Comparing Value-at-Risk Methodologies

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Abstract

In this paper, we perform a Monte Carlo experiment to compare four different Value-at-Risk (VaR) methodologies under five different Data Generating Processes (DGPs). We show that the ARCH(1) Quantile methodology is robust to DGP misspecification. In an empirical exercise, we use the four methodology to estimate VaR for returns of São Paulo stock exchange index, IBOVESPA, in a period of market turmoil, and we show that the ARCH(1) Quantile.VaR presents the least number of violations.

1 Introduction

Every day, financial institutions (like banks) estimate measures of market risk exposure, that are analyzed by the institutions's decision makers. These estimates are also analyzed by internal and external auditors and regulatory agencies, who enforce that those institutions set aside enough capital to cover their risk exposures. This concern about market risk exposure has been increasing since the stock market crash in 1987, when 1 trillion of dollars (23% drop in value) was lost in a single day, known as the Black Monday. The recent turbulence in emerging markets, starting in Mexico in 1995, continuing in Asia in 1997, and spreading to Russia and Latin America in 1998, has further extended the interest in risk management.

Imprecise measures of risk cause inefficiencies: on one hand, if the measure is too conservative, then too much capital, that could be used in a more profitable way, is set aside; on the other hand, if the measure is too risky, that is, if it gives rise to a large number of violations, then there is a higher probability that a loss may lead the institution to bankruptcy. Hence, researching for more and more reliable and accurate measure of risk methodologies is an active and growing literature.

Value-at-Risk (VaR) is probably the most used measure of risk since the 1996 amendment to the Basle Capital Accord proposed that commercial banks

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with significant[1] trade activity could use their own VaR measure to define how much capital they should set aside to cover their market risk exposure, and U.S. bank regulatory agencies could audit the VaR methodology employed by the banks. This amendment was adopted in 1998 (Lopez, 1999)[2]. In Brazil, the article 59 of the resolution N^o 2829, March 2001, of the Brazilian Central Bank brings "*Para os seguimentos de renda fixa e de renda variável deverá ser feito o cálculo do Valor em Risco (VaR) ...*", mandating the use of VaR to some markets.

Value-at-Risk as the loss in market value over a given time horizon that is exceeded with probability τ. That is, for a time series of returns r_t, find VaR_t such that

$$P\left[r_t < -VaR_t | I_{t-1}\right] = \tau, \qquad (1)$$

where I_{t-1} denotes the information set at time $t-1$. From this definition, it is clear that finding a VaR essentially is the same as finding a $100\tau\%$ conditional quantile. Note that, for convention, the sign is changed to avoid negative number in the $VaR_t(\tau)$ time series. For regulatory purpose, τ is generally set to 1%. It does not mean that the banks may not estimate VaRs under different significance levels for their risk managers.

Although VaR is a relatively simple concept, robust estimation of it is often ignored in practice. Indeed, one popular approach to estimate VaR assumes a conditionally normal return distribution. The estimation of VaR is, in this case, equivalent to estimating conditional volatility of returns. Another popular method is to compute the empirical quantile nonparametrically, for example, rolling historical quantiles or Monte Carlo simulations based on an estimated model[3].

However, these models are based on restricted assumptions about the distribution of returns. There has been accumulated evidence that portfolio returns (or log returns) are usually not normally distributed. In particular, it is frequently found that market returns display structural shifts, negative skewness and excess kurtosis in the distribution of the time series. This is particularly true in periods of market stress such as the financial crises faced by the Brazilian economy from 1997 to 2000. These market characteristics suggest that more robust methods are needed to estimate VaR.

In this paper, we estimate VaR using a robust method based on quantile regression model that allows for ARCH effect, and compare it to several other non-robust VaR methodologies that are based on GARCH type volatility models. It is important to mention that Engle and Manganelli (1999) consider a different quantile regression based method. In particular, they consider an au-

[1] Any bank or bank holding company whose trading activity equals greater than 10 percent of its total assets or whose trading activity equals greater than \$1 billion must hold regulatory capital against their market risk exposure.

[2] Not ony the American banks but also the Brazilian banks use VaR to measure their risk exposure. For example: Unibanco S.A., HSBC Bank Brasil S.A., Banco Real ABN AMRO S.A., Banco do Brasil S.A. and many others.

[3] This approach includes the weighted moving average method by J.P. Morgan´s Riskmetrics and the hybrid method by Boudoukh, Richardson, and Whitelaw (1998).

toregression of the estimated VaRs. Our approach, however, has the advantage of pursuing a well-developed distributional theory which facilitates statistical inference and computational optimization.

We are not the first ones to compute VaR using a quantile regression model that allows for ARCH effect. In fact, Wu and Xiao (2002) used this model to estimate VaR and left-tail measures that were next employed to construct a risk-managed index fund. The performance of the ARCH Quantile method were then evaluated according to the capacity of the risk-managed index fund in tracking the S&P500 index.

There are, however, other ways to assess the quality of a VaR methodology. In this paper, we follow Engle and Manganelli (2001) who compare VaR methodologies using descriptive statistics of the distributions of violations obtained via Monte Carlo simulations. Specifically, we simulate many trajectories of the return time series assuming different innovation distributions, and compute the number of violations[4] using different VaR methodologies. For each simulated trajectory of the return series, we save the amount of violations. At the end of the experiment, we will have a distribution of violations for each VaR methodology. Hence, we can compute descriptive statistics of the various distributions of violations and evaluate the quality of a VaR methodology according to these statistics.

