Welfare Gains from Monetary Commitment in a Model of the Euro-Area*

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January 30, 2006

Abstract

This paper sets out first, to quantify the stabilization gains from commitment in terms of household welfare and second, to examine how commitment to an optimal or approximately optimal rule can be sustained as an equilibrium in which reneging hardly ever occurs. We utilize an influential empirical micro-founded DSGE model, the Euro-Area model of Smets and Wouters (2003), and a quadratic approximation of the representative household's utility as the welfare criterion. We impose the effect of a lower zero nominal interest rate bound. In contrast with previous studies we find substantial stabilization gains from commitment – as much as a $5 - 6\%$ permanent increase in consumption. We also find that a simple optimized commitment rule with the nominal interest rate responding to current inflation and the real wage closely mimics the optimal rule.

JEL Classification: E52, E37, E58

Keywords: Monetary rules, commitment, discretion, welfare gains.

*Draft paper not to be quoted without permission. We acknowledge financial support for this research from the ESRC, project no. RES-000-23-1126 and from the European Central Bank's Research Visitors Programme for Paul Levine. Thanks are owing to the ECB for this hospitality and to numerous resident and visiting researchers for stimulating discussions. Views expressed in this paper do not necessarily reflect those of the ECB.

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1 Introduction

Following the pioneering contributions of Kydland and Prescott (1977) and Barro and Gordon (1983), the credibility problem associated with monetary policy has stimulated a huge academic literature that has been influential with policymakers. The central message underlying these contributions are the existence of significant macroeconomic gains, in some sense, from 'enhancing credibility' through formal commitment to a policy rule or through institutional arrangements for central banks such as independence, transparency, and forward-looking inflation targets, that achieve the same outcome.

In the essentially static model used in these seminal papers and in much of the huge literature they inspired, the loss associated with a lack of credibility takes the form of a long-run inflationary bias. For a dynamic models of the New Keynesian genre, such as the DSGE Euro-Area model employed in this paper, the influential review of Clarida *et al.* (1999) emphasizes the *stabilization gains* from commitment which exist whether or not there is a long-run inflationary bias. But what are the size of these stabilization gains from commitment? If they are small then the credibility problem is solely concerned with the credibility of long-run low inflation.

This paper sets out first, to quantify the stabilization gains from commitment in terms of household welfare and second, to examine how commitment to an optimal or approximately optimal rule can be sustained as an equilibrium in which reneging hardly ever occurs. Previous work has addressed these issues (see, for example, Currie and Levine (1993) and Dennis and Söderström (2005)), but only in the context of econometric models without micro-foundations and using an ad hoc loss function, or both. The credibility issue only arises because the decisions of consumers and firms are forward looking and depends on expectations of future policy. In the earlier generation of econometric models lacking micro-foundations, many aspects of such forward-looking behaviour were lacking and therefore important sources of time-inconsistency was missing. Although for simple New Keynesian models a quadratic approximation of the representative consumer's coincides with the standard ad hoc loss that penalizes variances of the output gap and inflation, in more developed DSGE models this is far from the case. By utilizing an influential empirical micro-founded DSGE model, the Euro-Area model of Smets and Wouters (2003), and using a quadratic approximation of the representative household's utility as the welfare criterion, we remedy these deficiencies of earlier estimates of commitment gains.

An important consideration when addressing the gains from commitment is the existence of a zero lower nominal interest rate bound. In an important contribution to the credibility literature, Adam and Billi (2005) show that ignoring this constraint on the setting of the nominal interest rate can result in considerably underestimating the stabilization gain from commitment. The reason for this is that under discretion the monetary authority cannot make credible promises about future policy. For a given setting of future interest rates the volatility of inflation is driven up by the expectations of the private sector that the monetary authority will re-optimize in the future. This means that to achieve a given low volatility of inflation the lower bound is reached more often under discretion than under commitment. Unlike these authors we stay within a more tractable linear-quadratic (LQ) framework[1] and follow Woodford (2003), chapter 6, in approximating the effects of a lower bound by imposing the requirement that the interest rate volatility in a discretionary equilibrium is small enough to ensure that the violations of the zero lower bound are very infrequent.

The rest of the paper is organized as follows. Section 2 begins by using a simple New Keynesian model to show analytically how a stabilization bias arises in models with structural dynamics. It goes on to generalize the treatment to any linear DSGE model with a quadratic loss function and to also take into account the lower interest rate bound. We derive closed-form expressions for welfare under optimal commitment, discretion and simple commitment rules and use these to derive a 'no-deviation condition' for commitment to exist as an equilibrium in which reneging on commitment takes place very infrequently.

Section 3 sets out a version of the Smets-Wouter model (henceforth SW) with two additional features: we incorporate external habit formation in labour supply in addition to that in consumption and we add distortionary taxes. A linearization of the model about a zero-interest steady state and a quadratic approximation of the representative household's utility (provided in section 5) sets up the optimization problem facing the monetary authority in the required LQ framework. Section 4 estimates the modified SW model and provides a comparison with the original.

In section 5 we examine a relatively neglected aspect of New Keynesian models that arises with the inclusion of external habit in consumption, namely that the natural rate of output and employment can actually be *below* the social optimum making the inflationary bias negative and taxes welfare-enhancing. In section 6 we address the central questions in the paper: how big are stabilization gains and how can the fully optimal commitment rule or a simple approximation be sustained as an equilibrium given the time-inconsistency problem? Section 7 concludes the paper.

[1]A LQ framework is convenient for a number of reasons: it allows closed-form expressions for the welfare loss under optimal commitment, discretion and simple commitment rules that enable us to study the incentives to renege on commitment. A linear framework allows us to characterize saddle-path stability and the possible indeterminacy of simple rules. A further benefit is that for very simple models we can express the all these rules as analytic solutions.

2 The Time Inconsistency Problem

2.1 The Stabilization Bias in Two Simple DSGE Models

We first demonstrate how a stabilization bias in addition to the better known long-run inflationary bias can arise using two simple and now very standard DSGE models. The first popularized notably by Clarida *et al.* (1999) and Woodford (2003) is 'New Keynesian' and takes the form.

$$\pi_t = \beta E_t \pi_{t+1} + \lambda y_t + u_t \quad (1)$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(r_t - E_t \pi_{t+1}) \quad (2)$$

In (1) and (2), π_t is the inflation rate, β is the private sector's discount factor, $E_t(\cdot)$ is the expectations operator and y_t is output measured relative to its flexi-price value, the 'output gap', which equals consumption measured relative to its flexi-price value in this closed-economy model without capital stock or government spending. (1) is derived as a linearized form of Calvo staggered price setting about a zero-inflation steady state and (2) is a linearized Euler equation with nominal interest rate r_t and a risk aversion parameter σ. u_t is a zero-mean shock to marginal costs. All variables are expressed as deviations about the steady state, π_t and r_t as absolute deviations, and y_t as a proportional deviation.

The second model simply replaces (1) with a 'New Classical Phillips Curve' (see Woodford (2003), chapter 3):

$$\pi_t = E_{t-1}\pi_t + \lambda y_t + u_t \quad (3)$$

This aggregate supply curve can be derived by assuming some firms fix prices one period in advance and others can adjust immediately.

Kydland and Prescott (1977) and Barro and Gordon (1983) employed the 'New Classical Phillips Curve' (3) and showed that a time-inconsistency or credibility problem in monetary policy arises when the monetary authority at time 0 sets a state-contingent inflation rate π_t to minimize a loss function

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t \left[w_y (y_t - k)^2 + \pi_t^2 \right] \right] \quad (4)$$

Having set the inflation rule, the Euler equation (2) then determines the nominal interest rate that will put the economy on a path with the implied interest rate trajectory. The constant k in the loss function arises because the steady state is inefficient owing to imperfect competition and other distortions. For this simple, essentially static model of the economy (it is really SGE rather than DSGE), optimal rules must take the form of a constant deterministic component plus a stochastic shock-contingent component. These rules depend on whether the policymaker can commit, or she exercises discretion and engages in period-by-period optimization. The standard results in these two cases are respectively:

$$\pi_t = \frac{w_y}{w_y + \lambda^2} u_t = \pi^C(u_t) \tag{5}$$

$$\pi_t = \frac{w_u k}{\lambda} + \frac{w_y}{w_y + \lambda^2} u_t = \pi^D(u_t) \tag{6}$$

Thus the optimal inflation rule with commitment, $\pi^C(u_t)$ consists of zero average inflation plus a shock-contingent component which sees inflation raised (i.e., monetary policy relaxed) in the face of a negative supply shock. The discretionary policy, $\pi^D(u_t)$, can be implemented as a rule with the *same* shock-continent component as the ex ante optimal rule. The *only* difference is now that it includes a non-zero average inflation or *inflationary* bias equal to $\frac{w_u k}{\lambda}$ which renders the rule time-consistent. The credibility or 'time-inconsistency' problem, first raised by Kydland and Prescott, was simply how to *eliminate the inflationary bias whilst retaining the flexibility to deal with exogenous shocks.*

We have established that there are no stabilization gains from commitment in a model economy characterized by the New Classical Philips Curve. This is no the case when we move to the New Keynesian Phillips Curve, (1). Then using general optimization procedures described below in section 2.2.2 and in Appendix A, (5) and (6) now become

$$\pi_t^C = \pi_t^C(u_t, u_{t-1}) = \delta \pi_{t-1}^C + \delta(u_t - u_{t-1}) \tag{7}$$

$$\pi_t^D = \pi_t^D(u_t) = \frac{w_u k}{\lambda} + \frac{w_u}{w_u + \lambda^2} u_t \tag{8}$$

where $\delta = \frac{1-\sqrt{1-4\beta b^2}}{2b\beta}$.[2] Comparing these two sets of results we see that the discretionary rule is unchanged, but the commitment rule now is a rule responding to past shocks (i.e., is a *rule with memory*) and therefore the stabilization component of the commitment rules now differs from that of the discretionary rule. Since the commitment rule is the ex ante optimal policy it follows that there are also now *stabilization gains from commitment.* The time-inconsistency problem facing the monetary authority in a New Keynesian economic environment now becomes the elimination of the inflationary bias whilst retaining the flexibility to deal with exogenous shocks *in an optimal way.*

2.2 The Stabilization Bias in General DSGE Models

The stabilization bias arose in our simple DSGE model by replacing a Phillips Curve based on one-period ahead price contracts with one based on Staggered Calvo-type price

[2] See also Clarida *et al.* (1999)

setting. In the DSGE model of the Euro Area presented in the next section there are a number of additional mechanisms that create price, wage and output persistence. The model also incorporates capital accumulation. All these features add *structural dynamics* to the model and these, together will forward-looking behaviour involving consumption, investment, price-setting and wage- setting add further sources of stabilization gains from commitment.

To examine this further a general linear state-space model

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ E_t\mathsf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Br_t + C\epsilon_{t+1} \tag{9}$$

$$\mathsf{o}_t = E \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{10}$$

where z_t is a $(n-m) \times 1$ vector of predetermined variables at time t with z_0 given, x_t, is a $m \times 1$ vector of non-predetermined variables and o_t is a vector of outputs. A, C and A are fixed matrices and ϵ_t as a vector of random zero-mean shocks. Rational expectations are formed assuming an information set $\{z_s, x_s, \epsilon_s\}$, $s \leq t$, the model and the monetary rule. The linearized euro-area model set out in the next section can be expressed in this form where z_t consists of exogenous shocks, lags in non-predetermined and output variables and capital stock; x_t consists of current inflation, the real wage, investment, Tobin's Q, consumption and flexi-price outcomes for the latter two variables, and outputs o_t consist of marginal costs, the marginal rate of substitution for consumption and leisure, the cost of capital, labour supply, output, flexi-price outcomes, the output gap and other target variables for the monetary authority. Let $\mathsf{s}_t = M[\mathsf{z}_t^T \mathsf{x}_t^T]^T$ be the vector of such target variables. For both ad hoc and welfare-based loss function discussed below, the inter-temporal loss function (4) generalizes to

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t L_t \right] \tag{11}$$

where the single-period loss function is given by $L_t = \mathsf{s}_t^T Q_1 \mathsf{s}_t = \mathsf{y}_t^T Q \mathsf{y}_t$ where $\mathsf{y}_t = [\mathsf{z}_t^T \mathsf{x}_t^T]^T$ and $Q = M^T Q_1 M$.

2.2.1 Imposing a Lower Interest Rate Bound Constraint

In the absence of a lower bound constraint on the nominal interest rate the policymaker's optimization problem is to minimize (11) subject to (9) and (10). Then complete stabilization of the output gap and inflation is possible, but if shocks are their variances are sufficiently large this will lead to a large nominal interest rate variability and the possibility of it becoming negative. To rule out this possibility and to remain within the convenient LQ framework of this paper, we follow Woodford (2003), chapter 6, and approximate the lower interest rate bound effect by introducing constraints of the form

$$E_0\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t r_t\right] \geq 0 \tag{12}$$

$$E_0\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t r_t^2\right] \leq K\left[E_0\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t r_t\right]\right]^2 \tag{13}$$

Then Woodford shows that the effect of these extra constraints is to follow the same optimization as before except that the single period loss function is replaced with

$$L_t = \mathsf{y}_t^T Q \mathsf{y}_t + w_r(r_t - r^*)^2 \tag{14}$$

where $w_r > 0$ if (13) binds (which we assume) and $r^* > 0$ if monetary transactions frictions are negligible, but $r^* < 0$ is possible otherwise (i.e., the interest rate must be lower that that necessary to keep inflation zero in the steady state). In what follows we put $r^* = 0$.

2.2.2 Commitment Versus Discretion

To derive the ex ante optimal policy with commitment following Currie and Levine (1993) we maximize the the Lagrangian

$$\mathcal{L}_0 = E_0\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t\left[\mathsf{y}_t^T Q\mathsf{y}_t + w_r r_t^2) + \mathsf{p}_{t+1}(A\mathsf{y}_t + Br_t - \mathsf{y}_{t+1})\right]\right] \tag{15}$$

with respect to $\{r_t\}$, $\{\mathsf{y}_t\}$ and the row vector of costate variables, p_t, given z_0. From Appendix A where more details are provided, this leads to a optimal rule of the form

$$r_t = D\begin{bmatrix}\mathsf{z}_t\\ \mathsf{p}_{2t}\end{bmatrix} \tag{16}$$

where

$$\begin{bmatrix}\mathsf{z}_{t+1}\\ \mathsf{p}_{2t+1}\end{bmatrix} = H\begin{bmatrix}\mathsf{z}_t\\ \mathsf{p}_{2t}\end{bmatrix} \tag{17}$$

and the optimality condition[3] at time $t = 0$ imposes $\mathsf{p}_{20} = 0$. In (16) and (17) $\mathsf{p}_t^T = \left[\mathsf{p}_{1t}^T\,\mathsf{p}_{2t}^T\right]$ is partitioned so that p_{1t}, the co-state vector associated with the predetermined variables, is of dimension $(n-m)\times 1$ and p_{2t}, the co-state vector associated with the non-predetermined variables, is of dimension $m \times 1$. The loss function is given by

$$\Omega_t^{OP} = -(1-\beta)\mathrm{tr}\left(N_{11}\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right) + N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T\right) \tag{18}$$

[3]Optimality from a 'timeless perspective' imposes a different condition at time $t = 0$ (see Appendix A.1.2), but this has no bearing on the stochastic component of policy, the focus of this paper.

where $Z_t = \mathsf{z}_t\mathsf{z}_t^T$, $\Sigma = \text{cov}(\epsilon_t)$,

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{19}$$

and S is the solution to the steady-state Ricatti equation. In (19) matrices S and N are partitioned conformably with $\mathsf{y}_t = [\mathsf{z}_t^T \mathsf{x}_t^T]^T$ so that S_{11} for instance has dimensions $(n-m) \times (n-m)$.

Note that in order to achieve optimality the policy-maker sets $\mathsf{p}_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $\mathsf{p}_{2t} = 0$. It can be shown that N_{22} is negative definite, so the incentive to renege exists at all points along the trajectory of the optimal policy. This essentially is the time-inconsistency problem facing stabilization policy in a model with structural dynamics.

To evaluate the discretionary (time-consistent) policy we write the expected loss Ω_t at time t as

$$\Omega_t = E_t\left[(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}L_\tau\right] = (1-\beta)(\mathsf{y}_t^T Q\mathsf{y}_t + w_r r_t^2) + \beta\Omega_{t+1} \tag{20}$$

The dynamic programming solution then seeks a stationary solution of the form $r_t = -Fz_t$, $\Omega_t = \mathsf{z}^T S\mathsf{z}$ and $\mathsf{x} = -N\mathsf{z}$ where matrices S and N are now of dimensions $(n-m) \times (n-m)$ and $m \times (n-m)$ respectively, in which Ω_t is minimized at time t subject to (1) in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.[4] Both the instrument r_t and the forward-looking variables x_t are now proportional to the predetermined component of the state-vector z_t and the equilibrium we seek is therefore *Markov Perfect*. In Appendix A we set out an iterative process for F_t, N_t, and S_t starting with some initial values. If the process converges to stationary values independent of these initial values,[5] F, N and S say, then the time-consistent feedback rule is $r_t = -Fz_t$ with loss at time t given by

$$\Omega_t^{TC} = (1-\beta)\text{tr}\left(S\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{21}$$

2.2.3 Simple Commitment Rules

There are two problems with the optimal commitment rule. First in all but very simple models it is extremely complex, with the interest rate feeding back at time t on the full state vector z_t and all past realizations of z_t back to the initiation of the rule at $t = 0$.[6]

[4]See Currie and Levine (1993) and Söderlind (1999).

[5]Indeed we find this is the case in the results reported in the paper.

[6]See Appendix A.1.1.

The second problem arises from this latter feature: although the optimal commitment rule achieves a low volatility of key target variables such as inflation it does so at the expense of a larger volatility of the interest rate than for discretion. For these reasons we seek to mimic the optimal commitment rule with simpler rules of the form

$$r_t = D\mathsf{y}_t = D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{22}$$

where D is constrained to be sparse in some specified way. In Appendix A we show that the loss at time t is given by

$$\Omega_t^{SIM} = (1-\beta)\mathrm{tr}\left(V\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{23}$$

where $V = V(D)$ satisfies a Lyapanov equation. Ω_t^{SIM} can now be minimized with respect to D to give an *optimized simple rule* of the form (22) with $D = D^*$. A very important feature of optimized simple rules is that unlike their optimal commitment or optimal discretionary counterparts they are *not certainty equivalent.* In fact if the rule is designed at time $t = 0$ then $D^* = D^*((Z_0 + \frac{\beta}{1-\beta}\Sigma))$ and so dependent on the displacement z_0 at time $t = 0$ and on the covariance matrix of innovations $\Sigma = \mathrm{cov}(\epsilon_{\mathsf{t}})$. From non-certainty equivalence it follows that if the simple rule were to be re-designed at ant time $t > 0$, since the re-optimized D^* will then depend on Z_t the new rule will differ from that at $t = 0$. This feature is true in models with or without rational forward-looking behaviour and it implies that *simple rules are time-inconsistent even in non-RE models.*

2.3 Sustaining the Commitment Outcome as An Equilibrium

Suppose that there are two types of monetary policymaker, a ‘strong’ type who likes to commit and perceives substantial costs from reneging on any such commitment, and a ‘weak’ type who optimizes in an opportunistic fashion on a period-by-period basis. The ‘strong type’ could be a policymaker with a modified loss function as in Rogoff (1985), Walsh (1995), Svensson and Woodford (1999), though for the case of Rogoff-delegation the outcome is second-best. In a complete information setting, these types would be observed by the public and the strong type would pursue the optimal commitment monetary rule or a simple approximation, and the weak type would pursue the discretionary policy. We assume there is uncertainty about the type of policymaker and the weak type is trying to build a reputation for commitment. The game is now one of incomplete information and we examine the possibility that commitment rules can be sustained as a Perfect Bayesian Equilibrium.

