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A VIABLE SOLUTION TO A SMALL OPEN-ECONOMY MONETARY POLICY PROBLEM

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A . Some real-life intertemporal economic problems are about maintaining the problem key variables' evolution within certain bounds rather than computing the variables' "optimal" trajectories. Problems of that kind include determination of a country's central-bank interest-rate and natural environment sustainability. For such problems, optimisation might be an unsuitable solution procedure in that it suggests a unique "optimal" solution while many solutions could be *satisficing*. This claim is in line with Herbert A. Simon's (1978 Economics Nobel Prize laureate) postulate that the economists need *satisficing* (his neologism) rather than *optimising* solutions. We aim to use viability theory that rigorously captures the essence of *satisficing* to study a monetary policy problem. The latter is defined as a *qualitative* game between a central bank, which wants to keep inflation under control and a "nuisance" (rather than evil) agent that represents the foreign exchange rate impact on the local economy. We show that *satisficing* adjustment rules can be endogenously obtained.

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1. I

The aim of this paper[1] is to use viability theory for the analysis and synthesis of an economic state-constrained control problem.

Viability theory is a relatively young area of continuous mathematics (see [1] and [2]), that rigorously captures the essence of *satisficing*. It was Herbert A. Simon,

[1]The paper draws from and extends [8].

1978 Economics Nobel Prize laureate, who talked about *satisficing* (his neologism) rather than *optimising*, solutions as being what the economists really need. We think that economic theory, which follows the Simon prescription, brings modelling closer to how people actually behave. Therefore, viability theory appears to be an appropriate tool of achieving a *satisficing* solution to many economic problems. We aim to demonstrate this by solving a stylised Central Bank macroeconomic problem. We believe that an evolutionary analysis enabled by viability analysis gives results that are less invasive (*i.e.*, attempting to change status quo) than those delivered through optimisation. In particular, a viability theory based analysis will advocate adjustment rules that may be “passive” [4] for a large number of the economy states and “active” for some critical states only.

We also believe that an evolutionary analysis enabled by viability analysis is more insightful regarding the system economics than just an equilibrium analysis. The insight is gained (mainly) because of a precautionary character of the advice. A satisficing policy is precautionary (or “preventative”) in that it is based on the economic dynamics *inertia* hence naturally forward looking and suitable for “any” future circumstances. This is because knowing the system’s inertia enables us to detect (and avoid) regions of economic conditions (like high output gap and accelerating inflation) where the control of the system is difficult or impossible.

The precautionary character of our policy advice links our analysis to the literature on robust policies see *e.g.*, [13] and [17]. However, our results are obtained through economic dynamics analysis rather than robust optimisation. Also, our policy advice requires less parameters to calibrate and estimate. In that it is more robust than the one, which is computed for a model that requires more parameters. This is why our policies are less vulnerable to the Lucas critique.

Notwithstanding the macroeconomic applications, of which we address one in this paper, the viability theory can be applied to other problems where uniqueness of the optimal strategy is not of major concern.

In the next section, we provide a brief introduction to viability theory and, in Section 3, we apply it to a simple macroeconomic model[2]. The paper ends with concluding remarks.

2. W ?

2.1. **Definitions.** Viability theory is an area of mathematics concerned with *viable evolution* of controlled dynamic systems.

Suppose a system evolution is expected to satisfy some normative constraints, which define a closed set K of the phase space. A controlled system evolution is called *viable* if there exists a system trajectory that remains in set K for as long as the evolution is concerned. The basic problem that viability theory attempts to solve is whether a control strategy exists that prevents the system from leaving the constraint set. The *viability kernel* for the closed set K is the subset of K that contains initial conditions, for which such a strategy exists. The kernel will formally be defined in Definition 2.1.

[2]For a viability theory application to environmental economics see [3]; also, see [14] for a viability analysis of an endogenous business cycle.

Consider a dynamic economic system with several stock (or state) variables. At time $t \in \Theta \equiv [0, T] \subset \mathcal{R}^+$, where T can be finite or infinite, the state variables are

$$x(t) \equiv [x_1(t), x_2(t), \dots x_N(t)]' \in \mathcal{R}^N, \qquad \forall t \in \Theta$$

and the instrument flows (or actions) are

$$i(t) \equiv [i_1(t), i_2(t), \dots i_M(t)]' \in \mathcal{R}^M, \qquad \forall t \in \Theta.$$

Imposition of normative restrictions on states and strategies means that

$$x(t) \in K \quad \text{and} \quad \{i(t)\}_{t \in \Theta} \equiv i \in \mathcal{I}$$

where symbols K, $\mathcal{I}$ represent sets of constraints[3] that the state and instrument variables need to satisfy.

The state evolves according to the system dynamics $f(\cdot, \cdot)$ and instruments $i(t)$ as follows

$$(1) \qquad \dot{x}(t) = f(x(t), i(t)) \qquad t \in \Theta, \quad x(t) \in K, \quad i \in \mathcal{I}.$$

Evidently, at every state $x(t)$, the system *velocity* $\dot{x}(t)$ depends on action $i(t)$. We may say that the velocity at $x(t) \in K$, for any $t \in \Theta$, is governed by the *set-valued* map (or correspondence)

$$(2) \qquad F(x) \equiv \{f(x, i), i \in \mathcal{I}\}$$

where we have dropped the time index. Combining the above formulae, the system dynamics can be rewritten in form of a *differential inclusion*:

$$(3) \qquad \dot{x}(t) \in F(x(t)), \quad \text{for almost all} \quad t \in \Theta,$$

which determines the range of velocities of the state variables at $x(t)$.

In economic terms, the last relationship tells us that at time t, for a given composition of x (capital, labour, technology, *etc.*), the extent of growth (or decline), or steady state stability, are all dependent on the map $F : \mathcal{R}^N \twoheadrightarrow \mathcal{R}^N$ whose values are limited by the scope of the system dynamics f and instruments contained in $\mathcal{I}$.

A viability theory analysis will tell us that system trajectories $x(t), t \in \Theta$ evolve *viably* if a *viability kernel* is non-empty.

Definition 2.1. *The T-**viability kernel** of the constraint set K for the instrument set $\mathcal{I}$ is the set of initial conditions $x_0 \in K$ denoted as $V_F^T(K)$ and defined as follows:*

$$(4) \qquad V_F^T(K) \equiv \{x_0 \in K : \exists x(t), t \in \Theta \, \text{solution to (1) with} \, x(0) = x_0, x(t) \in K, \forall t \in \Theta\}.$$

In other words, if a trajectory begins inside the viability kernel[4] $V_F^T(K)$ then we know that it remains in the constraint set K for $t \in [0, T]$ (we have denoted the latter set Θ). See Figure 1 for an illustration of the viability idea.