Our Monte Carlo simulations indicate that the robust model based on quantile regression dominates other models that requires distributional assumptions. In particular, the distribution of violations generated from non-robust models are right-skewed and presents excess kurtosis, meaning that these non-robust models have high probability to predict implausibly high VaRs. We illustrate our findings with an empirical application. We consider returns of the São Paulo stock exchange index, IBOVESPA. and show that the VaR estimated by the quantile regression approach tracks very well VaRs estimated from non-robust models during normal market conditions. However, during market turmoils, the robust method tend to predict VaRs more accurately.

The outline of this paper is as follows: In section 2, we describe the general framework and present the competing models. We describe our Monte Carlo experiment in section 3. An empirical illustration is provided in section 4 and section 5 concludes.

2 The Competing Models

Most of the VaR methodologies are GARCH type models. Hence, they can be described using a GARCH framework (Giot and Laurent, 2004). GARCH models are designed to model the conditional heteroskedasticity in the time series of returns y_t, that is,

[4] The number of violations is defined as the number of losses greater than $VaR(\tau)$

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t, \\
\varepsilon_t &= \sigma_t z_t, \\
\mu_t &= c\left(\eta|\Omega_{t-1}\right), \\
\sigma_t &= h\left(\eta|\Omega_{t-1}\right),
\end{aligned} \tag{2}
$$

where $c\left(\eta|\Omega_{t-1}\right)$ and $h\left(\eta|\Omega_{t-1}\right)$ are functions of the vector of parameters η and of the information set Ω_{t-1}; z_t is a independent and identically distributed process, independent of Ω_{t-1}, with $E\left[z_t\right] = 0$ and $Var\left[z_t\right] = 1$; μ_t is the conditional mean for y_t and σ_t^2 is its conditional variance. The volatility model (2) encompass a family of methodologies used to predict VaRs. We next describe some members of such family.

2.1 RiskMetrics[5]

RiskMetrics (J.P. Morgan, 1996) is the most simple analyzed methodology. However, it is still one of the most used to compute VaR, and it is available for free by J.P. Morgan. In fact, RiskMetrics is a gaussian Integrated GARCH(1,1) model where the autoregressive parameter is set at a pre-specified value of 0.94 (for daily VaR, in the United States) and the decay parameter (it can be viewed as an exponential filter in volatility) is set at 0.06, that is,

$$
\sigma_t^2 = 0.06\varepsilon_{t-1}^2 + 0.94\sigma_{t-1}^2. \tag{3}
$$

The conditional mean μ_t is estimated by OLS, running y_t against its own lags[6] and $z_t\tilde{}N\left(0,1\right)$.

2.2 Gaussian GARCH(1,1)

In spite of using RiskMetrics, we could use the same GARCH(1,1) model but, instead of setting prespecified values of the parameters, we estimate them. In other words, we estimate the model

$$
\sigma_t^2 = \omega + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2, \tag{4}
$$

and $z_t\tilde{}N\left(0,1\right)$.

This is the second model to be analyzed in our Monte Carlo experiment. The Gaussian (or Normal) GARCH(1,1) is expected to generate better forecasts than RiskMetrics, because the parameters are estimated rather than prespecified.

[5] RiskMetrics is a trademark by J.P. Morgan.

[6] The number of lags in the OLS regression can be choosen using Information Criteria. One can also add other conditioning variables.

Observe that these two first models do not capture neither the asymmetric dynamics[7] nor all the leptokurtosis that is generally present in macroeconomics and financial time series, due to the fact that they assume normality for z_t. Indeed, in VaR applications, the choice of a appropriate distribution for the innovation process z_t is an important issue as it directly affects the quality of the estimation of the required quantiles. One way to weaken the assumption on the distribution of z_t is to consider the Skewed Student-t APARCH model, which we describe next.

2.3 Skewed Student-t APARCH(1,1)

The APARCH (Ding, Granger and Engle, 1993) is an extension of the GARCH model that nests at least seven GARCH specifications. It can be described as

$$\sigma_t^\delta = \omega + \alpha_1 \left(|\varepsilon_{t-1}| - \gamma_1 \varepsilon_{t-1}\right)^\delta + \beta_1 \sigma_{t-1}^\delta, \tag{5}$$

where ω, α_1, γ_1, β_1 and δ $(\delta > 0)$ are parameters to be estimated. δ plays the role of a Box-Cox transformation of σ_t, while γ_1 $(-1 < \gamma_1 < 1)$ reflects the so-called leverage effect: the stylized fact that negative shocks impact volatility more than positive shocks.

Giot and Laurent (2003) and Giot (2003) use the above model considering a standardized version of the Skewed Student-t distribution - introduced by Fernández and Steel (1998) - for the z_t process. They show that such standardized version provides more accurate VaR forecasts than the GARCH model. This result is somehow expected, because the Skewed Student-t APARCH(1,1) nests the Gaussian GARCH(1,1)[8].

