taxation below.

**Exogenous primary surplus** Under the third regime *iii*) the sequence of lump-sum taxes is completely determined by (21), i.e.  $\tau_t = g_t$ . Consider again a sequence  $\{m_t\}_{t=0}^{\infty}$  with a unique associated sequence  $\{\pi_{t+1}\}_{t=0}^{\infty}$ . Thus, the right-hand side of (22) is determined and positive, implying that the price level  $P_0 > 0$  is uniquely determined, which in turn determines a unique sequence  $\{b_t\}_{t=0}^{\infty}$ . This result restates the *FTPL* derived by Woodford (1994) and Sims (1994). For convenience, the results of this subsection can be summarized as follows.

**Proposition 1** *Suppose that all taxes are lump-sum such that  $T_t/P_t = \tau_t$ . Then the equilibrium exhibits*

1. *nominal indeterminacy under a fiscal policy regime satisfying *i*) or *ii*),*
2. *nominal determinacy under a fiscal policy regime satisfying *iii*).*

The results of Proposition 1 confirm the main principle established by Schmitt-Grohé and Uribe (2000), namely that under an exogenously specified path of interest rates and lump-sum taxation the price level can only be uniquely determined if the primary surplus is exogenously given. By contrast, the price level is indeterminate if the primary surplus is endogenous or, more generally, if fiscal policy sets taxes such that government solvency is guaranteed for any paths of the price level and of the interest rate.

### 3.2 Distortionary taxation

We now turn to the case where the government does not have access to lump-sum taxes and all taxes are distortionary, i.e.  $T_t/P_t = \tau_t^d w_t l_t = \tau_t^d (m_t + g_t)$  and  $\tau_t = 0$ . The set of equilibrium sequences  $\{m_t, \pi_{t+1}, \tau_t^d, b_t\}_{t=0}^{\infty}$  and the price level  $P_0$  have to satisfy (10)-(12),

i.e.

$$m_t^\sigma (m_t + g_t)^\vartheta = (1 - \tau_t^d) \frac{R_t}{2R_t - 1} \quad (23)$$

$$\beta \frac{R_{t+1}}{2R_{t+1} - 1} m_{t+1}^{-\sigma} \pi_{t+1}^{-1} = \frac{1}{2R_t - 1} m_t^{-\sigma} \quad (24)$$

$$b_{t+1} + m_{t+1} + \tau_{t+1}^d (m_{t+1} + g_{t+1}) = g_{t+1} + R_t b_t \pi_{t+1}^{-1} + m_t \pi_{t+1}^{-1}, \quad (25)$$

$$\text{and } b_0 + m_0 + \tau_0^d (m_0 + g_0) = g_0 + (R_{-1} B_{-1} + M_{-1}) / P_0,$$

and

$$\text{for regime } ii) \quad \tau_{t+1}^d (m_{t+1} + g_{t+1}) = (R_t - 1) b_t \pi_{t+1}^{-1} + g_{t+1}, \quad (26)$$

$$\text{and } \tau_0^d (m_0 + g_0) = (R_{-1} - 1) B_{-1} P_0^{-1} + g_0,$$

$$\text{for regime } iii) \quad \tau_t^d (m_t + g_t) = g_t, \quad (27)$$

as well as

$$\frac{S}{P_0} = \sum_{t=1}^{\infty} \left( \prod_{i=1}^t \frac{\pi_i}{R_{i-1}} \right) \left( \tau_t^d (m_t + g_t) - g_t + \frac{R_{t-1} - 1}{\pi_t} m_{t-1} \right) + \tau_0^d (m_0 + g_0) - g_0, \quad (28)$$

with (28), as discussed above, to be satisfied under all three regimes. According to equation (23) the tax rate amounts to  $\tau_t^d = 1 - \frac{2R_t - 1}{R_t} m_t^\sigma (m_t + g_t)^\vartheta$ . Using this expression in (28) leads to the condition

$$\begin{aligned} \frac{S}{P_0} = & \sum_{t=1}^{\infty} \left( \prod_{i=1}^t \frac{\pi_i}{R_{i-1}} \right) \left( m_t - \frac{2R_t - 1}{R_t} m_t^\sigma (m_t + g_t)^{1+\vartheta} + \frac{R_{t-1} - 1}{\pi_t} m_{t-1} \right) \\ & + m_0 - \frac{2R_0 - 1}{R_0} m_0^\sigma (m_0 + g_0)^{1+\vartheta}. \end{aligned} \quad (29)$$