Consider the following strategy profile.

1. A strong type follows an optimal or simple commitment rule.

2. In period t a weak type acts as strong and follows the commitment rule with probability $1 - q_t$, if it has acted strong ($q_t = 0$) in all previous periods. Otherwise it has revealed its type and must pursue the discretionary rule.

3. Let ρ_t the probability assigned by the private sector to the event that the policymaker is of the strong type. We can regard ρ_t as a measure of reputation. At the beginning of period 0 the private sector chooses its prior $\rho_0 > 0$. In period t the private sector receives the 'signal' consisting of the regulated price or the inflation set by the policymaker. At the end of the period it updates the probability ρ_t, using Bayes rule, and then forms expectations of the next period's regulated price or inflation rate.

In principle there are three types of equilibria to these games. If both strong and weak governments send the same message (i.e. implement the same interest rate) we have a *pooling equilibrium*. If they send different messages this gives a *separating equilibrium*. If one or more players randomizes with a mixed strategy we have a *hybrid equilibrium*. Thus in the above game, $q_t = 0$ gives a pooling equilibrium, $q_t = 1$ a separating equilibrium and $0 < q_t < 1$ a hybrid equilibrium. *If $q_t = 0$ is a Perfect Bayesian Equilibrium to this game, we have solved the time-inconsistency problem.*

To show $q_t = 0$, it is sufficient to show that given beliefs by the private sector there is no incentive for a weak government to ever deviate from acting strong. To show this we must compare the welfare if the policymaker continues with the optimal commitment policy at time t with that if it reneges, re-optimizes and then suffers a loss of reputation.

Consider the optimal commitment rule first. At time t the single period loss function is $L(\mathsf{z}_t, \mathsf{p}_{2t})$ and the intertemporal loss function can be written

$$\Omega_t^{OP}(\mathsf{z}_t, \mathsf{p}_{2t}) = (1 - \beta)L(\mathsf{z}_t, \mathsf{p}_{2t}) + \beta\Omega_{t+1}^{OP}(\mathsf{z}_{t+1}^{OP}, \mathsf{p}_{2,t+1}) \tag{24}$$

where $(\mathsf{z}_{t+1}^{OP}, \mathsf{p}_{2,t+1})$ is given by (17). If the policymaker re-optimizes at time t the corresponding loss is

$$\Omega_t^{R}(\mathsf{z}_t, 0) = (1 - \beta)L(\mathsf{z}_t, 0) + \beta\Omega_{t+1}^{TC}(\mathsf{z}_{t+1}^{R}) \tag{25}$$

where from (17) we now have that $\mathsf{z}_{t+1}^{R} = H_{11}\mathsf{z}_t$.

The condition for a perfect Bayesian pooling equilibrium is that for all realizations of shocks to $(\mathsf{z}_t, \mathsf{p}_{2t})$ at every time t the no-deviation condition

$$\Omega_t^{OP}(\mathsf{z}_t, \mathsf{p}_{2t}) < \Omega_t^{R}(\mathsf{z}_t, 0) \tag{26}$$

holds. If the condition holds, then the weak authority always mimics the strong authority and follows the commitment rule thus sustaining average zero inflation coupled with optimal stabilization.

Using (24), (31), (18) and (21) the no-deviation condition (26) can be written as

$$\begin{aligned} L(\mathsf{z}_t, \mathsf{p}_{2t}) - L(\mathsf{z}_t, 0) \quad &- \quad \beta E_t \left[\mathrm{tr}(S Z^R_{t+1} + N_{11} Z^{OP}_{t+1} + N_{22} \mathsf{p}^T_{2,t+1} \mathsf{p}_{2,t+1}) \right] \\ &< \quad \frac{\beta^2}{1-\beta} tr((S + N_{11})\Sigma) \end{aligned} \tag{27}$$

The first term on the left-hand-side of (27) is the *single-period* gain from reneging and putting $p_{2t} = 0$. The second term on the left-hand-side of (27) are the possible *one-off stabilization gains* since the state of the economy after reneging reflected in z^R_{t+1} will be closer to the long-run than that along the commitment policy reflected in $\mathsf{z}^{OP}_{t+1}, \mathsf{p}_{2t+1}$. These two terms together constitute the *temptation* to renege. Since $tr((S + N_{11}) > 0$, the right-hand-side is always positive and constitutes the penalty in the shape of the stabilization loss when dealing with future shocks following a loss of reputation.

If the time-period is small then the single-period gains are also relatively small and we can treat the loss of reputation as if it were instantaneous. Then the no deviation condition becomes simply

$$\Omega^{OP}_t < \Omega^{TC}_t \tag{28}$$

for all realizations of exogenous stochastic shocks. From (18) and (21) this condition becomes

$$\mathrm{tr}((N_{11} + S)(Z_t + \frac{\beta}{1-\beta}\Sigma) > -\mathrm{tr}(N_{22} p_{2t} p^T_{2t}) \tag{29}$$

Note that both $-N_{22}$ and $(N_{11} + S)$ are positive see Currie and Levine (1993), chapter 5 for a continuous-time analysis on which the discrete-time analysis here is based). It follows that both the right-hand-side and the left-hand side are positive, so (29) is not automatically satisfied.

Finally we consider the no-deviation condition for a simple rule. Consider the optimized rule set at $t = 0$ which we take to be the steady state. Then $Z_0 = 0$ and $D^* = D^*(\Sigma)$. If the policymaker continues with this policy then in state z_t at time t the welfare loss is given by

$$\Omega^{SIM}_t(\mathsf{z}_t, D^*) = (1-\beta) L(\mathsf{z}_t, D^*) + \beta \Omega^{SIM}_{t+1}(\mathsf{z}^{SIM}_{t+1}, D^*) \tag{30}$$

where $\mathsf{z}^{SIM}_{t+1} = H(D^*)\mathsf{z}_t$ and H is given in Appendix A. If the policy deviates she does to a re-optimized reneging rule $D^R = D^R((Z_t + \frac{\beta}{1-\beta}\Sigma))$ which now depends on the realization of z_t at Time t. The welfare loss is then

$$\Omega^R_t(\mathsf{z}_t, D^R) = (1-\beta) L(\mathsf{z}_t, D^R) + \beta \Omega^{TC}_{t+1}(\mathsf{z}^R_{t+1}) \tag{31}$$

where $\mathsf{z}_{t+1}^{SIM} = H(D^R)\mathsf{z}_t$. Proceeding as before the no-deviation condition now becomes

$$\begin{aligned} L(\mathsf{z}_t, D^*) - L(\mathsf{z}_t, D^R) &- \beta E_t \left[\mathrm{tr}(SZ_{t+1}^R - VZ_{t+1}^{SIM}\right] \\ &< \frac{\beta^2}{1-\beta} tr((S-V)\Sigma) \end{aligned} \tag{32}$$

The intuition behind this condition is very similar to that of (27). In these three no-deviation conditions (27), (29) and (32), since Z_t or p_{2t} are unbounded stochastic variables there will inevitably be *some* realizations for which they are *not* satisfied. In other words the Bayesian equilibrium must be of the mixed-strategy type with $q_t > 0$. What we must now show that q_t is very small so we will only experience very occasional losses of reputation. This we examine in section 6.6.

3 The Model

3.1 The Smets-Wouters Model

The Smets-Wouters (SW) model in an extended version of the standard New-Keynesian DSGE closed-economy model with sticky prices and wages estimated by Bayesian techniques. The model features three types of agents: households, firms and the monetary policy authority. Households maximize a utility function with two arguments (goods and leisure) over an infinite horizon. Consumption appears in the utility function relative to a time-varying external habit-formation variable. Labor is differentiated over households, so that there is some monopoly power over wages, which results in an explicit wage equation and allows for the introduction of sticky nominal Calvo-type wages contracts (Calvo (1983)). Households also rent capital services to firms and decide how much capital to accumulate given certain capital adjustment costs. Firms produce differentiated goods, decide on labor and capital inputs, and set Calvo-type price contracts. Wage and price setting is augmented by the assumption that those prices and wages that can not be freely set are partially indexed to past inflation. Prices are therefore set as a function of current and expected real marginal cost, but are also influenced by past inflation. Real marginal cost depends on wages and the rental rate of capital. The short-term nominal interest rate is the instrument of monetary policy. The stochastic behavior of the model is driven by ten exogenous shocks: five shocks arising from technology and preferences, three cost-push shocks and two monetary-policy shocks. Consistent with the DSGE set up, potential output is defined as the level of output that would prevail under flexible prices and wages in the absence of cost-push shocks.

We incorporate three modifications to the SW model. First, we introduce external habit in labour supply in addition to consumption. As we will see this has an important bearing on the existence of an inflationary bias. This is also true of our second modification, the addition of distortionary taxes. Finally we adopt a *ratio* rather than the *difference* form for habit more usual in the literature (and in SW). The reason for this is a technical one discussed in section 2.5 below. In fact, as we show in section 2.7, the two forms of habit are observationally equivalent.

3.2 Households

There are ν households of which a representative household r maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U_{C,t} \left[\frac{(C_t(r)/H_{C,t})^{1-\sigma}}{1-\sigma} + U_{M,t} \frac{\left(\frac{M_t(r)}{P_t}\right)^{1-\varphi}}{1-\varphi} - U_{L,t} \frac{(L_t(r)/H_{L,t})^{1+\phi}}{1+\phi} + u(G_t) \right] \tag{33}$$

where E_t is the expectations operator indicating expectations formed at time t, β is the household's discount factor, $U_{C,t}$, $U_{M,t}$ and $U_{L,t}$ are preference shocks $C_t(r)$ is an index of consumption, $L_t(r)$ are hours worked, $H_{C,t}$ and $H_{L,t}$ represents the habit, or desire not to differ too much from other households, and we choose $H_{C,t} = C_{t-1}^{h_C}$, where $C_t = \frac{1}{\nu}\sum_{r=1}^{\nu} C_t(r) \simeq \int_0^1 C_t(r)dr$ (normalizing the household number ν at unity) is the average consumption index, $H_{L,t} = L_{t-1}^{h_L}$, where L_t is aggregate labour supply defined after (37) below and $h_C, h_L \in [0,1)$. When $h_C = 0$, $\sigma > 1$ is the risk aversion parameter (or the inverse of the intertemporal elasticity of substitution)[7]. $M_t(r)$ are end-of-period nominal money balances and $u(G_t)$ is the utility from exogenous real government spending G_t.

The representative household r must obey a budget constraint:

$$P_t(C_t(r) + I(r)) + E_t(P_{D,t+1}D_{t+1}(r))M_t(r) = (1-T_t)P_tY_t(r) + D_t(r) + M_{t-1}(r) \tag{34}$$

where P_t is a price index, $I_t(r)$ is investment, $D_{t+1}(r)$ is a random variable denoting the payoff of the portfolio $D_t(r)$, purchased at time t, and $P_{D,t+1}$ is the period-t price of an asset that pays one unit of currency in a particular state of period $t+1$ divided by the probability of an occurrence of that state given information available in period t. The nominal rate of return on bonds (the nominal interest rate), R_t, is then given by the relation $\mathcal{E}_t(P_{D,t+1}) = \frac{1}{1+R_t}$. Finally T_t is a tax on total income, $P_tY_t(r)$ where the latter given by

$$P_tY_t(r) = W_t(r)L_t(r) + (R_{K,t}Z_t(r) - \Psi(Z_t(r))P_tK_{t-1}(r) + \Gamma_t(r) \tag{35}$$

where $W_t(r)$ is the wage rate, $R_{K,t}$ is the real return on beginning-of period capital stock K_{t-1} owned by households, $Z_t(r) \in [0,1]$ is the degree of capital utilization with costs

[7]When $h_C \neq 0$, σ is merely an index of the curvature of the utility function.

$P_t\Psi(Z_t(r))K_{t-1}(r)$ where Ψ', $\Psi'' > 0$, and $\Gamma_t(r)$ is income from dividends derived from the imperfectively competitive intermediate firms plus the net cash inflow from state-contingent securities. We first consider the case of flexible wages and introduce wage stickiness at a later stage.

Capital accumulation is given by

$$K_t(r) = (1-\delta)K_{t-1}(r) + (1 - S(X_t(r)))\, I_t(r) \tag{36}$$

where $X_t(r) = \frac{U_{I,t}I_t(r)}{I_{t-1}(r)}$, $U_{I,t}$ is a shock to investment costs and we assume the investment adjustment cost function, $S(\cdot)$, has the properties $S(1) = S'(1) = 0$.

As set below, intermediate firms employ differentiated labour with a constant CES technology with elasticity of supply η. Then the demand for each consumer's labor is given by

$$L_t(r) = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} L_t \tag{37}$$

where $W_t = \left[\int_0^1 W_t(r)^{1-\eta}dr\right]^{\frac{1}{1-\eta}}$ is an average wage index and $L_t = \left[\int_0^1 L_t(r)^{\frac{\eta-1}{\eta}}dr\right]^{\frac{\eta}{\eta-1}}$ is average employment.

Household r chooses $\{C_t(r)\}$, $\{M_t(r)\}$, $\{K_t(r)\}$, $\{Z(r)\}$ and $\{L_t(r)\}$ (or $\{W_t(r)\}$) to maximize (C.2) subject to (34)– (37), taking habit $H_{C,t}$, $H_{N,t}$, $R_{K,t}$ and prices and as given. Imposing symmetry on households (so that $C_t(r) = C_t$, etc) and putting $P_{B,t} = \frac{1}{1+R_t}$ yields, by the standard Lagrangian method, the results:

$$\begin{aligned}
1 &= \beta(1+R_t)E_t\left[\frac{MU^C_{t+1}}{MU^C_t}\frac{P_t}{P_{t+1}}\right] \\
&= \beta(1+R_t)E_t\left[\left(\frac{U_{C,t+1}C_{t+1}^{-\sigma}H_{C,t+1}^{\sigma-1}}{U_{C,t}C_t^{-\sigma}H_{C,t}^{\sigma-1}}\right)\frac{P_t}{P_{t+1}}\right] && (38)\\
\left(\frac{M_t}{P_t}\right)^{-\varphi} &= \frac{(C_t - H_{C,t})^{-\sigma}}{\chi P_t}\left[\frac{R_t}{1+R_t}\right] && (39)\\
Q_t &= E_t\left[\beta\left(\frac{C_{t+1}^{-\sigma}H_{C,t+1}^{\sigma-1}}{C_t^{-\sigma}H_{C,t}^{\sigma-1}}\right)(Q_{t+1}(1-\delta) + R_{K,t+1}Z_t - \Psi(Z_{t+1}))\right] && (40)\\
1 &= Q_t[1-(1-S(X_t) - S'(X_t)X_t] \\
&+ \beta E_tQ_{t+1}\left(\frac{C_{t+1}^{-\sigma}H_{C,t+1}^{\sigma-1}}{C_t^{-\sigma}H_{C,t}^{\sigma-1}}\right)S'(X_t)\frac{U_{I,t+1}I_{t+1}^2}{I_t^2} && (41)\\
R_{K,t} &= \Psi'(Z_t) && (42)\\
\frac{W_t(1-T_t)}{P_t} &= -\frac{1}{(1-\frac{1}{\eta})}\frac{MU^L_t}{MU^C_t} = \frac{U_{L,t}}{(1-\frac{1}{\eta})}L_t^{\phi}H_{L,t}^{-1-\phi}C_t^{\sigma}H_{C,t}^{1-\sigma} && (43)
\end{aligned}$$

where MU^C_t and MU^L_t are the marginal utilities of consumption and work respectively. (38) is the familiar Keynes-Ramsey rule adapted to take into account of the consumption habit. In (39), the demand for money balances depends positively on consumption relative to habit and negatively on the nominal interest rate. Given the central bank's setting of the latter, (39) is completely recursive to the rest of the system describing our macro-model.In (40) and (41), Q_t is the real value of capital (Tobin's Q) and these conditions describe optimal investment behaviour. (42) describes optimal capacity utilization. In (43) the real disposable wage is proportional to the marginal rate of substitution between consumption and leisure (equal to -$\frac{MU_t^L}{MU_t^C}$) this constant of proportionality reflecting the market power of households arising from their monopolistic supply of a differentiated factor input with elasticity η.

3.3 Firms

Competitive final goods firms use a continuum of intermediate goods according to a constant returns CES technology to produce aggregate output

$$Y_t = \left(\int_0^1 Y_t(f)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)} \tag{44}$$

where ζ is the elasticity of substitution. This implies a set of demand equations for each intermediate good m with price $P_t(f)$ of the form

$$Y_t(f) = \left(\frac{P_t(f)}{P_t} \right)^{-\zeta} Y_t \tag{45}$$

where $P_t = \left[\int_0^1 P_t(f)^{1-\zeta} dm \right]^{\frac{1}{1-\zeta}}$. P_t is an aggregate intermediate price index, but since final goods firms are competitive and the only inputs are intermediate goods, it is also the domestic price level.