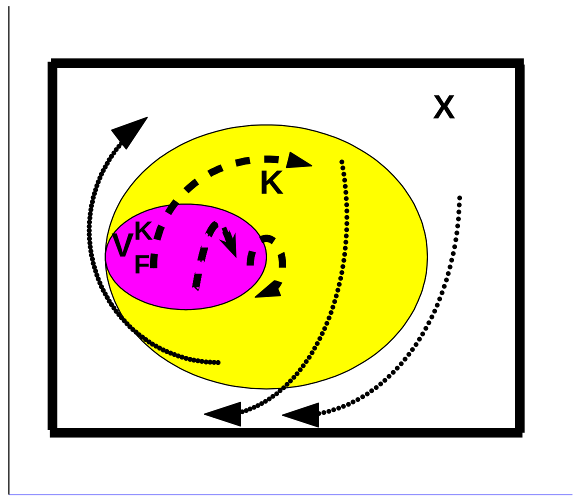
The state constraint set K is represented by the yellow (or light shadowed) oval contour contained in state space $X \equiv \mathcal{R}^2$. The dotted and dashed lines symbolise system evolution.

The viability kernel for the constraint set K, given instruments from set $\mathcal{I}$ and dynamics correspondence F, is the purple (darker) shadowed contour denoted $V_F^K(T)$. The system evolution represented by the trajectories that start inside the

[3]These sets will include local and global constraints and might comprise bounds on the rate of change of $i(t)$. Moreover, the instrument (or action) set $\mathcal{I}$ could be split into two (or more) parts if there were two (or more) players who would decide upon the actions.

[4]If $V_F^T(K) = \emptyset$ then we can say that K is a *repeller*.

kernel (dashed lines) is *viable* in K *i.e.*, the evolution remains in K. This is not the property of the other trajectories (dotted lines) that start outside the kernel. They leave K before T.



F 1. The viable and non viable trajectories.

With relationship to Figure 1, a question may be asked whether a point on a dashed line in K but outside V_F^K (or $V_F^K(T)$) is viable. If policy instruments are $i(x,t)$ rather than $i(x)$ *i.e.*, they depend[5] not only on x but also on time t then there is no reason to expect that a trajectory that starts at $x_0 \in K \setminus V_F^K(T)$ remains in K.

We can now define what we understand as a viability problem, and what we mean by its solution.

Definition 2.2. *Given the system correspondence $F(\cdot)$ (i.e., given the system dynamics f and sets of constraints K and $\mathcal{I}$), the associated viability problem consists of establishing existence of the viability kernel $V_F^K(T)$.*

Remark 2.1. *When the kernel is nonempty $V_F^K(T) \neq \emptyset$, we say that the viability problem has a solution; otherwise, the viability problem has no solution.*

In general, for non-stationary problems (which include finite horizon problems), it matters when one starts to control the process. Here, we can see that viability theory allows us to treat finite and infinite horizon problems uniformly.[6]

[5]For example, if an economy has inflation level $x(1) = \bar{\pi}$ in the first year after elections, we can apply $i(\bar{\pi}, 1) \in \mathcal{I}(1)$ that, presumably, will be sufficient to keep $x(2)$ in K. However, in the last year before new elections we need to select an instrument from $\mathcal{I}(3)$. If $\mathcal{I}(3)$ is "smaller" than $\mathcal{I}(1)$ then $i(\bar{\pi}, 3) \in \mathcal{I}(3)$ might be insufficient to keep $x(4)$ in K.

[6]Notice that in traditional monetary-policy optimisation models (*e.g.*, [15]), the horizon is infinite.

2.2. **Linear dynamics example.** There are theorems that characterise viability kernels for some typical dynamic systems (see [1], [2], [5]).

Of particular usefulness for numerical solutions to viability problems are the theorems that are using geometric properties of kernels, in which system evolutions are contained (*i.e.*, no system trajectory leaves the kernel at least until time T). It happens that it is easier to compute such kernels' boundaries [7] than the "general" kernels' boundaries.

One theorem that can be used (see [5]) to solve a viability problem says that if system dynamics $f(x(t), i(t))$ is linear, set $\mathcal{I}$ is compact and nonempty and map $F(x(t))$ is convex[8] then D is a viability kernel **iff**

$$\forall x \in D,\ \forall n \in \mathcal{N}(x), \exists i \in \mathcal{I} \text{ such that } \langle f(x, i), n\rangle \leq 0 \tag{5}$$

where $\mathcal{N}(x)$ is the set of *proximal normals* to D at x (and we have dropped the time argument).

Symbol $\langle\cdot,\cdot\rangle$ means scalar product, which is negative when the two arguments are vectors that form an *obtuse* angle (*i.e.*, greater than 90^0 or less than -90^0). As n is a normal vector pointing outside D, this condition means that the cone of evolution directions at x defined by $f(x, i)$ has to contain some vectors that point inside D.

Figure 2 provides an illustration of the geometric properties required for viability of a linear dynamic system defined as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \end{bmatrix}. \tag{6}$$

The state variables are x, y; the instrument set $\mathcal{I}$ is the unit ball

$$\mathcal{I} \equiv \left\{(v_x, v_y) : v_x^2 + v_y^2 \leq 1, \quad (v_x, v_y) \in \mathbf{R}^2\right\}. \tag{7}$$

Suppose we want to decide if the rectangle

$$Q \equiv \left\{(x, y) \ : \ \max(|x|, |y|) \leq 1,\ (x, y) \in \mathbf{R}^2\right\}$$

is a viability kernel. If it is, we will be certain that we can prevent the system to escape from Q using actions from $\mathcal{I}$.

Consider the boundary point **B** of the rectangle. The velocities from $\mathcal{I}$ generate evolution directions denoted by "x" (crosses); the normal at **B** is the thin black line. We see that there is no vector at **B**, which would form an obtuse angle with the normal; all angles are acute. This is a consequence of the constraint imposed on v_x, v_y in (7). In effect, inequality (5) is not satisfied at point **B**. This means that set Q cannot be a viability kernel.