Consider a sequence  $\{m_t > 0\}_{t=0}^{\infty}$  satisfying (23)-(25). According to Corollary 1, associated with this sequence there exists a unique sequence  $\{\pi_{t+1}\}_{t=0}^{\infty}$ . Hence, the right-hand side of (29) is uniquely determined, implying a unique value  $P_0$ . This logic is not related to the specifics of the three regimes *i*), *ii*), and *iii*), leading to the result:

**Proposition 2** *Suppose that all taxes are distortionary such that  $T_t/P_t = \tau_t^d w_t l_t$ . Then, the equilibrium exhibits nominal determinacy under a fiscal policy regime satisfying *i*), *ii*), or *iii*).*

The logic sketched so far shows that under distortionary taxation any equilibrium allocation will always be associated with a unique price level  $P_0$ . Yet, to establish the existence of an equilibrium is more challenging than under lump-sum taxation since equation (23) ensures that the fiscal variables  $\tau_t^d$  and  $b_t$  are not separable from the first-order conditions (23) and (24) which characterize the optimizing behavior of the private sector. Because of this non-separability feature within the set of equilibrium conditions it is not in all cases possible to follow a two-step procedure and to establish for a ‘given sequence’  $\{m_t > 0\}_{t=0}^{\infty}$  the existence of the remaining equilibrium sequences as well as  $P_0 > 0$ . Therefore, we leave this issue for Section 4.2 below.

## 4 Determination of equilibrium allocations

This section addresses the question whether the sequences of equilibrium allocations  $\{m_t, c_t, l_t\}_{t=0}^{\infty}$  are uniquely determined or not.

### 4.1 Lump-sum taxation

Under lump-sum taxation it can be immediately shown that equilibria always display real determinacy, irrespective of the choice of the fiscal regime. This result follows directly from the fact that tax rates do not appear in the first-order conditions of the household sector. Specifically, equilibria need to satisfy (see (17))

$$m_t^\sigma (m_t + g_t)^\vartheta = \frac{R_t}{2R_t - 1}. \quad (30)$$

Equation (30) – together with Corollary 1 – implies that there exists a unique equilibrium allocation  $\{m_t, c_t, l_t\}_{t=0}^{\infty}$ . Specifically, the right-hand side of (30) takes on a positive and finite number  $\forall t \geq 0$ . The left-hand side is continuous and increasing in  $m_t$ , with LHS ( $m_t = 0$ ) = 0 and  $\lim_{m_t \rightarrow \infty} \text{LHS}(m_t) \rightarrow \infty$ . Hence, there exists a unique equilibrium sequence  $\{m_t > 0\}_{t=0}^{\infty}$ .

**Proposition 3** *Suppose that all taxes are lump-sum such that  $T_t/P_t = \tau_t$ . Then the equilibrium displays real determinacy under a fiscal policy regime satisfying i), ii), or iii).*

Combining Proposition 1 and Proposition 3 leads to the well-known conclusion that the equilibrium defined in Definition 1 is not fully determined under lump-sum taxation and an endogenous primary surplus, i.e. under regime i.) and ii.), since the price level  $P_0$  is indeterminate.<sup>7</sup> By contrast, under regime iii) the equilibrium is fully determined.

### 4.2 Distortionary taxation

When taxes are distortionary  $T_t/P_t = \tau_t^d w_t l_t$ , the real (in)determinacy analysis has to be conducted separately for the three fiscal policy regimes.

**Solvent fiscal policies** Under regime i) the set of equilibrium sequences  $\{m_t, \pi_{t+1}, \tau_t^d, b_t\}_{t=0}^{\infty}$  and the price level  $P_0$  are characterized by conditions (23)-(25) and (29). One easily shows that regime i) is characterized by real indeterminacy. To see why consider *some* sequence  $\{m_t > 0\}_{t=0}^{\infty}$ . According to Corollary 1, equations (23) and (24) determine unique sequences  $\{\pi_{t+1} > 0, \tau_t^d\}_{t=0}^{\infty}$ . According to (29), to obtain a positive price level  $P_0 > 0$  it is sufficient to have  $m_t - \frac{2R_t - 1}{R_t} m_t^\sigma (m_t + g_t)^{1+\vartheta} > 0 \forall t \geq 0$ . Within any period  $t$

<sup>7</sup>This finding corresponds to the result in Carlstrom and Fuerst (2001), who show that a passive interest rate policy (which includes a peg) fails to pin down the inflation rate in period 0 when the stock of money held at the end of the period ( $M_t$ ) provides transactions services.

there exist for any given value  $g_t$  infinitely many values  $m_t > 0$  satisfying this inequality.<sup>8</sup> Hence, there exists an infinite number of equilibrium sequences  $\{m_t\}_{t=0}^{\infty}$ , each of them implying a different unique price level  $P_0 > 0$ , and, via the flow budget constraint (25), also a different unique sequence  $\{b_t\}_{t=0}^{\infty}$ . In sum this implies that the equilibrium displays real indeterminacy.