In the intermediate goods sector each good m is produced by a single firm m using differentiated labour and capital with another constant returns Cobb-Douglas-CES technology:

$$Y_t(f) = A_t(Z_t(f)K_{t-1}(f))^{\alpha} L_t(f)^{1-\alpha} - F \tag{46}$$

where F are fixed costs and

$$L_t(f) = \left(\int_0^1 L_t(r,f)^{(\eta-1)/\eta} dr \right)^{\eta/(\eta-1)} \tag{47}$$

is an index of differentiated labour types used by the firm, where $L_t(r,f)$ is the labour input of type r by firm m, and A_t is an exogenous shock capturing shifts to trend total factor productivity (TFP) in this sector. Minimizing costs $P_t R_{K,t} Z_t(f) K_{t-1}(f) + W_t L_t(f)$ gives

$$\frac{W_t L_t(f)}{Z_t P_t R_{K,t} K_{t-1}(f)} = \frac{1-\alpha}{\alpha} \tag{48}$$

Then aggregating over firms and denoting $\int_0^1 L_t(r,f)dm = L_t(r)$ leads to the demand for labor as shown in (37). In an equilibrium of equal households and firms, all wages adjust to the same level W_t and it follows that $Y_t = A_t(Z_tK_{t-1})^\alpha L_t^{1-\alpha}$. For later analysis we need the firm's minimum real marginal cost:

$$MC_t = \frac{\left(\frac{W_t}{P_t}\right)^{1-\alpha} R_K^\alpha}{A_t}\alpha^{-\alpha}(1-\alpha)^{1-\alpha} \tag{49}$$

Turning to price-setting, we assume that there is a probability of $1-\xi_P$ at each period that the price of each good mf is set optimally to $P_t^0(f)$. If the price is not re-optimized, then it is indexed to last period's aggregate producer price inflation.[8] With indexation parameter $\gamma_P \geq 0$, this implies that successive prices with no re-optimization are given by $P_t^0(f)$, $P_t^0(f)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_P}$, $P_t^0(mf)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_P}$, ... For each producer m the objective is at time t to choose $P_t^0(f)$ to maximize discounted profits

$$E_t\sum_{k=0}^{\infty}\xi_H^k D_{t+k}Y_{t+k}(f)\left[P_t^0(f)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_P} - P_{t+k}\mathrm{MC}_{t+k}\right] \tag{50}$$

where D_{t+k} is the discount factor over the interval $[t, t+k]$, subject to a common downward sloping demand given by (45). The solution to this is

$$E_t\sum_{k=0}^{\infty}\xi_P^k D_{t+k}Y_{t+k}(f)\left[P_t^0(f)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_P} - \frac{\zeta}{(\zeta-1)}P_{t+k}\mathrm{MC}_{t+k}\right] = 0 \tag{51}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi_P\left(P_t\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_P}\right)^{1-\zeta} + (1-\xi_P)(P_{t+1}^0(f))^{1-\zeta} \tag{52}$$

3.4 Staggered Wage-Setting

We introduce wage stickiness in an analogous way. There is a probability $1-\xi_W$ that the wage rate of a household of type r is set optimally at $W_t^0(r)$. If the wage is not re-optimized then it is indexed to last period's CPI inflation. With a wage indexation parameter γ_W the wage rate trajectory with no re-optimization is given by $W_t^0(r)$, $W_t^0(r)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_W}$, $W_t^0(r)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_W}$, $\cdots$. The household of type r at time t then chooses $W_t^0(r)$ to maximize

$$E_t\sum_{k=0}^{\infty}(\xi_W\beta)^k\left[W_t^0(r)(1-T_{t+k})\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_W}L_{t+k}(r)\frac{MU_{t+k}^C(r)}{P_{t+k}} - \frac{\left(L_{t+k}(r)/L_{t+k-1}^{h_L}\right)^{1+\phi}}{1+\phi}\right] \tag{53}$$

[8]Thus we can interpret $\frac{1}{1-\xi_P}$ as the average duration for which prices are left unchanged.

where $MU_t^C(r)$ is the marginal utility of consumption income and $L_t(r)$ is given by (37). The first-order condition for this problem is

$$E_t \sum_{k=0}^{\infty} (\xi_W \beta)^k \; W_{t+k}^{\eta} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{-\gamma_W \eta} L_{t+k} \Lambda_{t+k}(r) \Big[W_t^0(r)(1 - T_{t+k}) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_W}$$

$$- \frac{\eta}{(\eta - 1)} \frac{L_{t+k}^{\phi}(r)/L_{t+k-1}^{h_L(1+\phi)}}{\Lambda_{t+k}(r)} \Big] = 0 \tag{54}$$

Note that as $\xi_W \rightarrow 0$ and wages become perfectly flexible, only the first term in the summation in (B.10) counts and we then have the result (43) obtained previously. By analogy with (52), by the law of large numbers the evolution of the wage index is given by

$$W_{t+1}^{1-\eta} = \xi_W \left(W_t \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_W} \right)^{1-\eta} + (1 - \xi_W)(W_{t+1}^0(r))^{1-\eta} \tag{55}$$

3.5 Equilibrium

In equilibrium, goods markets, money markets and the bond market all clear. Equating the supply and demand of the consumer good we obtain

$$Y_t \quad = \quad A_t (Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} = C_t + G_t + I_t + \Psi(Z_t) K_{t-1} \tag{56}$$

Assuming the same tax rate levied on all income (wage income plus dividends) a balanced budget government budget constraint

$$P_t G_t = P_t T_t Y_t + M_t - M_{t-1} \tag{57}$$

completes the model. Given the interest rate R_t (expressed later in terms of an optimal or IFB rule) the money supply is fixed by the central banks to accommodate money demand. By Walras' Law we can dispense with the bond market equilibrium condition and therefore the government budget constraint that determines taxes τ_t. Then the equilibrium is defined at $t = 0$ by stochastic processes C_t, B_t, I_t, P_t, M_t, L_t, K_t, Z_t, $R_{K,t}$, W_t, Y_t, given past price indices and exogenous shocks and government spending processes.

In what follows we will assume a 'cashless economy' version of the model in which both seigniorage in (57) and the utility contribution of money balances in (C.2) are negligible. Then given the nominal interest rate, our chosen monetary instrument, we can dispense altogether with the money demand relationship (39).

3.6 Zero-Inflation Steady State

A deterministic zero-inflation steady state, denoted by variables without the time subscripts, $E_{t-1}(U_{C,t}) = 1$ and $E_{t-1}(U_{N,t}) = \kappa$ is given by

$$\begin{aligned}
1 &= \beta(1+R) && (58)\\
Q &= \beta(Q(1-\delta) + R_K Z - \Psi(Z)) && (59)\\
R_K &= \Psi'(Z) && (60)\\
Q &= 1 && (61)\\
\frac{W(1-T)}{P} &= \frac{\kappa}{1-\frac{1}{\eta}} L^{\phi - h_L(1+\phi)} C^{\sigma + h_C(1-\sigma)} && (62)\\
Y &= AK^{\alpha}L^{1-\alpha} - F && (63)\\
\frac{WL}{PZR_K K} &= \frac{1-\alpha}{\alpha} && (64)\\
1 &= \frac{P^0}{P} = \frac{\mathrm{MC}}{\left(1-\frac{1}{\zeta}\right)} && (65)\\
\mathrm{MC} &= \frac{\left(\frac{W}{P}\right)^{1-\alpha} R_K^{\alpha}}{A} \alpha^{-\alpha}(1-\alpha)^{1-\alpha} && (66)\\
Y &= C + (\delta + \Psi(Z))K + G && (67)\\
T &= \frac{G}{Y} && (68)
\end{aligned}$$

giving us 11 equations to determine R, Z, Q, $\frac{W}{P}$, L, K, R_K, MC, C, Y, and T. In our cashless economy the price level is indeterminate.

The solution for steady state values decomposes into a number of independent calculations. First from (58) the natural rate of interest is given by

$$R = \frac{1}{\beta} - 1 \tag{69}$$

which is therefore pinned down by the household's discount factor. Equations (59) to (61) give

$$1 = \beta[1 - \delta + Z\Psi'(Z) - \Psi(Z)] \tag{70}$$

which determines steady state capacity utilization. SW further assume that $Z = 1$ and $\Psi(1) = 0$ so that (70) and (60) imply that $R_K = \Psi'(Z) = \frac{1}{\beta} - 1 + \delta = R + \delta$ meaning that perfect capital market conditions apply in the deterministic steady state.

From (64) to (66) a little algebra yields the capital-labour ratio and the real wage $\frac{W}{P}$:

$$\begin{aligned}
\frac{K}{L} &= \left[A\left(1-\frac{1}{\zeta}\right)\frac{\alpha}{R_K}\right]^{\frac{1}{1-\alpha}} && (71)\\
\frac{W}{P} &= \frac{(1-\alpha)R_K}{\alpha}\frac{K}{L} && (72)
\end{aligned}$$

Then (62), (63) and (67) give

$$
\begin{aligned}
& Y^{(1+\phi)(1-h_L)+(\sigma-1)(1-h_C)} \left(1 - \frac{\delta \frac{K}{L}}{\left(A\left(\frac{K}{L}\right)^{\alpha} - F\right)} - \frac{G}{Y}\right)^{\sigma + h_C(1-\sigma)} \\
= \; & \frac{(1-\alpha)(1-T)\left(1-\frac{1}{\eta}\right)\left(1-\frac{1}{\zeta}\right) A^{(1+\phi)(1-h_L)} \left(\frac{K}{L}\right)^{\alpha(1+\phi)(1-h_L)}}{\kappa} \qquad (73)
\end{aligned}
$$

For $\sigma > 1$, which empirically proves to be the case, and given he capital-labour ratio $\frac{K}{L}$, the left-hand-side of (73) is increasing in output Y. Thus given government spending as a proportion of GDP, and given the capital-labour ratio, the natural rate of output falls as market power in output and labour markets increases (with decreases in ζ and η respectively), distortionary taxes T increase. Market power, distortionary taxes and external habit persistence are all sources of inefficiency, but as we shall now see, they do not impact in the same direction.

3.7 Linearization about the Zero-Inflation Steady State

We now linearize about the deterministic zero-inflation steady state. Define all lower case variables as proportional deviations from this baseline steady state except for rates of change which are absolute deviations.[9] Then the linearization takes the form:

$$
\begin{aligned}
(\sigma + (\sigma-1)h_C)c_t &= (\sigma-1)h_C c_{t-1} + \sigma E_t c_{t+1} \\
&\quad - (r_t - E_t\pi_{t+1} + E_t u_{C,t+1} - u_{C,t}) \qquad (74) \\
q_t &= \beta(1-\delta)E_t q_{t+1} - (r_t - E_t\pi_{t+1}) + \beta Z E_t r_{K,t+1} + \epsilon_{Q,t} \qquad (75) \\
z_t &= \frac{r_{K,t}}{Z\Psi''(Z)} = \frac{\psi}{R_K} r_{K,t} \quad \text{where } \psi = \frac{\Psi'(Z)}{Z\Psi''(Z)} \qquad (76) \\
i_t &= \frac{1}{1+\beta} i_{t-1} + \frac{\beta}{1+\beta} E_t i_{t+1} + \frac{1}{S''(1)(1+\beta)} q_t + \frac{\beta u_{I,t+1} - u_{I,t}}{1+\beta} \qquad (77) \\
\pi_t &= \frac{\beta}{1+\beta\gamma_P} E_t\pi_{t+1} + \frac{\gamma_P}{1+\beta\gamma_P}\pi_{t-1} + \frac{(1-\beta\xi_P)(1-\xi_P)}{(1+\beta\gamma_P)\xi_P} mc_t + \epsilon_{P,t} \qquad (78) \\
k_t &= (1-\delta)k_{t-1} + \delta i_t \qquad (79)
\end{aligned}
$$

[9] That is, for a typical variable X_t, $x_t = \frac{X_t - X}{X} \simeq \log\left(\frac{X_t}{X}\right)$ where X is the baseline steady state. For variables expressing a rate of change over time such as r_t, $x_t = X_t - X$.

$$
\begin{aligned}
mc_t &= (1-\alpha)wr_t + \frac{\alpha}{R_K} r_{K,t} - a_t && (80) \\
wr_t &= \frac{\beta}{1+\beta} E_t wr_{t+1} + \frac{1}{1+\beta} wr_{t-1} + \frac{\beta}{1+\beta} E_t \pi_{t+1} - \frac{1+\beta\gamma_W}{1+\beta}\pi_t + \frac{\gamma_W}{1+\beta}\pi_{t-1} \\
&+ \frac{(1-\beta\xi_W)(1-\xi_W)}{(1+\beta)\xi_W(1+\eta\phi)}(mrs_t - wr_t + t_t) + \epsilon_{W,t} && (81) \\
mrs_t &= \sigma c_t - h_C(\sigma-1)c_{t-1} + \phi l_t - h_L(1+\phi)l_{t-1} + u_{L,t} && (82) \\
l_t &= k_{t-1} + \frac{1}{R_K}(1+\psi)r_{K,t} - wr_t && (83) \\
y_t &= c_y c_t + g_y g_t + i_y i_t + k_y \psi r_{K,t} && (84) \\
y_t &= \phi_F[a_t + \alpha(\frac{\psi}{R_K} r_{K,t} + k_{t-1}) + (1-\alpha)l_t] \quad \text{where } \phi_F = 1 + \frac{F}{Y} && (85) \\
g_t - y_t &= t_t && (86) \\
u_{C,t+1} &= \rho_C u_{C,t} + \epsilon_{C,t+1} && (87) \\
u_{L,t+1} &= \rho_L u_{L,t} + \epsilon_{L,t+1} && (88) \\
u_{I,t+1} &= \rho_I u_{I,t} + \epsilon_{I,t+1} && (89) \\
g_{t+1} &= \rho_g g_t + \epsilon_{g,t+1} && (90) \\
a_{t+1} &= \rho_a a_t + \epsilon_{a,t+1} && (91)
\end{aligned}
$$

where "inefficient cost-push" shocks $\epsilon_{Q,t}$, $\epsilon_{P,t}$ and $\epsilon_{W,t}$ have been added to value of capital, the marginal cost and marginal rate of substitution equations respectively. Variables y_t, c_t, mc_t, $u_{C,t}$, $u_{N,t}$, a_t, g_t are proportional deviations about the steady state. $[\epsilon_{C,t}, \epsilon_{N,t}, \epsilon_{g,t}, \epsilon_{a,t}]$ are i.i.d. disturbances. π_t, t_t, $r_{K,t}$ and r_t are absolute deviations about the steady state.[10]

For later use we require the *output gap* the difference between output for the sticky price model obtained above and output when prices and wages are flexible, $\hat{y}_t$ say. Following SW we also eliminate the inefficient shocks from this target level of output. The latter, obtained by setting $\xi_P = \xi_W = \epsilon_{Q,t+1} = \epsilon_{P,t+1} = \epsilon_{W,t+1} = 0$ in (78) to (84), is in deviation

[10]Note that in the SW model they define $\hat{r}_{K,t} = \frac{r_{K,t}}{R_K}$. Then $z_t = \frac{\Psi'(Z)}{Z\Psi''(Z)}\hat{r}_{K,t} = \psi\hat{r}_{K,t}$. In our set-up $z_t = \frac{\psi}{R_K} r_{K,t}$ has been eliminated.

form given by[11]

$$
\begin{aligned}
(\sigma + (\sigma-1)h_C)\hat{c}_t &= (\sigma-1)h_C\hat{c}_{t-1} + \sigma E_t\hat{c}_{t+1} \\
&- (\hat{r}_t - E_t\hat{\pi}_{t+1} + E_t u_{C,t+1} - u_{C,t}) && (92) \\
\hat{q}_t &= \beta(1-\delta)E_t\hat{q}_{t+1} - (\hat{r}_t - E_t\hat{\pi}_{t+1}) + \beta Z E_t \hat{r}_{K,t+1} && (93) \\
\hat{i}_t &= \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}E_t\hat{i}_{t+1} + \frac{1}{S''(1)(1+\beta)}\hat{q}_t + \frac{\beta u_{I,t+1} + u_{I,t}}{1+\beta} && (94) \\
\hat{k}_t &= (1-\delta)\hat{k}_{t-1} + \delta\hat{i}_t && (95) \\
\widehat{mc}_t &= 0 = (1-\alpha)\hat{wr}_t + \frac{\alpha}{R_K}\hat{r}_{K,t} - a_t && (96) \\
\widehat{mrs}_t &= \widehat{wr}_t - \hat{t}_t = \sigma\hat{c}_t - h_C(\sigma-1)\hat{c}_{t-1} + \phi\hat{l}_t - h_L(1+\phi)\hat{l}_{t-1} + u_{L,t} && (97) \\
\hat{l}_t &= \hat{k}_{t-1} + \frac{1}{R_K}(1+\psi)\hat{r}_{K,t} - \hat{wr}_t && (98) \\
\hat{y}_t &= c_y\hat{c}_t + g_y g_t + i_y\hat{i}_t + k_y\psi\hat{r}_{K,t} && (99) \\
\hat{y}_t &= \phi_F[a_t + \alpha(\frac{\psi}{R_K}\hat{r}_{K,t} + \hat{k}_{t-1}) + (1-\alpha)\hat{l}_t] && (100) \\
g_t &= \hat{t}_t && (101)
\end{aligned}
$$

In this system (92) determines the trajectory for the natural real rate of interest $\hat{r}_t - E_t\hat{\pi}_{t+1}$ given the potential consumption $\hat{c}_t$. Then the remaining equations determine $\hat{q}_t$, $\hat{i}_t$, $\hat{k}_t$, $\hat{r}_{K,t}$, $\hat{wr}_t$, $\hat{l}_t$, $\hat{c}_t$, $\hat{y}_t$ and $\hat{t}_t$, given g_t. Eliminating superfluous variables, we can write the combined sticky and flexi price-wage model in the required state space form (9) and (10), where $\mathsf{z}_t = [u_{C,t}, u_{L,t}, u_{I,t}, a_t, g_t, \epsilon_{Q,t}, \epsilon_{P,t}, \epsilon_{W,t}, c_{t-1}, \hat{c}_{t-1}, c_{t-1}, \hat{c}_{t-1}, l_{t-1}, \hat{l}_{t-1}, k_{t-1}, \hat{k}_{t-1}, r_{t-1}, \pi_{t-1}, wr_{t-1}, l_{t-1}]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c_t, \pi_t, wr_t, i_t, q_t, \hat{i}_t, \hat{q}_t]$, are non-predetermined variables. Thus the output gap, $\hat{y}_t - y_t$ is obtained. Table 1 provides a summary of our notation.

[11] Note that the zero-inflation steady states of the sticky and flexi-price steady states are the same.

π_t	producer price inflation over interval $[t-1,t]$
r_t	nominal interest rate over interval $[t,t+1]$
$wr_t = w_t - p_t$	real wage
mc_t	marginal cost
mrs	marginal rate of substitution between work and consumption
l_t	employment
z_t	capacity utilization
k_t	end-of-period t capital stock
i_t	investment
$r_{K,t}$	return on capital
q_t	Tobin's Q
c_t	consumption
y_t, $\hat{y}_t$	output with sticky prices and flexi-prices
$o_t = y_t - \hat{y}_t$	output gap
t_t	tax rate
$u_{C,t+1} = \rho_a u_{C,t} + \epsilon_{C,t+1}$	AR(1) process for utility preference shock, $u_{C,t}$
$u_{L,t+1} = \rho_a u_{L,t} + \epsilon_{L,t+1}$	AR(1) process for utility preference shock, $u_{L,t}$
$u_{I,t+1} = \rho_a u_{I,t} + \epsilon_{I,t+1}$	AR(1) process for investment cost shock, $u_{I,t}$
$a_{t+1} = \rho_a a_t + \epsilon_{a,t+1}$	AR(1) process for factor productivity shock, a_t
$g_{t+1} = \rho_g g_t + \epsilon_{g,t+1}$	AR(1) process government spending shock, g_t
β	discount parameter
γ_P, γ_W	indexation parameters
h_C, h_L	habit parameters
$1-\xi_P$, $1-\xi_W$	probability of a price, wage re-optimization
σ	risk-aversion parameter
ϕ	disutility of labour supply parameter

Table 1. Summary of Notation (Variables in Deviation Form).

3.8 Habit, Observational Equivalence and Identification of Parameters:

Although it proves convenient for the linearization and quadratic approximation, there is an observational equivalence between the two linearized forms. If we denote the two forms by the superscripts D, R, then the relationship between the key parameters that differ is as follows:

$$\sigma^R = \frac{\sigma^D}{1-h_C^D} \qquad (\sigma^R - 1)h_C^R = \frac{h_C^D\sigma^D}{1-h_C^D} \qquad \phi^R = \frac{\phi^D}{1-h_L^D} \qquad (1+\phi^R)h_L^R = \frac{h_L^D\phi^D}{1-h_L^D} \tag{102}$$

There is an additional identification issue that has been addressed by Smets and Wouters. This relates to the parameter η. This parameter only occurs in the linearization within the context of the marginal utility of labour. Thus the three parameters η, h_L, ϕ can only be obtained from estimation of two parameters. Note that η also occurs within the definition of the steady state of the system, but this is the only equation that can be used for pinning down the parameter κ. This is the reason that Smets and Wouters pin down η in order to be able to identify h_L, ϕ.