According to Lambert and Laurent (2001) and provided that the degrees of freedom $\nu > 2$, the innovation process z_t is said to be standardized Skewed Student-t distributed, i.e. $z_t \sim SKST(0, 1, \xi, \nu)$ if:

$$f\left(z_t|\xi,\nu\right) = \begin{cases} \frac{2}{\xi+\frac{1}{\xi}} sg\left[\xi\left(sz_t+m\right)|\nu\right], \text{ if } z_t < -\frac{m}{s} \\ \frac{2}{\xi+\frac{1}{\xi}} sg\left[\frac{sz_t+m}{\xi}|\nu\right], \text{ if } z_t \geq -\frac{m}{s} \end{cases}, \tag{6}$$

where $g\left[\cdot|\nu\right]$ is a symmetric (unit variance) Student-t density and $\xi > 0$ is the asymmetry coefficient. The parameters m and s^2 are, respectively, the mean and the variance of the nonstandardized Skewed Student-t:

$$m = \frac{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\nu-2}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(\xi - \frac{1}{\xi}\right) \tag{7}$$

[7] Beaudry and Koop (1993) showed that positive shocks to US GDP are more persistent than negative shocks, indicating asymmetric business cycle dynamics. More recently, Nam et al. (2005) identified asymmetric dynamics for daily return on the S&P 500 and used that to develop optimal technical trading strategies. In 1992, Brock, Lakonishok and LeBaron showed that two of the simplest and most popular trading rules - moving average and trading range break - consistently generate buy signals with higher returns than sell signals, and further, the returns following buy signals are less volatile than returns following sell signals.

[8] Indeed, the GARCH(1,1) is an APARCH(1,1) with $\delta = 2$ and $\gamma_1 = 0$, and the Skewed Student-t distribution with the asymmetry coefficient $\xi = 1$ (no asymmetry) converges to the Gaussian distribution when the degrees of freedom ν tends to the infinity.

and

$$s^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1\right) - m^2. \tag{8}$$

In short, ξ models the asymmetry, while ν accounts for the tail thickness. See Lambert and Laurent (2001) for a discussion of the link between these two parameters and the skewness and the kurtosis.

2.4 ARCH(q) Quantile

Koenker and Zhao (1996) introduced a quantile regression model that allows for ARCH effect. The ARCH(q) Quantile methodology uses OLS estimator to estimate the conditional mean μ_t, but this is the only similarity with the first three methodologies. The ARCH(q) Quantile does not assume any particular distribution to the process z_t. The model can be described as follows

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, \\ \varepsilon_t &= \left(\gamma_0 + \gamma_1 \left|\varepsilon_{t-1}\right| + ... + \gamma_q \left|\varepsilon_{t-q}\right|\right) z_t. \end{aligned} \tag{9}$$

Thus, the ARCH(q) Quantile specification assumes that the errors follow an ARCH(q) type model[9], in which the fundamental innovation z_t is drawn fro a unknown distribution F_z.

In all the models presented in this paper, the $VaR(\tau)$ is defined as the τth conditional quantile of the return, that is

$$-VaR_t(\tau) = \mu_t + Q_\varepsilon(\tau|\Omega_{t-1}), \tag{10}$$

where $Q_\varepsilon(\tau|\Omega_{t-1})$ is the conditional quantile function of ε_t.

Given a known distribution for the process z_t, the computation of (10) is straightforward. When the distribution of z_t is unknown, we are led to the problem of quantile regression. The quantile regression method is an extension of the empirical quantile methods. While classical linear regression methods based on minimization sums of squared residuals enable one to estimate models for conditional mean functions, quantile regression methods offer a mechanism for estimating models for the conditional quantile functions, like the one appearing in (10). Thus, quantile regression is capable of providing a complete statistical analysis of the stochastic relationships among random variables.

Moreover, quantile regression method has the important property that it is robust to distributional assumptions. This property is inherited from the robustness property of the ordinary sample quantiles. Quantile estimation is only influenced by the local behavior of the conditional distribution of the response variable near the specified quantile. As a result, the estimated conditional quantile function is not sensitive to outlier observations. Such a property is specially attractive in financial applications since many financial data such as

[9] Observe both the similarities to and the differences from the classical ARCH specification introduced by Engle (1982).

IBOVESPA returns are usually heavy-tailed and thus are not (conditional) normally distributed

2.4.1 Quantile Regression

As we stated above, the idea of quantile regression provides a natural way of estimating Value at Risk. Quantile regression was introduced by Koenker and Basset (1978) and has received a lot of attention in econometrics research in the past two decades. To introduce quantile regression, let Y be a random variable with distribution function $F(y)$, the τ-th quantile of Y is defined by

$$Q_Y(\tau) = \inf\{y | F(y) \geq \tau\}. \tag{11}$$

Similarly, if we have a random sample $\{y_1, y_2, ..., y_n\}$ from the distribution F, the τ-th sample quantile is:

$$\hat{Q}_y(\tau) = \inf\left\{y | \hat{F}(y) \geq \tau\right\}, \tag{12}$$

where $\hat{F}$ is the empirical distribution function of the random sample. This sample quantile may be found by solving the minimization problem:

$$\min_{b \in \mathbb{R}} \left[\sum_{t \in \{t: y_t \geq b\}} \tau |y_t - b| + \sum_{t \in \{t: y_t < b\}} (1-\tau) |y_t - b| \right]. \tag{13}$$

Generalizing, if we consider the model:

$$y_t = x_t' b + \eta_t, \tag{14}$$

where x_t is a k x 1 vector of regressors including an intercept term. Then, conditional on the regressor x_t, the τ-th quantile of y:

$$Q_Y(\tau | x_t) = \inf\{y | F(y | x_t) \geq \tau\}, \tag{15}$$

is a linear function of x_t:

$$b_1 + x_{2t}' b_2 + ... + x_{kt}' b_k + F_\eta^{-1}(\tau), \tag{16}$$

where $F_\varepsilon(\cdot)$ is the cumulative distributional function of the residual. The τ-th conditional quantile of y can be estimated by an analogue of equation (13):

$$\hat{Q}_Y(\tau | x_t) = x_t' \hat{b}(\tau), \tag{17}$$

where

$$\hat{b}(\tau) = \arg\min_{b \in \mathbb{R}^k} \left[\sum_{t \in \{t: y_t \geq x_t' b\}} \tau |y_t - x_t' b| + \sum_{t \in \{t: y_t < x_t' b\}} (1-\tau) |y_t - x_t' b| \right] \tag{18}$$

is called the quantile regression. As a special case, the least absolute deviation (LAD) estimator (or l_1 regression) is the median regression, i.e., the quantile regression for $\tau = 0.5$ [10].