Intuitively, the real indeterminacy of the equilibrium occurs for the same reason which is responsible for the nominal indeterminacy of the equilibrium under lump-sum taxes. Again, variations in the initial price level lead to revaluations of outstanding government liabilities and ‘offsetting’ variations in the primary surplus can be found which respect all equilibrium conditions of the economy. Yet, under distortionary taxation these variations are now associated with different equilibrium allocations. Under a solvent fiscal policy, which is silent on the detailed structure of debt and taxes, there exists an infinite number of such variations. In other words, when moving from lump-sum to distortionary taxation, the ‘achievement’ of nominal determinacy comes at the expense of real indeterminacy of equilibria.

Finally, we point out that for the special case where  $g_t = g > 0 \forall t \geq 0$  and  $R_t = R > 1 \forall t \geq 0$ , regime *i*) is consistent with infinitely many steady-state values  $m > 0$ , each of them associated with unique values  $\pi > 0$ ,  $\tau^d$  and  $b$ . Thus, there exists a continuum of steady-state allocations.

**Balanced Budget** Under regime *ii*) the set of equilibrium sequences  $\{m_t, \pi_{t+1}, \tau_t^d, b_t\}_{t=0}^{\infty}$  and the price level  $P_0$  need to satisfy (23)-(26), while (29) will always be satisfied. After substituting out for the tax rate  $\tau_t^d$  and the inflation rate  $\pi_t$ , the four equations (23)- (26) can be arranged as a system of two non-linear difference equations in  $m_t$  and  $b_t \forall t \geq 0$ :

$$m_{t+1}^{1-\sigma} - (m_{t+1} + g_{t+1})^{1+\vartheta} \frac{2R_{t+1} - 1}{R_{t+1}} = \frac{1}{\beta} m_t^{-\sigma} b_t \frac{2R_{t+1} - 1}{R_{t+1}} \frac{R_t - 1}{2R_t - 1}$$

$$\left( m_{t+1}^{1-\sigma} - (m_{t+1} + g_{t+1})^{1+\vartheta} \frac{2R_{t+1} - 1}{R_{t+1}} \right) \left( 1 + \frac{m_t}{b_t} \right) m_{t+1}^{\sigma-1} = \left( 1 + \frac{b_{t+1}}{m_{t+1}} \right) (R_t - 1),$$

$$\text{and } \left( m_0^{1-\sigma} - (m_0 + g_0)^{1+\vartheta} \frac{2R_0 - 1}{R_0} \right) \left( 1 + \frac{M_{-1}}{B_{-1}} \right) m_0^{\sigma-1} = \left( 1 + \frac{b_0}{m_0} \right) (R_{-1} - 1).$$

Let  $a_t = m_t/b_t = M_t/B_t$ . Then, the system can be compactly rewritten as a two-dimensional dynamic system in  $m_t$  and  $a_t \forall t \geq 0$ :

$$\Phi(m_{t+1})a_t = \frac{1}{\beta} m_t^{1-\sigma} \frac{2R_{t+1} - 1}{R_{t+1}} \frac{R_t - 1}{2R_t - 1} \quad (31)$$

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<sup>8</sup>To verify this claim consider  $m_t^{1-\sigma} > \frac{2R_t-1}{R_t} (m_t + g_t)^{1+\vartheta}$ . The RHS of this inequality is continuous and increasing in  $m_t$ , with RHS ( $m_t = 0$ ) =  $\frac{2R_t-1}{R_t} g_t^{1+\vartheta} > 0$  and  $\lim_{m_t \rightarrow \infty} \text{RHS}(m_t) \rightarrow \infty$ . The LHS is continuous and decreasing in  $m_t$ , with  $\lim_{m_t \rightarrow 0} \text{LHS}(m_t) \rightarrow \infty$  and  $\lim_{m_t \rightarrow \infty} \text{LHS}(m_t) = 0$ , implying the existence of infinitely many values  $m_t > 0$  satisfying the inequality.