4 Estimation

The Bayesian approach itself combines the prior distributions for the individual parameters with the likelihood function to form the posterior density. This posterior density can then be optimized with respect to the model parameters through the use of the Monte-Carlo Markov Chain sampling methods. The model is estimated using the Dynare software, Juillard (2004).[12] Table 13 reports the posterior mean and the 5th and 95th confidence interval of the posterior distribution obtained through the Metropolis-Hasting (MH) sampling algorithm (using 100,000 draws from the posterior and an average acceptance rate of around 0.25) for the various model variants as well as the marginal likelihood (LL). Note, in re-estimating we use identical priors to those used in SW.

In the table we report results for five models: the unaltered SW model, then the SW model without indexing in wages, $\gamma_W = 0$, without indexing in prices, $\gamma_P = 0$, with neither and finally the SW model with habit in labour supply.[13]. From the LL values we can see that the model without any indexing performs the best, followed by the unaltered SW mode, followed by the model with only price indexing with the habit in labour supply model well behind the others. In the following sections we provide results for the straight SW model, but a future version will conduct sensitivity analysis drawing upon all these model variants.

[12]We are grateful to Gregory De Walque and Raf Wouters for providing the SW model in dynare code.

[13]Since we use the SW priors we estimate habit here in *difference* form. Our welfare calculations use the ratio form and the transformations (102) to convert these estimates. A model with tax distortions will be reported in the next version of the paper

	SW	$\gamma_w = 0$	$\gamma_p = 0$	$\gamma_w = \gamma_p = 0$	h_L
ρ_a	0.89 [0.81:0.96]	0.89 [0.82:0.97]	0.87 [0.80:0.95]	0.88 [0.81:0.96]	0.90 [0.83:0.99]
ρ_{pb}	0.84 [0.68:0.99]	0.84 [0.68:0.99]	0.86 [0.71:0.99]	0.86 [0.70:0.99]	0.85 [0.70:0.99]
ρ_b	0.83 [0.77:0.89]	0.83 [0.77:0.90]	0.84 [0.78:0.90]	0.84 [0.77:0.89]	0.81 [0.74:0.89]
ρ_g	0.95 [0.90:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]	0.95 [0.91:0.99]
ρ_l	0.91 [0.84:0.97]	0.93 [0.89:0.98]	0.92 [0.88:0.98]	0.93 [0.89:0.98]	0.86 [0.75:0.97]
ρ_i	0.91 [0.87:0.97]	0.92 [0.86:0.97]	0.92 [0.86:0.97]	0.92 [0.87:0.98]	0.92 [0.86:0.97]
ϕ_i	6.79 [5.08:8.55]	6.70 [5.04:8.44]	6.77 [4.96:8.50]	6.78 [5.13:8.65]	6.83 [5.02:8.70]
σ	1.40 [0.94:1.86]	1.44 [0.96:1.88]	1.43 [0.97:1.91]	1.45 [0.97:1.90]	1.47 [1.05:1.95]
h_C	0.57 [0.45:0.68]	0.57 [0.45:0.68]	0.57 [0.45:0.68]	0.56 [0.45:0.68]	0.60 [0.48:0.73]
ξ_w	0.73 [0.66:0.81]	0.71 [0.64:0.77]	0.74 [0.66:0.81]	0.71 [0.65:0.78]	0.75 [0.67:0.82]
ϕ	2.40 [1.37:3.35]	2.31 [1.29:3.23]	2.39 [1.44:3.39]	2.38 [1.39:3.33]	2.11 [1.13:3.15]
ξ_p	0.91 [0.89:0.92]	0.91 [0.89:0.92]	0.89 [0.87:0.91]	0.90 [0.88:0.92]	0.91 [0.89:0.93]
ξ_e	0.54 [0.47:0.61]	0.54 [0.45:0.60]	0.53 [0.46:0.61]	0.53 [0.46:0.60]	0.53 [0.45:0.61]
γ_w	0.69 [0.44:0.94]	-	0.66 [0.40:0.93]	-	0.72 [0.49:0.98]
γ_p	0.44 [0.26:0.60]	0.42 [0.25:0.59]	-	-	0.46 [0.30:0.63]
ψ^{-1}	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.32 [0.21:0.42]	0.33 [0.23:0.43]
ϕ_F	1.56 [1.39:1.73]	1.57 [1.40:1.74]	1.55 [1.37:1.72]	1.55 [1.39:1.72]	1.59 [1.40:1.75]
$\theta_{r_p i}$	1.69 [1.54:1.86]	1.70 [1.53:1.86]	1.69 [1.53:1.84]	1.69 [1.52:1.85]	1.69 [1.50:1.85]
$\theta_{\Delta\pi}$	0.15 [0.07:0.23]	0.17 [0.09:0.24]	0.17 [0.08:0.25]	0.17 [0.09:0.26]	0.17 [0.08:0.25]
ρ	0.96 [0.94:0.98]	0.96 [0.94:0.98]	0.96 [0.95:0.98]	0.96 [0.95:0.98]	0.96 [0.94:0.99]
θ_y	0.11 [0.04:0.18]	0.10 [0.03:0.17]	0.11 [0.04:0.19]	0.11 [0.04:0.18]	0.11 [0.03:0.18]
$\theta_{\Delta y}$	0.15 [0.11:0.19]	0.15 [0.12:0.19]	0.15 [0.12:0.19]	0.16 [0.13:0.20]	0.14 [0.10:0.18]
$sd(\epsilon_a)$	0.50 [0.39:0.59]	0.49 [0.38:0.58]	0.50 [0.38:0.61]	0.49 [0.39:0.59]	0.47 [0.36:0.57]
$sd(\epsilon_{\bar{\pi}})$	0.01 [0.00:0.06]	0.02 [0.00:0.02]	0.02 [0.00:0.03]	0.05 [0.00:0.03]	0.02 [0.00:0.03]
$sd(\epsilon_C)$	0.38 [0.20:0.56]	0.38 [0.20:0.54]	0.38 [0.19:0.56]	0.37 [0.21:0.54]	0.52 [0.23:0.72]
$sd(\epsilon_g)$	1.99 [1.73:2.26]	1.99 [1.74:2.26]	1.98 [1.73:2.25]	1.97 [1.73:2.23]	1.99 [1.73:2.26]
$sd(\epsilon_L)$	3.33 [1.80:4.88]	2.92 [1.58:4.17]	3.22 [1.93:4.55]	3.01 [1.77:4.13]	4.23 [1.49:6.66]
$sd(\epsilon_I)$	0.07 [0.03:0.10]	0.07 [0.03:0.10]	0.07 [0.03:0.11]	0.07 [0.03:0.10]	0.07 [0.04:0.11]
$sd(\epsilon_R)$	0.08 [0.04:0.11]	0.09 [0.06:0.13]	0.08 [0.04:0.11]	0.08 [0.05:0.12]	0.08 [0.05:0.13]
$sd(\epsilon_Q)$	0.61 [0.50:0.70]	0.61 [0.50:0.70]	0.61 [0.52:0.72]	0.61 [0.52:0.73]	0.62 [0.52:0.73]
$sd(\epsilon_P)$	0.16 [0.13:0.18]	0.16 [0.14:0.19]	0.21 [0.18:0.25]	0.22 [0.18:0.26]	0.16 [0.14:0.19]
$sd(\epsilon_W)$	0.29 [0.24:0.33]	0.27 [0.23:0.31]	0.29 [0.25:0.34]	0.27 [0.23:0.31]	0.29 [0.25:0.34]
h_L	-	-	-	-	0.52 [0.38:0.66]
LL	-298.72	-298.96	-299.02	-298.17	-303.78

Table 2. Bayesian Estimation of Parameters for the Smets-Wouters Euro Area Model

5 Is There a Long-Run Inflationary Bias?

As we have seen a long-run inflationary bias under discretion arises only if the steady state associated with zero inflation, about which we have linearized, is inefficient lo To examine the inefficiency of the steady state we consider the social planner's problem for the deterministic case obtained by maximizing

$$\Omega_0 = \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t/C_{t-1}^{h_C})^{1-\sigma}}{1-\sigma} - \kappa \frac{(L_t/L_{t-1}^{h_L})^{1+\phi}}{(1+\phi)} \right] \tag{103}$$

with respect to $\{C_t\}$, $\{K_t\}$, $\{L_t\}$ and $\{Z_t\}$ subject to

$$Y_t = A_t(Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} = C_t + G_t + K_t - (1-\delta)K_{t-1} + \Psi(Z_t)K_{t-1} \tag{104}$$

To solve this optimization problem define a Lagrangian

$$\mathcal{L} = \Omega_0 + \sum_{t=0}^{\infty} \beta^t \mu_t \left[A_t(Z_t K_{t-1})^{\alpha} L_t^{1-\alpha} - C_t - G_t - K_t + (1-\delta)K_{t-1} - \Psi(Z_t)K_{t-1} \right] \tag{105}$$

First order conditions are:

$$C_t: \quad C_t^{-\sigma} C_{t-1}^{h_C(\sigma-1)} - \beta h_C C_{t+1}^{1-\sigma} C_t^{h_C(\sigma-1)-1} - \mu_t = 0 \tag{106}$$

$$K_t: \quad -\mu_t + \left[(1-\delta)\beta + \alpha\beta A_t Z_{t+1} \left(\frac{L_{t+1}}{Z_{t+1}K_t} \right)^{1-\alpha} - \beta\Psi(Z_{t+1}) \right] \mu_{t+1} = 0 \tag{107}$$

$$\begin{aligned} L_t: \quad & -\kappa \left[L_t^{\phi} L_{t-1}^{-h_L(1+\phi)} - \beta h_L L_{t+1}^{1+\phi} L_t^{-h_L(1+\phi)-1} \right] \\ + \quad & (1-\alpha)A_t \left(\frac{Z_t K_{t-1}}{L_t} \right)^{\alpha} \mu_t = 0 \end{aligned} \tag{108}$$

$$Z_t: \quad \Psi'(Z_t) - \alpha A_t \left(\frac{L_t}{Z_t K_{t-1}} \right)^{1-\alpha} = 0 \tag{109}$$

The efficient steady-state levels of output $Y_{t+1} = Y_t = Y_{t-1} = Y^*$, say, is therefore found by solving the system:

$$C^{-\sigma+h_C(\sigma-1)}(1-\beta h_C) - \mu = 0 \tag{110}$$

$$-1 + (1-\delta)\beta + \alpha\beta A Z \left(\frac{L}{ZK} \right)^{1-\alpha} - \beta\Psi(Z) = 0 \tag{111}$$

$$-\kappa(1-\beta h_L)L^{\phi-h_L(1+\phi)} + (1-\alpha)A \left(\frac{ZK}{L} \right)^{\alpha} \mu = 0 \tag{112}$$

$$\Psi'(Z) - \alpha A \left(\frac{L}{ZK} \right)^{1-\alpha} = 0 \tag{113}$$

Solving as we did for the natural rate and denoting the social optimum by Z^*, Y^* etc we arrive at

$$1 = \beta[1 - \delta + Z^*\Psi'(Z^*) - \Psi(Z^*)] \tag{114}$$

Hence $Z^* = Z = 1$ and $R_K^* = R_K = R + \delta$. Thus the *natural rate of capacity utilization is efficient*. However,

$$\frac{K^*}{L^*} = \left[\frac{A\alpha}{R_K}\right]^{\frac{1}{1-\alpha}} > \frac{K}{L} \tag{115}$$

and the *natural capital-labour ratio is below the social optimum*. The socially optimal level of output is now found from

$$Y^{*(1+\phi)(1-h_L)+(\sigma-1)(1-h_C)} \left(1 - \frac{\delta\frac{K^*}{L^*}}{(A\left(\frac{K^*}{L^*}\right)^{\alpha} - F)} - \frac{G}{Y}\right)^{\sigma+h_C(1-\sigma)}$$
$$= \frac{(1-\alpha)(1-\beta h_C)A^{(1+\phi)(1-h_L)}\left(\frac{K^*}{L^*}\right)^{\alpha(1+\phi)(1-h_L)}}{\kappa(1-\beta h_L)} \tag{116}$$

The inefficiency of the natural rate of output can now be found by comparing (73) with (116). The case of a model without capital is straightforward. Putting $\alpha = \frac{K}{L} = \frac{K^*}{L^*} = 0$ (and $0^0 = 1$) in these two results, since the left-hand-side of (116) is an increasing function of Y, we arrive at

Proposition 1

In a model without capital, the natural level of output, Y, is below the efficient level, Y^*, if and only if

$$(1-T)\left(1-\frac{1}{\zeta}\right)\left(1-\frac{1}{\eta}\right) < \frac{1-h_C\beta}{1-h_L\beta} \tag{117}$$

In the case where there is no habit persistence for both consumption and labour effort, $h_C = h_L = 0$, then (117) always holds. In this case tax distortions and market power in the output and labour markets captured by the elasticities $\eta \in (0, \infty)$ and $\zeta \in (0, \infty)$ respectively drive the natural rate of output below the efficient level. If $T = 0$ and $\eta = \zeta = \infty$, tax distortions and market power disappear and the natural rate is efficient. Another case where (117) always holds is where habit persistence for labour supply exceeds that for consumption; i.e., $h_L \geq h_C$. In the SW model, $h_C > h_L = 0$ which leads to the possibility that the natural rate of output can actually be *above* the efficient level (see Choudhary and Levine (2005)). But to arrive at that conclusion we need to consider the full model with capital.

Now the analysis is not so straightforward. Denote the left-hand-sides of (73) and (116) by $f(Y)$ and $f^*(Y^*)$ respectively and the intercept terms on the right-hand-side by c and c^* respectively. Then we have

$$\frac{f(Y^*)}{f(Y)} = \frac{c^*}{c} \equiv \frac{(1-\beta h_C)}{(1-\beta h_L)(1-T)(1-\frac{1}{\eta})(1-\frac{1}{\zeta})}\left[\frac{\frac{K^*}{L^*}}{\frac{K}{L}}\right]^{\alpha(1+\phi)(1-h_L)} \tag{118}$$

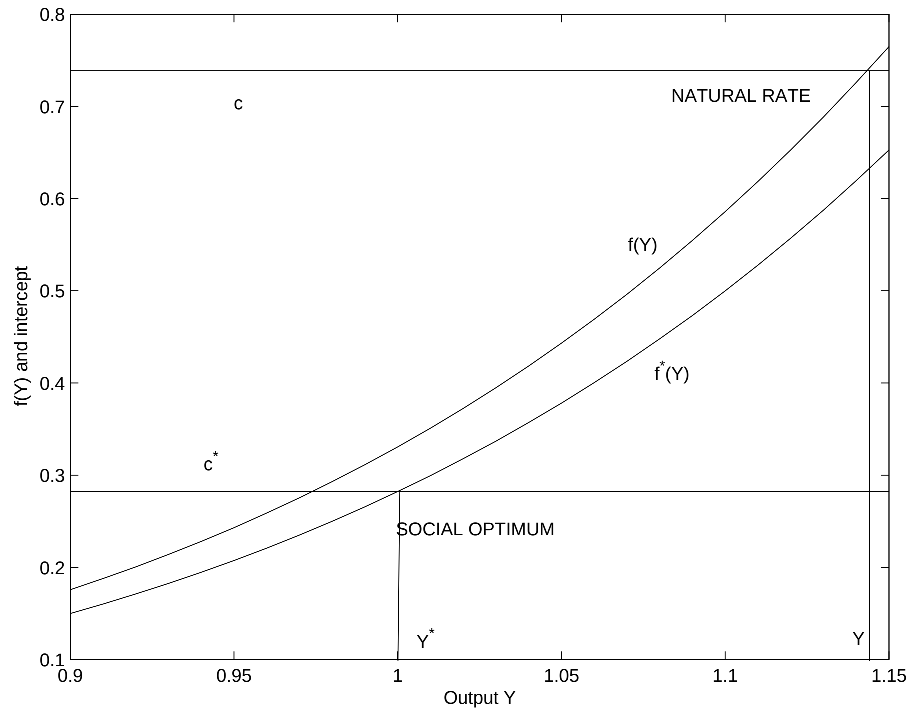

Figure 1: **The Inefficiency of the Natural Rate:** $h_L = 0$

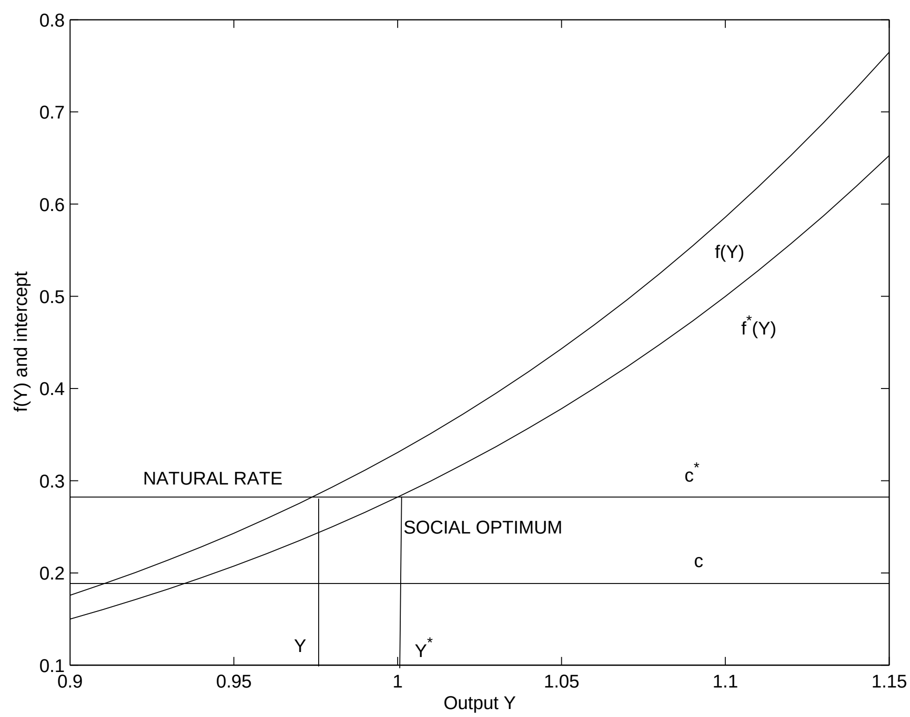

Figure 2: **The Inefficiency of the Natural Rate:** $h_L = 0.5$

Then because the $\frac{K}{L} < \frac{K^*}{L^*}$ we have that $f(Y) > f^*(Y)$, given Y and habit causes the upward-sloping $f(Y)$ curve to shift to the left. We can now see two opposite effects on the efficiency of the natural rate. If habit in consumption is much larger than habit in labour supply (as in the SW model) then despite other distortions it is possible that $c > c^*$ which has the effect of making the natural rate of output above the efficient level. On the other hand the capital-labour ratio is inefficient which has the opposite effect on output. Figures 1 and 2 below show two cases: $h_L = 0$ where the first effect dominates and $h_L = 0.5$ where the second dominates.

6 Optimal Monetary Stabilization Policy

6.1 Formulating the Policymaker's Loss Function

Much of the optimal monetary policy literature has stayed with the ad hoc loss function (4) which with a lower interest rate bound constraint becomes

$$\Omega_0 = E_0 \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t \left[(y_t - \hat{y}_t - k)^2 + w_\pi \pi_t^2 + w_r r_t^2 \right] \right] \tag{119}$$

Indeed Clarida *et al.* (1999) provide a stout defence of a hybrid research strategy that combines a loss function based on the stated objectives of central banks with a micro-founded macro-model. A normative assessment of policy rules requires welfare analysis and for this, given our linear-quadratic framework,[14] we require a quadratic approximation of the representative consumer's utility function.