[10] For more on quantile regression, see Koenker (2005).

2.4.2 Estimating the ARCH Quantile VaR

Given equation (9), we denote $(1, |\varepsilon_{t-1}|, ..., |\varepsilon_{t-q}|)'$ as X_t and the corresponding coefficient vector as γ. Then,

$$Q_\varepsilon(\tau|\Omega_{t-1}) = X_t'\alpha(\tau), \tag{19}$$

where

$$\alpha(\tau) = \left(\gamma_0 Q_z(\tau), \gamma_1 Q_z(\tau), ..., \gamma_q Q_z(\tau)\right)', \tag{20}$$

and $Q_z(\tau) = F_z^{-1}(\tau)$ is the quantile function of z. By definition, $VaR_t(\tau)$, the conditional Value-at-Risk at the τ-th quantile is just

$$-VaR_t(\tau) = \mu_t + X_t'\alpha(\tau). \tag{21}$$

So we need to estimate $\widehat{\alpha}(\tau)$[11]. It can be achieved by solving the problem:

$$\widehat{\alpha}(\tau) = \arg\min_{\gamma \in \mathbb{R}^{q+1}} \left[\sum_{t \in \{t: u_t \geq Z_t'\gamma\}} \tau |\varepsilon_t - X_t'\gamma| + \sum_{t \in \{t: u_t < Z_t'\gamma\}} (1-\tau)|\varepsilon_t - X_t'\gamma| \right]. \tag{22}$$

In practice, we can replace ε_t by their OLS estimators $\widehat{\varepsilon}_t = y_t - \widehat{\mu}_t$. For example, if $\mu_t = \alpha_o - \alpha_1 y_{t-1}$, then $\widehat{\varepsilon}_t = y_t - \widehat{\alpha}_o - \widehat{\alpha}_1 y_{t-1}$, where $\widehat{\alpha}_o$ and $\widehat{\alpha}_1$ are estimated by OLS. Under mild regularity conditions, Koenker and Zhao (1996) show that $\widehat{\alpha}(\tau)$ estimated based on $\widehat{\varepsilon}_t$ is still a consistent estimator of $\alpha(\tau)$.

Before we move to the empirical section, it is important to mention that the comparison of different VaR methodologies depends on the specification of μ_t (the conditional mean) and the specifications of the conditional volatility. In the empirical example that follows, we consider $\mu_t = \alpha_o - \alpha_1 y_{t-1}$[12]. As for the number of lags present in the definition of ε_t in equation (9), we follow Wu and Xiao (2002) and use a Wald test to determine the optimal lag choice[13]. As for the order of the GARCH and APARCH models, we follow Enders (2003, pp 136) and use adjusted information criteria.

3 Monte Carlo Simulations

The objective of this section is to compare the four aforementioned VaR methodologies. We perform Monte Carlo Simulations with 1000 replications for each Data Generating Process (DGP) and 1250 observations for each generated time series. We use a rolling window of 250 observations to estimate the parameters of the four methodologies and forecast the $VaR(1\%)$ associated to the 251^{st} observation. The result is a 1-day-ahead VaR time series, one for each methodology. At the end, we will have1001 forecast observations for each methodology.

[11] Remember that $\widehat{\mu}_t$ is estimated by OLS.

[12] First-order serial correlation in returns is not necessarily at odds with the efficient market hypothesis. See Campbell et al. (1997) for a detailed discussion.

[13] As all the codes used in this paper, the R code that computes the p-values of the Wald tests in this general-to-specific modelling strategy with $k = 1$ is available upon request.

We decide for a 250-observation window because it is the number of observations required to compute the multiplication factor F_t in the capital charge formula and because it is approximately 1 year, a reasonable time to be used by the banks, containing enough information for the parameters estimation, without loosing so many observations. To find the violations, we need to compare the last 1000 observations of the generated series with the first 1000 observations of the VaR forecasts[14]. The choice for the $VaR\,(1\%)$ is due to regulatory purpose.

The DGPs used in this experiment are

$$y_t = 0.5y_{t-1} + \varepsilon_t, \tag{23}$$

$$\varepsilon_t = \sigma_t z_t, \tag{24}$$

$$\sigma_t^2 = 1 + 0.5\varepsilon_{t-1}^2, \tag{25}$$

where z_t are independent and identically distributed fundamental innovations, $z_t \sim i.i.d.$ There are five different innovation distributions and, therefore 5 DGPs that are described below:

DGP	Distribution of z_t
1	$N\,(0,1)$
2	$t_{(3)}$
3	$\chi^2_{(1)} - \delta_{(1)}$
4	$\delta_{(2)} - Gamma\,(2,1)$
5	$\chi^2_{(1)} I_{\{\nu_t \leq 0.2\}} + \left(\chi^2_{(1)} + \delta_{(-4)}\right) I_{\{0.2<\nu_t \leq 0.8\}} + \delta_{(-4)} I_{\{\nu_t > 0.8\}}$

, (26)

where $\delta_{(x_0)}$ is the Dirac's Delta density, which distribution $F_{\delta_{(x_0)}}(x)$ which is given by

$$F_{\delta_{(x_0)}}(x) = \begin{cases} 1, & \text{if } x \geq x_0 \\ 0, & \text{if } x < x_0 \end{cases}, \tag{27}$$

$I_{\{.\}}$ is an indicator function that values 1 if the condition inside the braces is true and 0 otherwise, and ν_t is an independent and identically distributed standard uniform distribution, $\nu_t \sim U\,[0,1]$.