$$\begin{aligned}\Phi(m_{t+1})m_{t+1}^{\sigma-1}(1+a_t) &= (1+a_{t+1}^{-1})(R_t-1) \\ \text{and } \Phi(m_0)m_0^{\sigma-1}(1+a_{-1}) &= (1+a_0^{-1})(R_{-1}-1),\end{aligned}\tag{32}$$

where we have used the definition  $\Phi(m_t) = m_t^{1-\sigma} - (m_t + g_t)^{1+\vartheta} \frac{2R_t-1}{R_t}$ . The system is characterized by one backward-looking variable ( $a$ ) with initial condition  $a_{-1} = M_{-1}/B_{-1}$  and one forward-looking variable ( $m$ ), as one infers from (31).<sup>9</sup> Given  $a_{-1} = M_{-1}/B_{-1}$ , one may be tempted to conclude that the system (31)-(32) has one degree of freedom to determine infinitely many sets of equilibrium sequences  $\{m_t, b_t\}_{t=0}^{\infty}$ . However, this reasoning does not recognize that working time needs to satisfy  $l_t = m_t + g_t \in (0, \bar{l}) \forall t \geq 0$ , placing an upper bound on sequences  $\{m_t\}_{t=0}^{\infty}$ . Similarly, we maintain that consumption  $c_t = m_t > 0$  needs to hold in all periods, thereby ruling out sequences of  $\{m_t\}_{t=0}^{\infty}$  with a limit value of  $c_{\infty} = m_{\infty} = 0$ . These bounds provide terminal conditions for (31)-(32) which make the non-linear dynamics no longer tractable. However, as shown in Appendix B, clear-cut characterizations can be obtained if one considers first-order approximations of the dynamics of (31) and (32) around steady states.

Any steady state needs to satisfy the pair of equations

$$m - \frac{R-1}{R\beta}b = (g+m)^{1+\vartheta}m^{\sigma}\frac{2R-1}{R},\tag{33}$$

$$m + b = \frac{1}{R\beta}(m + b).\tag{34}$$

According to (34) there exist two different types of steady-state constellations which depend on how the nominal interest rate  $R$  relates to the critical value  $1/\beta$ . Equivalently, since  $R\beta = \pi$  the constellations can be related to the level of the steady-state inflation rate. In line with Definition 1 we consider only solutions satisfying  $m > 0$ .

1. Suppose that  $R \neq 1/\beta \Leftrightarrow \pi \neq 1$ . Then,  $m + b = 0$  and (33)-(34) imply

$$\begin{aligned}m^{1-\sigma}\left(1 + \frac{R-1}{R\beta}\right) &= (g+m)^{1+\vartheta}\frac{2R-1}{R}, \\ b &= -m,\end{aligned}$$

ensuring that there exists a unique steady state  $m > 0$ .<sup>10</sup> As derived in Appendix B, local dynamics around this steady state, which we obtain from the linearized versions of equations (31)-(32), are characterized by the pair of eigenvalues

$$\lambda_1 = \frac{1}{R\beta} > 0, \quad \lambda_2 = -\frac{(\sigma-1)}{(1+\vartheta)\frac{m}{m+g}\left(1 + \frac{R-1}{R\beta}\right) + (\sigma-1)}\frac{R-1}{R\beta} < 0.$$

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<sup>9</sup>Alternatively, to see why  $m$  is a forward-looking variable combine the  $t = 0$ -versions of (23) and (26) to obtain

$$m_0^{\sigma}(m_0 + g_0)^{1+\vartheta}\frac{2R_0-1}{R_0} = m_0 - (R_{-1}-1)\frac{B_{-1}}{P_0}.$$

In this equation  $m_0$  depends on the realization of  $P_0$  which, in line with Section 3, is not a predetermined variable.

<sup>10</sup>To verify this claim consider  $m^{1-\sigma}\left(1 + \frac{R-1}{R\beta}\right) = (g+m)^{1+\vartheta}\frac{2R-1}{R}$ . The RHS is continuous and increasing in  $m$ , with RHS ( $m = 0$ ) =  $g^{1+\vartheta}\frac{2R-1}{R} > 0$  and  $\lim_{m \rightarrow \infty} \text{RHS}(m) \rightarrow \infty$ . The LHS is continuous and decreasing in  $m$ , with  $\lim_{m \rightarrow 0} \text{LHS}(m) \rightarrow \infty$  and  $\lim_{m \rightarrow \infty} \text{LHS}(m) = 0$ , implying the existence of a unique value  $m > 0$ .

The classification of the first eigenvalue is straightforward. Assume  $R < 1/\beta \Leftrightarrow \pi < 1$ , i.e. there is deflation. Then,  $\lambda_1 > 1$ . Conversely, assume  $R > 1/\beta \Leftrightarrow \pi > 1$ , i.e. the steady state displays inflation. Then,  $\lambda_1 \in (0, 1)$ . The classification of the second eigenvalue is also straightforward if one exploits that  $\frac{R-1}{R\beta} < 1 \Leftrightarrow R < 1/(1-\beta)$  is a sufficient condition for  $\lambda_2 \in (0, -1)$ . For convenience, assume  $\beta > 1/2$  and restrict the analysis to  $R \in (1, 1/(1-\beta))$ .<sup>11</sup> Then, if  $R \in (1, 1/\beta)$  the steady state displays deflation ( $\pi < 1$ ) and the equilibrium exhibits locally real determinacy. Conversely, if  $R \in (1/\beta, 1/(1-\beta))$  the steady state displays inflation ( $\pi > 1$ ), and the equilibrium exhibits locally real indeterminacy.