A common procedure for reducing optimal policy to a LQ problem is as follows. Linearize the model about a deterministic steady state as we have already done. Then expand the consumer's utility function as a second-order Taylor series after imposing the economy's resource constraint. In general this procedure is incorrect unless the steady state is not to far from the efficient outcome (see Woodford (2003), chapter 6 and Benigno and Woodford (2003)). This we assume and for this case we show in Appendix C that a quadratic single-period loss function that approximates the utility takes the form

$$\begin{aligned} L &= w_c(c_t - h_C c_{t-1})^2 + w_l(l_t - h_L l_{t-1})^2 + w_\pi(\pi_t - \gamma_P \pi_{t-1})^2 + w_{\Delta w}(\Delta w_t - \gamma_W \Delta w_{t-1})^2 \\ &+ w_{lk}(l_t - k_{t-1} - z_t - \frac{1}{\alpha} a_t)^2 + w_z(z_t + \psi(1-\alpha)a_t)^2 - w_{ak} a_t k_{t-1} \end{aligned} \tag{120}$$

[14]Recent developments in numerical methods now allow the researcher to go beyond linear approximations of their models and to conduct analysis of both the dynamics and welfare using higher-order (usually second-order) approximations (see, Kim *et al.* (2003) and for an application to simple monetary policy rules, Juillard *et al.* (2004)). However as we have argued above there are costs as well as benefits from going down this path.

where weights w_c etc are defined in Appendix C. All variables are in log-deviation form about the steady state as in the linearization.[15] The first four terms in (120) give the welfare loss from consumption, employment, price inflation and wage inflation variability respectively. The remaining terms are contributions from arise from the resource constraint in our quadratic approximation procedure.

In what follows we provide numerical results for both the ad hoc and welfare-based loss functions with $k = 0$ so our focus now is exclusively on stabilization gains from commitment.

6.2 Imposing the Lower Interest rate Bound

We first calibrate the weight w_r so the $2sd(r_t) < i$ under discretion where $i = \frac{1}{\beta} - 1$ is the zero-inflation steady nominal interest rate. For a normal distribution this would give a probability of hitting the lower interest rate bound 0.025%. With $\beta = 0.99$ imposed this condition becomes $var(r_t) < 0.25(\%)^2$. Tables 2 and 3 show the effect on $var(r_t)$ of increasing the weight under discretion. By reporting the expected intertemporal loss at time $t = 0$ under both the time-consistent discretionary policy and optimal commitment, Ω^{TC} and Ω^{OP} respectively, we can also assess the stabilization gains from commitment as the lower interest rate bound takes greater effect. We compute these gains as equivalent permanent percentage increases in consumption and inflation, c_e and π_e respectively. From Appendix C these are given by

$$c_e = \frac{\Omega^{TC} - \Omega^{OP}}{1 - h_C} \times 10^{-2} \tag{121}$$

$$\pi_e = \sqrt{\frac{2(\Omega^{TC} - \Omega^{OP})}{w_\pi}} \tag{122}$$

For the ad hoc loss function the appropriate equivalent that repaces c_e is a permanent fall in the output gap given by

$$y_e = \sqrt{2(\Omega^{TC} - \Omega^{OP})} \tag{123}$$

Tables 3 and 4 report results for the ad hoc (with $w_\pi = 16$) and welfare-based (with w_π and other weights functions of fundamental parameters given in Appendix C) loss functions. A number of interesting points emerge from these tables. First, with the welfare-based loss function, using (121) the minimal cost of consumption fluctuations given by Ω^{OP} lie between 1.07% and 1.75% and are much larger than the welfare cost reported by Lucas (1987) which were of the order 0.05%. Our figures are of the order of

[15] Our quadratic approximation is along the lines of Onatski and Williams (2004) with some differences.

those reported in Levin *et al.* (2005) for a similar model and are much larger because of the welfare costs of imposing the lower interest rate bound and of price and wage inflation not included in the Lucas calculations. Our figure is also increased by the existence of internal habit which reduces the utility increase from a increase in consumption. Thus in our set-up the answer to the question posed by Lucas, "Is there a Case for Stabilization Policy?" is in the affirmative.

Our second point is that a welfare-based assessment of the stabilization gains from commitment leads to rather similar conclusions to those using the ad hoc formulation with a balance choice of weights penalizing the variability of the output gap and inflation. Choosing the latter without a lower interest rate bound ($w_r = 0.0001$)[16] and comparing y_e with c_e and the inflation costs in the two cases we see that these gains are of roughly of the same order and could be brought more into line by choosing a high weight w_π in (119). However as we will see the simple commitment rules that approximate the optimal rule are quite different in the two cases.

Finally the most important point from these tables endorses the conclusion reached by Adam and Billi (2005) discussed in the introduction, namely that the lower bound constraint on the nominal interest rate increases the gains from commitment. Both in terms of the output gap equivalent y_e for the ad hoc loss function, and the consumption equivalent for the welfare-based case c_e we can see that the stabilization gain from commitment rises dramatically until at the point where $var(r_t) < 0.25$ these gains are $y_e = 4.03\%$, $c_e = 5.8\%$ and $\pi_e = 0.99$, 1.49 for the 2 cases achieved when $w_r = 2.5$, 30 respectively.

Weight w_r	var(r_t)	Ω_0^{TC}	Ω_0^{OP}	y_e	π_e
0.0001	1.33	0.54	0.49	0.32	0.08
1.0	0.49	4.82	0.64	2.94	0.72
2.5	0.23	8.62	0.73	4.03	0.99
3.0	0.19	9.22	0.74	4.18	1.03
4.0	0.13	10.0	0.78	4.36	1.07

Table 3. Imposing the Lower Interest Rate Bound Using the Ad Hoc Loss Function: $w_\pi = 16$.

[16]The solution procedures set out in Appendix A require a very small weight is needed on the instrument. One can get round this without significantly changing the result by letting inflation be the instrument and then setting the interest rate at a second stage of the optimization to achieve the optimal path for inflation.

Weight w_r	var(r_t)	Ω_0^{TC}	Ω_0^{OP}	c_e	π_e
0.0001	12	22	19	0.17	0.26
5	0.46	39	23	1.81	0.59
20	0.27	84	27	3.22	1.12
30	0.25	130	28	5.76	1.49
50	0.24	269	31	13.5	2.28

Table 4. Imposing the Lower Interest Rate Bound Using the Welfare-Based Loss Function: $w_\pi = 92$**.**

6.3 Stabilization Gains with Simple Rules

Having calibrated the weight $w_r = 2.5$, 30 for the two forms of loss function, we now report results for simple commitment rules and discretionary policy. The general form of simple rule examined is

$$i_t = \rho i_{t-1} + \Theta_\pi E_t \pi_{t+j} + \Theta_y (y_t - \hat{y}_t) + \Theta_{\Delta w} \Delta w_t + \Theta_{wr} wr_t\,; \;\; \rho \in [0,1],\, \Theta_\pi, \Theta_y, \Theta_{\Delta w}, \Theta_{wr} > 0\, j \geq 0 \tag{124}$$

Putting $\Theta_{\Delta w} = \Theta_{wr} = 0$ gives the Taylor rule where the interest rate only to current price inflation and the output gap, $\Theta_{\Delta w} = \Theta_{wr} = \Theta_y = 0$ gives a price inflation rule, $\Theta_{pi} = \Theta_{wr} = \Theta_y = 0$ gives a wage inflation rule and finally $\Theta_{\Delta w} = \Theta_y = 0$ gives a current price inflation and real wage rule.

Results for these various simple are summarized in tables 5 and 6. There are two notable results that emerge from the table and the figures. First we assess the effect of using an arbitrary rather than an optimized simple commitment rule by examining the outcome when a minimal rule $i_i = 1.001\pi_t$ that just produces saddle-path stability. This is the worse case for an arbitrary choice and we see that the costs are substantial: $y_e = 2.84\%$ and $c_e = 12.2\%$. Interestingly in the former case this outcome is still better than that under discretion. Second, simple price inflation or wage inflation rules perform reasonably well in that they achieve about three-quarters of the commitment gains achieved by the optimal rule for the ad hoc loss function falling to about half for the welfare-based loss.[17]The simple rule that closely mimicked optimal commitment for the welfare-based case was the inflation and real wage rule. From table 4 almost all the gains from commitment are achieved by this rule though simplicity still leaves a nt insignificant cost of $c_e = 0.11$ and $\pi_e = 0.21$ or about $\pi = 0.84\%$ on an annual basis.

[17]This contrasts with the result in Levin *et al.* (2005) where the wage inflation rule performed a lot better than the price inflation and closely mimicked optimal commitment.

Rule	ρ	Θ_π	Θ_y	Ω_0	y_e	π_e	var(r_t)
Minimal Feedback on π_t	0	1.001	0	4.75	2.84	0.71	0.32
Price Inflation Rule	0.82	0.53	0	1.36	1.12	0.28	0.11
Taylor Rule	0.82	0.55	0.11	1.35	1.11	0.28	0.11
Optimal Commitment	n.a.	n.a.	n.a.	0.73	0	0	82
Optimal Discretion	n.a.	n.a.	n.a	8.62	3.97	0.99	0.24

Table 5. Optimal Commitment Rules Using the Ad Hoc Loss Function

Rule	ρ	Θ_π	$\Theta_{\Delta w}$	Θ_{wr}	Ω_0	c_e	π_e	var(r_t)
Minimal Feedback on π_t	0	1.001	0	0	244	12.2	2.17	0.31
Price Inflation Rule	0.33	1.17	0	0	79	2.9	1.05	0.32
Wage Inflation Rule	0.95	0	0.59	0	80	2.9	1.07	0.22
Real-Wage/Price Inflation Rule	0.95	0.16	0	0.28	30	0.11	0.21	0.34
Optimal Commitment	n.a.	n.a.	n.a.	n.a.	28	0	0	0.99
Optimal Discretion	n.a.	n.a.	n.a	n.a.	130	5.76	1.49	0.26

Table 6. Optimal Commitment Rules Using the Welfare-Based Loss Function

6.4 Impulse Responses Under Commitment and Discretion

Figures 6-13 concentrate on the welfare-based results and compare the responses under the optimal commitment, discretion and the optimized simple inflation/real wage rule following an unanticipated government spending shock ($g_0 = 1$) and an unanticipated productivity shock ($a_0 = 1$).

To interpret these graphs it is usefult to consider the three sources of the time-inconsistency problem in our model; from pricing behaviour and consumption behaviour together, from investment behaviour and from wage setting. Following a shock which diverts the economy from its steady state, given expectations of inflation, the opportunist policy-maker can increase or decrease output by reducing or increasing the interest rate which increases or decreases inflation. Consider the case where the economy is below the its steady state level of output. A reduction in the interest rate then causes consumption demand rise. Firms locked into price contracts respond to an increase in demand by increasing output and increasing the price according to their indexing rule. Those who can re-optimize increase only increase their price. Given inflationary expectations, a reduction in the interest rate sees Tobin's Q rise, and with it investment and capital

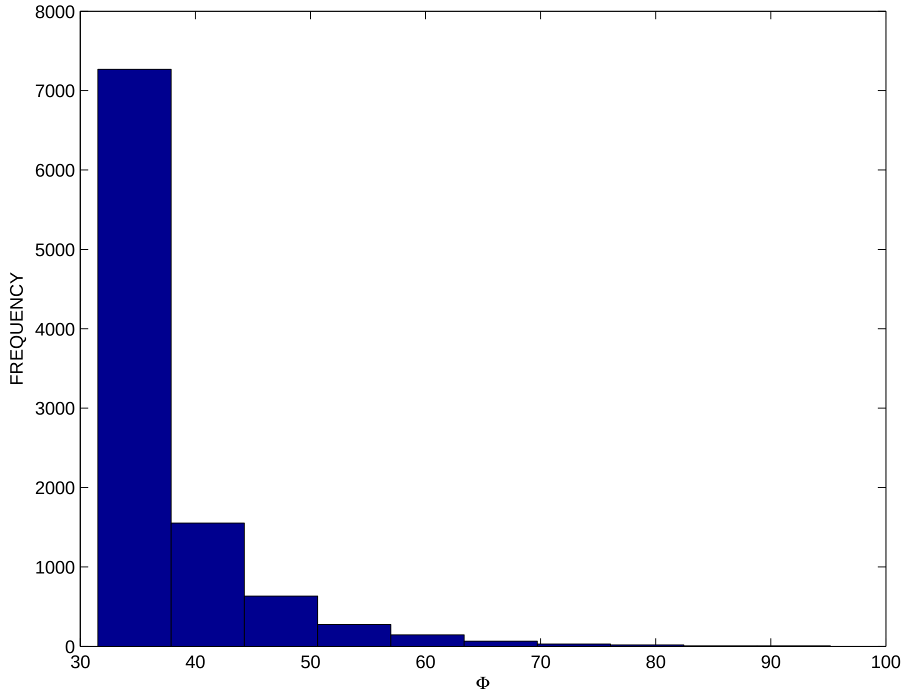

Figure 3: **The No-Deviation Condition:** $\Phi = \mathbf{tr}((N_{11} + S)(Z_t + \frac{\lambda}{1-\lambda}\Sigma) + \mathbf{tr}(N_{22}p_{2t}p_{2t}^T)$

stock. This increases output on the supply side. Given inflationary expectations an inflationary impulse results in a fall in the real wage and an increase in labour supply, adding further to the supply side boost to output. All these changes are for *given* inflationary expectations and illustrates the incentive to inflate when the output gap increases. In an non-commitment equilibrium however the incentive is anticipated and the result is higher inflation compared with the commitment case. This contrast between the commitment and discretionary cases is seen clearly in the figures. Finally comparing the optimal commitment and the simple inflation-real wage we see how the latter rule closely mimics the former.

6.5 Sustaining Commitment as an Equilibrium

We examine empirically the no-deviation condition for commitment to be a perfect Bayesian equilibrium. We confine ourselves to reporting results for the form of the condition given by (29) which assume an instantaneous loss of reputation following deviation. Experiment revealed this to give very similar results to those using (27), and this in turn implied that the condition relevant for our simple inflation/real wage, (32), was satisfied.

Figure 3 plots a histogram from $10,000$ draws of the sector $[z_t^T p_{2,t}^T]^T$ in the vicinity of the steady state of the economy under the optimal commitment rule. The probability of the weak government deviating from the optimal rule, q_t, is then the proportion of these draws for which (29) does not hold; i.e., $\Phi = \mathrm{tr}((N_{11} + S)(Z_t + \frac{\lambda}{1-\lambda}\Sigma) + \mathrm{tr}(N_{22}p_{2t}p_{2t}^T) < 0$. For our model and sample of $10,000$ draws we see that in fact $q_t = 0$ so that optimal commitment for a weak government turns out to be a perfect Bayesian equilibrium.

6.6 Sensitivity Analysis

To conduct a sensitivity exercise we again confine ourselves to the central variant of the SW model. We utilize the MCMC draws of underlying parameter values that are used in the Bayesian estimation of this model to compute the posterior distribution of parameters. For each draw the consumption and inflation equivalent gains from commitment, c_e and π_e respectively, are computed as before using (121) and (122). The histograms for a random sample of 100 draws are shown in figures 4 and 5. From these histograms we confirm the existence of substantial gains of at least $c_e = \frac{1}{2}\%$ with over 80% of the draw giving gains $c_e \geq 4\%$. The corresponding lower bound for the quarterly inflation-equivalent gain is $\pi_e = 0.75\%$ and over 90% of draws given a gain $\pi_e \geq 1.0\%$.

7 Conclusions

The main findings of this paper can be summarized as follows:

1. External habit in consumption reduces the inefficiency of the steady state, but external habit in labour supply has the opposite effect. If the former dominates sufficiently and labour market and product market distortions are not too big then in the absence of tax distortions the natural rate can be below the social optimum. This would then render the long-run inflationary bias negative.

2. In terms of an equivalent permanent increase in consumption, c_e for the welfare-based loss function or an equivalent permanent increase in the output gap, y_e for the ad hoc loss function, and a permanent decrease in inflation π_e, the stabilization gains from commitment rise considerably if the lower bound effect is taken into account and if there is habit in consumption. Using empirical estimates from the SW model we find gains as much as $c_e = 6\%$ and $\pi_e = 1.5\%$, the latter on a quarterly basis.

3. If the standard ad hoc loss function with 'balanced (equal) weights' on the output gap and annual inflation variance is employed, then y_e is of the same order as c_e. However the form of optimal simple rule differs substantially in these two cases.

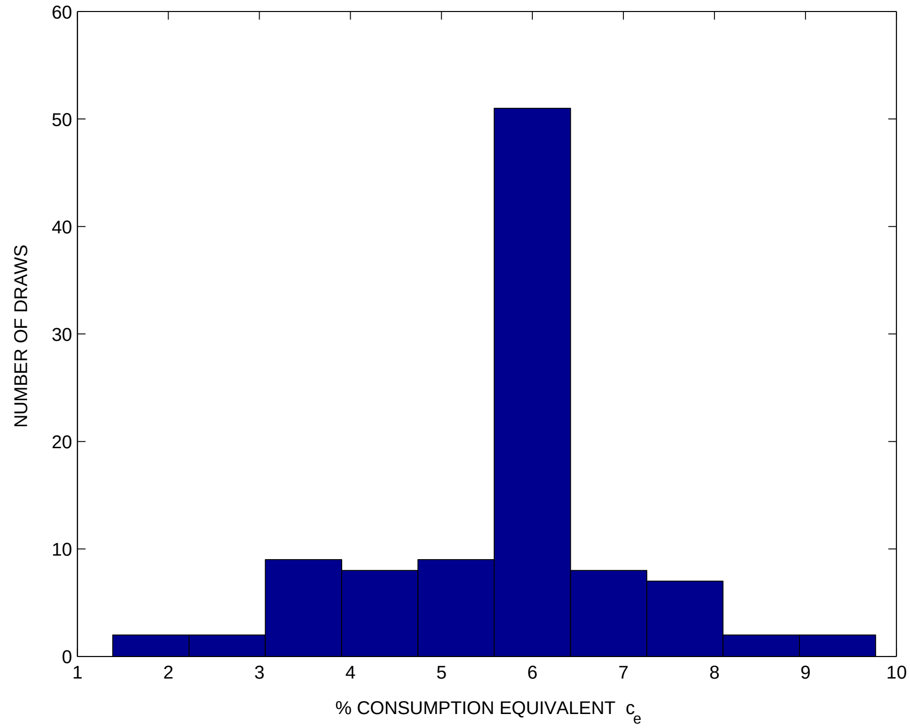

Figure 4: **Stabilization Gains from Commitment: Consumption Equivalent % Increase c_e.**

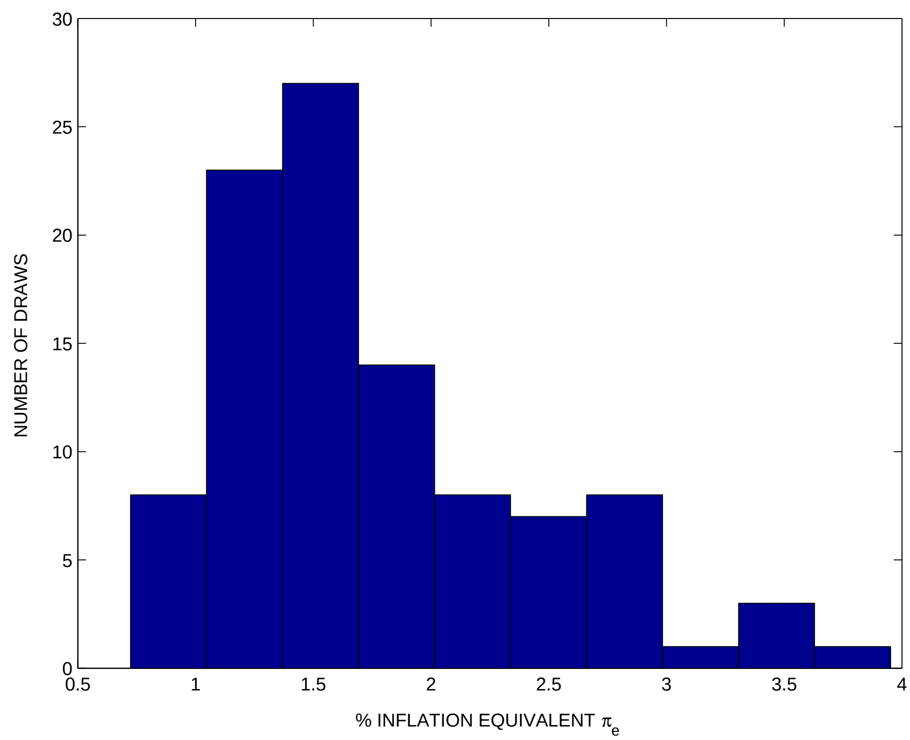

Figure 5: **Stabilization Gains from Commitment: Inflation Equivalent % Decrease π_e.**

4. Given these large gains from commitment, the incentives for central banks to avoid a loss of reputation for commitment is large. Consequently a commitment rule can be sustained as a perfect Bayesian equilibrium in which deviation from commitment hardly ever happens despite the possibility of large exogenous shocks.