Therefore, DGP_1 corresponds to the standard gaussian one. In the DGP_2, z_t are drawn from student-t distribution with 3 degrees of freedom. The distribution of z_t in DGP_3 is no longer symmetric. The distribution of z_t in DGP_4 and DGP_5 are nonstandard, and they are considered to verify the robustness of VaR methodologies against distributional misspecification.

3.1 Computational Details

We use R and Ox to conduct this experiment. The former is an open source computer-programming language, hence not only it can be freely downloaded from the Internet[15] but also its source codes under a GNU General Public

[14] Observe that the 1001-th observation is the VaR forecast for a day that there are no more observation in the sample (in the returns time series).

[15] www.r-project.org

License (GPL). This availability keeps R always updated to the most recent techniques in Statistics, Econometrics and Computer Science. The latter can be freely download from the Internet[16] for research purpose.

The time series are generated in R because its default[17] Random Number Generator (RNG) is the Mersenne-Twister (see Matsumoto and Nishimura, 1998), an impressive RNG with period $2^{19937}-1$ and equidistribution in 623 consecutive dimensions (over the whole period). The RNGs available in Ox are the Modified Park and Muller (see Park and Muller, 1988, and for Box-Muller Transformation, see Box and Muller, 1958), the Marsaglia-Multicarry (see Marsaglia and Zaman, 1994, and Marsaglia, 1997) and the L'Ecuyer (see L'Ecuyer, 1999), with approximate periods of 2^{32}, 2^{60} and 2^{113}, respectively. Only the last one is not yet implemented in R, but the user can supply it as well.

This Monte Carlo experiment is extremely computational intensive. For each observation in the VaR forecast, there are three likelihood maximizations - RiskMetrics, Gaussian GARCH(1,1) and Skewed Student-t APARCH(1,1) - with 250 observations (the window length) each. The third maximization occurs in a 7-dimensional hyperplane within a 8-dimensional space (5 parameters for the APARCH(1,1) specification and 2 parameters for the Skewed Student-t distribution). The R is supposed to take some months to conclude all the Monte Carlo, even in our server with 4 Intel Pentium IV Xeon at 2.8 GHz, a 4 GB RAM and a 100 GB SCSI Hard Disk running Linux Debian as Operating System. R is not so fast since it is an interpreted language: the interpreter executes the code line by line, so the user can enter a single line and see the results, which makes it more interactive and user-friendly. Ox is one order of magnitude faster than R since it is a compiled language: the compiler analyses the code as a whole, really optimizing it before executing it, which makes it much faster in large computations.

However, the ARCH(1) Quantile VaR must be estimated in R because the *quantreg* package for R, version 3.82, May 15, 2005, developed mostly by Roger Koenker himself, is very complete and operational[18]. Thus, we proceed as follows: R generates the time series, then it calls Ox to estimate the first three VaR methodologies[19]. Next, Ox returns these VaR forecasts to R that estimates the ARCH(1) Quantile VaR, computes the descriptive statistics and saves the results in the Hard Disk. R then generates another time series and the next replication begins. Using this hybrid solution (Ox and R), all the Monte Carlo experiment takes a couple of weeks. Every written code, for both R and Ox, used in this paper are available upon request.

[16] www.doornik.com

[17] Alternatively, the user can select one of the eight RNGs available, or to supply another one.

[18] Ox has also a code called *rq* to, at least, estimate quantile regression, but it is quite incomplete. It was written by a Roger Koenker's student, Daniel Morillo, but it has been abandoned in its version 1.0, August 1999.

[19] Our Ox code uses some function from the package G@RCH 4.0, by Laurent and Peters (see Laurent and Peters, 2005), for Ox.

3.2 Results

For each replication (with 1000 daily forecasts), the ideal number of violations of a $VaR\,(1\%)$ is 10, but there are replications with more violations and there are replications with less violations. Hence, we shall analyze the distribution of violations. Since there are 1000 replications, such a distribution of violations will have 1000 points (each point represents the number of violations that occurred at a trajectory). Recall that there are four VaR methodologies labelled as $VaR\ i$, $i = 1, 2, 3, 4$. Hence, VaR 1, VaR 2, VaR 3, and VaR 4 correspond to the RiskMetrics, GARCH(1,1), APARCH(1,1), and ARCH(1) Quantile VaR methodologies, respectively. We assess the performance of each VaR methodology under the 5 aforementioned DGPs. Table 1 and 2 presents location and scale parameter estimates of the various distributions of violations