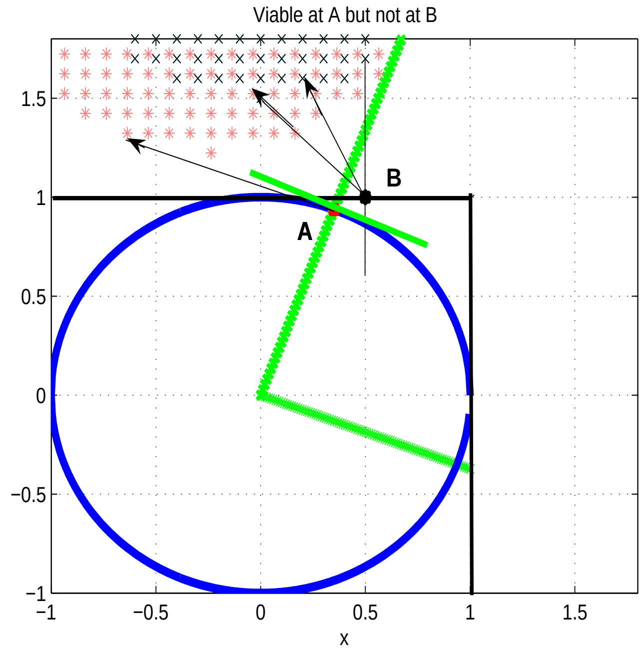
However, we can prove that set Q_B

$$Q_B \equiv \left\{(x, y) \ : \ x^2 + y^2 \leq 1,\ (x, y) \in \mathbf{R}^2\right\}$$

(delimited by the circle of radius 1 centred at origin) is a viability kernel. Indeed, there exist velocities at boundary point **A** that generate evolution directions represented by "*" (stars), which form an obtuse angle with the normal going through this point (thick green line). This means that inequality (5) is satisfied at point **A**

[7]Such kernels are called viability domains. However, we will not delve into the differences between viability domains and "general" kernels.

[8]Then F is called a *Marchaud map*.



F 2. A geometric characterisation of viability.

hence, one can select a pair of velocities from $\mathcal{I}$ that keeps the system trajectory inside the set Q_B.

The evolution direction, which is a consequence of use of the velocities, is marked by the black vector at **A** pointing left. Presumably, this instrument is "outermost" or *extreme* in that there is no other velocity vector, which would generate a more obtuse angle with the normal at **A**. In fact, the pair of velocities satisfies $v_x^2 + v_y^2 = 1$.

A comparison between sets Q and Q_B (the latter is a viability kernel the former is not) tells us that at **B**, the systems moves too "fast" to be controlled through $(v_x, v_y) \in \mathcal{I}$. However, the same instrument set contains elements that are sufficient to restrain the system, should we apply them "early"[9] *i.e.*, when the process is within Q_B.

Numerical procedures can be applied to distinguish between viable points (like **A**) and the non viable ones (like **B**), see [5].

2.3. **Policy advice.** In economic situations, in which a "planner" may be identified (*e.g.*, Central Bank), the establishment of a viability kernel can be used to select policies that keep the dynamic process x inside the closed constraint set K. Once the kernel is established, choosing a satisficing policy is a simple procedure, which can be illustrated using Figure 2. Before we explain the procedure let us briefly look at what kind of actions a Central Bank planner undertakes.

Routinely, every given time interval, the planner announces a cash interest rate. A *Taylor rule* or an optimising rule[10] might be used to determine the "new" interest rate. The latter usually equals the old interest rate *plus* or *minus* a fraction of a

[9]Suppose the evolution started at origin.

[10]See *e.g.*, [15].

percentage point. While this might look simple, the process leading to the rate determination is typically based on optimisation of a loss function, which contains a significant number of parameters calibrated and/or estimated.

Looking back at Figure 2 we can see that there are instruments $v_x^2 + v_y^2 \leq 1$ that keep trajectory x in $Q_B \subset Q$. In particular, we know that even if $x(t)$ is at the boundary of Q_B the "outermost" or *extreme* instrument is sufficient to prevent the system trajectory to leave Q_B.

We will term *extreme* an instrument from set $\Im$ that belongs to this set's boundary. We will make the meaning of this definition clear every time a viability kernel will be determined.[11]

Denote $\mathsf{fr}\, D$ the boundary of the viability kernel D. For deterministic models, the following *satisficing* policy prescription follows from the above:

$$(8)\qquad \begin{cases} \texttt{if} \quad x(t) \in D \backslash \mathsf{fr}\, D, & \texttt{apply} \quad \underline{\texttt{any}} \ \texttt{instrument} \quad \in \mathcal{I}; \\ \texttt{if} \quad x(t) \in \mathsf{fr}\, D, & \texttt{apply} \ \underline{\texttt{extreme}} \ \texttt{instrument} \quad \in \mathcal{I} \end{cases}$$

The effect of the Central Bank optimisation process is similar to the application of the satisficing policy: either maintains x (*e.g.*, inflation) in K. However, as it will be explained later, fewer (subjective) parameters are needed to establish D than to compute a minimising solution to the bank loss function. Also, the "relaxed" approach advocated by (8) (the first ``if") offers the planner a possibility to strive to achieve other goals (*e.g.*, political), which were not used for the specification of K. (Perhaps they were difficult to specify mathematically or they arose after the viability kernel had been established.) This is not the case of an optimal solution that remains optimal for the original problem formulation only.

Should there be uncertainties regarding the model parameters, a sensitivity analysis needs be performed to establish to what extent the system dynamics is affected by the uncertainties. Once established, the current position of the system will be generalised from $x(t)$ to $x(t)+b_1(x(t), \kappa(t))$ where $b_1(\cdot,\cdot)$ ball centred at $x(t)$ with radius $\kappa(t)$ that will result from a robustness analysis of (3).

When the model is subjected to shocks whose magnitude can be estimated (or their distribution is known), the viability kernel will have to be such that $x(t) + b_2(x(t), \varepsilon(t)) \in D$ where radius $\varepsilon(t)$ will depend on the shock[12]. Then, the above policy prescription can be followed.

3. A

3.1. A viability theory problem. Realistically, what a typical Central Bank wants to achieve is the maintenance of a few key macroeconomic variables within some bounds. Usually, the bank realises its multiple targets using *optimising* solutions that result from minimisation of the bank's loss function. Typically, the loss function includes penalties for violating an allowable inflation band and also for a non-smooth interest adjustment. The solution, which minimises the loss function, is unique for a given selection of the loss function parameters. In that, it does not allow for alternative strategies.

Our intention is to apply viability theory to a bank's problem, which we understand as to keep variables of interest in a constrained set. This sounds very much

[11]Element of $\mathcal{I}$ that guarantees viability of Q_B in Figure 2 is *extreme* because $v_x^2 + v_y^2 = 1$.

[12]This radius might equal an expected shock magnitude.It may also equal the size of the shock that occurs "once in 100 years". *Etc.*

like the viability theory problem illustrated in Figure 1. We will try to establish what the set of economy states is (*i.e.*, what the kernel V_F^K is), from which a Central Bank is able to keep the economy's evolution viable (*i.e.*, such that the key variables remain within some prescribed bounds K), given available instruments.