To obtain an intuition for these findings it is worth making two comments. First, with  $\pi \neq 1$ , the sum of real government liabilities ( $m + b$ ) must be zero, implying that at any steady state with  $m > 0$  the government is a net lender towards the private sector ( $b < 0$ ). The uniqueness of such a steady state follows from the low degree of substitution in consumption ( $\sigma > 1$ ), precluding the possibility of multiple steady states characterized by different debt levels.<sup>12</sup> Second, to see why the local (in)determinacy properties of the unique steady state are different under inflation and deflation it is instructive to look at the equation  $b_t + m_t = (b_{t-1} + m_{t-1}) \pi_t^{-1}$  which must hold under the balanced-budget regime. If  $\pi_t > 1$  total government liabilities have a stable root. This reflects that under inflation at any given level of real balances the government receives positive seigniorage revenues because of rising prices and, hence, of rising nominal balances. Under the balanced-budget requirement these revenues are used to reduce the outstanding amount of debt. These ‘benign’ dynamics converge locally to a unique steady state  $m = -b > 0$  for arbitrary initial conditions. By contrast, if  $\pi_t < 1$  total government liabilities have an unstable root and, at a given level of real balances, the government receives negative seigniorage revenues because of falling prices and falling nominal balances. Under the balanced-budget regime these losses need to be offset by the issuance of additional debt, leading to potentially unstable debt dynamics. For this process to be consistent with an equilibrium, the interest payments on debt, which are also potentially unstable, would need to be refinanced entirely out of distortionary tax revenues. Such revenues, however, cannot grow without bounds for feasible equilibrium allocations. Yet, there exists a unique price level  $P_0$  (associated with uniquely defined tax sequences and equilibrium allocations) that devalues the initial nominal liabilities in a manner consistent with the real determinacy of the equilibrium.<sup>13</sup>

2. Suppose that  $R = 1/\beta \Leftrightarrow \pi = 1$ , i.e. the steady state is characterized by price

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<sup>11</sup>The discount factor  $\beta$  realistically takes values which are close to one, e.g.  $\beta = 0.99$ , implying  $1/(1-\beta) = 100$ . Hence, the ‘restriction’  $R \in (1, 1/(1-\beta))$  can never be violated by reasonable values for the gross nominal interest rate  $R$ .

<sup>12</sup>As summarized at the end of this section, multiple steady states characterized by different debt levels become possible under the complementary assumption  $0 < \sigma < 1$ .

<sup>13</sup>This sketch bears some resemblance to the explanation given by Schmitt-Grohé and Uribe (2000, p. 221). However, the key difference is that in our economy with distortionary taxation revenues must be bounded. This precludes the possibility of ever increasing debt and ensures, at the same time, the local determinacy of equilibria.

stability. Then the steady-state configuration is described by

$$m - (1/\beta - 1)b = (g + m)^{1+\vartheta} m^\sigma (2 - \beta). \quad (35)$$

Equation (35) implies that there exists a unique steady state if  $b \leq 0$ . If  $b > 0$  there exist exactly two steady states if  $b$  remains below some threshold  $\bar{b} > 0$ , i.e. as long as  $b \in (0, \bar{b})$ .<sup>14</sup> The eigenvalues of the linearized dynamics at any steady state are given by

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{(\sigma - 1)}{(1 + \vartheta)(g + m)^\vartheta m^\sigma (2 - \beta) + (\sigma - 1)} \left( \frac{1}{\beta} - 1 \right).$$

Hence, if  $\pi = 1$  at any steady state equilibrium sequences are non-stationary because of one unit root. Suppose that  $\beta > 1/2$ . Then,  $\lambda_1 = 1$ ,  $\lambda_2 \in (0, -1)$ , implying that at any steady state the equilibrium exhibits locally real indeterminacy, irrespective of whether the steady state is unique or comes in pairs.

The intuition for these findings is as follows. Price stability ( $\pi = 1$ ) introduces a unit root into the process of total government liabilities ( $b_t + m_t$ ) and steady states with  $m \neq -b$  become possible. In particular, if  $b > 0$  there is scope for a Laffer-type constellation in which for given levels of  $b$  and  $g$  total steady-state expenditures  $(1/\beta - 1)b + g > 0$  can be financed either with a low tax rate ( $\tau^d$ ) and a high tax base ( $m + g$ ) or, alternatively, with a high tax rate and a low tax base. However, tax revenues cannot be arbitrarily high, placing an upper bound on the debt level ( $\bar{b}$ ) to be rolled over in steady state.