Table 1. Distributions of violations: mean, bias and variance

Methodology	DGP_1	DGP_2	DGP_3	DGP_4	DGP_5
Mean					
VaR 1	19.4	19.8	19.8	19.7	20.0
VaR 2	12.7	13.2	12.9	13.2	13.4
VaR 3	12.8	13.5	13.3	13.5	13.5
VaR 4	14.7	14.7	14.6	14.8	14.6
Bias					
VaR 1	9.4	9.8	9.8	9.7	10.0
VaR 2	2.7	3.2	2.9	3.2	3.4
VaR 3	2.8	3.5	3.3	3.5	3.5
VaR 4	4.7	4.7	4.6	4.8	4.6
Variance					
VaR 1	62.1	72.0	72.5	66.6	69.0
VaR 2	88.1	105.4	103.3	102.7	102.1
VaR 3	80.5	102.2	100.1	93.6	98.7
VaR 4	7.4	7.5	7.5	7.3	8.1

(28)

On one hand, we notice in Table 1 that all four methodologies present positive biases. The RiskMetrics methodology is the most biased and the Gaussian GARCH(1,1) has the least bias. However, as the distribution of z_t becomes different from the Gaussian one, the non-robust methods (VaR 1, VaR 2, and VaR 3) tend to exhibit larger bias. This does not happen to the robust ARCH(1) Quantile method, which exhibits a very stable bias across different innovation distributions. On the other hand, Table 1 also shows that the variance is much higher (one or two order of magnitude higher) in the first three methodologies than in the ARCH(1) Quantile.

We compute in Table 2 the range of the distribution of violations, i.e., the difference between the maximum and the minimum amount of violations. We notice that the fourth methodology has the lowest range. Indeed, its maximum value never exceeds 27 violations, which can be considered a good performance in a $VaR\,(1\%)$. The non-robust methodologies have maximum number of violations at least three times as large as the ARCH(1) Quantile qmethod. This excess dispersion invalidates the first three VaR methodologies, since they jeopardize the bank or institution that use them to compute VaR measures. It is not acceptable for a measure of risk be too *risky*, in the sense that its probability of having trajectories with a large number of violations is too high, as a bank may go belly-up if this trajectory is the true (realized) one.

Table 2. Distributions of violations: minimum, maximum and range.

Methodology	DGP_1	DGP_2	DGP_3	DGP_4	DGP_5
Minimum Value (Min)					
VaR 1	6	7	8	7	6
VaR 2	2	3	3	1	3
VaR 3	3	3	2	2	2
VaR 4	7	7	6	7	3
Maximum Value (Max)					
VaR 1	76	75	80	69	75
VaR 2	83	68	79	66	77
VaR 3	75	71	78	65	66
VaR 4	26	25	25	27	23
Range = Max - Min					
VaR 1	70	68	72	62	69
VaR 2	81	65	76	65	74
VaR 3	72	68	76	63	64
VaR 4	19	18	19	20	20

. (29)

The ARCH(1) Quantile VaR exhibits the second greatest bias, but displays the lowest variance and range. To assess the trade-off between bias and variance, we adopt the Mean Squared Error (MSE), abiding by the formula (see Engle and Manganelli, 2001)

$$MSE\left(\hat{X}\right) := \frac{1}{1000}\sum_{i=1}^{1000}\left(X_i - 10\right)^2, \qquad (30)$$

where X_i is the number of violation in the i-th replication, 1000 is the total number of replications and 10 is the ideal number of violations, for a $VaR(1\%)$, at each replication. It can be shown that the $MSE\left(\hat{X}\right) = Var\left(\hat{X}\right) + Bias\left(\hat{X}\right)^2$, where $Bias\left(\hat{X}\right) = \bar{X} - 10$ and $\bar{X} = \frac{1}{1000}\sum_{i=1}^{1000} X_i$. The bias and the MSE are show in the next table:

Table 3. Distributions of violations: Mean Squared Error

Methodology	DGP_1	DGP_2	DGP_3	DGP_4	DGP_5
Mean Squared Error					
VaR 1	150.7	168.8	168.0	160.9	169.3
VaR 2	95.3	115.4	110.9	112.7	113.5
VaR 3	88.1	114.6	110.9	106.1	111.0
VaR 4	29.6	29.5	29.0	30.3	29.2

. (31)

Notice that, even considering the bias, the ARCH(1) Quantile VaR methodology has, by far, the lowest MSE. Notice that the MSE of the non-robust methods tends to increase as we consider innovations distributions that are different from the Gaussian one. This unpleasant property does not appear in the robust ARCH(1) method since we can see that its MSE is pretty stable across different distributions. For completeness, we show in Table 4 estimates of skewness and excess kurtosis of the distributions of violations.

Table 4. Distributions of violations: skewness and excess kurtosis

Methodology	DGP_1	DGP_2	DGP_3	DGP_4	DGP_5
Skewness					
VaR 1	2.8	2.6	2.7	2.6	2.6
VaR 2	3.2	2.8	3.0	2.8	2.8
VaR 3	3.3	3.0	3.1	2.8	2.7
VaR 4	0.3	0.3	0.3	0.3	0.1
Kurtosis					
VaR 1	10.1	8.2	8.5	7.8	8.3
VaR 2	11.6	7.8	8.9	7.5	7.8
VaR 3	11.9	9.2	10.4	7.7	7.2
VaR 4	0.2	0.1	0.2	0.6	0.1

, (32)

We observe that the non-robust VaR methodologies give rise to distributions of violations that are skewed to the right and possess excess kurtosis. Again, the robust ARCH(1) Quantile method gives rise to an well-behaved distribution of violations, with almost none skewness nor excess kurtosis.

In sum, our Monte Carlo experiment suggests that the robust method dominates the other methods, since the former gives rise to a distribution of violations that present very low MSE, almost none skewness and excess kurtosis. More importantly, these nice properties are preserved over a wide range of innovation distributions. This result is expected because the robust method does not depend on distributional assumption.