In the next section we will use a stylised monetary rules model (inspired by [15] and [16]). We will then show that the solutions obtained through viability theory do not suffer from drawbacks typical of their optimising counterparts.

3.2. **A Central Bank problem.** Suppose a Central Bank is using nominal short-term interest rate $i(t)$ as an instrumental variable to control inflation $\pi(t)$ and, to a lesser extent, output gap y_t and to an even lesser extent, real exchange rate[13] $q(t)$. A model that relates these variables may look like follows (see [15], [16][14]; also see [7]):

$$y(t) = a_1\, y(t-h) - a_2\Big(i(t-h) - E_{t-h}\pi(t)\Big) + b_q E_{t-h} q(t) + u(t) \tag{9}$$

$$\pi(t) = \pi(t-h) + \gamma y(t) + a_q E_{t-h} q(t) + \eta(t) \tag{10}$$

$$E_{t-h} q(t) = q_{t-h} + (i_{t-h} - E_{t-h}\pi_t) - (i^*(t-h) - E_{t-h}\pi^*(t)) - \phi(t) \tag{11}$$

where $y(t)$ is output gap, $u(t), \eta(t)$ are serially uncorrelated disturbances (*shocks*: in aggregate demand and inflation, respectively) with means equal to zero and $\phi(t)$ is a stochastic variable that represents exchange risk premium; $a_1, a_2, \gamma, \alpha_q, \beta_q$ are estimated or calibrated parameters; E_{t-h} is the expectation operator. The variables marked by * correspond to the outside world.

Applying the expectation operator to (9), (10) and (11) yields

$$E_{t-h} y(t) = a_1\, E_{t-h} y(t-h) - a_2\Big(E_{t-h} i(t-h) - E_{t-h}\pi(t)\Big) + b_q E_{t-h} q(t) \tag{12}$$

$$E_{t-h}\pi(t) = E_{t-h}\pi(t-h) + \gamma E_{t-h} y(t) + a_q E_{t-h} q(t) \tag{13}$$

$$\begin{aligned} E_{t-h} q(t) = {} & E_{t-h} q(t-h) + \Big(E_{t-h} i(t-h) - E_{t-h}\pi(t)\Big) \\ & -\Big(E_{t-h} i^*(t-h) - E_{t-h}\pi^*(t)\Big) \end{aligned} \tag{14}$$

At time $t-h$, the expectations are identical with observations so,

$$E_{t-h} y(t) = a_1\, y(t-h) - a_2\Big(i(t-h) - E_{t-h}\pi(t)\Big) + b_q E_{t-h} q(t) \tag{15}$$

$$E_{t-h}\pi(t) = \pi(t-h) + \gamma E_{t-h} y(t) + a_q E_{t-h} q(t) \tag{16}$$

$$\begin{aligned} E_{t-h} q(t) = {} & q(t-h) + \Big(i(t-h) - E_{t-h}\pi(t)\Big) \\ & -\Big(i^*(t-h) - E_{t-h}\pi^*(t)\Big). \end{aligned} \tag{17}$$

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[13] The exchange rate $q(t)$ is defined as the *log* ratio of *nominal exchange rate* $\times$ *foreign price index* to *domestic price index* and can be viewed as an aggregate measure of strength of a country's currency. If the currency weakens, then $q(t)$ increases. Or, larger $q(t)$ means real depreciation so, that domestic goods become relatively cheaper .

[14] Basically, our model is a version of the Radebusch-Svensson model. However, (11) is inspired by the famous *interest parity condition* studied in international finance literature.

Assume differentiability of the inflation, and output gap and exchange rate processes. If so, for small h

$$E_{t-h}y(t) = y(t-h) + \dot{y}\big|_{t-h} h \tag{18}$$

$$E_{t-h}\pi(t) = \pi(t-h) + \dot{\pi}|_{t-h} h \tag{19}$$

$$E_{t-h}q(t) = q(t-h) + \dot{q}\big|_{t-h} h. \tag{20}$$

The above relationships tell us that agents forecast the expected values using extrapolations. This corresponds to the basic learning process (compare [12]).

Before we use (18), (19), (20) for substitutions, assume that the foreign Central Bank is also targeting inflation and is doing so successfully. Hence, the foreign control that enters our model is the foreign short term real interest rate

$$r^*(t-h) \equiv i^*(t-h) - E_{t-h}\pi^*(t). \tag{21}$$

We use (21) and substitute (18), (19), (20) in (15), (16), (17) (and omit the time index $t-h$); this yields:

$$y + \dot{y}h = a_1\, y - a_2\Big(i - (\pi + \dot{\pi}h)\Big) + b_q(q + \dot{q}h) \tag{22}$$

$$\pi + \dot{\pi}h = \pi + \gamma(y + \dot{y}h) + a_q(q + \dot{q}h) \tag{23}$$

$$q + \dot{q}h = q + \Big(i - (\pi + \dot{\pi}h)\Big) - r^*\,. \tag{24}$$

From (23),

$$\dot{\pi}h = \gamma(y + \dot{y}h) + a_q(q + \dot{q}h);$$

from (24),

$$\dot{q}h = \Big(i - (\pi + \dot{\pi}h)\Big) - r^*.$$

Allowing for these and for $\alpha\, h = a_1 - 1$, $\xi\, h = a_2$, $\zeta\, h = \gamma$, $\beta_q\, h = b_q$, $\alpha_q\, h = a_q$, then dividing by h and requesting $h \to 0$ we get the following inflation, output gap and exchange rate dynamics

$$\frac{dy}{dt} = \alpha\, y(t) - \xi\Big(i(t) - \pi(t)\Big) + \beta_q\, q(t) \tag{25}$$

$$\frac{d\pi}{dt} = \zeta\, y(t) + \alpha_q\, q(t) \tag{26}$$

$$\frac{dq}{dt} = \Big(i(t) - \pi(t)\Big) - r^*. \tag{27}$$

This model tells us that the expected output gap constitutes a "sticky" process driven by real interest rate. Moreover, exchange rate affects competitiveness of domestic goods in the world market so, it also affects the output gap changes: if the domestic currency appreciates ($q(t)$ diminishes) then the output gap growth slows. The expected speed of inflation (26) changes proportionally to the expected output gap; exchange rate affects the cost of imported goods and if the local currency appreciates the inflation rate diminishes. Equation (27) is the differential version of the (real) interest parity condition. It says that *expected depreciation* of the local currency equals to the *real interest rate differential*.