In sum, the existence of steady states is related to the level of public debt and real determinacy crucially depends on the level of the nominal interest rate. The analysis of regime *ii*) thus reveals that genuine interactions of monetary and fiscal policy matter for the determination of the equilibrium allocation when both fiscal policy (due to tax distortions) and monetary policy (due to the liquidity constraint) are non-neutral.

**Exogenous primary surplus** For regime *iii*) the set of equilibrium sequences  $\{m_t, \pi_{t+1}, \tau_t^d, b_t\}_{t=0}^\infty$  and the price level  $P_0$  are characterized by conditions (23)-(25), (27), and (29). Combining (23) and (27) leads to

$$(m_t + g_t)^{1+\vartheta} = \frac{R_t}{2R_t - 1} m_t^{1-\sigma}, \quad (36)$$

which defines a unique sequence  $\{m_t\}_{t=0}^\infty$ .<sup>15</sup> According to Corollary 1 this sequence is associated with unique sequences  $\{\pi_{t+1}, \tau_t^d\}_{t=0}^\infty$ . According to (28) and as established in

<sup>14</sup>The RHS of (35) is continuously increasing in  $m$ , with  $\text{RHS}(m=0) = 0$ ,  $\lim_{m \rightarrow \infty} \text{RHS}(m) \rightarrow \infty$ , and has a convex shape. The LHS is linearly rising in  $m$ . If  $b > 0$ , there exists a unique  $\bar{b}$ , giving rise to a unique point of tangency between the LHS and RHS which satisfies  $m > 0$ . The LHS and the RHS have two intersections  $m > 0$  if  $b \in (0, \bar{b})$  and there are zero intersections if  $b > \bar{b}$ . Finally, if  $b \leq 0$  there exists a unique intersection which satisfies  $m > 0$ .

<sup>15</sup>The LHS of this equation is continuous and increasing in  $m_t$ , with  $\text{LHS}(m_t=0) = g_t^{1+\vartheta} > 0$  and  $\lim_{m_t \rightarrow \infty} \text{LHS}(m_t) \rightarrow \infty$ . The RHS is continuous and decreasing in  $m_t$ , with  $\lim_{m_t \rightarrow 0} \text{RHS}(m_t) \rightarrow \infty$  and  $\lim_{m_t \rightarrow \infty} \text{RHS}(m_t) = 0$ , implying the existence of a unique equilibrium sequence  $\{m_t > 0\}_{t=0}^\infty$ .



Proposition 2, there exists a unique value  $P_0 > 0$ , and equation (25) can be used to determine a unique sequence  $\{b_t\}_{t=0}^{\infty}$ . Thus, the equilibrium displays real determinacy. Finally, consider  $g_t = g > 0 \forall t \geq 0$  and  $R_t = R > 1 \forall t \geq 0$ . The previous reasoning implies that there exists a unique steady-state value  $m > 0$ , associated with unique values  $\pi > 0$ ,  $\tau^d$  and  $b$ . In  $t = 0$ , for arbitrary initial conditions the economy immediately settles down at the steady-state value  $m > 0$ . Similarly, the unique steady-state values of  $c$  and  $l$  are immediately reached without adjustment dynamics.

The main results of this section are summarized in the following proposition.

**Proposition 4** *Suppose that all taxes are distortionary such that  $T_t/P_t = \tau_t^d w_t l_t$ .*

1. *Under a fiscal policy regime satisfying i) the equilibrium displays real indeterminacy and there exists a continuum of steady-state allocations.*
2. *Consider a fiscal policy regime satisfying ii) and assume  $g_t = g > 0 \forall t \geq 0$  and  $R_t = R > 1 \forall t \geq 0$ .*
  - (a) *If  $R \neq 1/\beta \Leftrightarrow \pi \neq 1$ , there exists a unique steady state, satisfying  $m = -b$ . Assume  $\beta > 1/2$  and  $R \in (1, 1/(1 - \beta))$ . If  $R \in (1, 1/\beta)$  the steady state displays deflation ( $\pi < 1$ ) and the equilibrium exhibits locally real determinacy. If  $R \in (1/\beta, 1/(1 - \beta))$  the steady state displays inflation ( $\pi > 1$ ), and the equilibrium exhibits locally real indeterminacy.*
  - (b) *If  $R = 1/\beta \Leftrightarrow \pi = 1$ , there exists a unique steady state if  $b \leq 0$  and two steady states if  $b \in (0, \bar{b})$ . At any such steady state equilibrium sequences are non-stationary because of one unit root. Assume  $\beta > 1/2$ . Then, at any steady state the equilibrium exhibits locally real indeterminacy.*
3. *Under a fiscal policy regime satisfying iii) the equilibrium exhibits real determinacy and there exists a unique steady-state allocation.*