5. The optimal commitment rule can be closely approximated in terms of its good stabilization properties by a interest rate rule that responds positively to current inflation and the current real wage.

A Details of Policy Rules

First consider the purely deterministic problem. The most general policy starts with a general model of the form

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t \tag{A.1}$$

where z_t is an $(n-m) \times 1$ vector of predetermined variables including non-stationary processed, z_0 is given, w_t is a vector of policy variables, x_t is an $m \times 1$ vector of non-predetermined variables and $\mathsf{x}^e_{t+1,t}$ denotes rational (model consistent) expectations of x_{t+1} formed at time t. Then $\mathsf{x}^e_{t+1,t} = \mathsf{x}_{t+1}$ and letting $\mathsf{y}^T_t = \begin{bmatrix} \mathsf{z}^T_t & \mathsf{x}^T_t \end{bmatrix}$ (A.1) becomes

$$\mathsf{y}_{t+1} = A\mathsf{y}_t + B\mathsf{w}_t \tag{A.2}$$

Define target variables s_t by

$$\mathsf{s}_t = M\mathsf{y}_t + H\mathsf{w}_t \tag{A.3}$$

and the policy-maker's loss function at time t by

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t [\mathsf{s}^T_{t+i} Q_1 \mathsf{s}_{t+i} + \mathsf{w}^T_{t+i} Q_2 \mathsf{w}_{t+i}] \tag{A.4}$$

which we rewrite as

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t [\mathsf{y}^T_{t+i} Q \mathsf{y}_{t+i} + 2\mathsf{y}^T_{t+i} U \mathsf{w}_{t+i} + \mathsf{w}^T_{t+i} R \mathsf{w}_{t+i}] \tag{A.5}$$

where $Q = M^T Q_1 M$, $U = M^T Q_1 H$, $R = Q_2 + H^T Q_1 H$, Q_1 and Q_2 are symmetric and non-negative definite R is required to be positive definite and $\beta \in (0,1)$ is discount factor. The procedures for evaluating the three policy rules are outlined in the rest of this appendix (or Currie and Levine (1993) for a more detailed treatment).

A.1 The Optimal Policy with Commitment

Consider the policy-maker's *ex-ante* optimal policy at $t = 0$. This is found by minimizing Ω_0 given by (A.5) subject to (A.2) and (A.3) and given z_0. We proceed by defining the Hamiltonian

$$\mathcal{H}_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2}\beta^t(\mathsf{y}_t^T Q\mathsf{y}_t + 2\mathsf{y}_t^T U\mathsf{w}_t + \mathsf{w}_t^T R\mathsf{w}_t) + \mu_{t+1}(A\mathsf{y}_t + B\mathsf{w}_t - \mathsf{y}_{t+1}) \quad \text{(A.6)}$$

where μ_t is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$\mathcal{L}_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} \mathcal{H}_t \quad \text{(A.7)}$$

with respect to the arguments of L_0 (except z_0 which is given). Then at the optimum, $\mathcal{L}_0 = \Omega_0$.

Redefining a new costate column vector $\mathsf{p}_t = \beta^{-t}\mu_t^T$, the first-order conditions lead to

$$\mathsf{w}_t = -R^{-1}(\beta B^T \mathsf{p}_{t+1} + U^T \mathsf{y}_t) \quad \text{(A.8)}$$

$$\beta A^T \mathsf{p}_{t+1} - \mathsf{p}_t = -(Q\mathsf{y}_t + U\mathsf{w}_t) \quad \text{(A.9)}$$

Substituting (C.27) into (A.2)) we arrive at the following system under control

$$\begin{bmatrix} I & \beta BR^{-1}B^T \\ 0 & \beta(A^T - UR^{-1}B^T) \end{bmatrix} \begin{bmatrix} \mathsf{y}_{t+1} \\ \mathsf{p}_{t+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}U^T & 0 \\ -(Q - UR^{-1}U^T & I \end{bmatrix} \begin{bmatrix} \mathsf{y}_t \\ \mathsf{p}_t \end{bmatrix} \quad \text{(A.10)}$$

To complete the solution we require $2n$ boundary conditions for (A.10). Specifying z_0 gives us $n-m$ of these conditions. The remaining condition is the 'transversality condition'

$$\lim_{t\to\infty} \mu_t^T = \lim_{t\to\infty} \beta^t \mathsf{p}_t = 0 \quad \text{(A.11)}$$

and the initial condition

$$\mathsf{p}_{20} = 0 \quad \text{(A.12)}$$

where $\mathsf{p}_t^T = \left[\mathsf{p}_{1t}^T\, \mathsf{p}_{2t}^T\right]$ is partitioned so that p_{1t} is of dimension $(n-m) \times 1$. Equation (A.3), (C.27), (A.10) together with the $2n$ boundary conditions constitute the system under optimal control.

Solving the system under control leads to the following rule

$$\mathsf{w}_t = -F \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} = -F \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_{2t} \end{bmatrix} \quad \text{(A.13)}$$

where

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{p}_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv H \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \quad \text{(A.14)}$$

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{A.15}$$

$$\mathsf{x}_t = -\begin{bmatrix} N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{A.16}$$

where $F = -(R + B^T SB)^{-1}(B^T SA + U^T), G = A - BF$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{A.17}$$

partitioned so that S_{11} is $(n-m) \times (n-m)$ and S_{22} is $m \times m$ is the solution to the steady-state Ricatti equation

$$S = Q - UF - F^T U^T + F^T RF + \beta(A - BF)^T S(A - BF) \tag{A.18}$$

The cost-to-go for the optimal policy (OP) at time t is

$$\Omega_t^{OP} = -\frac{1}{2}(\mathrm{tr}(N_{11}Z_t) + \mathrm{tr}(N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T)) \tag{A.19}$$

where $Z_t = \mathsf{z}_t\mathsf{z}_t^T$. To achieve optimality the policy-maker sets $\mathsf{p}_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $\mathsf{p}_{2t} = 0$. It can be shown that $N_{11} < 0$ and $N_{22} < 0$.[18], so the incentive to renege exists at all points along the trajectory of the optimal policy. This is the time-inconsistency problem.

A.1.1 Implementation

The rule may also be expressed in two other forms: First as

$$\mathsf{w}_t = D_1\mathsf{z}_t + D_2H_{21}\sum_{\tau=1}^{t}(H_{22})^{\tau-1}\mathsf{z}_{t-\tau} \tag{A.20}$$

where $D = [D_1\ D_2]$ is partitioned conformably with z_t and p_{2t}. The rule then consists of a feedback on the lagged predetermined variables with geometrically declining weights with lags extending back to time $t = 0$, the time of the formulation and announcement of the policy.

The final way of expressing the rule is express the process for w_t in terms of the target variables only, s_t, in the loss function. This in particular eliminates feedback from the exogenous processes in the vector z_t. Since the rule does not require knowledge of these processes to design, Woodford (2003) refers to this as "robust" in describing it as the *Robust Optimal Explicit* rule.

[18]See Currie and Levine (1993), chapter 5.

A.1.2 Optimal Policy from a Timeless Perspective

Noting from (A.16) that long the optimal policy we have $\mathsf{x}_t = -N_{21}\mathsf{z}_t - N_{22}\mathsf{p}_{2t}$, the optimal policy "from a timeless perspective" proposed by Woodford (2003) replaces the initial condition for optimality $p_{20} = 0$ with

$$J\mathsf{x}_0 = -N_{21}\mathsf{z}_0 - N_{22}\mathsf{p}_{20} \tag{A.21}$$

where J is some $1 \times m$ matrix. Typically in New Keynesian models the particular choice of condition is $\pi_0 = 0$ thus avoiding any once-and-for-all initial surprise inflation. This initial condition applies only at $t = 0$ and only affects the deterministic component of policy and not the stochastic, stabilization component. Since our focus here is on the latter, the timeless perspective has no bearing on the results of this paper.

A.2 The Dynamic Programming Discretionary Policy

The evaluate the discretionary (time-consistent) policy we rewrite the cost-to-go Ω_t given by (A.5) as

$$\Omega_t = \frac{1}{2}[\mathsf{y}_t^T Q\mathsf{y}_t + 2\mathsf{y}_t^T U\mathsf{w}_t + \mathsf{w}_t^T R\mathsf{w}_t + \beta\Omega_{t+1}] \tag{A.22}$$

The dynamic programming solution then seeks a stationary solution of the form $w_t = -Fz_t$ in which Ω_t is minimized at time t subject to (1) in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.

Suppose that the policy-maker at time t expects a private-sector response from $t+1$ onwards, determined by subsequent re-optimisation, of the form

$$\mathsf{x}_{t+\tau} = -N_{t+1}\mathsf{z}_{t+\tau}, \ \tau \geq 1 \tag{A.23}$$

The loss at time t for the *ex ante* optimal policy was from (A.19) found to be a quadratic function of x_t and p_{2t}. We have seen that the inclusion of p_{2t} was the source of the time inconsistency in that case. We therefore seek a lower-order controller

$$\mathsf{w}_t = -F\mathsf{z}_t \tag{A.24}$$

with the cost-to-go quadratic in z_t only. We then write $\Omega_{t+1} = \frac{1}{2}\mathsf{z}_{t+1}^T S_{t+1}\mathsf{z}_{t+1}$ in (A.22). This leads to the following iterative process for F_t

$$\mathsf{w}_t = -F_t\mathsf{z}_t \tag{A.25}$$

where

$$
\begin{aligned}
F_t &= (\overline{R}_t + \lambda \overline{B}_t^T S_{t+1} \overline{B}_t)^{-1} (\overline{U}_t^T + \beta \overline{B}_t^T S_{t+1} \overline{A}_t) \\
\overline{R}_t &= R + K_t^T Q_{22} K_t + U^{2T} K_t + K_t^T U^2 \\
K_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} B^1 + B^2) \\
\overline{B}_t &= B^1 + A_{12} K_t \\
\overline{U}_t &= U^1 + Q_{12} K_t + J_t^T U^2 + J_t^T Q_{22} J_t \\
\overline{J}_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} A_{11} + A_{12})
\end{aligned}
$$

$$
\begin{aligned}
\overline{A}_t &= A_{11} + A_{12} J_t \\
S_t &= \overline{Q}_t - \overline{U}_t F_t - F_t^T \overline{U}^T + \overline{F}_t^T \overline{R}_t F_t + \beta (\overline{A}_t - \overline{B}_t F_t)^T S_{t+1} (\overline{A}_t - \overline{B}_t \overline{F}_t) \\
\overline{Q}_t &= Q_{11} + J_t^T Q_{21} + Q_{12} J_t + J_t^T Q_{22} J_t \\
N_t &= -J_t + K_t F_t
\end{aligned}
$$

where $B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}$, $U = \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and Q similarly are partitioned conformably with the predetermined and non-predetermined components of the state vector.

The sequence above describes an iterative process for F_t, N_t, and S_t starting with some initial values for N_t and S_t. If the process converges to stationary values, F, N and S say, then the time-consistent feedback rule is $w_t = -F z_t$ with loss at time t given by

$$
\Omega_t^{TC} = \frac{1}{2} \mathsf{z}_t^T S \mathsf{z}_t = \frac{1}{2} \mathrm{tr}(S Z_t) \tag{A.26}
$$

A.3 Optimized Simple Rules

We now consider simple sub-optimal rules of the form

$$
\mathsf{w}_t = D \mathsf{y}_t = D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{A.27}
$$

where D is constrained to be sparse in some specified way. Rule can be quite general. By augmenting the state vector in an appropriate way it can represent a PID (proportional-integral-derivative)controller.

Substituting (A.3) into (A.5) gives

$$
\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \beta^t \mathsf{y}_{t+i}^T P_{t+i} \mathsf{y}_{t+i} \tag{A.28}
$$

where $P = Q + UD + D^T U^T + D^T R D$. The system under control (A.1), with w_t given by (A.3), has a rational expectations solution with $\mathsf{x}_t = -N \mathsf{z}_t$ where $N = N(D)$. Hence

$$
\mathsf{y}_t^T P \mathsf{y}_t = \mathsf{z}_t^T T \mathsf{z}_t \tag{A.29}
$$

where $T = P_{11} - N^T P_{21} - P_{12} N + N^T P_{22} N$, P is partitioned as for S in (A.17) onwards and

$$\mathsf{z}_{t+1} = (G_{11} - G_{12}N)\mathsf{z}_t \tag{A.30}$$

where $G = A + BD$ is partitioned as for P. Solving (A.30) we have

$$\mathsf{z}_t = (G_{11} - G_{12}N)^t \mathsf{z}_0 \tag{A.31}$$

Hence from (A.32), (A.29) and (A.31) we may write at time t

$$\Omega_t^{SIM} = \frac{1}{2} z_t^T V z_t = \frac{1}{2}\mathrm{tr}(V Z_t) \tag{A.32}$$

where $Z_t = z_t z_t^T$ and V satisfies the *Lyapunov* equation

$$V = T + H^T V H \tag{A.33}$$

where $H = G_{11} - G_{12}N$. At time $t = 0$ the optimized simple rule is then found by minimizing Ω_0 given by (A.32) with respect to the non-zero elements of D given z_0 using a standard numerical technique. An important feature of the result is that unlike the previous solution the optimal value of D, D^* say, is not independent of z_0. That is to say

$$D^* = D^*(z_0)$$

A.4 The Stochastic Case

Consider the stochastic generalization of (A.1)

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t + \begin{bmatrix} \mathsf{u}_t \\ 0 \end{bmatrix} \tag{A.34}$$

where u_t is an $n \times 1$ vector of white noise disturbances independently distributed with $\mathrm{cov}(\mathsf{u}_t) = \Sigma$. Then, it can be shown that certainty equivalence applies to all the policy rules apart from the simple rules (see Currie and Levine (1993)). The expected loss at time t is as before with quadratic terms of the form $\mathsf{z}_t^T X \mathsf{z}_t = \mathrm{tr}(X\mathsf{z}_t, Z_t^T)$ replaced with

$$E_t\left(\mathrm{tr}\left[X\left(z_t z_t^T + \sum_{i=1}^{\infty} \beta^t \mathsf{u}_{t+i}\mathsf{u}_{t+i}^T\right)\right]\right) = \mathrm{tr}\left[X\left(z_t^T z_t + \frac{\lambda}{1-\lambda}\Sigma\right)\right] \tag{A.35}$$

where E_t is the expectations operator with expectations formed at time t.

Thus for the optimal policy with commitment (A.19) becomes in the stochastic case

$$\Omega_t^{OP} = -\frac{1}{2}\mathrm{tr}\left(N_{11}\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right) + N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}^T\right) \tag{A.36}$$

For the time-consistent policy (A.26) becomes

$$\Omega_t^{TC} = -\frac{1}{2}\mathrm{tr}\left(S\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{A.37}$$

and for the simple rule, generalizing (A.32)

$$\Omega_t^{SIM} = -\frac{1}{2}\mathrm{tr}\left(V\left(Z_t + \frac{\beta}{1-\beta}\Sigma\right)\right) \tag{A.38}$$

The optimized simple rule is found at time $t = 0$ by minimizing Ω_0^{SIM} given by (A.38). Now we find that

$$D^* = D^*\left(\mathsf{z}_0\mathsf{z}_0^T + \frac{\beta}{1-\beta}\Sigma\right) \tag{A.39}$$

or, in other words, the optimized rule depends both on the initial displacement z_0 and on the covariance matrix of disturbances Σ.

B Dynamic Representation as Difference Equations

The linearizations in the main text, especailly that for the real wage equation, requires us to express the price and wage-setting first order conditions as stochastic non-linear difference equations.[19] To do this first define

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} = \pi_t + 1 \tag{B.1}$$
$$\Phi_t \equiv P_t^0/P_t \tag{B.2}$$
$$\tilde{\Pi}_t \equiv \frac{\Pi_t}{\Pi_{t-1}^{\gamma}} \tag{B.3}$$

and use $D_{t+k} = \beta^k \frac{MU_{t+1}^C}{P_{t+k}}$ where $MU_t^C = C_t^{-\sigma} H_{C,t}^{1-\sigma}$ is the marginal utility of consumption. Then we can write the first order condition for optimal price-setting, (51) as

$$\Phi_t \Xi = \Lambda_t \tag{B.4}$$

where new variables Ξ_t and Λ_t are defined by

$$\Xi_t - \xi\beta E_t[\tilde{\Pi}_{t+1}^{\zeta-1}\Xi_{t+1}] = Y_t MU_t^C \tag{B.5}$$
$$\Lambda_t - \xi\beta E_t[\tilde{\Pi}_{t+1}^{\zeta}\Lambda_{t+1}] = \frac{U_{L,t}(L_t/L_{t-1}^{h_L})^{1+\phi}}{(1-1/\zeta)(1-1/\eta)(1-T_t)} \tag{B.6}$$
$$\tag{B.7}$$

From our definitions (B.2) and (B.3), (52) can now be written as

$$1 = \xi_P \tilde{\Pi}_t^{\zeta-1} + (1-\xi_P)\Phi_t^{1-\zeta} \tag{B.8}$$

Five equations (B.2) to (B.8) in Π_t, Φ_t, $\tilde{\Pi}_t$, Ξ_t and Ω_t now provide the dynamics of optimal setting in a convenient form

[19] This is also necessary if one wants to set up and solve numerically in standard software the non-linear DSGE model.