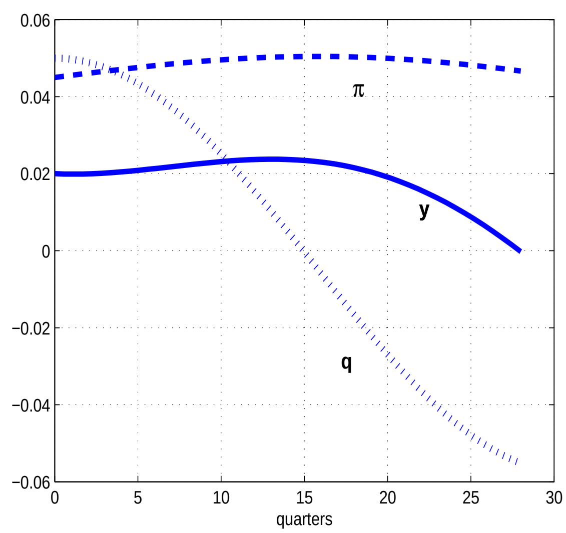
Among other phenomena, the latter equation captures the currency adjustment to the real interest rate differential. If domestic interest rate was increased then bonds would earn more in local currency; however, the exchange rate will adjust so that each country's bonds yield the same return.

3.3. **Parameter values.** We use the following parameter values (see [15]):

$$\xi = \frac{a_2}{h}\Big|_{h=1} = .35, \qquad \zeta = \frac{\gamma}{h}\Big|_{h=1} = .002\,.$$

Regarding α we follow [7] and choose $\alpha = \frac{a_1-1}{h}\big|_{h=1} = \frac{.98-1}{h}\big|_{h=1} = -.02$.

The remaining parameters β_q and α_q do not appear in [15] or [6]. They are given in [16] as $\beta_q = .039$ and $\alpha_q = .01$. However, [16] model is different than that of [15] and the use of the above β_q and α_q is not necessarily justified. Nonetheless, we propose to use them as they generate "reasonable" time profiles of all variables of interest, see Figure 3. In this figure, we have assumed a constant interest rate policy=.05 (5%) and a constant overseas real interest rate .005 (.5%). This presupposes that the foreign economy real interest rate is constant and low; it also says that domestic real interest rate will change even if output gap was stationary.



F 3. A control run of output gap, inflation and real exchange rate.

We can observe that such a combination of parameters and instruments produces a diminishing path of $q(t)$ (so, the domestic currency strengthens). Initially, with yet weak domestic currency, output increases but it falls subsequently. There are some inflationary pressures but they ease off with falling output and strong currency.

We believe that this path is plausible and attribute it to the selection[15] of the model parameters, which we will keep unchanged in the rest of the paper. We want to stress that we use thus calibrated model for illustrative purposes rather than a "real" policy analysis.

[15]Non unique, by far.

In summary, the macroeconomic model that we will analyse is

$$\frac{dy}{dt} = -0.02\,y(t) - 0.35\Big(i(t) - \pi(t)\Big) + .039q(t) \tag{28}$$

$$\frac{d\pi}{dt} = 0.002\,y(t) + .01q(t) \tag{29}$$

$$\frac{dq}{dt} = \Big(i(t) - \pi(t)\Big) - r^{*}(t). \tag{30}$$

4. V

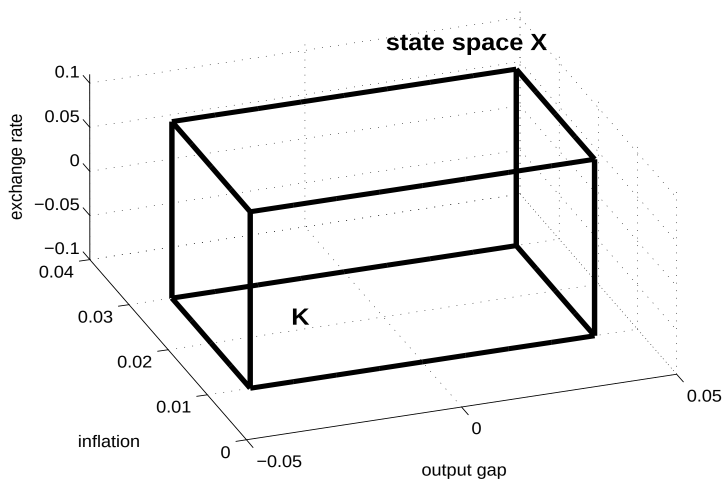
We will perform a viability theory analysis for a given constraint set, using model (28) - (30). This is *computational economics* and the results will be parameter specific; however, the procedure can be easily repeated for any plausible parameter selection.

4.1. **The state constraints.** Usually there is little doubt what the *politically* desired inflation bounds are. For example, in New Zealand, the inflation band has been legislated [.01, .03]. To reflect a lesser concern of the Central Bank for output gap we will model the interval for $y(t)$ rather wide: $y(t) \in [-.04, .04]$).

There is little agreement is as to what an *ideal* range of the real exchange rate is. We will assume a rather wide interval of acceptable $q(t) \in [-.1, .1]$. So, our constraint set is

$$K \equiv \{(y(t), \pi(t), q(t)) \;:\; -.04 \le y(t) \le .04,\;\; .01 \le \pi(t) \le .03 \;\;\text{and}\;\; -.1 \le q(t) \le .1\},$$

see Figure 4 for the set K.[16]



F 4. Constraint set K (viability domain) .

[16]Notice that this is a 3D-state space. A similar figure in [7] showed a 2D-state space *plus* a 1D instrument domain.

4.2. **The control and velocity constraints.** Similarly to the desired size of inflation, output gap and real exchange rate, the instrument set composition also depends on political decisions. We will assume that $i(t) \in [0, .07]$.

The above lower bound is obvious; the upper bound seems "historically" justified as it only infrequently violated in countries like the US or Japan.

Independently of keeping the interest range constrained, many central banks are worried about the interest rate *smoothness*. That concern is usually modelled by adding $w(i(t) - i(t-h))^2$, $w > 0$ to the loss function. In continuous time, limiting the interest rate "velocity" $\frac{di}{dt}$ will produce a smooth time profile of $i(t)$. Bearing in mind that the central bank's announcements are usually made every quarter and that the typical change is a $\frac{1}{4}$%, the *domestic* instrument set will be defined as

$$\Im \equiv \left\{ i \ : \ i(t) \in [0, .07], \quad \text{and} \quad \frac{di}{dt} \in [-.005, .005] \right\} \tag{31}$$

i.e., the interest rate can drop, or increase, between 0 and .5% per quarter.