4 An Empirical Illustration

4.1 The Data

We perform an empirical exercise using daily returns, in US dollars, of the Brazilian São Paulo Stock Exchange Index (IBOVESPA) from 08/07/1996 to

24/03/2000, summing up 920 observations. We choose this sample because we want to check the performance of each VaR methodology during periods of market turmoil, when market drops are followed by further drops or rebounds. Indeed, the above sample period covers the Korean Crisis in 1997, the Russian crisis in 1999, and the blast of the technology-stock market bubble in 2000. Figure 1 displays the behavior of the IBOVESPA return over the above sample period.

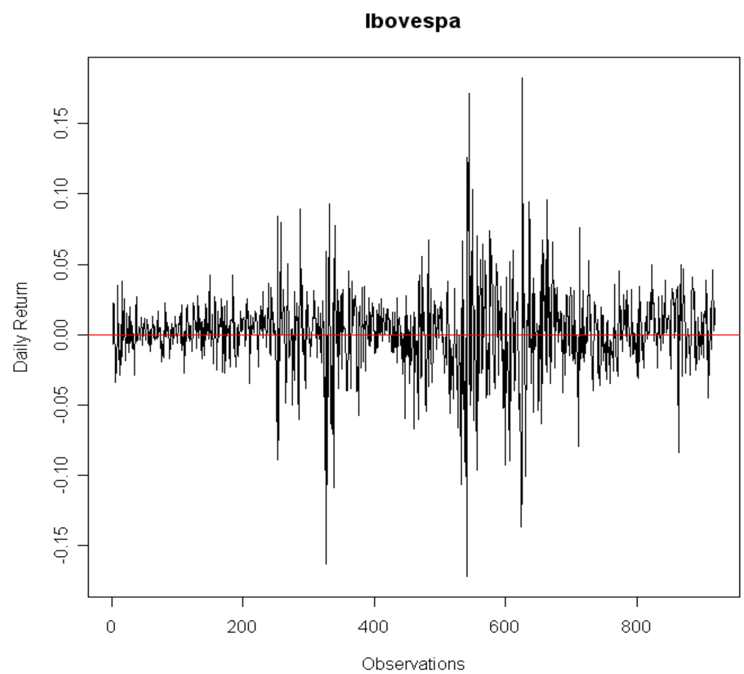

Figure 1

The above picture shows that during the considered sample period, market drops are followed by further drops or rebounds, characterizing what we called market turmoil. It is well known that GARCH volatility models tend to predict implausible high VaR during periods of market turmoil. This happens because GARCH models treat both large positive and large negative return shocks as indicators of high volatility, which only large negative return shocks indicate higher Value-at-Risk. In other words, volatility and VaR are not the same thing, and this is not taken into account by the non-robust GARCH volatility models[20]. In contrast, the robust ARCH Quantile, while predicting higher

[20]It is true, however, that the APARCH model assigns different weights to negative and positive shocks which helps avoiding estimation of high VaRs during periods of market stress.

volatility in the ARCH component, assigns a much larger weight to a big negative return shock than to a big positive return shock and, thus, we expect that the resulting estimated VaRs are closer to reality during periods of market turmoil (see similar argument in Wu and Xiao, 2002).

We next examine the distribution of the Ibovespa return. It was argued in this paper that, unlike the non-robust methods, the ARCH Quantile method has no need to specify the distribution of the innovation process, z_t. The importance of this robustness aspect is revealed by the Quantile-Quantile plots (QQ plot). Recall that the QQ plot graphs the quantiles of the observed variable (IBOVESPA return) against the quantiles of a specified distribution. Hence, if the returns are distributed according to that specified distribution, then the points in the QQ-plots should lie alongside a straight line. The next two graphs show the QQ plots against all the 5 innovation distributions used in our Monte Carlo experiment

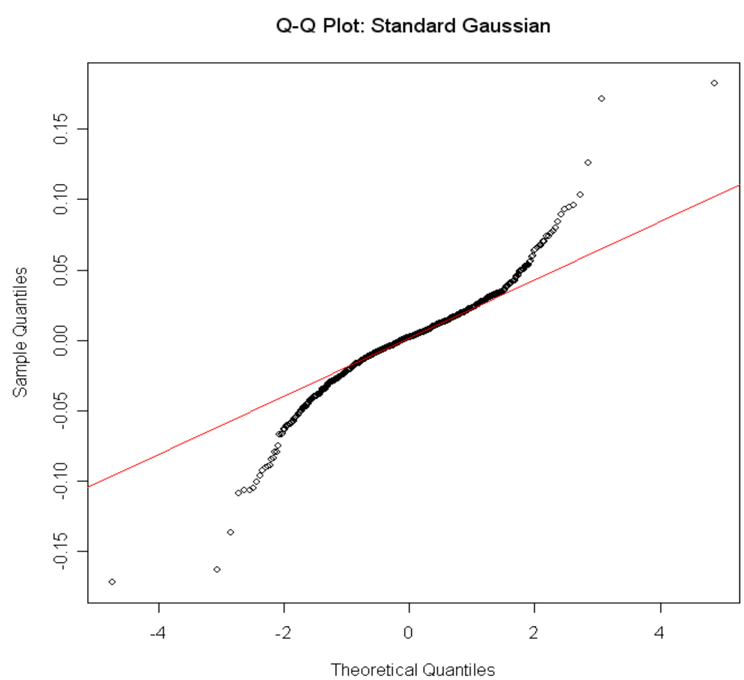

Figure 2

Figure 2 indicates that it is primarily large negative and positive shocks that are driving the departure from normality. In other words, the tail behavior of the distribution of the IBOVESPA return are far different from the tail behavior of a Gaussian distribution. Figure 3 exhibits the QQ plots against the 4 remaining distributions. It seems that the student-t distribution with 3 degrees of freedom approximates the data distribution reasonably well, but there still be extreme positive and negative observations that lie off the straight line suggesting that the student-t distribution with 3 degrees of freedom does not fit the tail of the data distribution pretty well, what is particularly bad for risk measures. Figures 3 also shows that the data distribution departures from the other 3 distributions considered in our Monte Carlo experiment, but they fit the data distribution even worse than the previous two distributions.