Similarly we can carry out the same exercise for wage setting. We can now use $\beta^k \Lambda_{t+k}(r) = D_{t+k}$, obtained from (38), and $\Lambda_t(r) = \frac{MU_t^C(r)}{P_t}$, and from (37) we have

$$L_{t+k}(r) = L_{t+k} \left(\frac{W_t^0(r) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_W}}{W_{t+k}} \right)^{-\eta} \tag{B.9}$$

to write (54) as

$$\left(\frac{W_t^0}{P_t}\right)^{1+\eta\phi} E_t \sum_{k=0}^{\infty} (\xi_W \beta)^k (1 - T_{t+k}) L_{t+k} MUC_{t+k} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\eta} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_W(1-\eta)} \left(\frac{P_t}{P_{t+k}}\right)^{1-\eta}$$
$$= \frac{\eta}{(\eta-1)} E_t \sum_{k=0}^{\infty} (\xi_W \beta)^k \left(\frac{L_{t+k}}{L_{t+k-1}^{h_L}}\right)^{1+\phi} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\eta(1+\phi)} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{-\gamma_W \eta(1+\phi)} \left(\frac{P_{t+k}}{P_t}\right)^{\eta(1+\phi)} \tag{B.10}$$

We can now see that for labour supply habit in ratio form[20] we can proceed as for the price dynamics. The following difference equations corresponding to (B.4) to (B.8) now apply:

$$\left(\frac{W_t^0}{P_t}\right)^{1+\eta\phi} \Upsilon_t = \Gamma_t \tag{B.11}$$

$$\Upsilon_t - \xi_W \beta E_t[\tilde{\Pi}_{t+1}^{\eta-1} \Upsilon_{t+1}] = \left(\frac{W_t}{P_t}\right)^{\eta} (1 - T_t) L_t MU_t^C \tag{B.12}$$

$$\Gamma_t - \xi_W \beta E_t[\tilde{\Pi}_{t+1}^{\eta(1+\phi)} \Gamma_{t+1}] = \left(\frac{W_t}{P_t}\right)^{\eta(1+\phi)} \frac{U_{L,t}(L_t/L_{t-1}^{h_L})^{1+\phi}}{(1-1/\eta)} \tag{B.13}$$

$$\left(\frac{W_{t+1}}{P_{t+1}}\right)^{1-\eta} = \xi_W \left(\frac{W_t}{P_t}\right)^{1-\eta} \tilde{\Pi}_{t+1}^{\eta-1} + (1-\xi_W) \left(\frac{W_{t+1}^0}{P_{t+1}}\right)^{1-\eta} \tag{B.14}$$

C Welfare Quadratic Approximation for the Case of An Approximately Efficient Steady State

We make a departure in notation from what appears in most of the literature in this area, in order to avoid confusion over labour $N_t(r)$ supplied by an individual r and the index of differentiated labour $L_t(f)$ employed by firm f. Defining $N_t(f, r)$ as the labour supplied to firm f by individual r, we have

$$N_t(r) = \int N_t(r, f) df \qquad L_t(r)^{(\eta-1)/\eta} = \int N_t(r, f)^{(\eta-1)/\eta} dr \tag{C.1}$$

To clarify the exposition we first consider the case without capital.

[20] This is reason we choose the ratio form over the difference form.

C.1 Labour The Only Factor

Ignoring the welfare implications of monetary frictions, the consumer's utility is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t(i)/C_{t-1}^{h_C})^{1-\sigma}}{1-\sigma} - \kappa \frac{(N_t(i)/N_{t-1}^{h_L})^{1+\phi}}{1+\phi} \right] \tag{C.2}$$

Since we assume complete risk-sharing within each bloc, we may regard each consumer as being identical with every other. From the point of view of leisure, to obtain the social welfare function, we need to sum over all workers. Before doing this, we obtain the expected value of $N_t(r)^{1+\phi}$. We note that

$$\begin{aligned} N_t(r) &= \int N_t(r,f)df = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \int L_t(f)df \\ &= \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \int \frac{Y_t(f)}{A_t} df = \left(\frac{W_t(r)}{W_t}\right)^{-\eta} \frac{Y_t}{A_t} \int \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} df \end{aligned} \tag{C.3}$$

Assuming that $lnW_t(r) \sim N(\mu_t^W, D_t^W)$ and $lnP_t(r) \sim N(\mu_t^P, D_t^P)$, from subsection C.4 we have that

$$\int \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} df \simeq 1 + \frac{1}{2}\zeta D_t^P \qquad \int \left(\frac{W_t(r)}{W_t}\right)^{-\eta} dr \simeq 1 + \frac{1}{2}\eta D_t^W \tag{C.4}$$

and in addition

$$\int \left(\frac{W_t(r)}{W_t}\right)^{-\eta(1+\phi)} di \simeq 1 + \frac{1}{2}\eta(1+\phi)(1+\eta\phi)D_t^W \tag{C.5}$$

It follows from this that summing over all r, we obtain the social welfare loss approximately as

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg[& \frac{(C_t/C_{t-1}^{h_C})^{1-\sigma}}{1-\sigma} - \kappa \frac{(Y_t/Y_{t-1}^{h_L})^{1+\phi}(A_t/A_{t-1}^{h_L})^{-(1+\phi)}}{1+\phi} \Bigg(1 + \\ & \frac{1}{2}(1+\phi)(\zeta D_t^P + \eta(1+\eta\phi)D_t^W - h_L(\zeta D_{t-1}^P + \eta(1+\eta\phi)D_{t-1}^W)) \Bigg) \Bigg] \end{aligned} \tag{C.6}$$

where

$$D_t^P = \xi_p D_{t-1}^P + \frac{\xi_p}{1-\xi_p}(\pi_t - \gamma_p \pi_{t-1})^2 \tag{C.7}$$

and

$$D_t^W = \xi_w D_{t-1}^W + \frac{\xi_w}{1-\xi_w}(\Delta w_t - \gamma_w \Delta w_{t-1})^2 = \xi_w D_{t-1}^W + \frac{\xi_w}{1-\xi_w}(\Delta wr_t + \pi_t - \gamma_w(\Delta wr_{t-1} + \pi_{t-1}))^2 \tag{C.8}$$

where for convenience we have written the *log* of the real wage relative to domestic producer prices $wr_t = w_t - p_t$, and π_t is the inflation rate for domestic producer prices.

We note that that the terms D_t^P, D_t^W occur in the utility terms at both t and $t+1$. From this it follows that the quadratic approximation to the welfare can be approximately written as

$$\begin{aligned}
& -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t[\sigma C^{(1-\sigma)(1-h_C)}(c_t-h_Cc_{t-1})^2+\kappa\left(\frac{Y}{A}\right)^{(1+\phi)(1-h_L)}\Bigg(\\
& \phi(y_t-a_t-h_L(y_{t-1}-a_{t-1}))^2+\frac{\zeta(1-\beta h_L)\xi_p}{(1-\beta\xi_p)(1-\xi_p)}(\pi_t-\gamma_p\pi_{t-1})^2 \\
+ \quad & \frac{\eta(1-\beta h_L)(1+\eta\phi)\xi_w}{(1-\beta\xi_w)(1-\xi_w)}(\Delta wr_t+\pi_t-\gamma_w(\Delta wr_{t-1}+\pi_{t-1}))^2\Bigg)]
\end{aligned} \tag{C.9}$$

C.2 Labour, Capital and Fixed costs F

With capital and fixed costs, the previous analysis changes to

$$Y_t(f)=A_tZ_t^{\alpha}K_{t-1}^{\alpha}L_t(f)^{1-\alpha}-F \qquad K_{t-1}(f)=\frac{1-\alpha}{\alpha}\frac{W_tL_t(f)}{P_tR_{K,t}} \qquad R_{K,t}=\Psi'(Z_t) \tag{C.10}$$

Hence we can write output as

$$Y_t(f)=\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}L_t(f)\left(\frac{W_t}{P_t}\right)^{\alpha}A_tZ_t^{\alpha}R_{K,t}^{-\alpha}-F \tag{C.11}$$

Hence we can calculate

$$\begin{aligned}
N_t(i) &= \int N_t(i,f)df=\left(\frac{W_t(i)}{W_t}\right)^{-\eta}\int L_t(f)df \\
&= \left(\frac{W_t(i)}{W_t}\right)^{-\eta}\int\frac{Y_t(f)+F}{A_tZ_t^{\alpha}}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{P_t}{W_t}\right)^{\alpha}R_{K,t}^{\alpha}df \\
&= \left(\frac{W_t(i)}{W_t}\right)^{-\eta}\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{P_t}{W_t}\right)^{\alpha}\frac{R_{K,t}^{\alpha}}{A_tZ_t^{\alpha}}(F+Y_t\int\left(\frac{P_t(f)}{P_t}\right)^{-\zeta}df)
\end{aligned} \tag{C.12}$$

and after defining $\frac{1}{B_t}=\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\frac{R_{K,t}^{\alpha}}{A_tZ_t^{\alpha}}$ we deduce from this that

$$N_t=\int N_t(r)dr=\left(\frac{P_t}{W_t}\right)^{\alpha}\left(\frac{Y_t}{B_t}(1+\frac{1}{2}(\eta D_t^W+\zeta D_t^P))+\frac{F}{B_t}(1+\frac{1}{2}\eta D_t^W)\right) \tag{C.13}$$

We also infer that when we sum over all individuals, we obtain

$$\begin{aligned}
\int N_t(i)^{1+\phi}di &= \left(\frac{P_t}{W_t}\right)^{\alpha(1+\phi)}\left(\frac{1}{B_t}\right)^{1+\phi}\left(F+Y_t(1+\frac{1}{2}\zeta D_t^P)\right)^{1+\phi}(1+\frac{1}{2}\eta(1+\eta\phi)(1+\phi)D_t^W) \\
&\cong \left(\frac{P_t}{W_t}\right)^{\alpha(1+\phi)}\left(\frac{F+Y_t}{B_t}\right)^{1+\phi}\left(1+\frac{Y_t}{F+Y_t}\frac{1}{2}\zeta(1+\phi)D_t^P\right)(1+\frac{1}{2}\eta(1+\eta\phi)(1+\phi)D_t^W)
\end{aligned} \tag{C.14}$$

Noting that

$$\left(\frac{P_t}{W_t}\right)^{\alpha}\left(\frac{F+Y_t}{B_t}\right)=\left(\frac{P_tK_{t-1}R_{K,t}}{W_tL_t}\right)^{\alpha}L_t\left(\frac{1-\alpha}{\alpha}\right)^{\alpha}=L_t \tag{C.15}$$

substituting for B_t and Y_t from (C.10), and then using the second second minimum cost condition in the same equation. It follows that the second-order terms in the Taylor-series approximation of the welfare loss is given by

$$
\begin{aligned}
& -\frac{1}{2}C^{(1-h_C)(1-\sigma)}E_0\sum_{t=0}^{\infty}\beta^t[\sigma C^{(1-\sigma)(1-h_C)}(c_t-h_Cc_{t-1})^2+\kappa L^{(1+\phi)(1-h_L)}\Bigg(\\
& \phi(l_t-h_Ll_{t-1})^2+\frac{Y}{F+Y}\frac{\zeta(1-\beta h_L)\xi_p}{(1-\beta\xi_p)(1-\xi_p)}(\pi_t-\gamma_p\pi_{t-1})^2 \\
+ \quad & \frac{\eta(1-\beta h_L)(1+\eta\phi)\xi_w}{(1-\beta\xi_w)(1-\xi_w)}(\Delta wr_t+\pi_t-\gamma_w(\Delta wr_{t-1}+\pi_{t-1}))^2\Bigg)]
\end{aligned} \tag{C.16}
$$

We can simplify this and eliminate κ using the steady state result $\frac{W(1-T)}{P}=\frac{\kappa}{1-\frac{1}{\eta}}L^{\phi-h_L(1+\phi)}C^{\sigma+h_c(1-\sigma)}$. Then writing

$$
\frac{\sigma C^{1-\sigma)(1-h_C)}}{\kappa L^{(1+\phi)(1-h_L)}}=\frac{\sigma PC}{(1-\frac{1}{\eta})WL(1-T)}=\frac{\sigma c_y}{(1-\frac{1}{\eta})(1-\alpha)} \tag{C.17}
$$

Thus we arrive a welfare loss of

$$
\begin{aligned}
& -\frac{1}{2}C^{(1-h_C)(1-\sigma)}E_0\sum_{t=0}^{\infty}\beta^t\Bigg[\quad\underbrace{\sigma c_y(c_t-h_Cc_{t-1})^2}_{\text{consumption variability}} \\
+ \quad & (1-\alpha)(1-\frac{1}{\eta})\Bigg(\quad\underbrace{\phi(l_t-h_Ll_{t-1})^2}_{\text{employment variability}}\quad+\underbrace{\frac{Y}{(F+Y)}\frac{\zeta(1-\beta h_L)\xi_p}{(1-\beta\xi_p)(1-\xi_p)}(\pi_t-\gamma_p\pi_{t-1})^2}_{\text{inflation variability}} \\
+ \quad & \underbrace{\frac{\eta(1-\beta h_L)(1+\eta\phi)\xi_w}{(1-\beta\xi_w)(1-\xi_w)}(\Delta w_t-\gamma_w\Delta w_{t-1})^2}_{\text{nominal wage variability}}\Bigg)\Bigg]
\end{aligned} \tag{C.18}
$$

noting that $\Delta wr_t+\pi_t=\Delta w_t$ is wage inflation.

However there are also some second-order terms in the quadratic approximation to the welfare that have so far been omitted. On cannot assume that the first-order deviations from the welfare sum to 0 even if the expansion takes place around the efficient steady state (i.e. even if we assume that the NK equilibrium is at the efficient equilibrium). We need to ensure that the first order deviations are evaluated about the resource constraints, so that the first order effects of these can be regarded as 0. The natural variables about which to expand to first order are L and K (and by implication investment I). Onatski and Williams use Y and K, but it is much more straightforward to stick with L and K. First note that we can write

$$
C_t=Y_t-G_t-I_t-\Psi(Z_t)K_{t-1} \tag{C.19}
$$

Now recall that $Y_t(f)=A_tZ_t^{\alpha}L_t(f)(K_{t-1}(f)/L_t(f))^{\alpha}-F$. But as we have seen earlier, $K_{t-1}(f)/L_t(f)$ is the same for all firms, and is therefore equal to the ratio of total capital to total output K_{t-1}/L_t. Summing over all firms then yields $Y_t = A_t Z_t^\alpha L_t (K_{t-1}/L_t)^\alpha - F$, so we can rewrite (C.19) as

$$C_t = A_t Z_t^\alpha L_t^{1-\alpha} K_{t-1}^\alpha - F - G_t - I_t - \Psi(Z_t) K_{t-1} \tag{C.20}$$

It follows that the contribution of second order terms to first-order terms from the consumption part of the utility $C^{(1-\sigma)(1-h_C)} c_t$ are given by (excluding t.i.p. terms)

$$\begin{aligned} - & \frac{1}{2} C^{(1-h_C)(1-\sigma)} E_0 \sum_{t=0}^{\infty} \beta^t c_y \Bigg[\frac{(Y+F)}{C} \Bigg(\alpha(1-\alpha)(l_t - k_{t-1})^2 - 2[(1-\alpha) l_t + \alpha k_{t-1}](a_t + \alpha z_t) \\ + & \alpha(1-\alpha) z_t^2 \Bigg) + \frac{K}{C} (\Psi''(1) z_t^2 + 2\Psi'(1) z_t k_{t-1}) \Bigg] \end{aligned} \tag{C.21}$$

Using the definition $\psi = \Psi'(1)/\Psi''(1)$ and the deterministic condition $\Psi'(1) = R_K$, and taking into account the habit effect of C_t at time $t+1$, the total contribution to loss function in the square brackets may be written as

$$\begin{aligned} & c_y(1-\beta h_C) \Bigg[\frac{(Y+F)}{C} \Bigg(\alpha(1-\alpha)(l_t - k_{t-1})^2 - 2[(1-\alpha) l_t + \alpha k_{t-1}](a_t + \alpha z_t) \\ + & \alpha(1-\alpha) z_t^2 \Bigg) + \frac{K}{C} R_K (\frac{1}{\psi} z_t^2 + 2 z_t k_{t-1}) \Bigg] \end{aligned} \tag{C.22}$$

Putting $\frac{Y+F}{C} = \frac{\phi_F}{c_y}$ and $\frac{K}{C} = \frac{i_y}{\delta c_y}$ this can be written

$$\begin{aligned} & (1-\beta h_C) \Bigg[\phi_F \alpha(1-\alpha)(l_t - k_{t-1})^2 + \left(\phi_F \alpha(1-\alpha) + \frac{i_y R_K}{\delta \psi} \right) z_t^2 \\ & +2 \left(\frac{i_y R_K}{\delta} - \phi_F \alpha^2 \right) k_{t-1} z_t \\ & -2\phi_F \left((1-\alpha) l_t a_t + \alpha(1-\alpha) l_t z_t + \alpha k_{t-1} a_t \right) \Bigg] \\ & = w_{k-l}(k_{k-1} - l_t)^2 + w_z z_t^2 + 2 w_{kz} k_{t-1} z_t \\ & +2 w_{la} l_t a_t + 2 w_{lz} l_t z_t + 2 w_{ka} k_{t-1} a_t \end{aligned} \tag{C.23}$$

Note that this can be rewritten as:

$$\begin{aligned} & (1-\beta h_C) c_y \Bigg[\frac{(Y+F)}{C} \alpha(1-\alpha)(l_t - k_{t-1} - z_t - \frac{1}{\alpha} a_t)^2 \\ + & \frac{R_K K}{\psi C} \left(z_t + \psi(1 - \frac{\alpha(Y+F)}{R_K K}) k_{t-1} + \frac{\psi(1-\alpha)(Y+F)}{R_K K} a_t) \right)^2 \\ - & \frac{\psi R_K K}{C} \left(1 - \frac{\alpha(Y+F)}{R_K K} \right)^2 k_{t-1}^2 \\ - & 2 \frac{Y+F}{C} \left(1 + (1-\alpha)\psi(1 - \frac{\alpha(Y+F)}{R_K K}) \right) a_t k_{t-1} \Bigg] \end{aligned} \tag{C.24}$$

Since in the steady state of the social optimum $\alpha(Y+F) = R_K K$ this expression simplifies considerably and we end up with (120). A number of points are worthy of note:

1. Combining (C.18) and (C.24) owing to the last in (C.24) our quadratic approximation to the utility is not quite negative definite. However without capital this terms disappears, so in that case the welfare function is positive definite.