Apparently, there is one more control in the model, which is the foreign nominal interest rate $i^*(t)$. However, if we assume that the foreign Central Bank is also targeting inflation and is so doing successfully, then it is the foreign Central Bank controls the short-term foreign real interest rate defined as

$$r^*(t) \equiv i^*(t) - \pi^*(t) \tag{32}$$

(compare (21)). In our study we assume that the *foreign* instrument set $\Re^*$ is defined as

$$\Re^* \equiv \left\{ r^* \ : \ r^*(t) \in [0, .07], \quad \text{and} \quad \frac{dr^*}{dt} \in [-.0025, .0025] \right\} \tag{33}$$

i.e., the foreign real interest rate can drop, or increase, between 0 and .25% per quarter (presumably, slower than the domestic rate) and that it can move between 0 and 7 %..

Hence, the dynamic system to analyse the relationship between the output gap, inflation, (real) exchange rate and control instruments augmented by the velocity constraints will now look as follows:

$$\frac{dy}{dt} = -0.02\, y(t) - 0.35\big(i(t) - \pi(t)\big) + .039 q(t) \tag{34}$$

$$\frac{d\pi}{dt} = 0.002\, y(t) + .01 q(t) \tag{35}$$

$$\frac{dq}{dt} = \big(i(t) - \pi(t)\big) - r(t). \tag{36}$$

$$\frac{di}{dt} \in [-.005, .005]\,. \tag{37}$$

$$\frac{dr^*}{dt} \in [-.0025, .0025]\,. \tag{38}$$

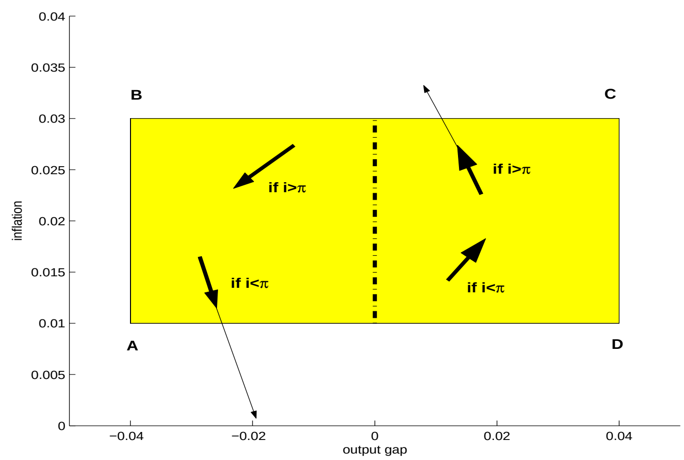
So, we have defined our macroeconomic problem dynamics as a differential inclusion (34)-(38) (compare (3)). We will call $F(y, \pi, q, i, r)$ the collective vector of right hand sides of (34)-(38). Now, our problem is

$$\text{given :} \quad K, F(y, \pi, q, i, r), \mathcal{I}, \Re^* \quad \text{determine} \quad V^K_{\Im, \Re^*}. \tag{39}$$

5. V

5.1. **A qualitative game.** Notice that problem (39) differs from the viability problem solved in ([7]). Here, we have two instrument sets $\Im$ and $\Re^*$, which are not controlled by the same agent. Hence, our problem is a *game*. However, problem (39) is about establishing a viability kernel $V^K_{\Im,\Re^*}$ and *not* about maximisation (or minimisation) of one or two objective functions. Problems that concern kernel establishment given multiple instruments sets are sometimes called *qualitative games*, see [5]. In fact, ours will be an easy case of a game considered in [5] because the "other" agent *i.e.*, that controls $r^*(t)$, does not aim to keep the state in a specific viability domain different from K. That agent is a nuisance agent rather than an *evil agent* as some economic papers tend to call them.

5.2. **Precarious situations.** From a steady state analysis conducted in [7] we can expect that the establishment of the viability kernel boundary will be critical at (at least) two situations. See Figure 5 which sketches approximated directions of the system evolution in the projected state space ${}^pX$ output-gap - inflation.



F 5. Approximated evolution of the economy on plane ${}^pX$.

With $q(t) = const$ and small, when output gap is positive and inflation is high (see corner C), increasing i, which helps turn the evolution *left*, must happen "early" otherwise the inflation upper boundary will be violated.

This is because we said interest rates need to move smoothly (see (31)). This means that the instrument speed is constrained and any sudden hike in i is impossible. If so, the Central Bank needs to know a collection of points from where the control from $\Im$ is *sufficient* to avoid leaving K. Such a collection is the viability kernel defined on page 3.

A different problem may occur if output gap is negative (see corner A). To avoid a *liquidity trap*[17] the economy evolution needs to avoid negative output gap and low inflation states. So the "arrow" in corner A needs must turn right. This can happen

[17] In a liquidity trap an economy remains in an area where output gap is negative and inflation is close to zero (positive or negative); see [10] for an analysis of a liquidity trap problem performed through an established method. Also notice that [11] is a recent publication where a liquidity trap problem is analysed in state space.

through lowering the nominal interest rate. However, if the bank starts lowering nominal interest rate when inflation is close to the boundary (here, 1%) then the real interest rate control will be close to zero and inefficient in stimulating growth. Consequently, the economy will further drift toward zero inflation with negative output gap. This situation also calls for establishing of a viability kernel, which will tell the Central Bank from where "turning" the economy away from a liquidity trap is possible, given the available instruments.

5.3. **The viability kernel for a "hot" economy.** We will show the viability kernel's boundaries for an economic situation that a country with a large output gap and inflation close to the upper limit (like New Zealand) might experience.

Given problem (39), we will determine the kernel's boundaries for the "north-east" corner (C) by running (34)-(38) backwards from $y(T) = 0, \pi(T) = .03, q(T) = 0$ where T is some final time[18]. The choice of this point may seem arbitrary but we believe that $y(T) = 0, \pi(T) = .03, q(T) = 0$ is an attractive point, toward which the Central Bank might want the economy to evolve. Small $q(T) \approx 0$ may reflect the fact that both importers and exporters are equally (un)happy. If $q(T)$ is close to zero, then the inflationary pressures ease for $y(T) = 0$. Finally, $\pi(T) = .03$ is the upper inflation limit prescribed by the legislator.