Finally, we point out that the assumption of  $\sigma > 1$  (i.e. of risk aversion and a low intertemporal elasticity of substitution) is important for some of our results. In particular, the assessment of exogenous primary surpluses under regime iii) requires an important modification. This can be most easily seen by making the complementary assumption  $0 < \sigma < 1$ . Concerning regime i), the analysis remains qualitatively unchanged. Concerning regime ii), steady states come in pairs if  $R \neq 1/\beta$  as one infers from  $m^{1-\sigma}(1 + \frac{R-1}{R\beta}) = (g + m)^{1+\vartheta} \frac{2R-1}{R}$ . Intuitively, a high elasticity of substitution generates the possibility of multiple steady states characterized by different debt levels (i.e. different levels of net lending towards the private sector), different activity levels and different levels of the (endogenously determined) distortionary tax rate. Local dynamics around these steady states are well identified and the local (in)determinacy of equilibria cannot be assessed without reference to the magnitude of  $R$ .<sup>16</sup> Concerning regime iii), it is no longer true that the real determinacy of the equilibrium under regime iii) remains preserved during

<sup>16</sup>The expressions of the eigenvalues derived above are valid for  $\sigma > 0$ .

the switch from lump-sum to distortionary taxation. This can be seen from equation (36) in which a high elasticity of substitution generically gives rise to two positive valued sequences  $\{m_t\}_{t=0}^{\infty}$ , provided  $g_t > 0$  and  $R_t > 1$  are not too high. Similarly, assume  $g_t = g > 0$  and  $R_t = R > 1$ . Then, for suitable values of  $g$  and  $R$  steady states with  $m > 0$  come in pairs. In  $t = 0$ , for arbitrary initial conditions either of the two steady states can be immediately reached without any adjustment dynamics.

## 5 Conclusion

This paper reconsiders the nominal and real determinacy of equilibria under an exogenously specified path of interest rates in an economy in which taxation is either lump-sum or distortionary and in which government liabilities are issued in nominal terms. Under lump-sum taxation, we confirm the well-known finding that equilibria display nominal (in)determinacy if the primary surplus is exogenous (endogenous). Under distortionary taxation this classification breaks down. Intuitively, distortionary taxes establish a link between equilibrium allocations and the paths of taxes and debt and, hence, the price level. This link is independent of whether the sequence of primary surpluses is exogenous or endogenous, implying that nominal determinacy of equilibria can always be ensured under distortionary taxation.

The paper also shows that, compared with lump-sum taxation, the non-neutrality of distortionary taxation increases the scope for the real indeterminacy of equilibria. With both fiscal and monetary policy being non-neutral, the determination of a unique equilibrium allocation requires a full characterization of government financing in terms of debt and taxes and it relies as well on the specification of monetary policy.

In sum, if fiscal policy is realistically assumed to be non-neutral due to distortionary taxation the nominal indeterminacy of equilibria is no longer an issue and genuine interactions of monetary and fiscal policy become relevant for the determination of equilibrium allocations.

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## Appendix A: The household problem

The optimization problem of households is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \left[ (1-\sigma)^{-1} c_t^{1-\sigma} - (1+\vartheta)^{-1} l_t^{1+\vartheta} \right] + \mu_t [m_t - c_t] \right. \\ \left. + \lambda_t \left[ \left(1 - \tau_t^d\right) w_t l_t + (1+i_{t-1}) \pi_t^{-1} b_{t-1} + \pi_t^{-1} m_{t-1} - c_t - b_t - m_t - \tau_t \right] \right\}.$$

For interior solutions, the first-order conditions are given by

$$\begin{aligned} \partial \mathcal{L} / \partial c_t &= 0 \Leftrightarrow c_t^{-\sigma} = \lambda_t + \mu_t, \\ \partial \mathcal{L} / \partial l_t &= 0 \Leftrightarrow l_t^{\vartheta} = \lambda_t \left(1 - \tau_t^d\right) w_t, \\ \partial \mathcal{L} / \partial b_t &= 0 \Leftrightarrow \beta (1+i_t) \lambda_{t+1} \pi_{t+1}^{-1} - \lambda_t = 0, \\ \partial \mathcal{L} / \partial m_t &= 0 \Leftrightarrow \beta \lambda_{t+1} \pi_{t+1}^{-1} - \lambda_t + \mu_t = 0, \end{aligned}$$