2. There are of course no second order contributions from first order changes in L_t.

3. When there is no capital stock, habit, wage-stickiness and government spending we have that $c_t = y_t = l_t + a_t$. Then putting $K = F = h_C = h_L = \eta = \alpha = 0$ the loss function reduces to the familiar results from Woodford (2003):

$$\Omega_0 = E_0 \left[\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[(y_t - \hat{y}_t)^2 + w_\pi (\pi_t - \gamma_P \pi_{t-1})^2 \right] \right] \tag{C.25}$$

where $\hat{y}_t = \frac{1+\phi}{\sigma+\phi} a_t$ is potential output achieved when prices are flexible and

$$w_\pi = \frac{\zeta \xi}{(1-\xi)(1-\beta\xi)(\sigma+\phi)} \tag{C.26}$$

4. Onatski and Williams (2004) express I_t in terms of K_t, K_{t-1}, I_{t-1} but there is no need to do this.

5. To work out the welfare in terms of a consumption equivalent percentage increase, expanding $U(C) = \frac{C^{(1-h_C)(1-\sigma)}}{1-\sigma}$ as a Taylor series, a 1% permanent increase in consumption yields a first-order welfare increase $(1-h_C)C^{(1-h_C)(1-\sigma)-1}\Delta C = (1-h_C)C^{(1-h_C)(1-\sigma)} \times 0.01$. Since standard deviations are expressed in terms of percentages, the welfare loss terms which are proportional to the covariance matrix (and pre-multiplied by 1/2) are of order 10^{-4}. Letting X be these losses reported in the paper. Then $c_e = \frac{X}{(1-h_C)} \times 0.01$ as given in (121). The expressions (122) and (123) are derived using only the quadratic terms.

C.3 Derivation of (C.4) and (C.5)

It is convenient though not essential to assume a normal distribution with $lnW_t(r) \sim N(\mu, \sigma^2)$. By definition,

$$W_t^{1-\eta} = \int W_t(r)^{1-\eta} dr = exp((1-\eta)\mu + (1-\eta)^2 \frac{1}{2}\sigma^2) \tag{C.27}$$

Hence

$$W_t = exp(\mu + (1-\eta)\frac{1}{2}\sigma^2) \tag{C.28}$$

Thus it follows that

$$\int W_t(r)^{-\eta} di = exp(-\eta\mu + \eta^2 \frac{1}{2}\sigma^2) \qquad W_t^{-\eta} = exp(-\eta\mu - \eta(1-\eta)\frac{1}{2}\sigma^2) \tag{C.29}$$

from which we obtain (C.4). Similarly

$$\int W_t(r)^{-\eta(1+\phi)} dr = exp(-\eta(1+\phi)\mu + \eta^2(1+\phi)^2 \frac{1}{2}\sigma^2) \tag{C.30}$$

$$W_t^{-\eta(1+\phi)} = exp(-\eta(1+\phi)\mu - \eta(1+\phi)(1-\eta)\frac{1}{2}\sigma^2) \tag{C.31}$$

and hence (C.5).

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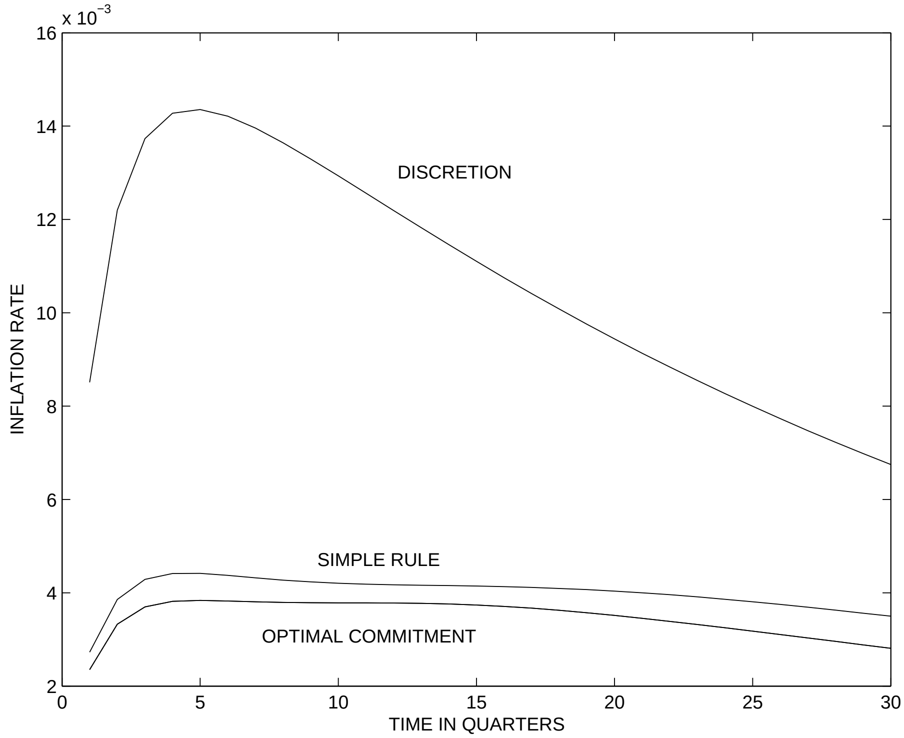

Figure 6: Price Inflation Rate Following a 1% Government Spending Shock

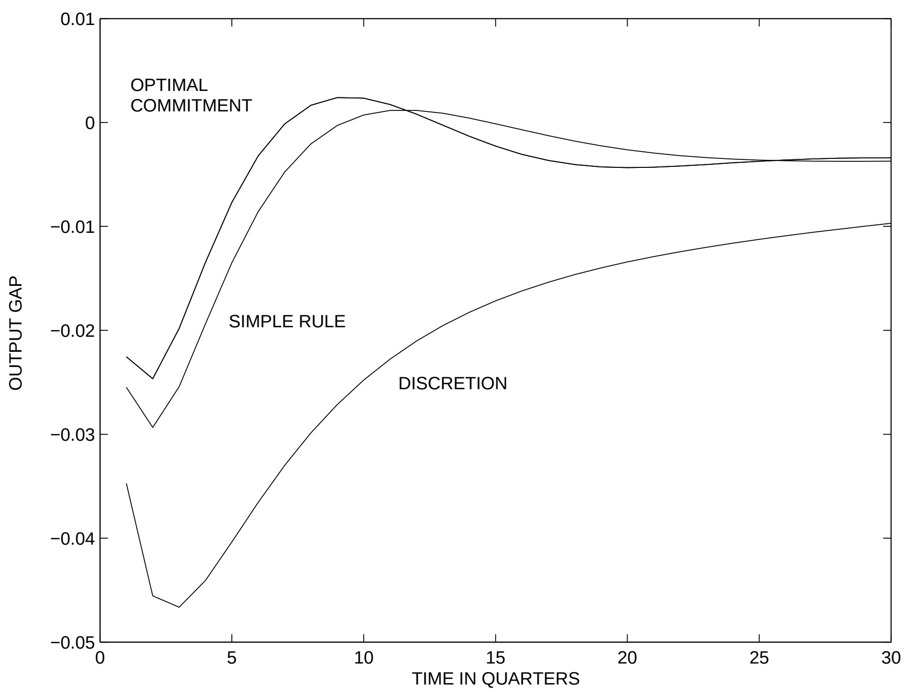

Figure 7: Output Gap Following a 1% Government Spending Shock

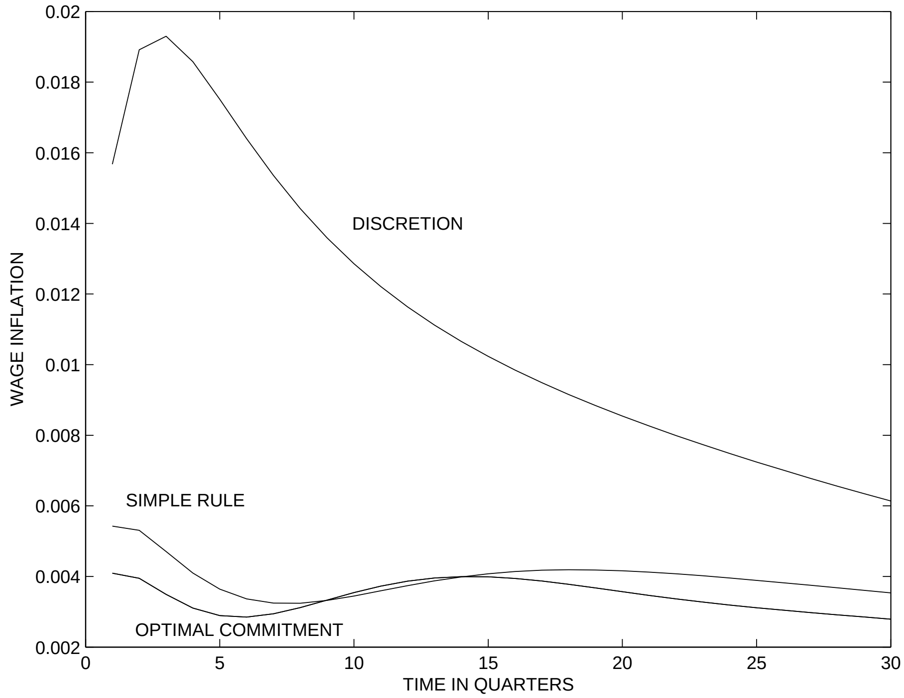

Figure 8: Wage Inflation Rate Following a 1% Government Spending Shock

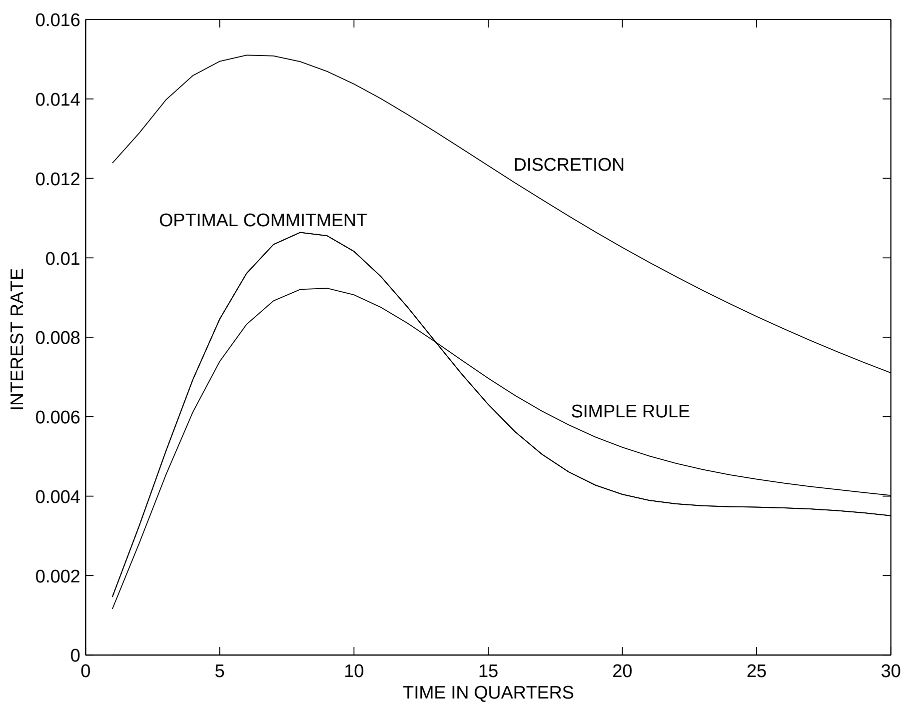

Figure 9: Interest Rate Following a 1% Government Spending Shock

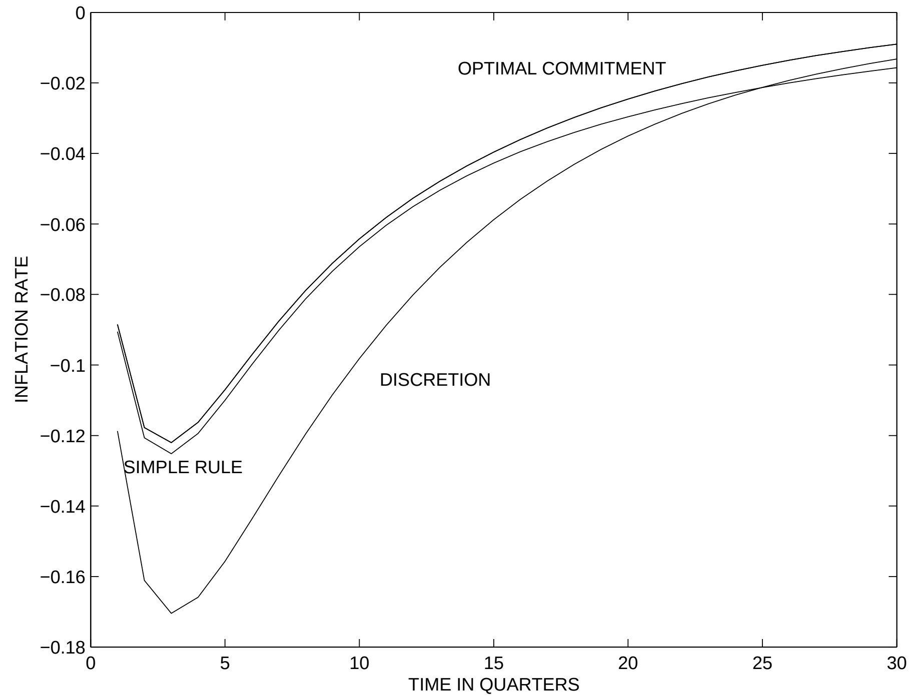

Figure 10: Price Inflation Rate Following a 1% Technology Shock

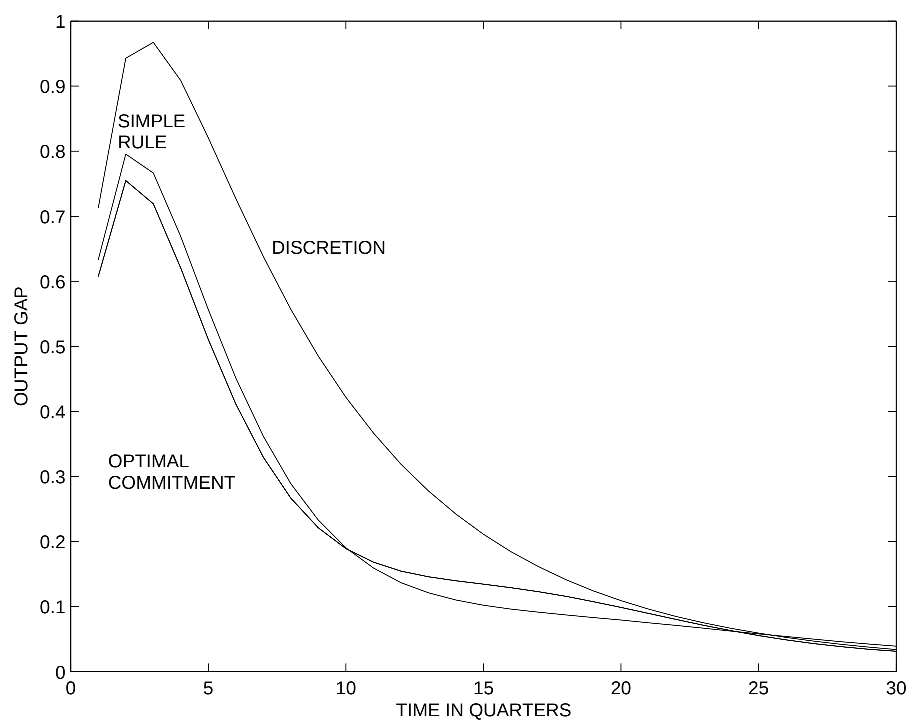

Figure 11: Output Gap Following a 1% Technology Shock

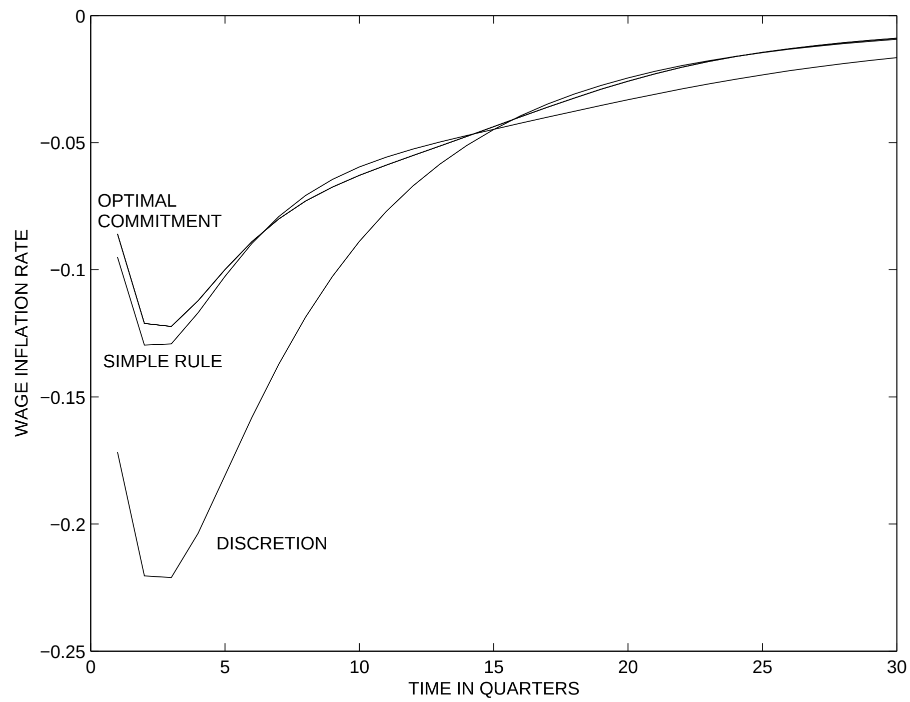

Figure 12: Wage Inflation Rate Following a 1% Technology Shock

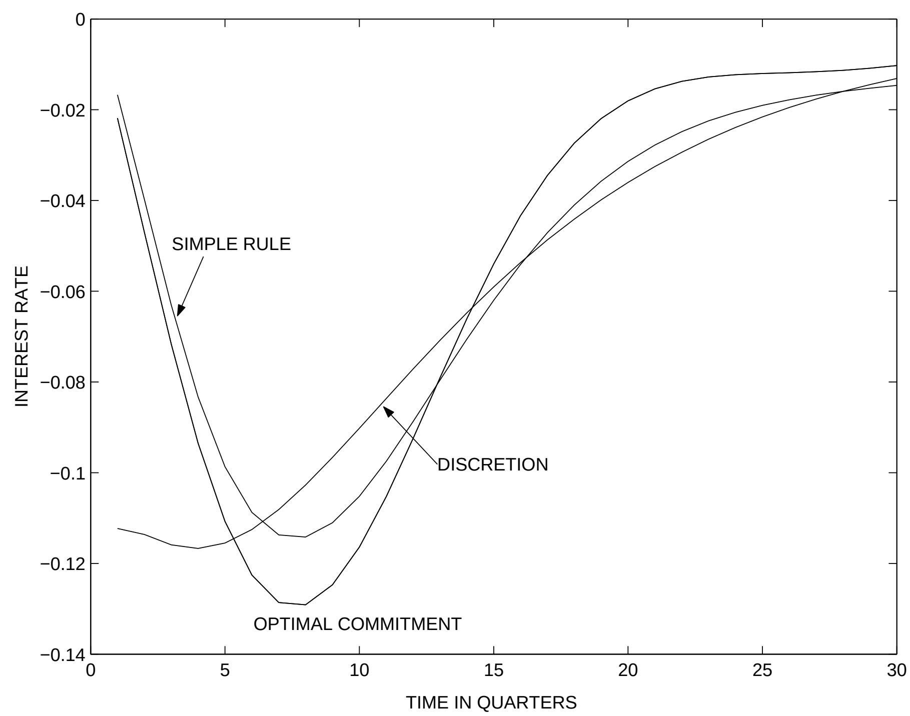

Figure 13: Interest Rate Following a 1% Technology Shock