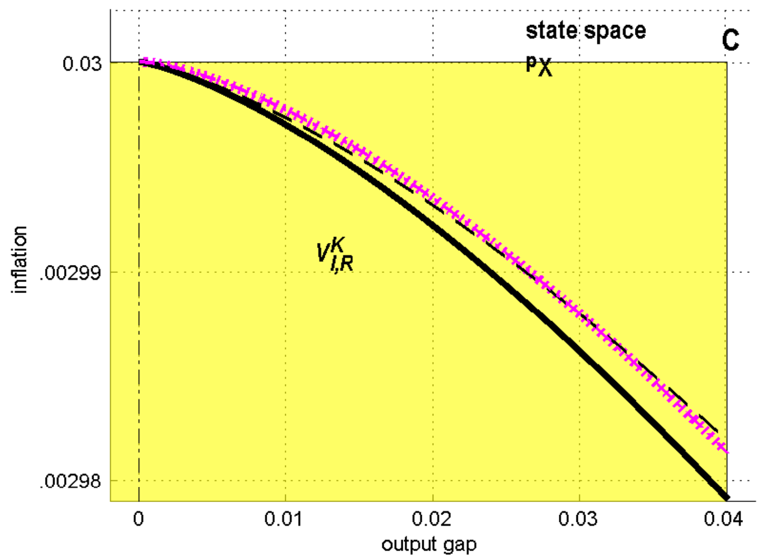
We need also to assume some values for the overseas real interest rate $r^*(T)$ and say whether this value is achieved when the rates are falling $\frac{dr^*}{dt} < 0$ or increasing $\frac{dr^*}{dt} > 0$. We will presuppose that, at time T when $[y(T), i(T) - \pi(T), q(T)]$ have reached $[0, 0, 0]$, the rate has converged to $r^*(T) = 1.5\%$. We will examine the economy's evolution when, first, $r^*(T) \to 1.5\%$ from above *i.e.*, when the rate is lowering fast with $\frac{dr^*}{dt} = -.0025$ and then from below *i.e.*, when the rate is rising fast with $\frac{dr^*}{dt} = .0025$. The results are shown in Figures 6 to 10.

Remember that from any point on the kernel's boundary the economy that is controlled with the maximal interest-rate rise "velocity", will be led to the "desired" point $y(T) = 0, \pi(T) = .03, q(T) = 0$ in minimum time. From every interior point of the kernel, the same can be achieved with even "slower" interest rate moves. From any point that lies outside the kernel, violation of the boundaries of K is imminent.

We see three lines in Figure 6. The one which is perpendicularly dashed (purple) is the boundary of a "2D" viability kernel, which was obtained in [7] for an open economy *i.e.*, with no allowance for exchange rate. This line almost coincides with the dashed line (black). The latter is obtained as the kernel's boundary for when the foreign real interest rate increases $\frac{dr^*}{dt} = .0025$. The solid line (black) delimits the kernel for the foreign real interest rate decreasing case $\frac{dr^*}{dt} = -.0025$. The respective viability kernels are (obviously) *left* from each boundary line defined above. The smaller the kernel the more difficult the economy to control.

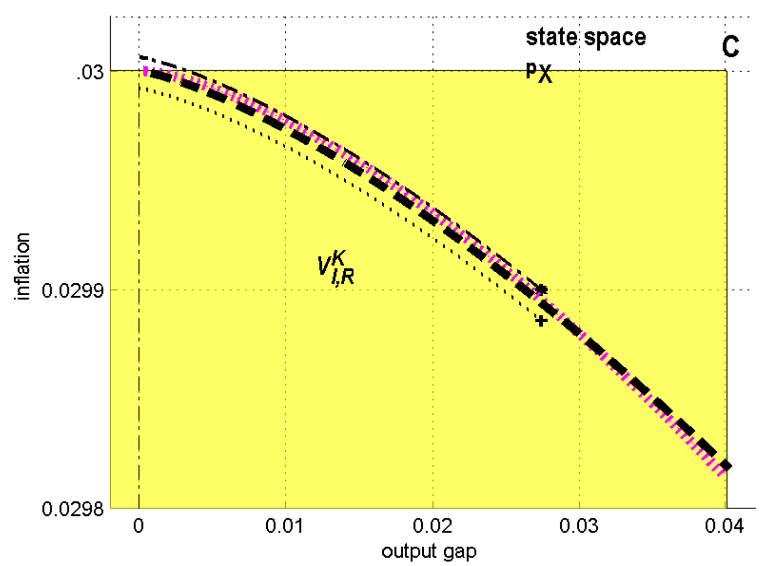
Figure 7 helps to understand an economy's evolution when foreign real interest rate is expected to increase. The dashed line is again the kernel's boundary (on the boundary: $\frac{dr^*}{dt} = .0025$). The dash dotted trajectory shows the evolution from a

[18]Rather than analysing the geometric properties of $K, F(\cdot), \mathfrak{I}$ and $\mathcal{R}$, see condition (5).



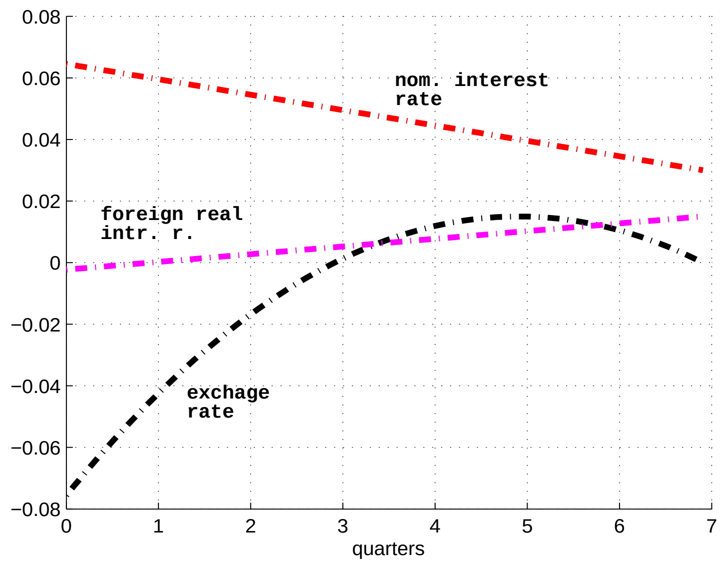
F 6. Viability kernels at corner C in the projected state space ${}^{p}X$.

point from outside the kernel. The dotted line shows the evolution when the bank did not let the economy leave the kernel. The latter evolution is *viable* the former is not.



F 7. Viable and non viable evolutions when foreign real interest rate increases.

We can also observe that there is little difference between this kernel and the one from [7] which was obtained without allowing for exchange rate, see the perpendicularly dashed (purple) line. Figure 8 sheds some light on a reason for the kernels' coincidence. It shows the time profiles of nominal interest rate, foreign real interest rate (increasing) and the resulting exchange rate that correspond to the economy controlled at the kernel's boundary.



F 8. Nominal interest rate, foreign real interest rate increasing and resulting exchange rate.