and the transversality condition  $\lim_{t \rightarrow \infty} (b_t + m_t) \prod_{i=1}^t \pi_i / R_{i-1} = 0$ . Combining the first-order conditions yields  $\beta \lambda_{t+1} \pi_{t+1}^{-1} = (1+i_t)^{-1} \lambda_t$  and  $(1+i_t)^{-1} \lambda_t - \lambda_t + \mu_t = 0$ , and thus  $c_t^{-\sigma} = \frac{2R_t-1}{R_t} \lambda_t$ , where  $R_t = 1+i_t$ . Eliminating  $\lambda_t$  and  $\mu_t$ , leads to (4)-(6).

## Appendix B: Dynamics under a balanced budget and distortionary taxation

Consider (31)-(32). For the steady-state analysis, assume  $g_t = g > 0$ ,  $R_t = R > 1 \forall t \geq 0$ , implying that the equilibrium sequences  $\{a_t, m_t\}_{t=0}^{\infty}$  need to satisfy

$$\Phi(m_{t+1}) a_t = m_t^{1-\sigma} \frac{R-1}{R\beta} \quad (37)$$

$$\Phi(m_t) m_t^{\sigma-1} (1+a_{t-1}) = (1+a_t^{-1}) (R-1), \quad (38)$$

with  $\Phi(m_t) = m_t^{1-\sigma} - (m_t + g)^{1+\vartheta} \frac{2R-1}{R}$ . (37) and (38) are characterized by one forward-looking variable ( $m_t$ ) and one backward-looking variable ( $a_{t-1}$ ). The steady-state conditions are given by  $\Phi(m)a = m^{1-\sigma} \frac{R-1}{R\beta}$  and  $i.) R \neq 1/\beta$  s.t.  $a = -1$  or  $ii.) R = 1/\beta$ . Log-linearizing (31)-(32) and using  $\Phi a = m^{1-\sigma} \frac{R-1}{R\beta}$  leads to

$$\begin{aligned} \phi \widehat{m}_{t+1} + \widehat{a}_t &= (1-\sigma) \widehat{m}_t \\ -\widehat{a}_t &= \frac{1}{R\beta} a \widehat{a}_{t-1} + \frac{a+1}{R\beta} (\sigma-1+\phi) \widehat{m}_t, \end{aligned}$$

where  $\phi = \frac{\Phi' m}{\Phi} = \frac{(1-\sigma)m^{1-\sigma} - m(\vartheta+1)(g+m)^{\vartheta} \kappa_1^{-1}}{m^{1-\sigma} - (m+g)^{1+\vartheta} \kappa_1^{-1}}$ , which can be written as

$$\begin{pmatrix} \widehat{m}_{t+1} \\ \widehat{a}_t \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi} (1-\sigma + \frac{a+1}{R\beta} (\sigma-1+\phi)) & \frac{a}{\phi R\beta} \\ -\frac{a+1}{R\beta} (\sigma-1+\phi) & -\frac{a}{R\beta} \end{pmatrix} \begin{pmatrix} \widehat{m}_t \\ \widehat{a}_{t-1} \end{pmatrix},$$

The associated characteristic equation is given by

$$\lambda^2 + (a - \sigma - \phi - R\beta - a\sigma + R\sigma\beta + 1) (R\beta\phi)^{-1} \lambda + a(\sigma-1) (R\beta\phi)^{-1} = 0 \quad (39)$$

1. Consider first steady states with  $R \neq 1/\beta$  and  $a = -1$ . Then (39) simplifies to  $\lambda^2 - \frac{\phi + R\beta(1-\sigma)}{R\beta\phi}\lambda + \frac{1-\sigma}{R\beta\phi} = 0$ , implying the root structure  $\lambda_1 = \frac{1}{R\beta}$  and  $\lambda_2 = \frac{1-\sigma}{\phi} = -\frac{\sigma-1}{(1+\vartheta)\frac{m}{m+g}(1+\frac{R-1}{R\beta})+(\sigma-1)}\frac{R-1}{R\beta}$ .
2. Now consider steady states with  $R = 1/\beta$ . Then (39) simplifies to  $\lambda^2 - (1 + \frac{(\sigma-1)a}{\phi})\lambda + \frac{(\sigma-1)a}{\phi} = 0$ , with roots  $\lambda_1 = 1$  and  $\lambda_2 = \frac{(\sigma-1)a}{\phi} = -\frac{(\sigma-1)}{(1+\vartheta)(g+m)^\vartheta m^\sigma (2-\beta) + (\sigma-1)}(\frac{1}{\beta} - 1)$ .