Thus, given this uncertainty about the specification of the innovation distribution, how could we go about computing Value-at-Risk correctly? A natural answer to it is to use a method robust against distribution misspecification, such as the method based on the ARCH Quantile model.

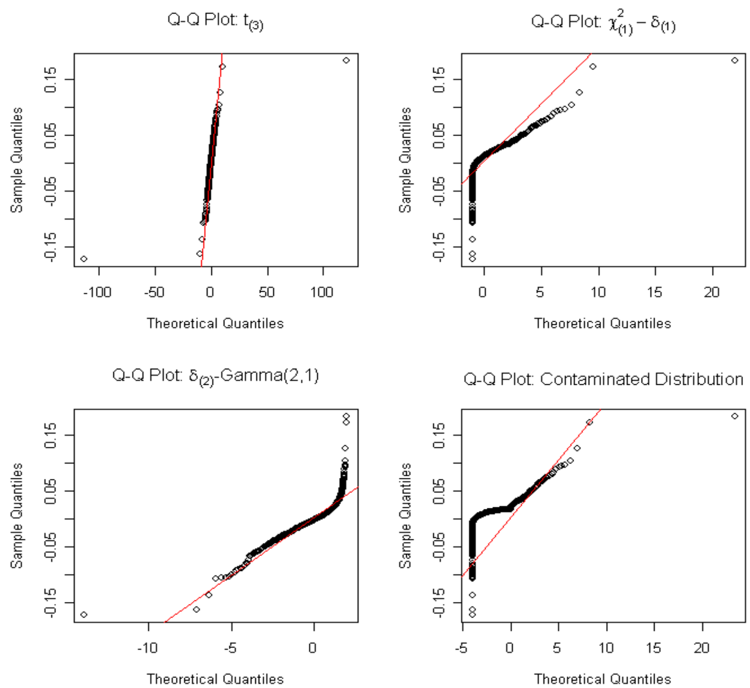

Figure 3

4.2 The Estimated VaRs

Notice that there is no VaR forecast for the first 250 observations, due to the first temporal window. Hence, there are 670 1-day-ahead forecast observations for each $VaR\,(1\%)$ methodology, ranging from 11/07/1997 to 24/03/2000. Since it is a 1% Value-at-Risk, we expect about 7 violations. Point estimates of the violations are reported in the table below

4.3 Backtest

The Unconditional Coverage backtest was proposed by Kupiec (1995). Under the null hypothesis that $P\left[r_t < -VaR_t(\tau)\,|I_{t-1}\right] = \tau$, $\forall t$, i.e., that the probability of occurrence of a violation is indeed τ, the number of violations, X, in a given time span, T, follows a binomial distribution: $X \sim Binomial\,(T, \tau)$. Define $\hat{\tau} := \frac{X}{T}$. Then, the likelihood ratio test statistic[21] is

$$LR_{uc} = 2\ln\left(\frac{\hat{\tau}^X (1-\hat{\tau})^{T-X}}{\tau^X (1-\tau)^{T-X}}\right). \tag{33}$$

Under the null hypothesis that $\tau = \hat{\tau}$, $LR_{uc} \sim \chi^2_{(1)}$.

The next table shows the results of the Unconditional Coverage Test:

Methodology	VaR 1	VaR 2	VaR 3	VaR 4
Unconditional Coverage Test				
Number of Violations	14	12	13	11
Test Statistic LR_{uc}	6.115232	3.429641	4.693915	2.335267
P-Value	0.013402	0.064036	0.030270	0.126473

(34)

Note that the number of violations in the RiskMetrics methodology is the greatest, while the ARCH(1) Quantile presents the greatest p-value in the Unconditional Coverage test, that is to say, we do not reject, even at a 10% significance level, the null hypothesis that the conditional probability of occurrence of a violation in this 1% Value-at-Risk estimated time series is indeed 1%.

5 Conclusion

We perform a Monte Carlo experimet to compare four different Value-at-Risk methodologies, RiskMetrics, Gaussian GARCH(1,1), Generalized Student-t APARCH(1,1), and ARCH(1) Quantile, under five different data generating processes. The ARCH(1) Quantile methodology does not assume any distribution for the returns, and this robustness is shown to avoid trajectories with too many violations. The number of violations tends to be higher in the non-robust methodologies when the distribution differs from the Gaussian one.

We also perform an empirical exercise applying the four Value-at-Risk methodologies to daily return of the IBOVESPA (measured in dollar values) in a period of market turmoil (1996-2000), when happens the Korean crisis, the Russian crisis and the blast of the technology-stock market bubble. We display that, again, the ARCH(1) Quantile methodology dominates the non-robust methodologies, in the sense that it presents the least number of violations.

[21] This is the uniformly most powerful (UMP) test for a given T.

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