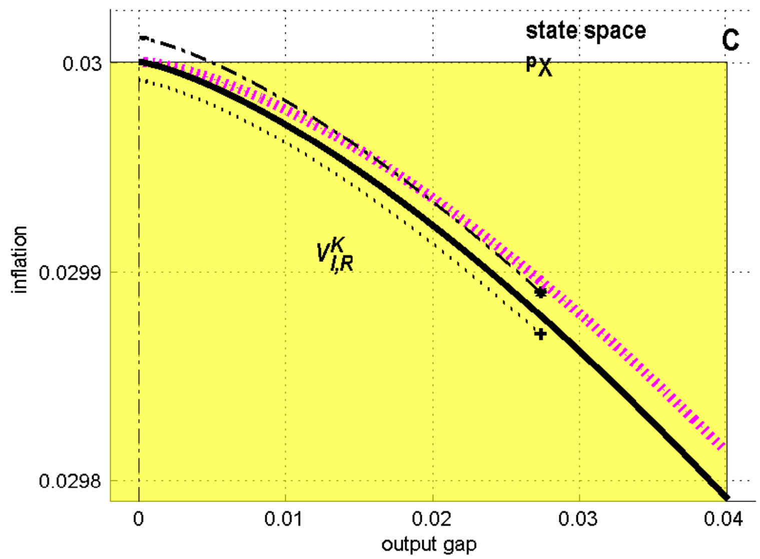
The increasing r^* causes the exchange rate to increase and then to decrease so, on average, the impact of this variable on the other variables appears moderated.

Figure 9 analyses the economy's evolution when the foreign real interest rate decreases $\left(\frac{dr^*}{dt} = -.0025\right)$. The solid line is the kernel's boundary (see Figure 6). The dash dotted trajectory shows the evolution from a point from outside the kernel. The dotted line shows the evolution when the bank did not let the economy leave the kernel. The latter evolution is *viable* the former is not. We can also see that a "viable" policy derived for the 2D viability kernel (perpendicularly dashed line see Figure 6; also [7]) is not viable in an open economy where we allow for exchange rate and if the foreign real interest rate is decreasing.

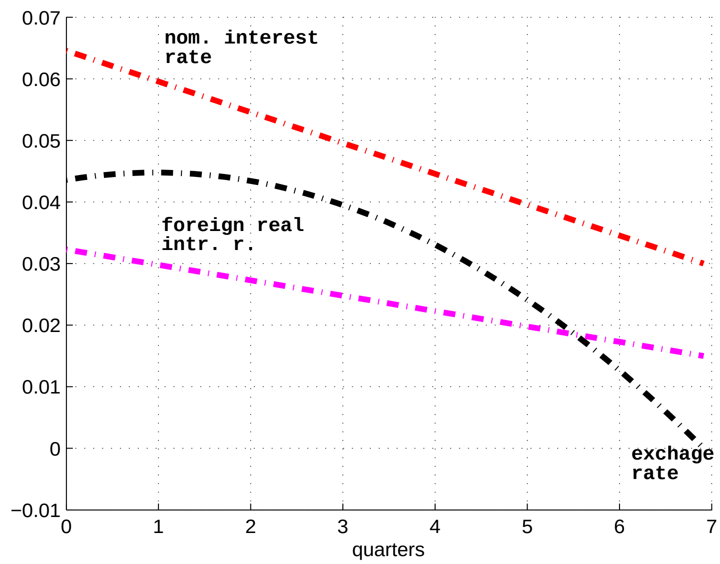
Figure 10 shows the time profiles of nominal interest rate, foreign real interest rate (dereasing) and the resulting exchange rate that correspond to the economy controlled at the kernel's boundary. We can see that in this case the exchange rate steadily decreases. This means that the domestic currency appreciates, which diminishes export and contributes to a faster drop in the output gap then when the foreign real interest rate was an increasing function of time (as in Figures 7 and 9). This provides an explanation of why this case may be more difficult[19] to control for the bank than when r^* increases.

In general, more numerical experiments (not shown) suggest that a "booming" economy is *viable* in that a relatively high interest policy can control the economy

[19]The smaller the kernel the more difficult to control the economy.



F 9. Viable and non viable evolutions when foreign real interest rate decreases.



F 10. Nominal interest rate, foreign real interest rate (decreasing) and resulting exchange rate.

to $y = 0$ and not violate the inflation upper bound. However, such policies might be from outside $\Im$ and non acceptable for this problem set up.

6. C

We have considered a simple macroeconomic model. An analysis based on viability theory enabled us to discuss how a Central Bank monetary policy can be established. We have endogenously derived some *satisficing* policy recommendations as follows:

(I) `if` $y(t), \pi(t)$ `are` *well inside* V_F^K `apply` $i(t)+\Delta i(t), \Delta i(t) \in [-.005, .005]$ `every time`
`interval` h[20]`;`
else

(II) `apply` $i(t) - .005$ `if` $y < 0$ `or` $i(t) + .005$ `if` $y > 0$.

These recommendations are in line with policy (8); in particular, (II) is *extreme* in that it calls for the "full speed" interest rate changes.

We explain "well inside" as follows. We believe there are two states an economy can be in: *well inside* the viability kernel and close to its boundaries. An assessment of in which of them the economy is, will obviously depend on the bank governor's judgment. Our graphs are helpful in the assessment. They tell the governor where the economy is *expected* to move to, given current conditions and the applied instruments. If at $t + h$ the economy is expected to remain in the kernel then the economy state at t is *well inside* the kernel.

The distinction between the two states of the economy is needed for the governor to decide which size of instrument $\Delta i(t) =$ to apply. With our model, the governor can assess where the economy is expected to be at time $t + h$ and what options he (or she) will then have. We believe that their choices made in this manner will be less arbitrary than the "optimal" ones that rely on the loss function weights and discount rate, which are subjective parameters.

The *satisficing* policy choices can be modified to allow for measurement errors, parameter uncertainty (like α, ζ *etc.*) and shocks even if the system dynamics is deterministic[21]. In broad terms, a ball around each point of the trajectory on the y, π plane (see *e.g.*, Figure 5) might be constructed where the ball size is "proportional" to uncertainty. The conditions to apply rules (I), (II) can be modified: if the ball does not intersect with the viability kernel's boundary, apply (I); else apply (II).

In general, the bank's policy established through a viability analysis may appear more credible to economic agents than its optimised counterpart. This might be so because the former depends on fewer arbitrary parameters than does the latter. More importantly, the former is precautionary and hence "naturally" forward looking; thus attractive for uncertainty conditions. We can say that policy advice based on viability analysis takes a compromise into account between the instrument timing and strength.

A

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[20] In our study, h is a quarter.

[21] Viability can also deal with explicit stochastic models.

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• All errors and omissions remain ours.

R

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